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Iteration One More Time

Roy T. Cook

Abstract A neologicist set theory based on an abstraction principle (*NewerV*) codifying the iterative conception of set is investigated, and its strength is compared to Boolos's *NewV*. The new principle, unlike *NewV*, fails to imply the axiom of replacement, but does secure powerset. Like *NewV*, however, it also fails to entail the axiom of infinity. A set theory based on the conjunction of these two principles is then examined. It turns out that this set theory, supplemented by a principle stating that there are infinitely many nonsets, captures all (or enough) of standard second-order ZFC. Issues pertaining to the axiom of foundation are also investigated, and I conclude by arguing that this treatment provides the neologicist with the most viable reconstruction of set theory he is likely to obtain.

1 Motivation

There are (at least) two reasons for investigating abstraction principles for set theory. The first concerns the technical feasibility of a neologicist foundation for all of mathematics. The second concerns the connection between the theory of Fregean extensions (as codified in various restrictions of *Basic Law V*) and the mathematical notion of set (as codified in various axiomatic set theories, such as ZFC).[1]

Neologicists argue that we can reproduce (the most important parts of) mathematics using abstraction principles. An abstraction principle is any second-order formula[2] of the form

$$(\forall P)(\forall Q)[@(P) = @(Q) \leftrightarrow E(P, Q)].$$

'@' here is a function from properties (or relations) to objects, and E is an equivalence relation on the properties (or relations). Abstraction principles are intended, in some sense, to be implicit definitions of the abstraction operator @ occurring on the left-hand side of the biconditional, and as a result allow us to take, as objects, characteristics that the properties (or relations) have in common.

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Frege's *Basic Law V* is

BLV $(\forall P)(\forall Q)[\text{EXT}(P) = \text{EXT}(Q) \leftrightarrow (\forall x)(Px \leftrightarrow Qx)]$.

Frege derives all of arithmetic from *BLV* plus second-order logic, but Russell's discovery that *BLV* is inconsistent with the second-order comprehension axiom renders this result less noteworthy. The resurrection of logicism stems from the observation that Frege's only ineliminable use of *BLV* occurs in his derivation of *Hume's Principle*:[3]

HP $(\forall P)(\forall Q)[\text{NUM}(P) = \text{NUM}(Q) \leftrightarrow P \approx Q]$.

($P \approx Q$ is the second-order formula asserting that there is a 1–1 correspondence between the Ps and the Qs.)

The 'NUM' operator is, in effect, a number generating function, mapping properties onto the number corresponding to the cardinality of the extension of the property. Unlike *BLV* above, *HP* is consistent. Frege's derivation of arithmetic in the *Grundgesetze* [8] can be reconstructed in second-order logic plus *HP*, thereby avoiding the troublesome *BLV*. This result, quite remarkable as a mathematical fact independent of any philosophical implications, has come to be called *Frege's Theorem*.

Given the success of *Hume's Principle*, neologicists have attempted to extend this treatment to more powerful mathematical theories. Although the results are somewhat promising in the case of real analysis (see Hale [9]), the attempts to capture set theory within the neologicist framework have so far been disappointing (see Shapiro and Weir [13]). The purpose of this paper is to further investigate such neo-Fregean treatments of set theory.

Two issues arise when one is reconstructing mathematical theories within the neologicist framework, one purely mathematical and one purely (or primarily) philosophical. First, one has to formulate abstraction principles which provide one with what is recognizably the mathematical theory in question. Second, one needs to defend these principles as neologicistically acceptable, where the notion of "acceptable" might be fleshed out in terms of analyticity, implicit definition, stipulation and so on. I shall have little to say here with regard to the second issue, and that only in passing. It is the first issue that is addressed by the results below.

Even if one is not amenable to the philosophy of mathematics espoused by neo-Fregeans, the framework provided by neologicist style variants of *Basic Law V* nevertheless provides an elegant and powerful setting within which to study and compare various intuitive notions of set (or of collection). Boolos's *NewV* [3] was formulated in order to capture one popular idea underlying attempts to provide a foundation for set theory (and thus for all of mathematics), the limitation of size conception of set. *NewerV*, the abstraction principle introduced below, is intended to codify its main rival, the iterative conception of set. As we shall see, *NewV* and *NewerV* provide quite different theories of Fregean extensions (i.e., set theories), and neither provides an account of sets as strong as second-order ZFC. As a result, we seemed forced to accept that the notion of set and the accompanying formal set theory accepted and studied by mathematicians and philosophers outstrips the content of both the limitation of size doctrine and the iterative conception of set.

2 Two Notions of Set

Historically there are (at least) two competing notions of set that have motivated mathematicians and philosophers studying the foundations of mathematics, the iterative conception and the limitation of size conception.[4]

Boolos, in his insightful comparison of the two notions in "Iteration Again", characterizes two versions of the limitation of size notion:

> On a stronger version of limitation of size, objects form a set if and only if they are not in one-one correspondence with all the objects there are. On a weaker, there is no set whose members are in one-one correspondence with all objects, but objects do form a set if they are in one-one correspondence with the members of a given set. (Under certain natural conditions, this last hypothesis can be weakened to: if there are no more of them than there are members of a given set.) The difference between the two versions is that the weaker does not guarantee that objects will always form a set if they are not in one-one correspondence with all objects. ([3], p. 90)

Boolos's *NewV*, which will be examined briefly in Section 3, corresponds to a neo-logicist reconstruction of the stronger version of the limitation of size conception of set.

The iterative notion of set, founded on the idea that each set is built up from other sets or objects that are simpler, or at least prior, is characterized by Boolos as follows:

> According to the iterative, or cumulative, conception of sets, sets are formed at stages; indeed, every set is formed at some stage of the following "process": at stage 0 all possible collections of individuals are formed.... The sets formed at stage 1 are all possible collections of sets formed at stage 0, ... The sets formed at stage 2 are all possible collections of sets formed at stages 0 and 1. The sets formed at stage 3 are all possible collections of sets formed at stages 0, 1, and 2, ... The sets formed at stage 4, ... In general, for any natural number n, the sets formed at stage n are all possible collections of sets formed at stages earlier than n, i.e., stages $0, 1, \ldots, n-1$. Immediately after all stages $0, 1, 2, \ldots$ there is a stage, stage ω. The sets formed at stage ω are, similarly, all possible collections of collections of sets formed at stages earlier ω, i.e., stages $0, 1, 2, \ldots$. After stage ω comes stage $\omega + 1$: at which .... In general, for each α, the sets formed at stage α are all possible collections of sets formed at stages earlier than α. There is no last stage: each stage is immediately followed by another. Thus there are stages $\omega + 2, \omega + 3, \ldots$. Immediately after all of these, there is a stage $\omega + \omega$, alias $\omega \cdot 2$. Then $\omega \cdot 2 + 1, \omega \cdot 2 + 2$, etc. Immediately after all $\omega, \omega \cdot 2, \omega \cdot 3, \ldots$ comes $\omega \cdot \omega$, alias ω^2. Then $\omega^2 + 1, \ldots$ and so it goes. ([3], p. 88)

Boolos gives a formal axiomatization of stages, and sets formed at stages, and investigates which set-theoretic axioms follow from this characterization. In many respects we will end up agreeing with Boolos's conclusions. There is one major point of (possible) disagreement, however, concerning the axiom of infinity. Thus, a brief look at Boolos's discussion of infinity is in order.

Boolos argues that the axiom of infinity follows from the iterative conception of sets, but this is only because, in characterizing sets as formed in stages, he assumes that there is a limit stage, that is, stage ω. After providing an axiom called *Inf* he writes:

> *Inf* states that there is a "limit" stage, a stage later than some stage but not immediately later than any stage earlier than it: the existence of stage ω and hence of such a stage as *Inf* claims to exist is a notable feature of the conception we have described. *Inf* is too weak to capture the full strength of the claims about the existence of infinite stages made in the rough description; a further axiom would be needed to guarantee the existence of a stage $\omega + \omega$,

> for example. It suffices, however, for the derivation of the sentence of set theory customarily called "the axiom of infinity". *Inf*, it should be noted, is used only in the derivation of the axiom of infinity. ([3], p. 92)

Even if too weak to capture all of the iterative conception as described in the passage quoted earlier, *Inf* is still, as Boolos puts it, quite "notable," since it amounts to nothing less than assuming the truth of the axiom of infinity. This is not to say that Boolos has given an incorrect description of (the intuitions behind) the iterative conception, rather, he has described one conception of set, which we might call Boolos-iterative set theory, that is codified by something like ZFC-replacement. In what follows, a more general conception of iteration (based on abstraction) will be presented, one that does not itself imply the axiom of infinity. Within this framework we can isolate additional principles of varying strength that imply (among other things) the existence of an infinite set. In particular, we will see exactly what assumptions are needed in order to arrive at a theory akin to Boolos-iterative set theory.

3 New V

As the first step toward a neologicist account of the limitation of size conception of set theory, a variation of *BLV* due to Boolos [3] called *NewV* has been proposed (where '$Big(P)$' is an abbreviation for the second-order formula asserting that the Ps are equinumerous with the entire domain):[5]

NewV: $$(\forall P)(\forall Q)[\text{EXT}(P) = \text{EXT}(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \vee (Big(P) \wedge Big(Q)))].$$

A set is the extension of a *small* property:

$$\text{Set}(x) \leftrightarrow (\exists P)[x = \text{EXT}(P) \wedge \neg Big(P)].$$

The membership relation is defined in terms of the EXT operator:

$$x \in y \leftrightarrow (\exists P)[Px \wedge y = \text{EXT}(P)].$$

Restricting the relevant quantifiers to sets, *NewV* entails the second-order extensionality, separation, empty set, pairing, and replacement axioms.[6] Oddly, however, *NewV* proves the negation of the *union axiom*:

Union: $$(\forall x)(\text{Set}(x) \rightarrow (\exists y)(\text{Set}(y) \wedge (\forall z)(z \in y \leftrightarrow (\forall w)(z \in w \ \wedge w \ \in x)))).$$

The reason for this failure is that the singleton of the '*Bad*' extension (i.e., the singleton of the extension of all '*Big*' properties) is a set, but its union, the '*Bad*' extension itself, is not.

We can reformulate the union axiom so that, for any set, the axiom asserts the existence of another set that contains exactly the elements of every set that is contained in the original set, that is, we ignore any elements of the original set that are not sets themselves:

*Union**: $$(\forall x)(\text{Set}(x) \rightarrow (\exists y)(\text{Set}(y) \wedge (\forall z)(z \in y \leftrightarrow (\forall w)(\text{Set}(w) \wedge (z \in w \ \wedge w \ \in x))))).$$

NewV entails this variant of the axiom, since $\cup\{\varnothing\} = \cup\{\text{EXT}(x = x)\} = \varnothing$. Here I do not wish to get embroiled in debates regarding which of these is the "correct" formulation of the union axiom, so in what follows we shall examine the behavior of both principles.

There are a number of ways that we might further restrict the notion of set. First, we lay down two conditions concepts might satisfy:

$$\begin{aligned} \mathit{BoolosClosed}(F) &\leftrightarrow (\forall y)((\mathrm{Set}(y) \wedge (\forall z)(z \in y \rightarrow Fz)) \rightarrow Fy) \\ \mathit{Transitive}(F) &\leftrightarrow (\forall y)((\mathrm{Set}(y) \wedge Fy)(\forall z)(z \in y \rightarrow Fz) \end{aligned}$$

We can then define several useful conditions that sets might satisfy:

$$\begin{aligned} \mathit{BoolosPure}(x) &\leftrightarrow (\forall F)(\mathit{BoolosClosed}(F) \rightarrow Fx) \\ \mathit{Transitive}(x) &\leftrightarrow (\forall F)((\forall y)(Fy \leftrightarrow y \in x) \rightarrow \mathit{Transitive}(F)) \\ \mathit{Hereditary}(x) &\leftrightarrow (\exists F)(\mathit{Transitive}(F) \wedge (\forall y)(Fy \rightarrow \mathrm{Set}(y)) \wedge Fx). \end{aligned}$$

Intuitively, Boolos-pure[7] sets are those that we can "build up" from the empty set. A set is hereditary if its members are sets, and the members of its members are sets, and the members of the members of its members are sets . . . ad infinitum. We can straightforwardly prove that *NewV* (in fact, any consistent restriction of *Basic Law V*) implies that all Boolos-pure sets are hereditary. The possibility, within *NewV* set theory, of hereditary sets that are not Boolos-pure has been extensively studied in Uzquiano and Jané [14].

