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Generic Models of the Theory of Normal Z-rings

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Abstract A normal **Z**-ring M is a discretely ordered ring, integrally closed in its fraction field and such that for each positive integer n, $M/nM \simeq \mathbf{Z}/n\mathbf{Z}$ as rings. Here we study some properties of finite generic normal **Z**-rings. We give a uniform universal definition of **N** in them. And we separate existentially closed normal **Z**-rings via generics.

1 Introduction and preliminaries

Let $\mathcal{L}$ denote the first order language of ordered rings based on the symbols 0, 1, +, −, ·, <. The theory of normal **Z**-rings (NZR) consists of the following axioms:

(i) OR: the theory of ordered rings;
(ii) D: $\forall x \neg (0 < x < 1)$ (the discreteness of the order);
(iii) N: for each $n \in \mathbf{N}$

$$\forall z_1, \ldots, z_n xy (x, y \neq 0 \wedge x^n + z_1 x^{n-1} y + \cdots + z_n y^n = 0 \rightarrow \exists w (x = wy))$$

and the **Z**-ring axioms:

(iv) Z: for each $n \in \mathbf{N}$, $n \neq 0$ $\forall x \forall yz (x = ny + z \wedge 0 \leq z < n)$.

The theory of normal **Z**-rings plays a relevant role in the study of the fragment of arithmetic Normal Open Induction (NOI). NOI is the $\forall\exists$-theory in the language $\mathcal{L}$ which consists of NZR together with

$$\forall \mathbf{x}((\theta(\mathbf{x},0) \wedge \forall y \geq 0 (\theta(\mathbf{x},y) \rightarrow \theta(\mathbf{x},y+1)) \rightarrow \forall y \geq 0 \theta(\mathbf{x},y))$$

for every quantifier-free $\mathcal{L}$-formula $\theta(\mathbf{x},y)$ ($\mathbf{x}$ denotes an n-tuple $(x_1, \ldots, x_n)$).

In [7] Shepherdson gave the following useful characterization of models of NOI:

> Let M be a normal discretely ordered ring. Then M is a model of NOI if and only if for every element α of the real closure of the fraction field of M there is an element a in M such that $|a - \alpha| < 1$.

From this several corollaries are deduced. Let us mention some of them. Let M be a model of NOI. Then every quantifier-free definable set in M is a finite

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union of intervals and points; for every x and $y \neq 0$ in M there are q and r in M such that $x = qy + r$ and $0 \leq r < y$; and the fraction field of M, $\mathrm{F}(M)$ is dense in its real closure $\mathrm{RC}(M)$.

Using Shepherdson's characterization Wilkie proved in [8] that every model of NZR can be extended to a model of NOI. From this and the well-known characterization of substructures of **Z**-rings we get that *if M is a normal discretely ordered ring for which there is defined a ring homomorphism $\varphi : M \to \hat{\mathbf{Z}}$ (where $\hat{\mathbf{Z}}$ is the product of p-adic integers $\prod_p \mathbf{Z}_p$), then M can be embedded in a model of NOI.*

One of the main problems in NOI is to know which Diophantine equations have solutions in a model of NOI. Wilkie's result tells us it suffices to study the problem for NZR. Another property of NOI (and also of NZR) which makes the study of this problem easier is that normal **Z**-rings have the joint embedding property (JEP).

The aim of this paper is on one hand to study some properties of generic models (see definition below) of NZR. The class of generics is a subclass of the existentially closed models of NZR. Asking whether a Diophantine equation is consistent with an $\mathcal{L}$-theory T, that is, if it has a solution in a model of T or not, it suffices to consider existentially closed models of T because if there is a model solving a given equation, then (and only then) there is an existentially closed model also solving it. JEP in NOI gives us that in this case the given equation has a solution in every existentially closed model.

On the other hand we shall see that there is not a unique theory of existentially closed models via the generics.

Notation If M is a domain, $\mathrm{F}(M)$ denotes its fraction field, and if M is an ordered domain, $\mathrm{RC}(M)$ denotes the real closure of $\mathrm{F}(M)$.

A boldface letter such as $\mathbf{x}$ denotes an n-tuple $(x_1, \ldots, x_n)$ where n should be clear from the context or arbitrary. And in this case, if M is any set $\mathbf{x} \in M$ denotes $\mathbf{x} \in M^n$.

