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Probabilistic Canonical Models for Partial Logics

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Abstract The aim of the paper is to develop the notion of partial probability distributions as being more realistic models of belief systems than the standard accounts. We formulate the theory of partial probability functions independently of any classical semantic notions. We use the partial probability distributions to develop a formal semantics for partial propositional calculi, with extensions to predicate logic and higher order languages. We give a proof theory for the partial logics and obtain soundness and completeness results.

1 Motivation

It is commonplace to use classical logic and classical probability theory to try to model rational belief. However, real believers (e.g., humans) do not have deductively closed belief states; for example, there are deductive consequences of my beliefs to which I would be reluctant to assent, just because they are so complex I am not aware of whether or not they are deductive consequences of my beliefs. Similarly, although classical probability distributions are total functions, there are many statements to which I feel unable to assign any probability at all; and I certainly do not want to say that all of such statements about whose probability value I am uncertain have the same probability value (e.g., 0.5). Real believers are just *not* probabilistically omniscient. So both the absolute consequence relation of classical logic and the graded consequence relation of classical probability distributions are at odds with belief systems of real believers. Thus there seems to be a need to develop formal tools that allow consequence relations, both absolute and graded, to be sometimes undefined, corresponding to a state of no opinion.

In this paper we wish to merge some of the ideas of Lapierre and Lepage [1] and [2] on partial logics with some of the ideas of Morgan [3] and [4] on canonical probability distributions. There are two major aspects to this research.

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(1) Morgan showed that there is a probabilistic semantics for (almost) every logic which includes the principles of classical sentence logic as a core. In this paper, we show that this result can be generalized to include logics which contain only the (weaker) partial logics of Lapierre and Lepage as a core. This extension is accomplished by developing a theory of partial probability distributions. Partial probability distributions more closely model real belief systems than do total probability distributions.

(2) Completeness results are frequently proven by contraposition. One begins by assuming that some formula A is not a theorem of some set of premises Γ. One then establishes the existence of an appropriate semantic structure in which the members of Γ are satisfied but in which A is not. That is, *for each* set Γ and nontheorem A, *there is* a semantic structure that shows that A is not semantically implied by Γ. In certain modal logics one can show that there is a semantic structure, called the canonical model, such that for each set Γ and nontheorem A, that structure shows that A is not semantically implied by Γ. Note the interchange of the quantifiers, from '*for each there is*' to '*there is for each*'. It is interesting to note that there is no canonical model for the standard semantics for propositional or first-order logics. However, Morgan in [4] has shown that in probabilistic semantics, there are canonical probability distributions, analogous to the canonical models of modal logic. In a canonical distribution P_c, for every set Γ and nontheorem A, there is some additional evidence Δ such that $P_c(A, \Gamma \cup \Delta) \neq 1$. In short, the intuition is: If A is not a logical consequence of what you believe, then there is some reason to doubt it. In this paper we show that canonical probability distributions also exist for probabilistic semantics based on partial probability theory.

One interesting feature about Morgan's proposals is that very little is said about the systems of logic themselves. Morgan just assumes that

1. the logics contain classical sentential logic;
2. the logics have rules of the following form:

IR If $\vdash A_0$ and $\cdots$ and $\vdash A_j$, and the A_i and A satisfy conditions COND, then $\vdash A$.

The notation COND just stands for any set of conditions on the A_i and A which are well founded; for a standard example, just think of the rule for universal quantifier introduction.

What we propose here is to weaken clause (1) in the following sense. Lapierre and Lepage, following Thijsse [5], have developed a kind of partial semantics in order to model epistemic states. Roughly, the semantics results from introducing the undefined truth value and the undefined object, and then restricting all the semantic values to partial monotonic functions instead of classical total functions; the sense of monotonicity is with respect to the order 'more defined or as defined as'. Interestingly, this approach provides partial interpretations not only for classical sentential logic, but also for first-order functional predicate calculus, for modal logic, for propositional type theory, for standard type theory and for intensional logic. It seems that this way to partialize semantics is quite universal. These partial logics are not really nonstandard logics, that is, they are not outside standard logics; rather, classical logics are just limiting cases of partial logics, namely, those in which the semantic values are totally defined.

The relationship between the partial logics and their total correlates is formally expressed by the following: If $\Gamma \Vdash_P A$, then $\Gamma \Vdash_T A$ (where $\Vdash_P$ and $\Vdash_T$ are, respectively, the relation of semantic consequence for the partial and the total logic under consideration), that is, the valid consequences in partial logic are classically valid consequences (the converse is certainly not true).

The question arises: Can we Morganize partial logic? More precisely, is it possible to define a kind of strongly canonical partial probability distribution P_c such that if $\Gamma \nvdash A$, there is some Δ such that $P_c(A, \Gamma \cup \Delta) \neq 1$ or is undefined? The aim of this paper is to show that the answer is yes.

As noted above, real agents simply cannot assign a probability value to every sentence; real agents are not probabilistically omniscient. The partial probability distributions introduced here go some way to answering the problem of probabilistic omniscience. For example, let q be any atomic formula, and let Γ be any set of sentences. If $P_P(q, \Gamma)$ is undefined and A is any sentence (even a classically logical truth or classically logical falsehood) involving only the letter q, then $P_P(A, \Gamma)$ is also permitted to be undefined. Thus an agent need not be able to assign probability values to every statement. In Section 5 below, we will discuss the problem of probabilistic omniscience in more detail.

2 Two New Tools

In the classical case, the method used by Morgan [4] in constructing canonical probability distributions relies on building up a sequence of maximally consistent sets. The sets must be "coherent" with each other in the sense that, as the sequence is built up, minimal changes are made when moving from one set to the next in the sequence. In order to adapt Morgan's proof for partial logic, we will need two notions slightly different from those used by Morgan.