If we restrict the quantifiers to Boolos-pure sets or hereditary sets we can still derive the axioms of extensionality, separation, empty set, pairing, union (the original formulation, in addition to union*), and replacement. *NewV* also proves the axiom of foundation when relativized to the Boolos-pure sets, although foundation may fail for the hereditary sets (see [14]). The failure of foundation for hereditary sets will become important in the discussion of the iterative conception below.

Neither the axiom of infinity nor the powerset axiom (nor either of them relativized to any of the restrictions discussed above) follow from *NewV* alone, however.[8] We should note that the failure of these axioms does not depend on the particular way in which we interpreted 'set' and '∈' within the theory of *NewV*, since the conditions relevant to the satisfaction of infinity and powerset can be formulated independently of these definitions (namely, that the universe contain a non-'*Big*' extension holding of infinitely many extensions on the one hand, and that the collection of extensions must be either countably infinite or of size $\beth_\alpha$ for a limit α on the other). Thus, if we wish to formulate a neologicist set theory of a strength similar to that of ZFC, *NewV* is mathematically inadequate.

4 The Basic Formal Theory

The first step in formulating the iterative conception of set within the neologicist framework is to generate, in some neologicistically acceptable way, an ordering of some definite collection of objects that can serve to enumerate stages. We achieve this by utilizing a variant of the *Order-Type Abstraction Principle*:[9]

$$\mathit{OAP}: \quad (\forall R)(\forall S)[\mathrm{OT}(R) = \mathrm{OT}(S) \leftrightarrow R \cong S].$$

Of course, *OAP* is inconsistent—the *Burali-Forti Paradox* can be derived from it. Consider, however, the *Size-Restricted Ordinal Abstraction Principle*:[10]

$$\begin{aligned} \mathit{SOAP}: \quad (\forall R)(\forall S)[\mathrm{ORD}(R) = \mathrm{ORD}(S) \leftrightarrow (((\neg \mathrm{WO}(R) \vee \mathit{Big}(R)) \\ \wedge(\neg \mathrm{WO}(S) \vee \mathit{Big}(S)) \vee (\mathrm{WO}(R) \wedge \mathrm{WO}(S) \\ \wedge R \cong S \wedge \neg \mathit{Big}(R) \wedge \neg \mathit{Big}(S)))]. \end{aligned}$$

We first note that *SOAP* is satisfiable (our metatheory throughout the paper will be first-order ZFC-foundation):

Theorem 4.1 *SOAP can be satisfied on any infinite set.*

Proof Given an infinite set X, we can construct a model of *SOAP* with X as domain: Let κ be the cardinality of X. Then there is a 1–1 mapping f from κ onto X. For each non-*Big* well-ordering R on X, $\mathrm{ORD}(R)$ is $f(\gamma+1)$ where $\gamma < \kappa$ is the ordinal such that R is isomorphic to γ. For any relation R on X where R either is not a well-ordering or is *Big*, $\mathrm{ORD}(R)$ is $f(0)$. □

Additionally, *SOAP* is only satisfied on infinite models.

Theorem 4.2 *Any model of SOAP has an infinite domain.*

Proof Assume that M is a model of *SOAP* with domain D where $|D| = n$ for some finite n. Then there are distinct objects given by *SOAP* for each of the well-ordering types $0, 1, \ldots, n-1$, and there is an object that is the value of $\mathrm{ORD}(R)$ for any R that is *Big* or not a well-ordering. Since this latter object is distinct from the objects given by *SOAP* for each of the non-*Big* ordering types, D contains at least $n+1$ distinct objects. Contradiction. □

The following abbreviation will be useful:

$$\mathrm{ON}(\alpha) \leftrightarrow (\exists R)(\alpha = \mathrm{ORD}(R) \wedge \neg Big(R) \wedge \mathrm{WO}(R)).$$

It is important to emphasize that ordinal numbers (i.e., the objects in the range of the ORD operator), upon which we will be building our neologicist account of set theory, are not (or are not necessarily) identical to the sets that we usually call ordinals (i.e., $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}, \ldots, \omega$, etc.). Thus, in what follows we will be careful to distinguish between ordinal numbers (i.e., $\mathrm{ORD}(R)$ for some R) and ordinals (i.e., transitive pure sets well-ordered by membership). Nevertheless, there is a clear correspondence between the class of ordinals and the class of ordinal numbers, and we shall, for convenience, use lower case Greek letters for both.

We define the ordering on the ordinal numbers generated by *SOAP* in the usual way:

$$\begin{aligned} \mathrm{ORD}(R) < \mathrm{ORD}(S) \quad &\leftrightarrow \quad (\exists f)((\forall x)(\forall y)((R(x,y) \rightarrow S(f(x), f(y))) \\ &\wedge \quad (\exists z)(\forall w)((\exists v)(R(v,w) \rightarrow S(f(w), z))). \end{aligned}$$

The common theorems about well-orderings can be proved to hold of the ordinal numbers generated by *SOAP* by standard proofs and are assumed in what follows. Most important is the fact that Theorems 4.1 and 4.2 imply the following corollary.

Corollary 4.3 *For any model of SOAP, the collection of ordinal numbers, ordered by <, is isomorphic to an infinite* ZFC *cardinal number.*

This implies that, in any model of *SOAP*, there is no last ordinal number.

Next, we have a principle telling us what objects we have access to prior to "applying" the iterative operation of set formation. This *Basis Axiom* will be some instance of the following schema:

BA: $\quad BASE(x) \leftrightarrow \Phi.$

The strength of our iterative set theory will depend greatly on what formula we select for Φ, as we shall see in Sections 8 and 11 below when we examine some particular candidates. There is no restriction that the members of the basis are not sets, and we allow that the '*Bad*' extension ('*Bad*' within the present context is defined below) might be contained in the basis.

We now define the notion of 'stage'. (In what follows, three different membership symbols will appear. $\in_S$ is used when defining our notion of stage. $\in_N$ is the set-theoretic membership relation defined within the neologicist set theory. Finally, $\in$ without subscripts is to be understood as the membership relation of first-order ZFC-foundation, used when we are working in the metatheory. Subscripts—or their lack thereof—will also be used to label notions defined in terms of the various notions of membership.)

$$\begin{aligned} x \in_S \text{Stg}(\alpha) \quad &\leftrightarrow \quad \text{ON}(\alpha) \wedge BASE(x) \vee (\exists P)(x = \text{EXT}(P) \\ &\wedge \quad (\exists \beta)(\text{ON}(\beta) \wedge \beta < \alpha \wedge (\forall y)(Py \rightarrow y \in_S \text{Stg}(\beta)))). \end{aligned}$$

The first stage consists of the elements of basis, and each succeeding stage contains the basis (if any) plus the extension of every property all of whose instances are contained in some prior stage. This definition guarantees that if $x \in_S \text{Stg}(\alpha)$ for some ordinal number n, then $x \in_S \text{Stg}(\beta)$ for all $\beta > \alpha$. The following abstraction principle "generates" extensions of properties within the iterative hierarchy:

NewerV: $$\begin{aligned} &(\forall P)(\forall Q)[\text{EXT}(P) = \text{EXT}(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \\ &\quad \vee(\neg(\exists \alpha)(\text{ON}(\alpha) \wedge (\forall x)(Px \rightarrow x \in_S \text{Stg}(\alpha))) \\ &\quad \wedge\neg(\exists \alpha)(\text{ON}(\alpha) \wedge (\forall x)(Qx \rightarrow x \in_S \text{Stg}(\alpha)))))]. \end{aligned}$$

To clarify things, we can reword *NewerV* along the lines of Boolos's *NewV*:

NewerV: $$\begin{aligned} &(\forall P)(\forall Q)[\text{EXT}(P) = \text{EXT}(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \\ &\quad \vee(Bad(P) \wedge Bad(Q)))] \end{aligned}$$

where

$$Bad(P) \leftrightarrow \neg(\exists \alpha)(\text{ON}(\alpha) \wedge (\forall x)(Px \rightarrow x \in_S \text{Stg}(\alpha))).$$

Boolos's definitions of set and membership can be reformulated in the present context to obtain similar notions within *NewerV* set theory (*NewerV* set theory should be understood to denote the theory that follows from the conjunction of *NewerV* and *SOAP*):[11]

$$\begin{aligned} \text{Set}(x) \quad &\leftrightarrow \quad (\exists P)[x = \text{EXT}(P) \wedge \neg Bad(P)]\,, \\ x \in_N y \quad &\leftrightarrow \quad (\exists P)[Px \wedge y = \text{EXT}(P)]. \end{aligned}$$

Along these lines, we can define notions of Boolos-pure sets, hereditary sets, and so on just as was done for *NewV* above.

A clarification at this point is useful to avoid confusion. We can define the notion of urelement in the standard way as follows:

$$\text{Ure}(x) \leftrightarrow \neg\text{Set}(x).$$

Again, there is no guarantee that the elements of the basis are all urelements, or vice versa.

If we restrict the relevant quantifiers to sets, then *NewerV* entails the extensionality, empty set, separation, union* (but not union), pairing, and powerset axioms. Derivations are given in the appendix. *NewerV*, like *NewV*, proves the axiom of foundation if restricted to Boolos-pure sets, although foundation may fail to hold of the hereditary sets or sets in general. Similarly, the union axiom holds when restricted to Boolos-pure or hereditary sets.

5 *NewerV* and Abstraction

NewerV, as formulated above, is circular—it contains reference to stages on the right-hand side of the biconditional yet our definition of stage contains explicit use of the extensions forming operator supposedly being defined. On the face of it this objection does not seem overly compelling—as is well known, for the implicit definitions codified in abstraction principles such as *Hume's Principle* and *NewV* to do the work intended, the quantifiers on the right-hand side of the biconditional must range over all objects, including the abstracts being introduced (and defined) on the left. Once this is accepted, there seems little reason not to explicitly refer to extensions in our definition of the identity conditions for extensions, since we are already forced to quantify over them in such a definition. Nevertheless, a method by which to avoid this outright circularity would no doubt be welcomed, and fortunately such a method exists.

In order to avoid such circularity, we could have our extensions forming operator 'EXT' apply, not to concepts, but to pairs (P, α) where P is a concept and α is an ordinal number. We then define the notion of stage* as

$$\begin{aligned} x \in_S \text{Stg}^*(\alpha) \quad &\leftrightarrow \quad \text{ON}(A) \wedge \mathit{BASE}(x) \vee (\exists\beta)(\exists P)(\text{ON}(\beta) \wedge \beta < \alpha \\ &\wedge \quad x = \text{EXT}^*(P, \beta) \wedge (\forall y)(Py \rightarrow y \in_S \text{Stg}^*(\beta)). \end{aligned}$$

The appropriate abstraction principle would be

*NewerV**: $$(\forall P)(\forall\alpha)(\forall Q)(\forall\beta)[\text{EXT}^*(P,\alpha) = \text{EXT}^*(Q,\beta) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \\ \vee(\neg(\text{ON}(\alpha) \wedge (\forall x)(Px \rightarrow x \in_S \text{Stg}^*(\alpha))) \\ \wedge\neg(\text{ON}(\beta) \wedge (\forall x)(Qx \rightarrow x \in_S \text{Stg}^*(\beta)))))],$$

and we would define set and membership as

$$\begin{aligned} \text{Set}(x)^* \quad &\leftrightarrow \quad (\exists P)(\exists\alpha)[x = \text{EXT}^*(P,\alpha) \wedge \neg \mathit{Bad}(P,\alpha)], \\ x \in_N^* y \quad &\leftrightarrow \quad (\exists P)(\exists a)[Px \wedge y = \text{EXT}^*(P,\alpha)], \end{aligned}$$

where

$$\mathit{Bad}(P,\alpha) \leftrightarrow \neg(\text{ON}(\alpha) \wedge (\forall x)(Px \rightarrow x \in_S \text{Stg}^*(\alpha))).$$

In order to insure that this method works, we need to verify that

$$(\forall P)(\forall\alpha)(\forall\beta)((\neg \mathit{Bad}(P,\alpha) \wedge \neg \mathit{Bad}(P,\beta)) \rightarrow \text{EXT}^*(P,\alpha) = \text{EXT}^*(P,\beta)),$$

that is,

$$\begin{aligned} (\forall P)(\forall Q)(\forall\alpha)(\forall\beta)((\neg(\text{ON}(\alpha) \wedge (\forall x)(Px \quad &\rightarrow \quad x \in_S \text{Stg}^*(\alpha))) \wedge \\ \neg(\text{ON}(\beta) \wedge (\forall x)(Qx \rightarrow x \in_S \text{Stg}^*(\beta)))) \quad &\rightarrow \quad \text{EXT}^*(P,\alpha) = \text{EXT}^*(Q,\beta)). \end{aligned}$$

This can be straightforwardly derived from *SOAP* + *NewerV**.