2 Existentially closed models of NZR

Definitions Let T be a theory, Σ_T denotes *the class of substructures of models of T.* An element A of Σ_T is said to be an *existentially closed structure of T* if for any $B \in \Sigma_T$ extending A, any existential formula $\varphi(\mathbf{x})$ of the language of T, and any $\mathbf{a} \in A$, $B \models \varphi(\mathbf{a})$ implies $A \models \varphi(\mathbf{a})$. E_T denotes *the class of existentially closed structures for T.*

Note first that Σ_{NZR} has the following explicit universal axiomatization:

Let M^ν denote the normalization of M, that is, the elements of $\mathrm{F}(M)$ which are roots of a monic polynomial with coefficients in M.

Then $M \in \Sigma_{\mathrm{NZR}}$ is equivalent to $M^\nu \models \mathrm{NDOR}$ and there is a ring homomorphism $\varphi : M^\nu \to \hat{\mathbf{Z}}$.

Now we see that the last assertion is universally axiomatizable. First, for each $l \geq 1$ let ψ_l express that every positive element of $\mathrm{F}(M)$ which satisfies a monic polynomial over M of degree l is ≥ 1, i.e.,

$$\psi_l = \forall x_1 \ldots x_l uv(u, v > 0 \wedge u^l + x_1 u^{l-1} v + \cdots + x_l v^l = 0 \to v \leq u).$$

Then, clearly $M \models \mathrm{OR} + \{\psi_l\}_{l\geq 1}$ if and only if $M^\nu \models \mathrm{NDOR}$. Making use of the usual characterization of the above ring homomorphism we get the following:

First, let $\nu_{m,l}(\mathbf{u}, \mathbf{v})$ (where $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_m)$) express that each $(u_i/v_i)(1 \leq i \leq m)$ satisfies a monic polynomial of degree $\leq l$, i.e.,

$$\nu_{m,l}(\mathbf{u}, \mathbf{v}) = \bigvee_{1\leq l_1,\ldots,l_m\leq l} \left[\bigwedge_{i=1}^{m} (v_i \neq 0 \wedge \exists x_1^i \ldots x_{l_i}^i) \times \left(\left(\frac{u_i}{v_i}\right)^{l_i} + x_1^i \left(\frac{u_i}{v_i}\right)^{l_i - 1} + \cdots + x_{l_i}^i = 0 \right) \right].$$

Next we want to express that for a fixed $n \geq 2$ and each $m \geq 1$ if $(u_1/v_1), \ldots, (u_m/v_m) \in M^\nu$ then there is a map from $\{u_1/v_1, \ldots, u_m/v_m\}$ to $\mathbf{Z}/n\mathbf{Z}$ preserving $+$ and $\cdot$, and then by a compactness argument we get a ring homomorphism $\varphi : M^\nu \to \hat{\mathbf{Z}}$. So let $T_m^n(u_1, \ldots, u_m, v_1, \ldots, v_m)$ express the existence of such a map and $S_{m,l}^n(\mathbf{u}, \mathbf{v}) = \nu_{m,l}(\mathbf{u}, \mathbf{v}) \to T_m^n(\mathbf{u}, \mathbf{v})$.

Therefore $M \in \Sigma_{\mathrm{NZR}}$ if and only if

$$M \models \{\psi_m : m \geq 1\} \cup \mathrm{OR} \cup \{S_{m,l}^n : m \geq 1,\ l \geq 1,\ n \geq 2\}.$$

Note also that because of Wilkie's result above and the fact that NOI is an $\forall\exists$-theory, existentially closed normal **Z**-rings are (existentially closed) models of NOI.

Let us mention here two results which we shall apply to existentially closed models of NZR.

In their paper [3] on the behavior of primes in models of NOI, Macintyre and Marker prove that the concepts of irreducible, prime, and maximal element do not coincide in models of NOI. Recall that an element q of a domain M is said to be irreducible (respectively prime or maximal) if a has no proper factorization (M/qM is respectively a domain or a field). On this line they prove the following.

Theorem 2.1 (Macintyre–Marker) *Let M be a normal* **Z***-ring. Then M can be extended to a normal* **Z***-ring in which the only irreducible elements are the standard ones.*

They also prove that there are models of NOI which do not satisfy the theorem of Lagrange which says that every nonnegative element can be represented as a sum of four squares. However we have the following (see Otero [5]).