Instead of maximally consistent sets, we will use *saturated deductively closed consistent sets* (SDCCS), and instead of total probability distributions, we will use partial probability distributions . Let us first present the partial sentential logic.

2.1 Partial logic The partial sentential logic is a kind of hard core of all the partial logics, and it is sufficient for the present purpose because the soundness and completeness proofs will only use very basic features of partial logics. As a consequence, the results will hold for (almost) any extension of partial sentential logic in a sense that will be specified below. The syntax is the ordinary syntax or sentential logic using only $\neg$ and $\wedge$. We will use a sequent formulation for presenting the logic.

We use only one rule, which we will call reiteration, that does not require knowledge of previous consequence relationships.

R If $A \in \Gamma$ then $\Gamma \vdash A$.

In all other cases, A is obtained from already derived sentences by using the following rules.

D.1 If $\Gamma \vdash B \wedge \neg B$, then $\Gamma \vdash A$.
D.2 If $\Gamma \vdash \neg\neg A$, then $\Gamma \vdash A$.
D.3 If $\Gamma \vdash A$, then $\Gamma \vdash \neg\neg A$.
D.4 If $\Gamma \vdash A \wedge B$, then $\Gamma \vdash A$.
D.5 If $\Gamma \vdash A \wedge B$, then $\Gamma \vdash B$.
D.6 If $\Gamma \vdash A$ and $\Gamma \vdash B$, then $\Gamma \vdash A \wedge B$.

D.7 If $\Gamma \vdash A$, then $\Gamma \vdash \neg(\neg A \wedge \neg B)$.
D.8 If $\Gamma \vdash A$, then $\Gamma \vdash \neg(\neg B \wedge \neg A)$.
D.9 If $\Gamma \vdash \neg(\neg\neg A \wedge \neg\neg B)$, then $\Gamma \vdash \neg(A \wedge B)$.
D.10 If $\Gamma \vdash \neg(A \wedge B)$, then $\Gamma \vdash \neg(\neg\neg A \wedge \neg\neg B)$.
D.11 If $\Gamma \cup \{B\} \vdash A$ and $\Gamma \cup \{C\} \vdash A$, then $\Gamma \cup \{\neg(\neg B \wedge \neg C)\} \vdash A$.

This system has no theorems; even $(A \vee \neg A)$ may be undefined. But the system is proved to be strongly complete according to the three-valued semantics when the values of $\neg$ and of $\wedge$ are those of Kleene strong connectors. As mentioned earlier, our results will hold for almost any extension. Following Morgan, we can specify stronger (partial or total) logics by adding rules of the following form.

IRP If $\Gamma \vdash A_0$ and $\cdots$ and $\Gamma \vdash A_j$, and the $A_i$and A satisfy conditions COND, then $\Gamma \vdash A$.

We will sometime write $(A \vee B)$ for $\neg(\neg A \wedge \neg B)$ and $(A \supset B)$ for $(\neg A \vee B)$. The following lemmas are easily proved.

L.1 If $\Gamma \vdash A$, then $\Gamma \vdash (A \vee \neg A)$.
L.2 If $\Gamma \vdash A$, then for some finite subset Δ of Γ, $\Delta \vdash A$.
L.3 If $\Gamma \cup \{A\} \vdash B$ and $\Gamma \vdash A$, then $\Gamma \vdash B$.
L.4 If $\Gamma \vdash A$, then $\Gamma \cup \Delta \vdash A$.
L.5 If $\Gamma \cup \{A\} \vdash B$, then $\Gamma \cup \{A \vee \neg A\} \vdash A \supseteq B$.

2.2 Saturated, deductively closed, consistent sets An SDCCS Γ is a set which is

1. saturated, that is, $(A \vee B) \in \Gamma$ if and only if $A \in \Gamma$ or $B \in \Gamma$;[1]
2. deductively closed, that is, $A \in \Gamma$ if and only if $\Gamma \vdash A$;
3. consistent, that is, there is an A such that $\Gamma \nvdash A$.

Clearly, in any logic which contains classical logic, to be an SDCCS and to be a maximally consistent set are equivalent properties.

It can be shown, following Thijsse, that every SDCCS provides a partial model for partial sentential logic. Lapierre and Lepage have extended this result to a large class of logics as mentioned above. In fact, for any SDCCS Γ we can define a valuation V^Γ such that

$$\begin{aligned} V^\Gamma(A) &= 1 \quad \text{iff } A \in \Gamma \\ V^\Gamma(A) &= 0 \quad \text{iff } \neg A \in \Gamma \\ V^\Gamma(A) & \quad\quad\ \text{is undefined otherwise.} \end{aligned}$$

So for each set Γ there is a valuation that simultaneously satisfies the members of Γ but fails to satisfy each nontheorem of Γ. But there will not be a single valuation that works for every set Γ. Our goal is to show that there will be a single canonical probability distribution that works for every set Γ and nontheorem A.

Let Δ be a consistent set. There are many ways to construct an SDCCS which is a superset of Δ. The following construction will be used in the completeness proof. Δ being consistent, there is a formula A_t(called the test formula) such that $\Delta \nvdash A_t$. We will define Γ, a superset of Δ, such that Γ is an SDCCS and $\Gamma \nvdash A_t$. The definition of Γ needs only very few features of the logic as we will see.

Let $E_0, \ldots, E_k, \ldots$ be any specific fixed enumeration of all the sentences of the language in which every sentence appears denumerably many times. We will call this enumeration the canonical enumeration.