Essentially, we have replaced the circular definition of extensions with a recursive definition, where at each "level" we introduce new extensions defined in terms of the ones at previous levels. To make things more intuitive, we can think of the recursive formulation *NewerV** as schematic for infinitely many noncircular definitions of infinitely many extension-forming operators. First, we obtain the level-0 extensions:

NewerV$_0$: $$(\forall P)(\forall Q)[\text{EXT}_0(P) = \text{EXT}_0(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \\ \vee((\exists x)(Px \wedge \neg \mathit{BASE}(x)) \wedge (\exists x)(Qx \wedge \neg \mathit{BASE}(x))))],$$

that is, level-0 extensions correspond to those collections whose members are members of the basis. We then define level-1 extensions in terms of level-0 extensions:

$$NewerV_1: \quad (\forall P)(\forall Q)[\mathrm{EXT}_1(P) = \mathrm{EXT}_1(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \\ \vee((\exists x)(Px \wedge \neg BASE(x) \wedge \neg(\exists F)(x = \mathrm{EXT}_0(F))) \\ \wedge(\exists x)(Qx \wedge \neg BASE(x) \wedge \neg(\exists F)(x = \mathrm{EXT}_0(F))))],$$

where level-1 extensions correspond to those collections whose members are either members of the basis or are level-0 extensions. We can continue in this way, explicitly defining more general extension operators, where the level-*n* extensions correspond to those sets whose members are either members of the basis or are level-*m* extensions for some $m < n$.

While this method will only take us (at best, assuming both enough objects and enough abstraction principles) as far as level-α extensions for $\alpha < \varepsilon_0$, we should note that each instance of $NewerV_n$ is, within the neologicist framework, an abstraction principle implicitly defining an abstraction operator EXT_n in terms of previously defined operators.[12] $NewerV^*$ is a generalization of this process, allowing us to handle all cases simultaneously (including ranks numbered by ordinals for which we might not have names) and thus does not, perhaps, deserve the title of *abstraction principle* in a literal sense. Nevertheless, $NewerV^*$ is a natural generalization of such a piecemeal process of abstraction and seems well within the spirit, if not the letter, of the neologicist approach.

With $NewerV^*$ in place, we can define the notion of extension simpliciter:

$$\begin{aligned} x = \mathrm{EXT}(P) \quad \leftrightarrow \quad & (\exists\alpha)(\mathrm{ON}(\alpha) \wedge (\forall y)(Py \rightarrow y \in_S \mathrm{Stg}^*(\alpha)) \\ \wedge \quad & x = \mathrm{EXT}^*(P,\alpha)) \vee (\forall\alpha)(\mathrm{ON}(\alpha) \rightarrow (\exists y)(Py \\ \wedge \quad & y \notin_S \mathrm{Stg}^*(\alpha)) \wedge x = \mathrm{EXT}^*(x = x, 0)). \end{aligned}$$

In other words, the extension of a concept P is the extension* of P and any ordinal α such that (P, α) is not *Bad*, and is the *Bad* extension $(\mathrm{EXT}^*(x = x, 0))$ if there is no such ordinal. We can then define stages in terms of extensions as before:

$$\begin{aligned} x \in_S \mathrm{Stg}(\alpha) \quad \leftrightarrow \quad & \mathrm{ON}(\alpha) \wedge BASE(x) \vee (\exists P)(x = \mathrm{EXT}(P) \\ \wedge \quad & (\exists\beta)(\mathrm{ON}(\beta) \wedge \beta < \alpha \wedge (\forall y)(Py \rightarrow y \in_S \mathrm{Stg}(\beta)))). \end{aligned}$$

The resulting derivation of *NewerV* (using this definition) from $NewerV^*$ is left to the reader.

This way of proceeding, using bounded quantification, accomplishes what the simpler formulation *NewerV* does, and in addition makes the recursive nature of iterative extensions more explicit: $\mathrm{EXT}^*(P, \alpha)$ is defined in terms of $\in_S \mathrm{Stg}^*$, and $\in_S \mathrm{Stg}^*$ is defined in terms of $\mathrm{EXT}^*(Q, \beta)$ for $\beta < \alpha$. Of course, it is possible that neo-Fregeans (or their opponents) might find $NewerV^*$ as objectionable as the explicitly circular *NewerV*. At this point I have no additional positive argument for the acceptability of $NewerV^*$ other than its intuitive plausibility.

What can be noted, however, is that if the neo-Fregean refuses to accept both *NewerV* and $NewerV^*$, then he will most likely find himself unable to formulate any version of the iterative conception of set. There seems to be no means by which one can formulate a general iterative principle for abstractions within the neologicist framework other than by providing identity criteria for the two extensions occurring on the left-hand side of the biconditional in terms of conditions being imposed (on the right-hand side of the biconditional) on other extensions that might be members of the original pair of extensions.

The reader should note that *NewerV* is meant to be the "official" formulation of the neologicist iterative conception of sets and will be used below. The "recursive" reformulation *NewerV** is provided only to assuage worries regarding circularity.

6 Standard Models

Given a particular *BASE*, we construct *ranks* within standard second-order ZFC-foundation as follows:

$$\begin{aligned} V_{BASE}(0) &= \{x : BASE(x)\} \\ V_{BASE}(\alpha+1) &= V(\alpha) \cup (\wp(V(a))) \\ V_{BASE}(\gamma) &= \cup_{\lambda<\gamma} V(\lambda)(\gamma \text{ a limit ordinal}). \end{aligned}$$

(The intuitive idea is that the members of the basis that we do not want to be sets in the model based on $V_{BASE}(\kappa)$ can be represented, in the model, by sets of cardinality greater than the cardinality of $V_{BASE}(\kappa)$.) Letting $\otimes$ be an arbitrary set not in $V_{BASE}(\kappa)$, to serve as the *Bad* extension, we can now construct what we will call standard models (consisting of domain and interpretation function) of *NewerV* set theory:[13]

$$M_{(BASE,\kappa)} = \langle V_{BASE}(\kappa) \cup \{\otimes\}, I\rangle$$

where for all relation symbols R,

$$\begin{aligned} I(\mathrm{ORD}(R)) &= \alpha \text{ if } \alpha \text{ is an ordinal in } V_{BASE}(\kappa) \\ &\quad\ \text{and } \langle\alpha, \in\rangle \text{ is isomorphic to } I(R) \\ I(\mathrm{ORD}(R)) &= \otimes \quad \text{otherwise;} \end{aligned}$$

and for any predicate P,

$$\begin{aligned} I(\mathrm{EXT}(P)) &= \{x \in V_{BASE}(\kappa) : x \in I(P)\} \\ &\quad\ \text{if } \{x \in V_{BASE}(\kappa) : x \in I(P)\} \in V_{BASE}(\kappa) \\ I(\mathrm{EXT}(P)) &= \otimes \quad \text{otherwise.} \end{aligned}$$

Note that it might be the case that $\otimes \in BASE$, in which case $V_{BASE}(\kappa) \cup \{\otimes\} = V_{BASE}(\kappa)$. The fact that models of our neologicist iterative set theory "look" like the standard iterative hierarchy is unsurprising. Let us call two structures M and N *extension-isomorphic* if there is a one-one onto function f from the domain of M to the domain of N such that f is an isomorphism with respect to EXT (but not necessarily ORD).

Theorem 6.1 *Any model of SOAP + NewerV of cardinality κ contains a substructure that is extension-isomorphic to $M_{(\varnothing,\kappa)}$.*

Proof Given a model M of cardinality κ with domain D and interpretation function I, let $O \subset D$ be the domain of the 'ORD' operator under I. Since D is of cardinality κ, and M is a model of *SOAP*, O (with its ordering) is isomorphic to κ. We can then construct the copy of $V_{\varnothing}(\kappa)$ (and thus $M_{(\varnothing,\kappa)}$) recursively using O, where the αth rank is just the collection containing the extension of every property all of whose instances occur in ranks less than α (where $V_{\varnothing}(\kappa) = \varnothing$). □

The following result is useful in what follows.

Theorem 6.2 *A standard model $M_{(BASE,\kappa)}$ is a model of SOAP + NewerV if and only if $|V_{BASE}(\kappa)| = \kappa$ and κ is infinite.*

Proof (⇒) Assume $M_{(BASE,\kappa)}$ is a model of *SOAP* + *NewerV*. If $M_{(BASE,\kappa)}$ is a model *SOAP*, then $M_{(BASE,\kappa)}$ must be infinite, but again by *SOAP*, there must be infinitely many ordinal numbers, so κ must be infinite. By an easy induction, $| V_{BASE}(\gamma) | \geq | \gamma |$ for any γ. Assume $| V_{BASE}(\kappa) | > \kappa$. Then, by *SOAP*, there would be $| V_{BASE}(\kappa) |$ many ordinal numbers, but then $M_{(BASE,\kappa)}$ would not be a model of *NewerV* (since *NewerV* entails a rank for each ordinal number). So $| V_{BASE}(\kappa) | = \kappa$.

(⇐) Assume $| V_{BASE}(\kappa) | = \kappa$ and κ infinite. For any κ, $M_{(BASE,\kappa)}$ is a model of *NewerV*. If $| V_{BASE}(\kappa) | = \kappa$, then *SOAP* generates κ many ordinal numbers, the right amount for κ ranks, so $M_{(BASE,\kappa)}$ is a model of *SOAP*. □

Restricting our attention further, to models with empty basis, we have the following theorem.

Theorem 6.3 *For infinite cardinals κ, $| V_{\varnothing}(\kappa) | = \kappa$ if and only if either $\beth_\kappa = \kappa$ or $\kappa = \omega$.*

Proof Evident from the fact that, first, $V_{\varnothing}(\kappa)$ is the hereditarily finite sets, and second, for $\kappa > \omega$, $| V_{\varnothing}(\kappa) | = \beth_\kappa$. □

In other words, $M_{(\varnothing,\kappa)}$ is a model of *SOAP* + *NewerV* if and only if $V_{\varnothing}(\kappa)$ is the collection of hereditarily-less-than-κ sets.

7 The Axiom of Replacement

Our next step is to examine the axiom of replacement:[14]

Replacement: $(\forall x)(\forall f)(\exists y)(\forall z)(z \in_N x \rightarrow f(z) \in_N y)$.

In the specific case of standard models of *SOAP* + *NewerV* where $BASE = \varnothing$ we have the following.

Theorem 7.1 *For any cardinal κ, if $M_{(\varnothing,\kappa)}$ is a model of SOAP + NewerV, then $M_{(\varnothing,\kappa)}$ satisfies the axiom of replacement if and only if κ is regular.*

Proof (⇒) Assume that $M_{(\varnothing,\kappa)}$ is a model of *SOAP* + *NewerV* and the axiom of replacement and assume, for reductio, that κ is not regular, that is, $cf(\kappa) < \kappa$. Then there is an ordinal $\gamma < \kappa$ and a function f such that f maps γ unboundedly into κ. Let S be the range of f restricted to γ. Then there is no ordinal number α such that, for all x in S, $x \in_S \text{Stg}(\alpha)$, so S is *Bad*, that is, not a set. Contradiction, so $cf(\kappa) = \kappa$ and κ is regular.