Theorem 2.2 *Every normal* **Z***-ring can be embedded in a normal* **Z***-ring which satisfies Lagrange's theorem.*

This last result is based on the following lemma which we shall also use later on.

Lemma 2.3 *Let M be a discretely ordered ring. Let $a > 0$ be a nonstandard element of M. Then the ring $M[x, y]$ with x transcendental over the fraction field of M and $x^2 + y^2 = a$ can be discretely ordered with an order extending that of M.*

Returning to existentially closed models we have the following.

Corollary 2.4 *Let M be an existentially closed model of* NZR. *Then*
(i) *the only irreducible elements of M are the standard ones, and*
(ii) *M is uniquely orderable.*

Proof:
(i) Suppose q is an infinite prime in M. Then by Theorem 2.1 there is a model M' of NZR extending M such that $M' \models \exists x, y\ (1 < x, y < q \wedge xy = q)$. Now q is in M and the above sentence is existential, therefore must be true in M.

(ii) By Theorem 2.2 those elements which are a sum of (four) squares are the nonnegative part of any order.

Remarks 1. An ordered field F has the Hilbert property if any rational function f with coefficients from F and nonnegative on F is a sum of squares of rational functions with coefficients on F. By a result of McKenna (see [4]) we know that an ordered field has the Hilbert property if and only if it is uniquely orderable and dense in its real closure. Therefore (ii) of the corollary gives us that the fraction field of any existentially closed model of NZR has the Hilbert property.

2. The above (i) gives us a uniform definition of $\mathbf{N}$ in E_{NZR}: for, if $M \in E_{\mathrm{NZR}}$ a nonegative element of M is standard if it is bounded by an irreducible element, i.e., for every a in M

$$a \in \mathbf{N} \Leftrightarrow \exists x \forall z, y (a \geq 0 \wedge x > 1 \wedge (1 < z, y < x \rightarrow zy \neq x \wedge a < x)).$$

This is a $\exists\forall$ (existential bounded universal) definition; modifying Lemma 2.3 we can improve this to get a (bounded) universal formula uniformly defining $\mathbf{N}$ in existentially closed models of NZR. (This is clearly the lowest possible complexity.)

Proposition 2.5 *Let $M \in E_{\mathrm{NZR}}$, then $\mathbf{N}$ is definable in M by a bounded universal formula without parameters.*

Proof: It suffices to prove that, given $M \in E_{\mathrm{NZR}}$, there is $M' \models$ NOI, extending M and a bounded universal parameter free formula $\theta(u)$ such that for all a in M, $a \in \mathbf{N}$ if and only if $M' \models \theta(a)$; for $M \in E_{\mathrm{NZR}}$ and $\theta(u)$ universal implies for all a in M, $M \models \theta(a)$ if and only if $M' \models \theta(a)$.

To get this it suffices to prove that given $M \models$ NOI, $a \in M$ with $a > n$ for all $n \in \mathbf{N}$, there is $M'' \models$ NZR extending M such that

$$M'' \models \exists x, y, z \leq a (x^2 + y^2 = 3z^2 \wedge x, y, z \neq 0).$$

Then by the usual union of chains of argument we get M' as above with

$$\theta(u) = \forall xyz \leq u (u \geq 0 \wedge (x, y, z > 0 \rightarrow x^2 + y^2 \neq 3z^2)).$$

For $x^2 + y^2 = 3z^2$ clearly has no nontrivial solution in $\mathbf{N}$.

So, fix $M \models$ NOI, $a \in M$ with $a > n$ for all $n \in \mathbf{N}$. First we extend M to $M[z]$ with z transcendental over M and the order given by $(a/2) - b < z < (a/2) - n$ for all $n \in \mathbf{N}$, and $b \in M$ with $b > m$ for all $m \in \mathbf{N}$.