Starting from Δ we define the sequence $\Delta_0, \ldots, \Delta_k, \ldots$ such that

$$\begin{array}{rcl}
\Delta_0 & = & \Delta \\
\Delta_{2k+2} & = & \Delta_{2k+1} = \Delta_{2k} \text{ if } \Delta_{2k} \cup \{E_k\} \vdash A_t \\
\Delta_{2k+1} & = & \Delta_{2k} \cup \{E_k\} \text{ if } \Delta_{2k} \nvdash A_t \text{ and} \\
\Delta_{2k+2} & = & \Delta_{2k+1} \text{ if } E_k \text{ is not } (A \vee B) \text{ and if } E_k \text{ is } (A \vee B) \text{ then} \\
\Delta_{2k+2} & = & \Delta_{2k+1} \cup \{A\} \text{ if } \Delta_{2k+1} \cup \{A\} \nvdash A_t \text{ and} \\
\Delta_{2k+2} & = & \Delta_{2k+1} \cup \{B\} \text{ if } \Delta_{2k+1} \cup \{A\} \vdash A_t.
\end{array}$$

One can easily check that $\Gamma = \cup_i \Delta_i$ is an SDCCS and $\Gamma \nvdash A_t$. It is worth emphasizing that the only logical resources of the system which are used in the proof are the definition of consistency, the rule R of reiteration, the cut rule (L.3) and $\vee$-elimination (D.11).

Carefully note that if A_t is a classical tautology, then Γ is not a maximally consistent set.

Further note that if in the construction process we replace the phrases '$\vdash A_t$' and '$\nvdash A_t$' by 'is inconsistent' and 'is consistent', respectively, then Γ would be maximally consistent.

Finally, note that we could choose the test formula A_t any way we like. Each selection of test formula yields a potentially distinct SDCCS. In particular, for any given set Δ, we could select A_t to be the first formula E_i in the enumeration such that $\Delta \nvdash E_i$.

The following are some more elementary results.

L.5 If Γ is any consistent set of formulas, then there is at least one SDCCS which is a superset of Γ.

L.6 If $\Gamma \nvdash A$ then there is an SDCCS Δ such that $\Gamma \subseteq \Delta$ and $\Delta \nvdash A$.

L.7 $\Gamma \vdash A$ if and only if for every SDCCS Δ which is a superset of Γ, $A \in \Delta$.

2.3 Partial probability functions We now turn our attention to the problem of characterizing partial probability distributions. Of course the main idea is that a partial probability distribution should be just like a total probability distribution, only with some of the values undefined. But our notion is slightly more complicated than that. While we want to allow gaps in the distributions, we do not allow completely arbitrary gaps. Essentially, while we allow gaps, we require that any value that can be computed from values that are given cannot be left undefined.

Definition 2.1 A partial probability function P is a partial function

$$P : L \times 2^L \to [0, 1]$$

such that the following constraints are always satisfied:

P.1 If $A \in \Gamma$, then $P(A, \Gamma)$ is defined;

P.2 If $P(\neg A, \Gamma)$ is defined, then $P(A, \Gamma)$ is defined;

P.3 If $P(A, \Gamma)$ is defined, then $P(\neg A, \Gamma)$ is defined;

P.4 If $P(A \wedge B, \Gamma)$ is defined, then $P(B \wedge A, \Gamma)$ is defined;

P.5 If $P(A \wedge B, \Gamma)$ is defined and > 0, then $P(A, \Gamma)$ and $P(B, \Gamma \cup \{A\})$ are defined;

P.6 If $P(A, \Gamma)$ is defined and $P(B, \Gamma \cup \{A\})$ is defined, then $P(A \wedge B, \Gamma)$ is defined;

P.7 If $P(A, \Gamma) = 0$, then $P(A \wedge B, \Gamma) = 0$;

P.8 If $P(A, \Gamma)$ is not defined, then $P(A \wedge B, \Gamma)$ is undefined unless $P(B, \Gamma) = 0$;
P.9 If $P(A, \Gamma)$ is defined and > 0 and $P(A \wedge B, \Gamma) = 0$, then $P(B, \Gamma \cup \{A\}) = 0$.

And when all the appropriate values of P are defined, the following (classical) constraints are satisfied:

NP.1 $0 \leq P(A, \Gamma) \leq 1$;
NP.2 If $A \in \Gamma$, then $P(A, \Gamma) = 1$;
NP.3 $P(A \vee B, \Gamma) = P(A, \Gamma) + P(B, \Gamma) - P(A \wedge B, \Gamma)$;
NP.4 $P(A \wedge B, \Gamma) = P(A, \Gamma) \times P(B, \Gamma \cup \{A\})$;
NP.5 $P(\neg A, \Gamma) = 1 - P(A, \Gamma)$ provided Γ is P-normal (i.e., it is not the case that $P(A, \Gamma) = 1$ for all A);
NP.6 $P(A \wedge B, \Gamma) = P(B \wedge A, \Gamma)$;
NP.7 $P(C, \Gamma \cup \{A \wedge B\}) = P(C, \Gamma \cup \{A, B\})$.

We now define the semantic consequence relation, based on partial probability distributions. We use the standard intuition that Γ semantically implies A, if and only if no matter what your belief system is like, there is no evidence in addition to Γ which you could obtain that would make you doubt A.

Definition 2.2 We say that A is a semantic consequence of Γ written $\Gamma \Vdash A$, if and only if for all probability distributions P, $P(A, \Gamma \cup \Delta) = 1$ for all Δ.

3 Some Useful Intermediate Results

In this section we state and prove a number of results which will be needed in order to establish soundness and completeness. We will give a brief comment on each lemma.

Our first lemma states a rule of detachment for the conditional of conditional probability.