(⇐) Assume that $M_{(\varnothing,\kappa)}$ is a model of *SOAP* + *NewerV* and that κ is regular. Let x be any set and f any function on $V_{\varnothing}(\kappa)$, and let S be the range of f restricted to x. Clearly, $| S | \leq | x |$. $x \in V_{\varnothing}(\kappa)$, and, since $M_{(\varnothing,\kappa)}$ is a model of *SOAP* + *NewerV*, $\wp(x) \in V_{\varnothing}(\kappa)$, so it follows that $\wp(x) \subseteq V_{\varnothing}(\kappa)$. Thus, $| x | < | V_{\varnothing}(\kappa) |$, and, since $M_{(\varnothing,\kappa)}$ is a model of *SOAP* + *NewerV*, $| x | < \kappa$. So $| S | < \kappa$. Thus, since κ is regular and there is no function from S into κ whose range is unbounded in κ, there must be a $\gamma < \kappa$ such that, for all y in S, $y \in_S \text{Stg}(\gamma)$. Thus S is not '*Bad*', so $\text{EXT}(S)$ is a set. □

This allows us to prove that the axiom of replacement does not follow from *SOAP* + *NewerV*. Define π as follows:[15]

$$\pi_0 = \omega$$
$$\pi_{n+1} = \beth_{\pi_n}$$
$$\pi = \sup\{\pi_i : i < \omega\}.$$

$M_{(\varnothing,\pi)}$ is a model of *SOAP* + *NewerV* since $\beth_\pi = \pi$[16] (in fact, it is the least such cardinal) but π is not regular, since $cf(\pi) = \omega$. Thus the axiom of replacement does not follow from *SOAP* + *NewerV*. The failure of replacement is, in this context, equivalent to the failure of the following *Same Size Principle*:

SSP: $(\forall P)(\forall Q)(\neg Bad(P) \wedge P \approx Q) \rightarrow \neg Bad(Q))$.

On the other hand, the following *Size Restriction Principle* does hold:

SRP: $(\forall P)(\neg Bad(P)) \rightarrow \neg(\exists Q)((\forall x)(Qx) \wedge P \approx Q))$.

That is, no sets are equinumerous with the entire domain.

The failure of replacement should not come as too much of a surprise on the iterative conception of set theory, however. Boolos, in "The Iterative Conception of Set," writes that

> There is an extension of the stage theory from which the axioms of replacement could have been derived. We could have taken as axioms all instances . . . of a principle which may be put, "If each set is correlated with at least one stage (no matter how), then for any set z there is a stage s such that for each member w of z, s is later than some stage with which w is correlated." This *bounding* or *cofinality* principle is an attractive further thought about the interrelation of sets and stages, but it does seem to us to be a further thought, and not one that can be said to have been meant in the rough description of the iterative conception . . . . Thus the axioms of replacement do not seem to us to follow from the iterative conception. (Boolos [2], p. 26–27)[17]

In the later paper [3], after considering ways in which the iterative conception might be strengthened to secure replacement, he writes that

> Whether some such strengthening. . . can be plausibly thought not to involve a new principle that is not really part of the iterative conception seems doubtful. ([3], p. 97)

Although Boolos's examination is not conducted within the neologicist framework, his comments regarding replacement agree both with the results obtained here and with intuition. Unlike the limitation of size conception, the iterative conception regards the size of a collection as irrelevant to whether or not it receives the honorific 'set'. All that matters is whether the objects contained in the collection are formed at some point in the hierarchy.

8 The Basis and Infinity

SOAP + *NewerV* plus the following *Basis Axiom*,

$BA_\varnothing$: $BASE(x) \leftrightarrow x \neq x$,

that is, *NewerV* set theory with an empty basis, or what we might call *pure NewerV* set theory, is extremely weak. Of the standard axioms, the only ones that hold here are those proved above. In addition to the failure of the axiom of replacement, the axiom of infinity,[18]

Infinity: $(\exists x)(\varnothing_N \in_N x \wedge (\forall y)(y \in_N x \rightarrow y \cup \{y\}_N \in_N x))$

fails.

Theorem 8.1 *$M_{(\varnothing,\omega)}$ is a model of SOAP + NewerV but fails to satisfy infinity.*

Proof $M_{(\varnothing,\omega)}$ is just the collection of hereditarily finite sets (plus the '*Bad*' extension) which is countable, so $|V_{\varnothing}(\omega)| = \aleph_0$, and $M_{(\varnothing,\omega)}$ is a model of *SOAP* + *NewerV*. There are no infinite sets in $M_{(\varnothing,\omega)}$, however, so the axiom of infinity is not satisfied. □

Theorem 8.1, combined with the results of the previous section, suffice to show that, relative to *SOAP* + *NewerV*, the axiom of infinity and the axiom of replacement are independent, since $M_{(\varnothing,\omega)}$ satisfies replacement but fails to satisfy infinity, and $M_{(\varnothing,\pi)}$ (with π defined as in the last section) satisfies infinity but fails to satisfy replacement. As far as providing the infinite sets needed to construct real and complex analysis, *NewerV* set theory is no better off than *NewV* was.

There seems to be no principled reason why we should not allow ourselves access to some preliminary collection of objects which we can then collect into sets, sets of sets, and so on, however. The elements of the basis are, on more traditional approaches to set theory, often ignored since they do not add anything substantially new to the theory. From the present perspective, however, this is not the case.

In order to guarantee that we have an infinite set, we need only assume that the basis contains infinitely many objects. Consider

BA_{ω}: $\quad BASE(x) \leftrightarrow (\mathrm{ON}(x) \wedge x < \omega)$.

We might justify this axiom by noting that the finite ordinal numbers were guaranteed to exist by *SOAP* alone, prior to any set theoretic theorizing, so there is no reason why we cannot form sets of these, or sets of sets, and so on. If we let *FO* be the (countable) collection of finite ordinal numbers, then we have the following.

Theorem 8.2 *For infinite* κ, $|V_{FO}(\kappa)| = \kappa$ *if and only if* $\beth_{\kappa} = \kappa$.

Proof Similar to that of Theorem 7.1 above. □

With π defined as before, the smallest model[19] of *SOAP* + *NewerV* + BA_{ω} is $M_{(FO,\pi)}$.

Theorem 8.3 *SOAP* + *NewerV* + BA_{ω} *implies the axiom of infinity.*

Proof For any finite ordinal number α, (i.e., any object in the range of ORD), $\alpha \in_S \mathrm{Stg}(0)$. So for any collection of finite ordinal numbers y, $y \in_S \mathrm{Stg}(1)$. In particular, $\varnothing_N \in_S \mathrm{Stg}(1)$. Since the ordinal numbers are infinite (see Theorem 4.2), the collection of collections of finite ordinal numbers (and thus Stg(1)) is uncountably infinite, and therefore so is the universe (this can be expressed in second-order logic, see Shapiro [12], p. 104). Thus all countably infinite well-orderings are not *Big*, so there must be a limit ordinal number β (i.e., an ordinal number β such that, for every $\gamma < \beta$, there is a δ such that $\gamma < \delta < \beta$.) So there is a set z containing all sets formed before β (since $\beta > 1, \varnothing_N \in_N z$, and, since β is a limit ordinal number, for any w, if $w \in_N z$ then $w \cup \{w\}_N \in_N z$, since if $w \in_S \mathrm{Stg}(n)$, then $w \cup_N \{w\}_N \in_S \mathrm{Stg}(n+2)$). □

Note that we did not construct the set that is intuitively associated with the axiom of infinity, that is, ω, the set containing exactly the finite ordinals. The axiom of infinity does not assert the existence of this particular (infinite) set, however, but asserts the existence of a particular kind of infinite set, specifically, some set containing $\varnothing_N$ and closed under the operation mapping x onto $x \cup_N \{x\}_N$. Here we have constructed a set (much) larger that ω satisfying the relevant constraints. ω can be obtained immediately, however, by an application of separation.

There are other variants of the axiom of infinity, and the fact that *SOAP* + *NewerV* + BA_ω implies these variants is nontrivial. Uzquiano [15] has shown that, if the axiom of infinity is formulated as above, then second-order Zermelo set theory[20] does not imply the following variant of the axiom of infinity:[21]

Zermelo Infinity: $(\exists x)(\varnothing_N \in_N x \wedge (\forall y)(y \in_N x \rightarrow \{y\}_N \in_N x))$.

If we formulate Zermelo set theory using *Zermelo Infinity* to express the idea that there is an infinite set, then the original axiom of infinity does not follow. One can easily prove that *Zermelo Infinity* follows from *SOAP* + *NewerV* + BA_ω, however. Thus, *SOAP* + *NewerV* + BA_ω is stronger than Zermelo set theory if the axiom of infinity is formulated in either of these ways.

Once we accept *SOAP* + *NewerV* + BA_ω and see that every model contains at least π ordinal numbers (again with π defined as before), we might be tempted to argue that, since we are guaranteed π ordinal numbers, we should allow all ordinal numbers less than π to be elements of the basis, adopting the following *Basis Axiom*:

BA_π: $BASE(x) \leftrightarrow (\mathrm{ON}(x) \wedge x < \pi)$.

Of course, all models of *SOAP* + *NewerV* + BA_π will be much larger than $M_{(FO,\pi)}$. We could continue, formulating stronger and stronger set theories by allowing more and more of the ordinal numbers into the basis. Once we have started down this route it becomes difficult to know when to stop.

Additionally, it seems reasonable to require that we only add instances of the *Basis Axiom* of the form

BA_β : $BASE(x) \leftrightarrow (\mathrm{ON}(x) \wedge x < \beta)$

where the ordinal β is definable in terms of purely logical vocabulary supplemented, if need be, by the abstraction operators ORD and EXT. Unlike the case of the finite ordinal numbers, however, the proof that π exists cannot be formulated within *NewerV* set theory (even when supplemented by BA_ω), since it relies on replacement. Thus BA_α does not seem to be a promising candidate for a *Basis Axiom*.[22]

There is another option, however—since the ordinal numbers are generated by *SOAP* which is, in some sense, theoretically prior[23] to *NewerV*, why not just allow all ordinal numbers to be contained in the basis? In other words,

BA_{ORD}: $BASE(x) \leftrightarrow \mathrm{ON}(x)$.

We can, in this case, derive a version of the Burali-Forti paradox.

Theorem 8.4 *SOAP* + *NewerV* + BA_{ORD} *is inconsistent.*

Proof If every ordinal number is contained in the basis, and thus in Stg(0), then every collection of ordinal numbers is a set and is in Stg(1). Let us call the set of ordinal numbers O. Let $X = \{S : S$ is a set of ordinal numbers and $(\forall n)(n \in_N S \rightarrow (\forall m)(m < n \rightarrow m \in_N S))\}$. X is a set by the powerset and separation axioms. There is an isomorphism between $\{O, <\}$ and $\{X, \subseteq\}$, $f\colon O \rightarrow X$ and, for $n \in_N O$, $f(n) = \{m : m < n\}$. Thus X ordered by $\subseteq$ is a well-ordering, and, since X is a set, this relation is not *Big*, so there is an ordinal number corresponding to it. But this ordinal number must be greater than any ordinal number in O, since f provides a mapping of the order type of each ordinal number in O into, but not onto, $\{X, \subseteq\}$. Contradiction. □

Corollary 8.5 *SOAP + NewerV implies* ‘$\mathrm{ON}(x)$’ *is Bad.*

Thus the best that we can do (if, in fact, BA_ω is acceptable) is to accept $SOAP + NewerV + BA_\omega$ and as a result we obtain all of second-order ZFC except for the axiom of replacement. It is worth noting that $SOAP + NewerV + BA_\omega$ seems to provide us with a good approximation to the iterative conception of set attributed to Boolos in Section 2 above.

9 *NewV + NewerV*

Neither *NewV* nor $SOAP + NewerV$ alone suffices to capture enough of second-order ZFC for us to claim unequivocally that either provides a mathematically adequate abstractionist account of contemporary set theory. Our next task is to determine whether combining the two principles suffices to provide the neologicist with such a reconstruction. We will consider the conjunction $SOAP + NewerV + NewV$ where we understand occurrences of ‘EXT’ in each principle to be distinct occurrences of the same operator.