Then $M[z] \models$ DOR for, if $0 < f(z) < 1$ for some $f(X) \in M[X]$, then by quantifier elimination for RCF, there are $r_1, r_2 \in \mathrm{RC}(M)$, $r_1, r_2 > 0$, $r_1 < n$ for some $n \in \mathbf{N}$ and $r_2 > n$ for all $n \in \mathbf{N}$ such that $\forall t \in [(a/2) - r_1, (a/2) - r_2]$ $0 < f(t) < 1$. Let now $m \in \mathbf{N}$ and $b \in M$ with $(a/2) - r_1 < b < (a/2) - m$ (given by Shepherdson characterization) and get a contradiction.

Now recall that given a discretely ordered ring M and a ring homomorphism $\varphi : M \to \hat{\mathbf{Z}}$, the ring

$$M_\varphi = \left\{ \frac{a}{n} : a \in M,\ n \in \mathbf{N}\ n \neq 0 \text{ and } n \mid \varphi(a) \textit{ in } \hat{\mathbf{Z}} \right\}$$

is a model of NZR extending M.

Recall also that *if M and φ are as above and furthermore the localization of M at $\mathbf{Z}$, $\mathbf{Z}^{-1}M$ is normal (integrally closed in its fraction field), then M_φ is also normal.*

To finish the proof we need the following further result. *Let R be a normal domain. Let a be a nonzero element of R and m a nonzero integer. Suppose* 2 *and m are units in R. Then $R[x,(a + mx^2)^{1/2}]$ is also normal.* (See chapter 2 in Otero [6] for proofs of these two last results.)

We go back to our proof. Since M is a Z-ring there is $\varphi : M \to \hat{\mathbf{Z}}$ attached to it. We extend φ by sending z to 0 in $\hat{\mathbf{Z}}$. Now consider the normal $\mathbf{Z}$-ring $M_0 = M[z]\varphi$ and the positive infinite element $3z^2$ of it. By Lemma 2.3 $M_1 = M_0[x,(3z^2 - x^2)^{1/2}]$ can be discretely ordered. By the results above we get first $\mathbf{Z}^{-1}M_1$ normal and hence $(M_1)\varphi$ a normal $\mathbf{Z}$-ring, where φ is the extension of the homomorphism from M_1 to $\hat{\mathbf{Z}}$ obtained by sending x to 0 in $\hat{\mathbf{Z}}$.

Finally note that since $z < (a/2)$ hence $3z^2 < 3a^2/4 < a^2$ so $x^2 + y^2 < a^2$, we can indeed take $x, y, z \geq 0$, therefore $x, y, z < a$; also $z \neq 0$ by construction, x transcendental over $\mathrm{F}(M)(z)$ hence $x, y \neq 0$.

Note that we can interpret $\mathrm{Th}(\mathbf{N})$ in every $M \in E_{\mathrm{NZR}}$ (as in any class of models where $\mathbf{N}$ is uniformly definable): for any (first order) sentence φ we have

$$\mathbf{N} \models \varphi \Leftrightarrow M \models \varphi^\theta$$

where θ is any formula uniformly defining $\mathbf{N}$ in E_{NZR}, and φ^θ relativizes φ to θ.

Next we are going to apply the fact we have JEP in NZR. We begin with an obvious corollary of JEP, which is actually equivalent to it.

Corollary 2.6 *Let M be an existentially closed normal $\mathbf{Z}$-ring. Then M solves all the Diophantine equations consistent with* NZR.

Proof: Indeed, for every $f(\mathbf{x}) \in \mathbf{Z}[\mathbf{x}]$ and every $M \in E_{\mathrm{NZR}}$

$$f(\mathbf{x}) = 0 \text{ is consistent with NOI} \Leftrightarrow M \models \exists \mathbf{x} f(\mathbf{x}) = 0.$$

Fix such f and M. The *if* part is clear. For the *only if* get $M' \models \mathrm{NOI}$ such that $M' \models \exists \mathbf{x} f(\mathbf{x}) = 0$, jointly embed M and M' in $M'' \models \mathrm{NOI}$ then $M \models \exists \mathbf{x} f(\mathbf{x}) = 0$, for $M \in E_{\mathrm{NZR}}$.

The corollary above gives us that all the existentially closed models satisfy the same existential sentence. This implies (in general for $\forall\exists$-theories), that if $M_1, M_2 \in E_{\mathrm{NZR}}$ then $M_1 \equiv_{\forall\exists} M_2$ (see Macintyre [1]). However, as we shall see later, not all existentially closed models of NZR are elementarily equivalent.