Lemma 3.1 *If $P(A, \Gamma \cup \{B\}) = 1$ and $P(B, \Gamma) = 1$, then $P(A, \Gamma) = 1$.*

Proof

$P(A, \Gamma \cup \{B\}) = 1$	hypothesis
$P(B, \Gamma) = 1$	hypothesis
$P(B \wedge A, \Gamma) = 1$	P.6 and NP.4
$P(A \wedge B, \Gamma) = 1$	P.4 and NP.6
$P(A, \Gamma) = 1$	P.4, P.5, and NP.4

□

Our second lemma essentially establishes that a set of assumptions that makes certain any contradiction must be an abnormal set of assumptions that certifies any sentence whatever.

Lemma 3.2 *If $P(B \wedge \neg B, \Gamma) = 1$, then Γ is P-abnormal.*

Proof

$1 = P(B \wedge \neg B, \Gamma)$	hypothesis
$= P(\neg B \wedge B, \Gamma)$	NP.6
$= P(\neg B, \Gamma) \times P(B, \Gamma \cup \{\neg B\})$	P.5 and NP.4
$= P(\neg B, \Gamma)$	NP.1

Thus
$P(B, \Gamma) = 0$ P.2 and NP.5
unless Γ is P-abnormal. But,

$$\begin{aligned} 1 &= P(B \wedge \neg B, \Gamma) && \text{hypothesis} \\ &= P(B, \Gamma) \times P(\neg B, \Gamma \cup \{B\}) && \text{P.5 and NP.4} \\ &= P(B, \Gamma) && \text{NP.1} \end{aligned}$$

Thus Γ is P-abnormal. □

Our third lemma is like a limited natural deduction rule of negation elimination.

Lemma 3.3 *$P(C \wedge B, \Gamma) = 0$ if and only if $P(\neg\neg C \wedge B, \Gamma) = 0$.*

Proof ($\Rightarrow$)

$$\begin{aligned} &P(C \wedge B, \Gamma) = 0 && \text{hypothesis} \\ &P(B \wedge C, \Gamma) = 0 && \text{P.4 and NP.6} \end{aligned}$$

(i) If $P(B, \Gamma)$ is not defined,

$$\begin{aligned} &P(C, \Gamma) = 0 && \text{P.8} \\ &P(\neg\neg C, \Gamma) = 0 && \text{NP.5} \\ &P(\neg\neg C \wedge B, \Gamma) = 0 && \text{P.7} \end{aligned}$$

(ii) If $P(B, \Gamma)$ is defined,

(a) if $P(B, \Gamma) = 0$,
$P(\neg\neg C \wedge B, \Gamma) = 0$ P.7, P.4, and NP.6

(b) if $P(B, \Gamma) > 0$,
$P(C, \Gamma)$ is defined P.8

But then we have

$$\begin{aligned} 0 &= P(C \wedge B, \Gamma) && \text{hypothesis} \\ &= P(B \wedge C, \Gamma) && \text{P.4 and NP.6} \\ &= P(C, \Gamma \cup \{B\}) && \text{P.9} \\ &= P(\neg\neg C, \Gamma \cup \{B\}) && \text{NP.5} \end{aligned}$$

But

$$\begin{aligned} P(\neg\neg C \wedge B, \Gamma) &= P(B \wedge \neg\neg C, \Gamma) && \text{P.4 and NP.6} \\ &= P(B, \Gamma) \times P(\neg\neg C, \Gamma \cup \{B\}) && \text{because all the expressions involved} \\ &= 0 && \text{are defined} \end{aligned}$$

($\Leftarrow$) Similar to ($\Rightarrow$) □

Essentially, our fourth lemma states that, given a set of background assumptions, if A is certified by separately adding a sentence and also adding its negation, then A must be certified by the background assumptions alone.

Lemma 3.4 *If* $P(A, \Gamma \cup \{B\}) = P(A, \Gamma \cup \{\neg B\}) = 1$*, then* $P(A, \Gamma) = 1$*, provided* $P(B, \Gamma)$ *is defined and* Γ *is* P*-normal.*

Proof

$P(A \wedge B, \Gamma) = P(B, \Gamma) \times P(A, \Gamma \cup \{B\})$	NP.4 and NP.6
$P(A \wedge \neg B, \Gamma) = P(\neg B, \Gamma) \times P(A, \Gamma \cup \{\neg B\})$	NP.4 and NP.6
$P(A \wedge B, \Gamma) = P(B, \Gamma)$	hypothesis
$P(A \wedge \neg B, \Gamma) = P(\neg B, \Gamma)$	hypothesis
$P(A \wedge B, \Gamma) + P(A \wedge \neg B, \Gamma) = 1$	NP.5
$P(A, \Gamma) \times P(B, \Gamma \cup \{A\}) + P(A, \Gamma) \times P(\neg B, \Gamma \cup \{A\}) = 1$	NP.4
$P(A, \Gamma) \times (P(B, \Gamma \cup \{A\}) + P(\neg B, \Gamma \cup \{A\})) = 1$	algebra
$P(A, \Gamma) = 1.$	NP.5

□

4 Soundness and Completeness

We now want to show that the semantic consequence relation based on partial probability distributions exactly captures the syntactic consequence relation of partial logics. To this end, we will prove soundness and completeness results for the propositional case. The result for more complex logics can be accomplished by adding restrictions on the probability distributions corresponding to the generalized inference rules of the logics, in the manner indicated in [4]. We begin by establishing soundness.

Theorem 4.1 (Soundness) *If* $\Gamma \vdash A$*, then* $\Gamma \Vdash A$*.*

Proof Let us suppose that $\Gamma \vdash A$. Then there is a proof from $\Gamma : \{A_0, \ldots, A_{n-1}\} \vdash A$. We prove by induction on i that $\Gamma \Vdash A_i$.

Basis: $A_0 \in \Gamma$. In that case $P(A_0, \Gamma) = 1$ and $P(A_0, \Gamma \cup \Delta) = 1$.

Induction step: $\Gamma \Vdash A_i$ for all $i < n$.