Before looking at the formal attributes of $SOAP + NewerV + NewV$, however, we should take note of an obvious line of objection. For the neo-Fregean, at least, an abstraction principle for extensions is meant to provide something like an implicit definition of the abstraction operator ‘EXT’. In the investigation above, *NewV* was understood as providing one candidate for such a definition, and *NewerV* as providing an alternative such definition. In considering a theory containing both *NewV* and *NewerV*, however, we are faced with a situation in which we have, in effect, simultaneously accepted two such definitions. We might legitimately question whether this is coherent, much less desirable. More pointedly, we might wonder which of the two principles is responsible for truly defining the abstraction operator in question, that is, for introducing the new piece of mathematical language and providing its meaning. If one of the abstraction principles amounts to such an implicit definition, what is the role of the other within the neologicist framework?

It is tempting to think that we can avoid the objection by replacing the conjunction of *NewV* and *NewerV* with a single abstraction principle that combines the formal features of both. For example, we might conclude that sets are extensions that are both reachable in the iterative heirarchy and not too ‘*Big*’. Let ‘Bad_{NewV}” be the predicate asserting that a property is *Big*, that is,

$$\begin{aligned} Bad_{NewV}(P) \quad &\leftrightarrow \quad (\exists f)((\forall x)(\forall y)((f(x) = f(y) \rightarrow x = y) \\ &\wedge \quad (\forall x)(\exists y)(Py \wedge f(y) = x)), \end{aligned}$$

and let ‘Bad_{NewerV}’ be the corresponding condition for the iterative conception as developed above, that is,

$$Bad_{NewerV}(P) \leftrightarrow \neg(\exists\alpha)(\mathrm{ON}(\alpha) \wedge (\forall x)(Px \rightarrow x \in_S \mathrm{Stg}(\alpha))),$$

with $\in_S$ Stg defined as before. Then the idea that sets are the extensions of concepts that are neither too ‘*Big*’ nor unreachable in the iterative heirarchy can be formulated as

NewestV: $$\begin{aligned} &(\forall P)(\forall Q)[\mathrm{EXT}(P) = \mathrm{EXT}(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \\ &\vee((Bad_{NewV}(P) \vee Bad_{NewerV}(P)) \\ &\wedge(Bad_{NewV}(Q) \vee Bad_{NewerV}(Q))))]. \end{aligned}$$

Unfortunately, this principle is no more powerful than *NewerV* alone.

Theorem 9.1 *Any model of NewerV is a model of NewestV.*

Proof A consequence of the fact that *NewerV* implies the size restriction principle from Section 7, that is, in any model of *NewerV*, every '*Big*' concept is '*Bad*'. □

Similar problems plague

*NewestV**: $$(\forall P)(\forall Q)[\mathrm{EXT}(P) = \mathrm{EXT}(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \vee (Bad_{NewV}(P) \wedge Bad_{NewerV}(P) \wedge Bad_{NewV}(Q) \wedge Bad_{NewerV}(Q)))].$$

Thus some other strategy must be adopted in order to obtain a theory combining the strength of *NewV* + *NewerV*.

Another option might be simultaneously to adopt the two principles but to treat them as implicit definitions of two distinct abstraction operators, EXT_{NewV} and EXT_{NewerV}, that is, we rewrite *NewV* and *NewerV* as

NewV: $$(\forall P)(\forall Q)[\mathrm{EXT}_{NewV}(P) = \mathrm{EXT}_{NewV}(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \vee (Bad_{NewV}(P) \wedge Bad_{NewV}(Q)))];$$

NewerV: $$(\forall P)(\forall Q)[\mathrm{EXT}_{NewerV}(P) = \mathrm{EXT}_{NewerV}(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \vee (Bad_{NewerV}(P) \wedge Bad_{NewerV}(Q)))].$$

This approach will, from a formal perspective, accomplish what we want, since the constraint on the cardinality of the domain imposed by *NewV* will guarantee that the replacement axiom holds when interpreted in terms of EXT_{NewerV}, and the constraints imposed by *NewerV* will imply that the powerset axiom will hold when interpreted in terms of EXT_{NewV}.[24]

There is philosophical problem, however. Do we identify sets as *NewV* extensions or as *NewerV* extensions? The problem is exacerbated by the following.

Theorem 9.2 *There is a model M such that M is a model of NewV, NewerV, and* $(\forall P)(\mathrm{EXT}_{NewV}(P) \neq \mathrm{EXT}_{NewerV}(P))$.

Proof Let $f : \omega \to V_{\{\otimes_1,\otimes_2\}}(\omega)$ be any enumeration of $V_{\{\otimes_1,\otimes_2\}}(\omega)$. Then (extending the notion of standard model in the obvious way) $M = \langle V_{\{\otimes_1,\otimes_2\}}(\omega), I\rangle$ where[25]

$$\begin{aligned} I(\mathrm{EXT}_{NewV}(P)) &= I(P) \text{ if } I(P) \in V_{\{\otimes_1,\otimes_2\}}(\omega) \\ I(\mathrm{EXT}_{NewV}(P)) &= \otimes_1 \quad \text{otherwise.} \end{aligned}$$

and

$$\begin{aligned} I(\mathrm{EXT}_{NewerV}(P)) &= (f(2n+1) \text{ if } I(P) \in V_{\{\otimes_1,\otimes_2\}}(\omega) \text{ and } I(P) = f(2n) \\ I(\mathrm{EXT}_{NewerV}(P)) &= f(2n) \text{ if } I(P) \in V_{\{\otimes_1,\otimes_2\}}(\omega) \text{ and } I(P) = f(2n+1) \\ I(\mathrm{EXT}_{NewerV}(P)) &= \otimes_2 \quad \text{otherwise.} \end{aligned}$$

□

What we have here is a particularly vicious version of the Caesar problem: given two distinct abstraction principles for two distinct extension operators, we cannot even determine whether the empty extension $(\mathrm{EXT}_{NewV}(x \neq x))$ arising from *NewV* is identical to the empty extension $(\mathrm{EXT}_{NewerV}(x \neq x))$ provided by *NewerV*.

What we need are necessary and sufficient conditions for the identity of two abstracts that are generated by different abstraction principles. In *The Limits of Abstraction* [7] Fine discusses this problem at length, and his solution offers us a way out of our present difficulty. After considering and rejecting the idea that abstracts provided by different abstraction principles must be distinct (since, for example, the finite numbers generated by *Hume's Principle* and restrictions of it, such as *Finite Hume*,[26] ought to be the identical), he suggests that we

> . . . face the possibility that the criteria of identity [of distinct abstraction principles] might be different in a way that is not relevant to the identities of the abstracts in question. And this might lead one to . . . take two abstracts to be the same when their associated equivalence classes are the same, regardless of the means of abstraction by which they were obtained.[27] ([7], p. 49)

The idea is simple: each abstraction principle divides up the collection of concepts on the domain into one or more equivalence classes, and each one of these corresponds to an abstract. If we think of the abstracts as going proxy for the equivalence classes, then any two abstracts that correspond to the same collection of concepts should be identical regardless of what abstraction principle was used to "generate" them.[28] We can formalize this as the *General Abstract-Identity Schema*.[29] Given any two (legitimate) abstraction operators $@_1$ and $@_2$:

$$GAS: \quad (\forall P)(\forall Q)(@_1(P) = @_2(Q) \leftrightarrow (\forall F)(@_1(F) = @_1(P) \leftrightarrow @_2(F) = @_2(Q))).$$

Of interest at present is the instance governing our two extension operators:

$$\begin{aligned}(\forall P)(\forall Q)(\text{EXT}_{NewV}(P) &= \text{EXT}_{NewerV}(Q)\\ \leftrightarrow (\forall F)(\text{EXT}_{NewV}(F) &= \text{EXT}_{NewV}(P)\\ \leftrightarrow \text{EXT}_{NewerV}(F) &= \text{EXT}_{NewerV}(Q))).\end{aligned}$$

If we have *GAS*, we can define sethood and membership in terms of those abstracts that are the EXT_{NewV} and EXT_{NewerV} of the same (dually) non-*Bad* concept:

$$\begin{aligned}\text{Set}(x) &\leftrightarrow (\exists P)[x = \text{EXT}_{NewV}(P) \wedge x = \text{EXT}_{NewerV}(P)\\ &\wedge \neg Bad_{NewV}(P) \wedge \neg Bad_{NewerV}(P)]\\ x \in y &\leftrightarrow (\exists P)[Px \wedge x = \text{EXT}_{NewV}(P) \wedge x = \text{EXT}_{NewerV}(P)].\end{aligned}$$

On these definitions, the conjunction of *NewV*, *NewerV*, and the relevant instance of *GAS* entails the axioms of extensionality, separation, empty set, pairing, union* (but not union), powerset, and replacement axioms. Thus *GAS* provides us with a means for combining two abstraction principles for extensions that is consistent with neo-Fregean ideas about implicit definition yet delivers the desired results.

In what follows, however, we shall as a matter of convenience use *NewV* + *NewerV*, assuming that the same abstraction operator occurs in both principles, since this allows us to straightforwardly adopt results that were proved earlier (or elsewhere) for one or the other of these two principles. The reader should keep in mind that this conjunction of 'definitions' of the extension operator 'EXT' can be replaced by a richer (but formally less tractable) account of identity conditions across distinct abstraction principles.[30]

10 *NewV* + *NewerV* and Infinity

With *SOAP* + *NewV* + *NewerV* in place, we define notions such as set, membership, ordinal, Boolos-pure set, and hereditary set as before. First off, we note that *SOAP* + *NewerV* + *NewV* is consistent.

Theorem 10.1 *SOAP* + *NewerV* + *NewV is satisfied by* $M_{(\{\otimes\},\omega)}$.

Proof Straightforward, left to the reader. □

This also shows that the axiom of infinity fails to follow from *SOAP* + *NewerV* + *NewV*. The results of previous sections suffice to show that *SOAP* + *NewerV* + *NewV* proves the axioms of extensionality, separation, empty set, pairing, union*, powerset, and replacement.

As one might expect, interesting consequences follow from the conjunction of these two "definitions" of set that do not follow from either alone. As an example we have the following.

Lemma 10.2 *SOAP* + *NewerV* + *NewV implies that, for all* P, P *is Big if and only if* P *is Bad.*

Proof Straightforward, left to the reader. □

More significantly, *SOAP* + *NewerV* + *NewV* proves that every object is either a set or is in the basis (although as we shall see some objects might be both).

Theorem 10.3 *SOAP* + *NewerV* + *NewV implies* $(\forall x)(UR(x) \rightarrow BASE(x))$.

Proof Assume for arbitrary a that a is an urelement, that is, a is not a set. Since both *NewV* and *NewerV* are only satisfied on infinite models, the property corresponding to '$x = a$' is not *Big*, and thus not *Bad*, that is, $(\exists\alpha)(\text{ON}(\alpha) \wedge (\forall x)(x = a \rightarrow x \in_S \text{Stg}(\alpha)))$. Let β be the least ordinal such that $(\forall x)(x = a \rightarrow x \in_S \text{Stg}(\beta)))$, that is, β is the least ordinal such that $a \in_S \text{Stg}(b))$. Assume that $\beta > 0$. Then by the definition of stages $(\exists P)(a = \text{EXT}(P) \wedge (\exists\delta)(\delta < \beta \wedge (\forall y)(Py \rightarrow y \in_S \text{Stg}(\delta))))$. So $(\exists P)(a = \text{EXT}(P) \wedge (\exists\delta)(\text{ON}(\delta) \wedge (\forall y)(Py \rightarrow y \in_S \text{Stg}(\delta))))$. Thus a is a set. Contradiction, so $\beta = 0$ and a is in the basis. □

As a result of this we have the following.