We end this section by considering coding properties of existentially closed models of NZR which we shall use later on.

Definition Let P denote the set of prime elements in $\mathbf{N}$. Given $X \subset P$ and a characteristic zero domain M, we say *M codes X* if there is $a \in M$ such that for all $p \in P$, $p \in X$ if and only if $M \models \exists z(pz = a)$.

Lemma 2.7 *Let M be a normal $\mathbf{Z}$-ring.*
(a) *For every $X \subset P$ there is $M' \models$ NZR extending M and coding X.*
(b) *For every family of subsets of P, there is an $M' \models$ NZR extending M and coding all the elements of the family.*

Proof: It is trivial by taking an elementary extension of M. Below we give an alternative construction we shall use later on.

1. Extend M to $M[x]$ with $x > a$ for all $a \in M$; then clearly $M[x] \models$ NDOR. Extend the attached $\varphi: M \to \hat{\mathbf{Z}}$ in the following way: for each $p \in P$ $\varphi(x) = 0$ if $p \in X$ and $\varphi(x) = 1$ if $p \notin X$ hence $M[x]\varphi \models$ NZR and also $M[x]\varphi$ codes X. For, if $p \in X$, $\varphi(x) = 0$ hence $(x/p) \in M[x]\varphi$, and if $p \notin X$, $\varphi(x) = 1$ hence $(x/p) \notin M[x]\varphi$ since $1/p \notin \mathbf{Z}_p$.

2. NZR is a $\forall\exists$-theory, hence the union of a chain of models of NZR is also a model of NZR. Thus by iterating 1 we get the result.

3 Generic models of NZR

Our next aim is to prove, via finite generic models, that the theory of existentially closed models of NZR is not complete.

We begin by recalling the concept of finite forcing and some basic properties (see [1] for proofs). $\mathcal{L}$ will denote a *countable* language and T a first order theory in $\mathcal{L}$. We assume T to be $\forall\exists$ so we have $E_T \subset \mathrm{Mod}(T)$.

Definitions Let C be a set of new constants for $\mathcal{L}$. We say that a sentence is *basic* if it is either atomic or the negation of an atomic sentence in $\mathcal{L}(C)$. A *T-condition* (or a condition if T is clear from the context) is a finite set of basic sentences q such that $q \cup T$ is consistent. In what follows we fix T and $\mathcal{L}$.

Given a condition q and $\varphi \in \mathrm{Sent}(\mathcal{L})$ we say *q forces φ* and write $q \Vdash \varphi$ if either (i) φ atomic and $\varphi \in q$, or (ii) $\varphi = \neg\psi$ and $\forall q' \supset q\ q' \nVdash \psi$, or (iii) $\varphi = \psi_1 \vee \psi_2$ and $(q \Vdash \psi_1$ or $q \Vdash \psi_2)$, or (iv) $\varphi = \exists x\psi(x)$ and $q \Vdash \psi(t)$ for some closed term of $\mathcal{L}(C)$.

Basic Lemma *For any condition q and any $\varphi \in \mathrm{Sent}(\mathcal{L}(C))$ we have either $q \nVdash \varphi$ or $q \nVdash \neg\varphi$, and if $q \Vdash \varphi$ and $q' \supset q$ then $q' \Vdash \varphi$.*

Definitions Let M be an $\mathcal{L}$-structure and C a set of new constants such that each element of M is named by a closed term (infinitely many of them) of $\mathcal{L}(C)$, as usual we write a for the name of a.

Let $M \in \Sigma_T$, $\varphi(\mathbf{x})$ a formula of $\mathcal{L}$ and $\mathbf{a} \in M$. We say *M forces $\varphi(\mathbf{a})$* and write $M \Vdash \varphi(\mathbf{a})$ if there is $q(\mathbf{a}, \mathbf{b}) \subset \Delta(M)$ (open diagram of M) such that $q(\mathbf{a}, \mathbf{b}) \Vdash \varphi(\mathbf{a})$.