1. $A_n \in \Gamma$. As basis.
2. $\Gamma \vdash (B \wedge \neg B)$ so $\Gamma \vdash A_n$. By the induction hypothesis, $P(B \wedge \neg B, \Gamma \cup \Delta) = 1$ for all Δ. By Lemma 3.2, $\Gamma \cup \Delta$ is P-abnormal for all Δ. Hence, $P(A_n, \Gamma \cup \Delta) = 1$ for all Δ.
3. $\Gamma \vdash \neg\neg A_n$ so $\Gamma \vdash A_n$. By the induction hypothesis, $P(\neg\neg A_n, \Gamma \cup \Delta) = 1$ for all Δ. If $\Gamma \cup \Delta$ is P-abnormal, the desired result is trivial, so suppose it is not. $P(\neg A_n, \Gamma \cup \Delta) = 0$ for all Δ (unless P-abnormal) by NP.5. $P(A_n, \Gamma \cup \Delta) = 1$ for all Δ by NP.5.
4. A_n is $\neg\neg B$. $\Gamma \vdash B$, so $\Gamma \vdash \neg\neg B$. Similar to (3).
5. $\Gamma \vdash (A_n \wedge B, \Gamma)$, so $\Gamma \vdash A_n$. By the induction hypothesis, $P(A_n \wedge B, \Gamma \cup \Delta) = 1$ for all Δ. Then by P.5 and NP.4, $P(A_n, \Gamma \cup \Delta) = 1$ for all Δ.
6. $\Gamma \vdash (B \wedge A_n, \Gamma)$, so $\Gamma \vdash A_n$. Similar to (5).
7. A_n is $B \wedge C$. $\Gamma \vdash B$ and $\Gamma \vdash C$, so $\Gamma \vdash (B \wedge C, \Gamma)$. By the induction hypothesis, $P(B, \Gamma \cup \Delta) = 1$ for all Δ and $P(C, \Gamma \cup \Delta) = 1$ for all Δ. Then $P(C, \Gamma \cup \{B\} \cup \Delta) = 1$ for all Δ. So, by NP.4 $P(B \wedge C, \Gamma \cup \Delta) = 1$ for all Δ.
8. A_n is $B \vee C$. $\Gamma \vdash B$, so $\Gamma \vdash B \vee C$. By the induction hypothesis, $P(B, \Gamma \cup \Delta) = 1$ for all Δ.

(i) If $\Gamma \cup \Delta$ is P-abnormal, then $P(\neg(\neg B \wedge \neg C), \Gamma \cup \Delta) = 1$ and $P(\neg(\neg C \wedge \neg B), \Gamma \cup \Delta) = 1$.

(ii) If $\Gamma \cup \Delta$ is P-normal, then $P(\neg B, \Gamma \cup \Delta) = 0$ and, by P.7, $P(\neg C \wedge \neg B, \Gamma \cup \Delta) = 0$. Thus $P(\neg(\neg C \wedge \neg B), \Gamma \cup \Delta) = 1$.

9. A_n is $C \vee B$. $\Gamma \vdash B$, so $\Gamma \vdash C \vee B$. Similar to (8).
10. A_n is $\neg(C \wedge B)$. $\Gamma \vdash \neg(\neg\neg C \wedge \neg\neg B)$, so $\Gamma \vdash \neg(C \wedge B)$. By the induction hypothesis, $P(\neg(\neg\neg C \wedge \neg\neg B), \Gamma \cup \Delta) = 1$.

(i) If $\Gamma \cup \Delta$ is P-abnormal, then trivial.

(ii) If $\Gamma \cup \Delta$ is P-normal, then $P(\neg\neg C \wedge \neg\neg B, \Gamma \cup \Delta) = 0$. We have

$P(C \wedge \neg\neg B, \Gamma \cup \Delta) = 0$	Lemma 3.3
$P(\neg\neg B \wedge C, \Gamma \cup \Delta) = 0$	P.4 and NP.6
$P(B \wedge C, \Gamma \cup \Delta) = 0$	Lemma 3.3
$P(C \wedge B, \Gamma \cup \Delta) = 0$	P.4 and NP.6
$P(\neg(C \wedge B), \Gamma \cup \Delta) = 1$	NP.5

11. As (10), but bottom up.
12. $\Gamma = \Sigma \cup \{B \vee C\} = \Sigma \cup \{\neg(\neg B \wedge \neg C)\}$ and $\Sigma \cup \{B\} \vdash A_n$ and $\Sigma \cup \{C\} \vdash A_n$.

By the induction hypothesis,

(*) $P(A_n, \Sigma \cup \Delta \cup \{B\}) = 1$ for all Δ and

(**) $P(A_n, \Sigma \cup \Delta \cup \{C\}) = 1$ for all Δ.

We have

$P(\neg(\neg B \wedge \neg C), \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) = 1$ for all Δ.

(i) If $\Sigma \cup \Delta \cup \{\neg(\neg B \wedge \neg C)\}$ is abnormal, then $P(A_n, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) = 1$ for all Δ.

(ii) If $\Sigma \cup \Delta \cup \{\neg(\neg B \wedge \neg C)\}$ is normal, then $P(\neg B \wedge \neg C, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) = 0$. NP.5

There are three possible cases.

(a) $P(\neg B, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) = 0$ or

(b) $P(\neg B, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) > 0$

(c) $P(\neg B, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta)$ is undefined.