Corollary 10.4 *SOAP* + *NewerV* + *NewV implies the Urelement Axiom:*

$$(\exists x)(\text{Set}(x) \wedge (\forall y)(UR(y) \leftrightarrow y \in_N x)).$$

Corollary 10.5 *Every model of SOAP* + *NewerV* + *NewV is extension-isomorphic to* $M_{(BASE,\kappa)}$ *for some cardinal* κ *such that either* $\kappa = \omega$ *and BASE is finite or* κ *is inaccessible and* $\kappa >\mid BASE \mid$.

Corollary 10.6 *If* $M_{(BASE,\kappa)}$ *is a model of SOAP* + *NewerV* + *NewV, then* $\otimes \in BASE$.

Thus, *SOAP* + *NewerV* + *NewV* captures all of ZFC except for the axiom of infinity and the axiom of foundation.[31] We set aside foundation until Section 11.

As already noted, *SOAP* + *NewerV* plus the claim that there are infinitely many elements in the basis implies the axiom of infinity. *NewV* plus the claim that there are uncountably many objects implies the axiom of infinity. Here we consider a principle that is independent of each of these assumptions:[32]

InfNonSets: "There are infinitely many urelements."

NewV has models with uncountably many sets but only finitely many nonsets, and models with infinitely many nonsets but only a countable infinity of sets. Similarly, *SOAP* + *NewerV* has models with infinitely many nonsets but only finitely many objects in the basis and, as we shall see in Section 11, it also has models with finitely many nonsets but infinitely many objects in basis. Thus the following is not trivial, even if its proof is.

Theorem 10.7 *SOAP + NewerV + NewV + InfNonSets implies the axiom of infinity.*

Proof Assume that there are infinitely many urelements. Then, by Theorem 9.2, there are infinitely many objects in *BASE*. *NewerV* plus the claim that *BASE* has infinitely many members implies the axiom of infinity. □

This provides the following.

Corollary 10.8 *SOAP + NewerV + NewV + InfNonSets is satisfied by* $M_{(\{\otimes\},\kappa)}$ *if and only if* κ *is an inaccessible cardinal.*

Thus every model of *SOAP* + *NewerV* + *NewV* + *InfNonSets* is also a model of second-order ZFC-foundation.

Of course, the crucial question is not what principles can be added to *SOAP* + *NewerV* + *NewV* in order to derive the axiom of infinity, but what additional neologicistically acceptable principle will provide the axiom of infinity. Provided that *GAS* provides the correct account of identity between abstracts arising from distinct abstraction principles, however, a neologicist justification of *InfNonSets* is straightforward.

The neologicist need only accept, in addition to *SOAP* + *NewerV* + *NewV*, a restricted version of *Hume's Principle* such as *Finite Hume*:[33]

FHP: $$(\forall P)(\forall Q)[\mathrm{NUM}(P) = \mathrm{NUM}(Q) \leftrightarrow (P \approx Q \vee(\neg\,\mathrm{Finite}(P) \wedge \neg\,\mathrm{Finite}(Q)))].$$

('$\mathrm{Finite}(P)$' is an abbreviation of the second-order formula asserting the nonexistence of a 1–1 correspondence from the Ps into, but not onto, the Ps).[34]

If we combine *FHP* with *SOAP* + *NewerV* + *NewV* and the relevant instance of *GAS*,

$$\begin{aligned}(\forall P)(\forall Q)(\mathrm{NUM}(P) &= \mathrm{EXT}(Q)\\ \leftrightarrow (\forall F)(\mathrm{NUM}(F) &= \mathrm{NUM}(P)\\ \leftrightarrow \mathrm{EXT}(F) &= \mathrm{EXT}(Q)))\end{aligned}$$

we obtain the following.

Theorem 10.9 *FHP + SOAP + NewerV + NewV + GAS implies InfNonSets.*

Proof Combine the standard proof of (part of) *Frege's Theorem* (i.e., the claim that there are infinitely many numbers) with the fact that, since for each *FHP* number x where $x \neq 0$, $(\exists P)(\exists Q)(x = \mathrm{NUM}(P) = \mathrm{NUM}(Q) \wedge \neg(\forall y)(Py \leftrightarrow Qy))$, all numbers (other than 0) are not extensions.[35] □

This provides the necessary corollary.

Corollary 10.10 *FHP + SOAP + NewerV + NewV + GAS implies the axiom of infinity.*

11 Foundation and Non-well-founded Sets

In this section we will examine the status of the second-order axiom of foundation:

Foundation: $(\forall P)((\forall x)(Px \rightarrow \text{Set}(x)) \rightarrow ((\exists y)(Py) \rightarrow (\exists y)(Py \wedge \neg(\exists z)(Pz \wedge z \in_N y))))$,

and the prima facie weaker axiom of regularity:

Regularity: $(\forall x)(\text{Set}(x) \rightarrow ((\exists y)(y \in_N x) \rightarrow (\exists y)(y \in_N x \wedge \neg(\exists z)(z \in_N x \wedge z \in_N y))))$

within *NewerV* + *NewV* set theory.[36]

As was the case with *NewV* or *NewerV* alone, *SOAP* + *NewerV* + *NewV* proves foundation when the quantifiers are restricted to Boolos-pure sets, but foundation, and the weaker axiom of regularity, can fail to hold of all sets, or even all hereditary sets. To show that regularity restricted to the hereditary sets fails to follow from *SOAP* + *NewerV* + *NewV* (and thus the unrestricted versions fail to follow as well), it suffices to show that the ΩAxiom,[37]

ΩAxiom: $(\exists x)(\forall y)(y \in x \leftrightarrow y = x)$,

can be consistently added to *SOAP* + *NewerV* + *NewV*.

To show this we will construct models within Aczel's (first-order) Non-well-founded set theory (see [1]). Since Aczel's systems are all interpretable within first-order ZFC-foundation, the results below can be proven within ZFC-foundation directly, although the presentation is less straightforward. Thus our adoption of non-well-founded set theory is a matter of convenience only, and our "official" metatheory remains first-order ZFC-foundation. (It should be noted that none of the results used below depend on the particular formulation of the Anti-foundation Axiom—any of the variants discussed in the literature will suffice. See Rieger [11] for a nice discussion of the popular variants).

The following demonstrates that we can, in a sense, have arbitrarily many non-well-founded sets in *NewV* + *NewerV* set theory.

Theorem 11.1 *SOAP + NewerV + NewV + InfNonSets is satisfied by* $M_{(BASE,\kappa)}$ *where BASE is the pairwise union of any transitive set of non-well-founded sets and* $\{\otimes\}$ *and* κ *is any inaccessible such that* $\kappa > \mid BASE \mid$.

Proof The transitivity of *BASE* guarantees that for any set that is also in the basis, all of its members are in the basis. The remainder is straightforward. □

The consistency of Ω*Axiom* is immediate.

Corollary 11.2 *SOAP + NewV + NewerV + InfNonSets + ΩAxiom is consistent.*[38]

Proof Since, letting Ω be a set such that $\Omega = \{\Omega\}$, $\{\Omega\}$ is a transitive set of non-well-founded sets, $M_{(\{\Omega,\otimes\},\kappa)}$ is a model of *SOAP* + *NewerV* + *NewV* + *InfNonSets* + Ω*Axiom*. □

Since Ω is a hereditary set, we have the desired corollary.

Corollary 11.3 *SOAP + NewV + NewerV fails to imply foundation or regularity restricted to hereditary sets.*

If we combine Theorem 10.9 with the following lemma we obtain a corollary promised in Section 10.

Lemma 11.4 *SOAP + NewerV + NewV implies that if $x \in_N x$, then x is an element of the basis.*

Proof Assume for an arbitrary a that $a \in_N a$. Either a is a set or a is not a set. By Theorem 9.2, if a is not a set, then a is in the basis. So assume that a is a set. Thus the property corresponding to '$x \in_N a$' is not *Bad*, so $(\exists\alpha)((\alpha \text{ is an ordinal number}) \wedge (\forall x)(x \in_N a \rightarrow x \in_S \mathrm{Stg}(\alpha)))$. Let β be the least ordinal such that $(\forall x)(x \in_N a \rightarrow x \in_S \mathrm{Stg}(\beta)))$. Assume that $\beta > 0$. It follows from the definition of stages $(\forall y)(y \in_N a \rightarrow (\exists\delta)(\delta < \beta \wedge y \in_S \mathrm{Stg}(\delta))$. Since $a \in_N a$, we have $(\exists\delta)(\delta < \beta \wedge a \in_S \mathrm{Stg}(\delta))$. Contradiction, so $\beta = 0$ and a is in the basis. □

Corollary 11.5 *SOAP + NewerV + NewV has models with finitely many nonsets but infinitely many members of the basis.*

Proof Let *BASE* be any infinite transitive set of non-well-founded sets and κ an inaccessible cardinal such that $\kappa >| BASE |$. Then $M_{(BASE\cup\{\otimes\},\kappa)}$ is a model of *SOAP + NewerV + NewV*. □

Thus, *SOAP + NewerV + NewV* does not rule out the existence of non-well-founded sets. *SOAP + NewerV + NewV* does rule out the simultaneous existence of all non-well-founded sets, however. The following is a theorem of Non-well-founded Set Theory:

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (z = y \vee z \in x)).$$

In other words, given any set x there is a set y (not necessarily distinct) that contains exactly the members of x and itself. This can be expressed within *NewV + NewerV* set theory as

WeakNWF: $$(\forall P)(\neg Bad(P) \rightarrow (\exists Q)(\neg Bad(Q) \wedge(\forall x)(Qx \leftrightarrow (Px \vee x = \mathrm{EXT}(Q))).$$

This principle, far weaker than any of the popular formulations of the *Anti-foundation Axiom*, is nevertheless incompatible with *NewerV + NewV* set theory.

Theorem 11.6 *SOAP + NewerV + NewV + WeakNWF is inconsistent.*

Proof There is an obvious correspondence between the ordinal numbers as provided by *SOAP* and the sets usually referred to as 'ordinals' (i.e., the transitive pure sets well-ordered by $\in$). By previous results, the property corresponding to 'x is an ordinal number' is *Bad*, and thus *Big*, so the collection of ordinals is *Big*. Consider a function f such that $f(x) = y$ if and only if for all z, $z \in_N y$ if and only if either $z \in_N x$ or $z = y$ (the existence of such a function is guaranteed by *WeakNWF* and choice). Note that if $f(x) = y$, it follows that $y \in_N y$. Assume that $f(x) = y = f(z)$ for x, y ordinals. Then for all w, $w \in_N y$ if and only if $w \in_N x$ or $w = y$ if and only if $w \in_N z$ or $w = y$. So, for an arbitrary w, if $w \in_N x$ then either $w \in_N z$ or $w = y$. But since foundation holds of the ordinals, $w \neq y$. Thus $x \subseteq_N z$, and similarly $z \subseteq_N x$. Thus, $x = z$, so f restricted to the ordinals is

1–1. So the image of the ordinals under f is *Big*. But for any y, if y is in the image of the ordinals under f, then y is in the basis. Thus the property corresponding to '$x \in BASE$' is *Big*. Contradiction. □

Thus although set theory based on *NewV* plus *NewerV* is consistent with the existence of (some) non-well-founded sets, it cannot tolerate the addition of *all* of them.[39]

12 Philosophical Lessons

There are four main areas of interest that arise in the comparison of the limitation of size conception of set (as codified in *NewV*) and the iterative conception of set (as codified in *SOAP* + *NewerV*). Each is associated with the status of one of the standard set-theoretic axioms. The axioms at issue are the axioms of powerset, replacement, infinity, and foundation.

As we have seen, *NewV* implies the replacement axiom but fails to secure the truth of powerset; alternatively *SOAP* + *NewerV* implies the powerset axiom but fails to secure replacement. Thus, if we are forced to choose a single abstraction principle that provides the "definition" of set, then we are left here with a dilemma—given a choice, would we rather have the powerset axiom and forego replacement, or have replacement and forego powerset?