M is said to be *finite generic* if for every formula of $\mathcal{L}$, $\varphi(\mathbf{x})$ and every $\mathbf{a} \in M$ $M \models \varphi(\mathbf{a})$ if and only if $M \Vdash \varphi(\mathbf{a})$. F_T denotes the class of finite generic structures for T. We shall see the elements of F_T are also models of T, and hence they are called *finite generic models* of T. In general we have the following:

Theorem 3.1 *Let T be an $\mathcal{L}$-theory. Then $F_T \subset E_T$ and $F_T \neq \emptyset$. Moreover, we can construct $M \in F_T$ satisfying some fixed existential sentence φ consistent with T.*

The proof is based on the construction of *a complete sequence of conditions*, that is, an ascending chain of conditions $\{q_n : n \in \omega\}$ such that for every $\varphi \in \text{Sent}(\mathcal{L}(C))$, $q_n \Vdash \varphi$ or $q_n \Vdash \neg\varphi$ for some $n < \omega$.

As usual associated with $\Vdash$ we have a notion of *weak forcing* $\Vdash^*$ which is closed under deduction. Let φ be an $\mathcal{L}(C)$-sentence and q a condition $q \Vdash^* \varphi$ if and only if $q \Vdash \neg\neg\varphi$.

The *forcing-companion* of T is the theory $T^f = \{\varphi \in \text{Sent}(\mathcal{L}) \,.\, \varnothing \Vdash^* \varphi\}$. Then one can easily prove that $\text{Th}(F_T) = T^f$.

Next we state an easy result which dominates the whole theory of finite forcing.

Lemma 3.2 *Let C be a set of new constants. Let $q(\mathbf{c}_0, \mathbf{c}_1)$ be a condition and $\varphi(\mathbf{c}_1)$ a sentence in $\mathcal{L}(C)$. Then*

$$q \Vdash^* \varphi(\mathbf{c}_1) \Leftrightarrow T^f \vdash \forall \mathbf{v}\mathbf{w}[\textstyle\bigwedge q(\mathbf{v}, \mathbf{w}) \to \varphi(\mathbf{w})]$$

where $\bigwedge q$ denotes the conjunction of the formulas in q.

Finally, for a theory T, JEP is equivalent to saying that the union of two T-conditions with no new constants in common is a T-condition:

Proposition 3.3 *Let T be a theory. T has* JEP *if and only if T^f is complete.*

Using the methods of Macintyre [2] we are now ready to prove the following.

Proposition 3.4 *Let $X \subset P$ (= primes in $\mathbf{N}$). If X is not recursive then there is $M \in F_{\text{NZR}}$ which does not code X. Furthermore, if X is definable in $\mathbf{N}$ then it is not coded in any finite generic model.*

The proof is based on the following:

Lemma 3.5 *Let $X \subset P$ be nonrecursive. Let C be a new set of constants, $t(\mathbf{c})$ a closed term of $\mathcal{L}(C)$ and q a* NZR*-condition. Then there is $q' \supset q$ and $l \in P$ such that: either*

(i) *$l \in X$ and $q' \Vdash \forall w(lw \neq t(\mathbf{c}))$*

or

(ii) *$l \notin X$ and $q' \Vdash \exists w(lw = t(\mathbf{c}))$.*

Proof: Let

$$\sigma_1(l, t(\mathbf{c})) = \forall xy(0 < x < l \to ly + x \neq t(\mathbf{c})), \quad \sigma_2(l, t(\mathbf{c})) = \forall z(lz \neq t(\mathbf{c}))$$

and

$$A_i = \{l \in P : q \Vdash^* \sigma_i(l, t(\mathbf{c}))\} \quad i = 1,2.$$

First note that

$$\begin{aligned} A_i &= \{l \in P : \text{NZR}^f \vdash \forall \mathbf{v}\mathbf{w}(\textstyle\bigwedge q(\mathbf{v}, \mathbf{w}) \to \sigma_i(l, t(\mathbf{w})))\} \\ &= \{l \in P : \text{NZR} \vdash \forall \mathbf{v}\mathbf{w}(\textstyle\bigwedge q(\mathbf{v}, \mathbf{w}) \to \sigma_i(l, t(\mathbf{w})))\}. \end{aligned}$$

The first equality is by Lemma 3.2, the second one because the relevant sentence is universal and the theory of finite generics has the same universal consequences as NZR.