(a)	$P(\neg B, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) = 0$	hypothesis
	$P(B, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) = 1$	P.2 and NP.5
	$P(A_n, \Sigma \cup \{B\} \cup \Delta \cup \{\neg(\neg B \wedge \neg C)\}) = 1$	(*)
	$P(A_n, \Sigma \cup \Delta \cup \{\neg(\neg B \wedge \neg C)\}) = 1$	Lemma 3.1
(b)	$P(\neg B, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) > 0$	
	$P(\neg C, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta \cup \{\neg B\}) = 0$	P.9
	$P(C, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta \cup \{\neg B\}) = 1$	P.2 and NP.5
	$P(A_n, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta \cup \{\neg B\} \cup \{C\}) = 1$	(**)
	$P(A_n, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta \cup \{\neg B\}) = 1$	Lemma 3.1
	$P(A_n, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta \cup \{B\}) = 1$	(*)
	$P(A_n, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) = 1$	Lemma 3.4
(c)	$P(\neg B, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta)$	is undefined
	$P(\neg C, \Sigma \cup \{\neg(\neg B \wedge \neg C)\} \cup \Delta) = 0.$	P.8

The rest of the proof is similar to (a). □

That completes the induction, so we have demonstrated soundness. We now turn our attention to strong completeness.

Theorem 4.2 (Completeness) *If* $\Gamma \Vdash A$*, then* $\Gamma \vdash A$.

Proof We proceed by contraposition. Let us suppose that for some particular set Γ^* and sentence A^*, that $\Gamma^* \nvdash A^*$. We need to find a probability distribution P such that there is some Δ^* such that $P(A^*, \Gamma^* \cup \Delta^*) \neq 1$ or is undefined. Instead of constructing a different function P for each set Γ^* and sentence A^*, we will instead construct one canonical distribution that will work for every set Γ^* and every nontheorem A^*.

For each set Γ, let $S(\Gamma)$ be the saturated set containing Γ defined by using the canonical enumeration $E_0, \ldots, E_n, \ldots$ of the formulas of the language, and using as test formula the first formula E_i in the canonical enumeration such that $\Gamma \nvdash E_i$. Then we define a function as follows:

$P : L \times L^2 \rightarrow [0, 1]$ such that

$$\begin{array}{lcl} P(B, \Gamma) & = & 1 \text{ if } B \in S(\Gamma); \\ P(B, \Gamma) & = & 0 \text{ if } \neg B \in S(\Gamma); \\ P(B, \Gamma) & & \text{is undefined otherwise.} \end{array}$$

It is clear that if $\Gamma \nvdash A$, then $A \notin S(\Gamma \cup \{\neg A\})$. So there is some Δ such that $P(A, \Gamma \cup \Delta) \neq 1$ or is undefined. Thus, provided P so defined is a legitimate partial probability function, it will follow that $\Gamma \nVdash A$ if $\Gamma \nvdash A$, for all Γ and A.

That was the easy part. We have now to check that P is a partial function satisfying P.1 – P.9 and NP.1 – NP.7. In order to do that, we will need the following lemma in order to deal with P.5, P.6, P.9 as well as NP.4. The problem is to ensure that if $A \in \Gamma$, then $S(\Gamma) = S(\Gamma \cup \{A\})$. For example, P.6 makes no sense if we do not have this property because we would not even define a unique SDCCS, and consequently, we would not define a partial probability function.

Lemma 4.3 *If* $A \in S(\Gamma)$*, then* $S(\Gamma) = S(\Gamma \cup \{A\})$.

Proof Let us consider the definitions of $S(\Gamma)$ and $S(\Gamma \cup \{A\})$ according to the same canonical enumeration $E_0, \ldots, E_n, \ldots$. We prove that they have exactly the same members. To do this, we prove that, for any i, $\Gamma_i \subseteq \Delta_i$. The conclusion that $S(\Gamma) = S(\Gamma \cup \{A\})$ then follows easily because for any B, if $\Delta_i \cup \{B\} \nvdash A_t$ then $\Gamma_i \cup \{B\} \nvdash A_t$ and so, by the definition of the SDCCS, everything that will be added or belong to some Δ_i will be sooner or later added to some Γ_i. Let us prove that, for any i, $\Gamma_i \subseteq \Delta_i$. We proceed by induction.

Basis: $\Gamma_0 \subseteq \Delta_0 = \Gamma_0 \cup \{A\}$.

Induction step: Suppose that $\Gamma_{2i} \subseteq \Delta_{2i}$. We prove that $\Gamma_{2i+2} \subseteq \Delta_{2i+2}$.

(a) Suppose that E_i is such that $\Gamma_{2i} \cup \{E_i\} \vdash A_t$. In that case, by the induction hypothesis, $\Delta_{2i} \cup \{E_i\} \vdash A_t$, and $\Gamma_{2i+2} = \Gamma_{2i+1} = \Gamma_{2i} \subseteq \Delta_{2i} = \Delta_{2i+1} = \Delta_{2i+2}$.

(b) Suppose that E_i is such that $\Gamma_{2i} \cup \{E_i\} \nvdash A_t$.
   (i) If $\Delta_{2i} \cup \{E_i\} \nvdash A_t$, then $\Gamma_{2i+1} = \Gamma_{2i} \cup \{E_i\} \subseteq \Delta_{2i\,i} \cup \{E_i\} = \Delta_{2i+1}$.
      (α) E_i is not $(C \vee D)$. In that case $\Gamma_{2i+2} = \Gamma_{2i+1} \subseteq \Delta_{2i+1} = \Delta_{2i+2}$.
      (β) E_i is $(C \vee D)$. If $\Gamma_{2i+1} \cup \{C\} \nvdash A_t$, then

(1) If $\Delta_{2i+1} \cup \{C\} \nvdash A_t$, then $\Gamma_{2i+2} = \Gamma_{2i+1} \cup \{C\} = \Delta_{2i+1} \cup \{C\} = \Delta_{2i+2}$.