Of course, this choice is not just a matter of personal preference. The powerset axiom is necessary for the formalization of much of contemporary mathematics within set theory, taking us from the naturals (modeled, e.g., by the finite ordinals) to the reals (modeled by the sets of finite ordinals), from the reals to the theory of functions on the reals (modeled by sets of sets of finite ordinals) and so on. The axiom of replacement, however, is for the most part only used in rather obscure and esoteric branches of pure set theory such as accounting for the behavior of transfinite ordinals and cardinals.[40] If we are interested in using our set theory to provide a foundation for much or all of mathematics, including but not limited to the mathematics necessary for doing science, then when faced with the option of having powerset or replacement but not both, the appropriate choice seems clear—powerset is more crucial to formulating modern mathematics within set theory. Nevertheless, in failing to imply one or the other of these central axioms, neither abstraction principle seems to satisfy the demand for mathematical adequacy.

Adopting *SOAP* + *NewerV* + *NewV* (or, perhaps more plausibly, by adopting *SOAP* + *NewerV* + *NewV* +*GAS* as discussed in Section 9), the worries about replacement and powerset evaporate—both are derivable from the conjunction of these abstraction principles. Additionally, this approach dovetails nicely with the historical development of axiomatic set theory, motivated as it was by two competing conceptions of set each corresponding to one of the abstraction principles in question. Of course, we have not dissolved all problems for a neologicist set theory, or even all problems relating to replacement and powerset, in this paper. Nevertheless, the theory based on *NewV* + *NewerV* seems to be the most promising candidate so far for a neologicist account of sets.[41]

The most pressing problem for a neologicist reconstruction of set theory, however, whether based on one abstraction principle or many, is the axiom of infinity. Unfortunately, the axiom of infinity does not follow from *NewV*, *SOAP* + *NewerV*, or *SOAP* + *NewerV* + *NewV*. Thus the neologicist needs to find some additional principle that implies that there is an infinite set. Some possibilities have been explored above. Again, *NewV* + *NewerV* set theory comes out on top, as the axiom of infinity follows merely from the additional assumption that there are infinitely many nonsets (i.e., *InfNonSets*), which in turn follows from other neologicist principles plus our identity principle *GAS*. Even if *GAS* turns out not to be neologicistically acceptable, *InfNonSets* certainly seems less problematic than the assumptions needed to obtain the axiom of infinity when working within *NewV* set theory or the theory of *SOAP* + *NewerV* alone. *SOAP* + *NewerV* + *NewV* + *InfNonSets* is a promising candidate for a sufficiently powerful neologicist account of sets.

With regard to the foundation axiom, however, *SOAP* + *NewerV* + *NewV* is, in one sense, little better than the weaker theories. Foundation holds of the Boolos-pure sets but can fail on the hereditary sets. Perhaps this is as it should be, since the axiom of foundation is of a different character than the other standard axioms of ZFC. Each of the other axioms is of one of two forms. First we have straight existential claims:

$$(\exists y)(\forall z)(z \in y \leftrightarrow \Phi(z)).$$

Infinity and empty set are axioms of this type. Second, we have conditional existence claims:

$$(\forall x_1)(\forall x_2)\ldots(\forall x_n)(\exists y)(\forall z)(z \in y \leftrightarrow \Phi(z, x_1, x_2, \ldots, x_n))$$

These axioms state that, given any sequence of sets (or objects), a second set with a certain relation to the given sequence of sets (objects) also exists. Separation, powerset, replacement, union, and pairing are all conditional existence axioms. Foundation, on the other hand, is of a very different logical character, displaying the following logical form:

$$(\forall x)\Phi(x).$$

Foundation does not imply the existence of any new sets but instead imposes a restriction on what sorts of characteristics sets can have and thereby restricts what sets can in fact exist. This restriction was originally motivated by a certain view about how sets should be structured, a view motivated by the idea that the set theoretic paradoxes were caused by circularity. Even if this restriction is motivated by an intuitive picture that also underlies the iterative conception of set, a conception that we have accepted in the form of *NewerV*, we need not feel forced to thereby accept a wholesale ban on circularity.

Instead we can view the neologicist as replacing one response to the set theoretic paradoxes (an overwhelming fear and avoidance of anything circular) with another response (the idea that acceptable abstraction principles provide secure foundations for mathematical theories irrespective of circularity). In reconstructing contemporary set theory he is forced to adopt some of the restrictions that evolved from the prior view of the nature of sets (i.e., those following from the iterative picture of sets as codified by *NewerV*), but he is free to countenance circular sets to the extent that his new set theory allows. On this way of viewing things it is perfectly reasonable that neologicist set theory implies that certain sorts of sets obey the axiom of foundation but leaves open the question of whether all do.

Another way of making the same point is to note that the neologicist can construct a set theory that implies all of the axioms of second-order ZFC if he merely modifies his definition of 'set'. Instead of having sets be the extensions of properties that are not '*Big*' (or *Bad*), let sets be those objects that are both extensions of properties that are not '*Big*' and contained in every Boolos-closed concept (i.e., the Boolos-pure sets). As we have seen, *SOAP* + *NewerV* + *NewV* + *InfNonSets* proves all of the axioms of second-order ZFC restricted to these objects. Thus, assuming that all of these principles are neologicistically acceptable, *NewerV* + *NewV* set theory (with a principle guaranteeing an infinite set) in a sense "contains" full second-order ZFC, but also leaves room for the existence of other non-well-founded sets.

To sum up, *SOAP* + *NewerV* + *NewV* + *InfNonSets* provides the neologicist with a set theory that is (roughly) as strong as full second-order ZFC. As already noted, detailed philosophical defense of the acceptability of these principles is still necessary. Nevertheless, the mathematical problem—determining whether there is a mathematically adequate neologicist set theory—seems to be solved.

Appendix A

Although I have given these proofs (and the ones in the main body of the text) rather informally, each of them can be straightforwardly (though tediously) rewritten as a formal deduction within second-order logic.

Lemma A.1 $(\neg Bad(P) \wedge (\forall x)(Qx \rightarrow Px)) \rightarrow \neg Bad(Q)$.

Proof Assume that P is not *Bad* and that $(\forall x)(Qx \rightarrow Px)$ holds. Then $(\exists \alpha)((\alpha \text{ is an ordinal number}) \wedge (\forall x)(Px \rightarrow x \in_S \text{Stg}(\alpha)))$. So $(\exists \alpha)((\alpha \text{ is an ordinal number}) \wedge (\forall x)(Qx \rightarrow x \in_S \text{Stg}(\alpha)))$. So Q is not *Bad*. □

Lemma A.2 $(\neg Bad(P) \wedge \text{EXT}(P) = \text{EXT}(Q)) \rightarrow \neg Bad(Q)$.

Proof Assume P is not *Bad* and $\text{EXT}(P) = \text{EXT}(Q)$. Then either P is *Bad* and Q is *Bad*, or $(\forall x)(Px \leftrightarrow Qx)$, so by Lemma A.1, since P is not *Bad*, Q is not *Bad*. □

Lemma A.3 $\neg Bad(P) \rightarrow (\forall x)(x \in_N \text{EXT}(P) \leftrightarrow Px)$.

Proof Assume that P is not *Bad*. Given an arbitrary x, if $x \in_N \text{EXT}(P)$ then $(\exists Q)[Qx \wedge \text{EXT}(P) = \text{EXT}(Q)]$. Since P is not *Bad*, this implies that $(\exists Q)[Qx \wedge (\forall y)(Py \leftrightarrow Qy)]$, that is, Px. Similarly, given an arbitrary x such that Px, it follows that $[Px \wedge \text{EXT}(P) = \text{EXT}(P)]$, so $(\exists Q)[Qx \wedge \text{EXT}(P) = \text{EXT}(Q)]$, that is, $x \in_N \text{EXT}(P)$. □

Theorem A.4 *NewerV entails Extensionality:*

$$(\forall x)(\text{Set}(x) \rightarrow (\forall y)(\text{Set}(y) \rightarrow (\forall z)((z \in_N x \leftrightarrow z \in y) \rightarrow x = y)).$$

Proof Let x and y be sets. Then $x = \text{EXT}(P)$ and $y = \text{EXT}(Q)$ where P and Q are not *Bad*. It follows, by Lemma A.3, that $(\forall z)(z \in_N \text{EXT}(P) \leftrightarrow Pz)$ and $(\forall z)(z \in_N \text{EXT}(Q) \leftrightarrow Qz)$. Assume that $(\forall z)(z \in x \leftrightarrow z \in y)$ holds, that is, $(\forall z)(z \in_N \text{EXT}(P) \leftrightarrow z \in_N \text{EXT}(Q))$. Then $(\forall z)(Pz \leftrightarrow Qz)$, so $\text{EXT}(P) = \text{EXT}(Q)$, or $x = y$. □

Theorem A.5 *NewerV entails Empty Set:*

$$(\exists x)(\text{Set}(x) \wedge (\forall y)(y \notin_N x)).$$

Proof Let $x = \text{EXT}(y \neq y)$. Then $\text{Set}(x)$, since 1 is an ordinal number and $(\forall y)(y \neq y \rightarrow y \in_S \text{Stg}(1))$. Additionally, $(\forall y)(y \notin x)$, since for no z is it the case that $z \neq z$. □

Theorem A.6 *NewerV entails Separation:*

$$(\forall P)(\forall x)(\text{Set}(x) \rightarrow (\exists y)(\text{Set}(y) \wedge (\forall z)(z \in_N y \leftrightarrow (z \in_N x \wedge Pz)).$$

Proof Let P be a property and x a set. Then $x = \text{EXT}(Q)$ where Q is not *Bad*. Let $y = \text{EXT}(P \wedge Q)$. Then $\text{Set}(y)$, since $P \wedge Q$ is not *Bad* by Lemma A.1. Also, for any z, $z \in_N y$ if and only if $(Pz$ and $Qz)$ if and only if Pz and $z \in_N x$. □

Theorem A.7 *NewerV entails Union*:*

$$\begin{aligned}&(\forall x)(\text{Set}(x) \rightarrow (\exists y)(\text{Set}(y)\\&\quad\wedge(\forall z)(z \in_N y \leftrightarrow (\exists w)(w \textit{ is a set } \wedge (z \in_N w \wedge w \in_N x))))).\end{aligned}$$

Proof Let x be a set, so that $x = \text{EXT}(P)$. Thus $(\exists\alpha)((\alpha$ is an ordinal number$)$ $\wedge$ $(\forall x)(Px \rightarrow x \in_S \text{Stg}(\alpha)))$. If $w \in_N x$, then by Lemma A.3, Pw, so $w \in_S \text{Stg}(\alpha)$. In other words, either w is in the basis $(w \in_S \text{Stg}(0))$ or $(\exists Q)(w = \text{EXT}(Q) \wedge (\exists\beta)(\beta < \alpha \wedge (\forall y)(Qy \rightarrow y \in_S \text{Stg}(\beta))))$. So if w is a set, then for any $z \in_N w$, Qz, so $z \in_S \text{Stg}(\beta)$, and thus, for all z and w such that $z \in w \in x$, $z \in_S \text{Stg}(\alpha)$. Let S be the property holding of just the members of members of x. Then $(\forall y)(Sy \rightarrow y \in_S \text{Stg}(\alpha))$, so $\text{EXT}(S)$ is a set and is the union of x. □

Theorem A.8 *NewerV entails Pairing:*

$$(\forall x)(\text{Set}(x) \rightarrow (\forall y)(\text{Set}(y) \rightarrow (\exists z)(\text{Set}(z)\wedge(\forall w)(w \in_N z \leftrightarrow (w = x \vee w = y))))).$$

Proof Let x and y be sets. Then there is a P such that $x = \text{EXT}(P)$ where $(\exists\alpha)((\alpha$ is an ordinal number$) \wedge (\forall x)(Px \rightarrow x \in_S \text{Stg}(\alpha)))$ and there is a Q such that $y = \text{EXT}(Q)$ where $((\exists\beta)((\beta$ is an ordinal number$)\wedge(\forall x)(Qx \rightarrow x \in_S \text{Stg}(\beta)))$. Let $\delta = \max(\alpha, \beta)$. Then $x \in_S \text{Stg}(\delta + 1)$ and $y \in_S \text{Stg}(\delta + 1)$. Let F be the property that holds of exactly x and y. Then $\text{EXT}(F)$ is a set and is the pair set of x and y. □

Theorem A.9 *NewerV entails Powerset:*

$$(\forall x)(\text{Set}(x) \rightarrow (\exists y)(\text{Set}(y) \wedge (\forall z)(z \in_N y \leftrightarrow (\forall w)(w \in_N z \rightarrow w \in_N x)).$$

Proof Let x be a set, that is, $x = \text{EXT}(P)$ where $(\exists\alpha)((\alpha$ is an ordinal number$)$ $\wedge$ $(\forall x)(Px \rightarrow x \in_S \text{Stg}(\alpha)))$, and let y be a subset of x, that is, there is a Q where $y = \text{EXT}(Q)$ and $(\forall x)(Qx \rightarrow Px)$. So $(\forall x)(Qx \rightarrow x \in_S \text{Stg}(\alpha)))$ and it follows that $y \in_S \text{Stg}(\alpha + 1)$. Let S be the property holding of exactly the subsets of x. $\text{EXT}(S)$ is a set and is the powerset of x. □

Notes

1. This paper is intended, among other things, to further the comparison of the iterative and limitation-of-size conceptions of set begun by Boolos in [2] and [3].