Hence both A_1 and A_2 are recursively enumerable, since hypothesis X is not recursive so either $X \neq A_1$ or $X^c \neq A_2$. There are four cases:

$$(1)\ X \not\subset A_1;\quad (2)\ A_1 \not\subset X;\quad (3)\ X^c \not\subset A_2;\quad \text{and } (4)\ A_2 \not\subset X^c.$$

(1): Take $l \in X$, $l \notin A_1$. Hence $q \nVdash^* \sigma_1(l, t(\mathbf{c}))$ so there is $r \supset q$ such that $r \Vdash \neg\sigma_1(l, t(\mathbf{c}))$. On the other hand in any generic we have

$$M \models \forall \mathbf{w}(\neg\sigma_1(l, t(\mathbf{w})) \rightarrow \forall z(lz \neq t(\mathbf{w})))$$

hence the sentence belongs to NZR^f, therefore by JEP there is $q' \supset r$ satisfying (i) above.

(2): Take $l \in A_1$, $l \notin X$. Hence $q \Vdash^* \sigma_1(l, t(\mathbf{c}))$. Also

$$M \models \forall \mathbf{w}(\sigma_1(l, t(\mathbf{w})) \rightarrow \exists z(lz = t(\mathbf{w})))$$

(as in (1)), so by JEP we have $q' \supset q$ and satisfying (ii) above.

(3): Take $l \notin X$, $l \notin A_2$. Hence

$$q \nVdash^* \forall w(lw \neq t(\mathbf{c})),$$

so there is $q' \supset q$ satisfying (ii) above.

(4): Take $l \in A_2$, $l \in X$. Then there is a $q' \supset q$ such that

$$q' \Vdash \forall z(lz \neq t(\mathbf{c})),$$

and hence q' satisfies (i) above.

Proof of Proposition 3.4: Suppose X is not recursive. We shall construct a complete sequence of conditions generating a generic model which will not have a code for X.

Let $\{\sigma_m : m < \omega\}$ be an enumeration of the sentences of $\mathcal{L}(C)$, and $\{t_m : m < \omega\}$ an enumeration of all closed terms of $\mathcal{L}(C)$. Assume first we have already constructed $\{q_m : m < \omega\}$ a complete sequence of conditions and $\{l_m : 0 < m < \omega\}$ an enumeration of P such that

(a) either $q_{m+1} \Vdash \sigma_m$ or $q_{m+1} \Vdash \neg\sigma_m$; and
(b) either

$$l_m \in X \quad \text{and} \quad q_{m+1} \Vdash \forall z(l_m z \neq t_m),$$

or

$$l_m \notin X \quad \text{and} \quad q_{m+1} \Vdash \exists z(lz = t_m).$$

Then, let $M \in F_{\text{NZR}}$ be generated by $\{q_m : m < \omega\}$ (hence the universe of M is generated by the constant terms), suppose M codes X and let t_m be a code of X in M. If $l_m \in X$ then $M \models \exists z(l_m z = t_m)$ by definition of code, then (a) implies there is an s such that $q_s \Vdash \exists z(l_m z = t_m)$. And (b) implies $q_{m+1} \Vdash \forall z(l_m z = t_m)$.

Now consider $q_{s'}$ where $s' = \max(m + 1, s)$ and get a contradiction. The case $l_m \notin X$ is similar.

To prove the first assertion of the proposition, it remains to construct the above complete sequence of conditions and $\{l_m : 0 < m < \omega\}$. Take $q_0 = \varnothing$ and $l_0 = 1$. Suppose we have found q_m and l_m with the required conditions, consider first q_m and σ_m. Define $q \supset q_m$ as follows: if $q_m \nVdash \sigma_m$ and $q_m \nVdash \neg\sigma_m$ get $q \supset q_m$

such that $q \Vdash \sigma_m$ otherwise let $q = q_m$. Now consider q and t_m and apply the lemma to get $l_m = l$ and $q_{m+1} = q'$.

This finishes the construction of

$$\{q_m : m < \omega\} \quad \text{and} \quad \{l_m : 0 < m < \omega\}.$$

Finally, to get the second assertion, note that if $\varphi(u)$ defines X in **N** then $\varphi^\theta(u)$ defines X in M (where θ defines **N** in E_{NZR}). *To say M codes X* is equivalent to

$$M \vDash \exists x \forall u(pr(u) \to (\varphi^\theta(u) \leftrightarrow u \mid x))$$

where $pr(u)$ is $\forall xy(u > 1 \wedge 1 < x, y < u \to xy \neq u)$. Hence, for X definable, there is $M \in F_{\text{NZR}}$ coding X if and only if all $M \in F_{\text{NZR}}$ code X, by completeness of NZR^f.