(2) If $\Delta_{2i+1} \cup \{C\} \vdash A_t$. In this case, there is a derivation of A_t, $A_0, \ldots, A_n \vdash A_t$. If all of the A_i are consequences of $\Gamma_0 \cup \{A\}$ then for some p, $\Gamma_{2p} \vdash A$ and thus $\Gamma_{2p} \vdash A_t$, which contradicts the hypothesis. Therefore, there is at least one of the A_j that was introduced in the process of defining the Δ_js. A_j was not introduced because $\Delta_{2j} \cup \{A_j\} \nvdash A_t$, in that case $A_j \in \Gamma_{2j+1} \subseteq \Gamma_{2i}$. Thus A_j was introduced by $\vee$-saturation after the treatment of some $A_k = (E \vee A_j)$, with $\Delta_{2k} \cup \{E \vee A_j\} \nvdash A_t$ thus $\Gamma_{2k} \cup \{E \vee A_j\} \nvdash A_t$ with $\Gamma_{2k+1} \cup \{E\} \nvdash A_t$ and $\Delta_{2k+1} \cup \{E\} \vdash A_t$. This is just the starting point of $(\beta)(2)$ and, as the number of formula introduced in Δ_{2i} is finite, after a finite number of steps, this case is no more an option.

(ii) If $\Delta_{2i} \cup \{E_i\} \vdash A_t$. The argument is the same as in (i)(β)(2).

Thus, $\Gamma_i \subseteq \Delta_i$ for all i; as proved above, this implies that $S(\Gamma) = S(\Gamma \cup \{A\})$. □

That is the end of the proof of Lemma 4.3, and we now return to the proof of the main theorem. We have now to prove that P is a partial probability function.

From the definition it follows that P is a partial function from $L \times 2^L$ into [0, 1]. We have to check that P.1 – P.9 and NP.1 – NP.7 are satisfied by a P so defined.

P.1 If $A \in \Gamma$, $S(\Gamma) \vdash A$, $A \in S(\Gamma)$ and thus $P(A, \Gamma) = 1$.

P.2 If $P(\neg A, \Gamma)$ is defined, it is either 1 or 0.

(i) If $P(\neg A, \Gamma) = 1$, then $\neg A \in S(\Gamma)$ and $P(A, \Gamma) = 0$.

(ii) If $P(\neg A, \Gamma) = 0$, then $\neg\neg A \in S(\Gamma)$ and, by D.2, $A \in S(\Gamma)$ and thus $P(A, \Gamma) = 1$.

P.3 Similar to P.2

P.4 If $P(A \wedge B, \Gamma)$ is defined, it is either 1 or 0.

(i) If $P(A \wedge B, \Gamma) = 1$, then $A \wedge B \in S(\Gamma)$. Using D.4, D.5, and D.6, we get $B \wedge A \in S(\Gamma)$ and thus $P(B \wedge A, \Gamma) = 1$.

(ii) Similar to (i).

P.5 If $P(A \wedge B, \Gamma)$ is defined and > 0, then $P(A \wedge B, \Gamma) = 1$. Then $A \wedge B \in S(\Gamma)$ and by D.4, $A \in S(\Gamma)$. By Lemma 4.3, $S(\Gamma) = S(\Gamma \cup \{A\})$ and so $P(B, \Gamma \cup \{A\}) = P(B, \Gamma)$. Thus $B \in S(\Gamma)$ and by D.5, $P(B, \Gamma) = 1$.

P.6 If $P(A, \Gamma)$ is defined, it is either 1 or 0.

(i) If $P(A, \Gamma) = 1$, then $A \in S(\Gamma)$. By Lemma 4.3, $S(\Gamma) = S(\Gamma \cup \{A\})$ and as $P(B, \Gamma \cup \{A\})$ is defined, $P(B, \Gamma\{A\}) = P(B, \Gamma)$ is either 1 or 0. If $P(B, \Gamma) = 1$, $B \in S(\Gamma)$ and by D.6, $A \wedge B \in S(\Gamma)$, $P(A \wedge B, \Gamma) = 1$ and thus is defined.

(ii) If $P(A, \Gamma) = 0$ then $\neg A \in S(\Gamma)$. By D.7, $\neg(\neg\neg A \wedge \neg\neg B) \in S(\Gamma)$, and by D.9 $\neg(A \wedge B) \in S(\Gamma)$. Thus $P(A \wedge B, \Gamma) = 0$ and it is defined.

P.7 Similar to (ii) of P.6.

P.8 If $P(A, \Gamma)$ is undefined, then $P(A \wedge B, \Gamma)$ is undefined unless $P(B, \Gamma) = 0$. Let us suppose that $P(A \wedge B, \Gamma)$ is defined and $P(B, \Gamma) \neq 0$.

(i) If $P(A \wedge B, \Gamma) = 1$ then $A \wedge B \in S(\Gamma)$ and thus $A \in S(\Gamma)$. In that case, $P(A, \Gamma) = 1$ and it is defined.

(ii) If $P(A \wedge B, \Gamma) = 0$, then $\neg(A \wedge B) \in S(\Gamma)$ and by $\vee$-saturation $\neg B \in S(\Gamma)$ or $\neg A \in S(\Gamma)$. In the first case $P(B, \Gamma) = 0$ and in the second $P(A, \Gamma)$ is defined.

P.9 If $P(A, \Gamma)$ is defined and > 0, then $P(A, \Gamma) = 1$ and $A \in S(\Gamma)$. If $P(A \wedge B, \Gamma) = 0$, then $\neg(A \wedge B) \in S(\Gamma)$. By $\vee$-saturation, $\neg A \in S(\Gamma)$ or $\neg B \in S(\Gamma)$. Thus $\neg B \in S(\Gamma)$. But $A \in S(\Gamma)$ implies by Lemma 4.3 that $S(\Gamma) = S(\Gamma \cup \{A\})$ and thus $\neg B \in S(\Gamma \cup \{A\})$. This implies that $P(B, \Gamma \cup \{A\}) = 0$.