Corollary 3.6 *There are $\exists\forall\exists\forall$-sentences of $\mathcal{L}$ which separate existentially closed models of* NZR, *i.e., which hold true in some of those models and fails to hold in others.*

Proof: Let X be a Σ_1-subset of P and nonrecursive. Let ψ_1 be an existential formula defining X in **N**. Let θ be the universal formula defining **N** in E_{NZR} (see Proposition 2.5). Then the relativization ψ_1^θ defines X in each existentially closed model of NZR. Let

$$\psi = \exists x \forall u(pr(u) \to (\psi_1^\theta(u) \leftrightarrow u \mid x)).$$

Note that ψ is $\exists\forall\exists\forall$.

X is arithmetical and nonrecursive, hence by the theorem, X is not coded in any generic. Let $M_1 \in F_{\text{NZR}}$. On the other hand, by Lemma 2.7 X is coded in some $M_2 \in E_{\text{NZR}}$. Hence $M_1 \nvDash \psi$ and $M_2 \vDash \psi$.

Therefore $M_1 \not\equiv_{\exists\forall\exists\forall} M_2$.

(My thanks to R. Kaye for drawing my attention to an error in the proof of this corollary.)

We end this section with an easy extension of Theorem 1 in [2] for $\forall\exists$-theories in a countable language satisfying JEP.

Definition Let L be a countable language and T a theory in L. Let $\exists^n F\mathcal{L}$ denote the set of existential formulas in $\mathcal{L}$ with at most n free variables. A subset $\tau(\mathbf{v})$ of $\exists^n F\mathcal{L}$ is said to an $\exists^n$*-type* of T if $\tau(\mathbf{v})$ is consistent with T and any subset of $\exists^n F\mathcal{L}$ extending $\tau(\mathbf{v})$ is inconsistent.

Proposition 3.6 *Let T be an $\forall\exists$-theory in a countable language $\mathcal{L}$ and $\tau(\mathbf{v})$ an $\exists^n$-type. Then, if $\tau(\mathbf{v})$ is omitted in some $M \in E_T$ then it is also omitted in some finite generic M'.*

Proof: This is similar to the proof of Proposition 3.5; here the key result is the following: assume there is an M in E_{NZR} omitting $\tau(\mathbf{v})$. Then for any n-tuple of $\mathcal{L}(C)$-terms $\mathbf{t}(\mathbf{c}) = (t_1(\mathbf{c}), \ldots, t_n(\mathbf{c}))$ (where C is a countable set of new constants) and any T-condition q, there is a T-condition $q' \supset q$ and $\varphi(\mathbf{v}) \in \tau(\mathbf{v})$ such that $q' \Vdash \neg\varphi(\mathbf{t}(\mathbf{c}))$.

To see this let $A = \{\varphi(\mathbf{v}) \in \exists^n F\mathcal{L} : q(\mathbf{t}(\mathbf{c}), \mathbf{d}) \Vdash^* \varphi(\mathbf{t}(\mathbf{c}))\}$ Hence $A = \{\varphi(\mathbf{v}) \in \exists^n F\mathcal{L} : T^f \vdash \forall \mathbf{v}\mathbf{w}(\bigwedge q(\mathbf{t}(\mathbf{v}), \mathbf{w}) \to \varphi(\mathbf{t}(\mathbf{v})))\}$.

The above sentence in $\mathcal{L}$ is $\forall\exists$ so if it is true in all finite generics M', it is also true in M and vice versa.

Hence $\tau(\mathbf{v}) \neq A$ for, q is a T-condition so $M \models \bigwedge q(\mathbf{t}(\mathbf{a}), (\mathbf{b}))$ for some $\mathbf{a}, \mathbf{b}$ in M (by JEP) hence $\tau(\mathbf{v}) = A$ would imply M realizes $\tau(\mathbf{v})$. By maximality of $\tau(\mathbf{v})$ it must be $\varphi(\mathbf{v}) \in \tau(\mathbf{v})$ with $\varphi(\mathbf{v}) \notin A$ and reasoning as in Lemma 3.5 we get the required q'.

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