NP.1 Trivial.
NP.2 Trivial.
NP.3 Like in the classical case.
NP.4 Two cases.

1. $P(A \wedge B, \Gamma) = 1$ and thus $P(A, \Gamma) = 1$. By Lemma 4.3, $S(\Gamma) = S(\Gamma \cup \{A\})$ and $1 = P(B, \Gamma) = P(B, \Gamma \cup \{A\})$. Thus $P(A \wedge B, \Gamma) = P(A, \Gamma) \times P(B, \Gamma \cup \{A\})$.
2. $P(A \wedge B, \Gamma) = 0$. If $P(A, \Gamma) = 0$ done by P.7. If $P(A, \Gamma) = 1$, $A \in S(\Gamma)$ and if $P(A \wedge B, \Gamma) = 0$, $\neg(A \wedge B) \in S(\Gamma)$ and thus $\neg(\neg\neg A \wedge \neg\neg B) \in S(\Gamma)$. Either $\neg A \in S(\Gamma)$ or $\neg B \in S(\Gamma)$. Thus $\neg B \in S(\Gamma)$ and $0 = P(B, \Gamma) = P(B, \Gamma \cup \{A\})$ by Lemma 4.3.

NP.5 Trivial.
NP.6 Trivial.
NP.7 $\Gamma \cup \{A \wedge B\}$ has exactly the same consequences as $\Gamma \cup \{A, B\}$.
Thus one can easily prove that $S(\Gamma \cup \{A \wedge B\}) = S(\Gamma \cup \{A, B\})$.

Thus P is a strongly canonical partial probability function, and that finishes our completeness proof. □

An examination of our completeness proof yields two interesting corollaries, which we state here without further proof.

Corollary 4.4 *If $\Gamma \not\vdash A$ and $\Gamma \not\vdash \neg A$ and A is not a classical tautology nor a classical contradiction, then for any P there is Δ and Δ' such that $P(A, \Gamma \cup \Delta) = 1$ and $P(A, \Gamma \cup \Delta') = 0$ (unless $\Gamma \cup \Delta'$ is P-abnormal).*

Corollary 4.5 *If $\Gamma \not\vdash A$ and $\Gamma \not\vdash \neg A$ there is a P such that $P(A, \Gamma)$ is undefined.*

5 Conclusion

We have shown that there is a probabilistic semantics that is both sound and complete for (almost) every extension of the partial logics of Lapierre and Lepage. Unlike the case with classical probability theory, we permit probability distributions to be only partial functions. The degree of "partiality" allowed is dictated by P.1 – P.9 of Section 2.3 above.

In order to give some examples, suppose we have three coins. Each coin has two sides. We will use H_i to stand for the assertion that coin i is showing heads. Further suppose we make the following assignments:

(e.1) $P(H_1, \varnothing) = 0.5$
(e.2) $P(H_2, \varnothing) = 1.0$
(e.3) $P(H_3, \varnothing) = \text{undefined}$

Recall that we are using $(A \vee B)$ as shorthand for $\neg(\neg A \wedge \neg B)$ and $(A \supset B)$ as shorthand for $(\neg A \vee B)$.

Suppose $\varnothing$ is P-normal, and let B be any sentence whatsoever, even one whose probability value is undefined. Then we may proceed as follows:

(e.4 $P(\neg H_2, \varnothing) = 0$ NP.5
(e.5) $P(\neg H_2 \wedge \neg B, \varnothing) = 0$ P.7
(e.6) $P(H_2 \vee B, \varnothing) = 1$ NP.5

In short, knowing that H_2 has a probability of 1 allows us to compute the value of the disjunction of H_2 with any other proposition, even one whose probability value is undefined. Using H_1 instead of H_2, we could not carry out a similar sequence of calculations to determine the value of $(H_1 \vee B)$ if the value of B is undefined. To begin with we would have

(e.4′) $P(\neg H_1, \varnothing) = 0.5$ NP.5

But then we could not apply P.7. In fact, P.8 and P.4 would require that the value of the disjunction $H_1 \vee B$ must be undefined if the value of B is undefined.

Similarly, P.8 and P.4 would require that unless B has a value of 1, the value of the disjunctions $H_3 \vee B$ and $B \vee H_3$ must be undefined. Thus even the value of $H_3 \vee \neg H_3$ must be undefined. And then the value of $H_1 \vee (H_3 \vee \neg H_3)$ must also be undefined.

In a similar way, it will be easy for the reader to verify the following, for the case in which the value of B is undefined:

(e.7) $P((H_1 \wedge \neg H_1) \supset B, \varnothing) = 1$
(e.8) $P((B \wedge \neg B) \supset H_1, \varnothing) = \text{undefined}$

However, the partial distributions we have defined here are not the final answer to the problem of probabilistic omniscience, and more research is needed. For example, if $P(r, \Gamma)$ is defined and A is any sentence involving only the letter r, then $P(A, \Gamma)$ will also have to be defined. But even this minimal degree of probabilistic omniscience does not seem to correspond to real agents.

In short, we would like to emphasize that our restrictions on partial probability distributions are still rather stringent, insofar as modeling actual belief systems is concerned. In particular, we required that any probability value that *could* be computed from defined values must not be left undefined. Of course, real people do not even approximate computational completeness. So although we have made some progress, there is still room for research on this topic.

Note

1. If the logic contains quantifiers, the notion of saturated set is a little bit more complex: $\exists x A \in \Gamma$ iff $A(y/x) \in \Gamma$ for some y such that bla bla bla and $\forall x A \notin \Gamma$ iff $A(y/x) \notin \Gamma$ for some y such that bla bla bla, where bla bla bla are specific conditions of no interest here. For sake of simplicity, we will just consider $\vee$.

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