2. I assume standard set theoretic semantics for second-order logic, where the second-order predicate variables range over the full powerset of the domain and which contains the second-order axiom of choice and the full comprehension scheme. For details see [12].

3. Every abstraction principle considered in this paper can be expressed using only the resources of second-order logic (plus, in some cases, previously defined abstraction operators). I give the formal expressions in the notes as necessary. The second-order formula expressing that there is a one-to-one correspondence between the Ps and the Qs is

$$(\exists R)((\forall x)(Px \rightarrow (\exists! y)(Qy \wedge Rxy)) \wedge (\forall z)(Qz \rightarrow (\exists! x)(Px \wedge Rxz)))$$

where $(\exists! x)(\Phi x)$ is an abbreviation for

$$(\exists x)(\Phi x \wedge (\forall y)(\Phi y \rightarrow y = x)).$$

4. There have been a number of careful and thorough studies of the historical development of, and interactions between, these two notions of set, Hallett [10] being one of best. I do not propose to make any contribution to this historical project here, but intend rather to examine the technical merits of the iterative conception as formulated within the neo-logicist framework.

5. $(\exists f)((\forall x)(\exists y)((f(x) = f(y) \rightarrow x = y) \wedge (\forall x)(\exists y)(Py \wedge f(y) = x))$. In light of the Schröder-Bernstein theorem, which can be proved in second-order logic (see [12], pp. 102–103), this can be simplified to $(\exists f)(\forall x)(\exists y)(Py \wedge f(y) = x)$.

6. The axiom of choice for sets follows immediately from the fact that we have assumed that choice holds in second-order logic.

7. Boolos-closed and Boolos-pure are the conditions Boolos calls "closed" and "pure" on p. 100 of [3]. The formulation of the notion of hereditary sets, and the observation that the class of hereditary sets need not be coextensive with the class of Boolos-pure sets, appears for the first time in Uzquiano and Jané [14] (as does the terminology "Boolos-pure").

8. To see that the axiom of infinity does not follow from *NewV* we need merely note that the model $\langle V_{\{\otimes\}}(\omega), I\rangle$ satisfies *NewV* where (for $\otimes$ an arbitrary object, not a hereditarily finite set):

$$\begin{aligned}
V_{\{\otimes\}}(0) &= \{\otimes\} \\
V_{\{\otimes\}}(n+1) &= V_{\{\otimes\}}(n) \cup (\wp(V_{\{\otimes\}}(n))) \\
V_{\{\otimes\}}(\omega) &= \cup V_{\{\otimes\}}(n)(n \in \omega) \\
I(\text{EXT}(P)) &= \{x \in V_{\{\otimes\}}(\omega) : x \in I(P)\} \text{ if } \{x \in V_{\{\otimes\}}(\omega) : x \in I(P)\} \text{ finite.} \\
I(\text{EXT}(P)) &= \otimes \quad \text{otherwise.}
\end{aligned}$$

Demonstrating the independence of the powerset axiom is a bit more complicated. (See Shapiro and Weir [13].)

9. $R \cong S$ abbreviates the claim that the relation R is isomorphic to S. We can give the second-order formula for R and S being isomorphic as follows. First, the following abbreviations:

$$\begin{aligned}
A(x) &\leftrightarrow (\exists y)(R(x, y) \vee R(y, x) \\
B(y) &\leftrightarrow (\exists y)(S(x, y) \vee S(y, x).
\end{aligned}$$

We then have

$$\begin{aligned}
R \cong S \leftrightarrow\ & (\exists f)((\forall x)(A(x) \rightarrow B(f(x))) \wedge (\forall x)(B(x) \rightarrow (\exists y)(f(y) = x) \\
\wedge\ & (\forall x)(\forall y)(f(x) = f(y) \rightarrow x = y) \\
\wedge\ & (\forall x)(\forall y)(R(x, y) \leftrightarrow S(f(x), f(y)))).
\end{aligned}$$

10. WO(R) abbreviates the claim that R is a well-ordering. Define $A(x)$ as in footnote 9 above. Then

$$\begin{aligned} \text{WO}(R) \quad &\leftrightarrow \quad (\forall x)(\neg R(x,x)) \wedge (\forall x)(\forall y)(\forall z)((R(x,y) \wedge R(y,z)) \\ &\rightarrow \quad R(x,z)) \wedge (\forall P)(((\exists x)(Px) \wedge (\forall x)(Px \rightarrow A(x))) \\ &\rightarrow \quad (\exists y)(Py \wedge (\forall z)(Pz \rightarrow (z = y \vee R(y,z))))). \end{aligned}$$

11. Although it is tempting to identify $\in_S$ and $\in_N$, the reader should be careful not to. '$x \in_S$ Stg(α)' is a defined binary relation holding between an object x and an ordinal number α, and asserts, intuitively, that x is "formed" by the αth stage. In other words, '$\in_S$' and 'Stg(α)' are not separable pieces of vocabulary. On the other hand, '$\in_N$' expresses a relation holding between an object and an extension. As a result '$x \in_N$ Stg(α)' is not well-formed in the language given above. Nevertheless,

$$(\forall x)(\forall \alpha)(\text{ON}(\alpha) \rightarrow (x \in_S \text{Stg}(\alpha) \leftrightarrow x \in_N \text{EXT}(y \in_S \text{Stg}(\alpha)))).$$

is a theorem.

12. Providing abstraction principles whose right-hand sides contain abstraction operators defined previously is (at least accepted as) neologicistically acceptable methodology, as the literature on reconstructing the reals attests. See, for example, Hale [9].

13. Note that '$\in_N$ Stg(α)' is defined in terms of 'EXT', so we need not include an additional clause.

14. That is, $(\forall x)(\text{Set}(x) \rightarrow (\forall f)(\forall y)((y \in_N x \rightarrow \text{Set}(f(y))) \rightarrow (\exists z)(\text{Set}(z) \wedge (\forall w)(w \in_N x \rightarrow f(w) \in_N z)$.

15. Note that $M_{(\varnothing,\omega+\omega)}$, although failing to satisfy replacement, also fails to satisfy *SOAP*.

16. This construction shows that we can prove, in second-order ZFC, that *SOAP* + *NewerV* has uncountable models, since π, and thus $M_{(\varnothing,\pi)}$, can be constructed within second-order ZFC. This is significant since Shapiro and Weir [13] prove that the claim that *NewV* has uncountable models is independent of second-order ZFC.

17. Boolos writes of "axioms of replacement" in the plural since he is considering first-order set theory.

18. That is, $(\exists x)(\text{Set}(x) \wedge (\varnothing_N \in_N x \wedge (\forall y)(\text{Set}(y) \rightarrow (y \in_N x \rightarrow y \cup_N \{y\}_N \in_N x))))$.

19. The statement that $M_{(FO,\pi)}$ is the smallest model of *SOAP* + *NewerV* + BA_ω should not be understood to imply that there are no other nonisomorphic models of the same cardinality since $M_{(FO\cup\{\otimes\},\pi)}$ clearly is such a model. Instead, all models have domains at least as large as that of $M_{(FO,\pi)}$.

20. Zermelo set theory is just first-order ZFC without replacement and choice. Second-order Zermelo set theory is second-order ZFC without replacement.

21. That is, $(\exists x)(\text{Set}(x) \wedge (\varnothing_N \in_N x \wedge (\forall y)(\text{Set}(y) \rightarrow (y \in_N x \rightarrow \{y\}_N \in x))))$.

22. This does not rule out the possibility that π could be definable within some extended, neologicistically acceptable language.

23. One needs to be careful here since the number of ordinal numbers generated by *SOAP* depends on the number of objects that exist, which could depend on *NewerV*.

24. This is a result of the fact that whether or not powerset (replacement) holds in the context of *NewV* (*NewerV*) is a function of the cardinality of the domain.

25. In order to get the result in its full generality we require that there be two '*Bad*' objects, one for each principle.

26. *Finite Hume* is a restricted version of *Hume's Principle* which provides each finite positive integer and a single pre-Cantorian infinite number (the abstract of all finite concepts). For the exact formulation see Section 10.

27. Fine immediately rejects this analysis in favor of one that identifies abstracts whose equivalence classes are necessarily identical. He spends the remainder of the chapter exploring the difficulties such a modalized account must face. Fortunately, the simpler nonmodal formulation suffices for our present purposes. For further discussion of this aspect of the Caesar problem, see Cook and Ebert [6].

28. I do not intend this to be read as a defense of this particular route to solving (this variant of) the Caesar problem but am content merely to briefly sketch how such a solution might proceed.

29. While acceptance of this principle has no impact on the standard version of the Caesar problem, which concerns identities between abstracts and nonabstracts, its acceptance completely solves the analogous problem of determining when two abstracts generated by different abstraction principles are identical. We can call this latter problem the *C*-*R* problem, since determining whether the real numbers are a subcollection of the complex numbers is a particular case of the problem, and the term '*C*-*R*' has a convenient similarity to the word 'Caesar'.

30. In addition, we can in the present context eliminate *SOAP* altogether, reformulating *NewerV* (or *NewestV*) so that the stages are ordered by the ordinals (i.e., the transitive pure sets well-ordered by $\in$) provided by *NewV* (or *NewestV*). Although this would be a bit more elegant than the methods employed in the text, ironing out the details of the relevant reformulation of *NewerV* (or *NewestV*) would add considerable length to this paper without any significant gain.

31. In fact, *SOAP* + *NewerV* + *NewV* is strictly stronger than second-order ZFCU minus the axioms of infinity and foundation since the former, but not the latter, implies that there is a set containing all urelements. Thanks go to an anonymous referee for pointing this out.

32. That is, $(\exists P)((\forall x)(Px \rightarrow \neg \text{Set}(x)) \wedge (\exists f)((\forall x)(\forall y)(f(x) = f(y) \rightarrow x = y) \wedge (\forall x)(P(x) \rightarrow P(f(x))) \wedge (\exists x)(Px \wedge (\forall y)(f(y) \neq x))))$.

33. One drawback to this general approach is that *Hume's Principle* or even *Small Hume*,
$$(\forall P)(\forall Q)[\text{NUM}(P) = \text{NUM}(Q) \leftrightarrow (P \approx Q \vee (Big(P) \wedge Big(Q)))],$$
is inconsistent with *NewV* + *NewerV* + *GAS*: If all the numbers (or all the small numbers) other than 0 are not extensions, then they must be urelements, but then there must be a set of all numbers, and further, the powerset of this set must exist. But since the universe must be the size of a strong inaccessible, there must be exactly as many numbers as there are objects in the universe. Contradiction.

34. That is, $(\exists f)((\forall x)(Px \rightarrow Pf(x) \wedge (\forall y)(\forall z)(f(y) = f(z) \rightarrow y = z) \wedge (\exists w)(Pw \wedge (\forall n)(Pn \rightarrow f(n) \neq w)))$.

35. Interestingly, *GAS* implies that $\text{NUM}(x \neq x) = \text{EXT}(x \neq x)$, that is, $0 = \varnothing$. Significantly, anti-zero, the number of the universe, which has been a topic of controversy since Boolos [4], (p. 314), is (provably) not identical (modulo *GAS*) to the *Bad* extension $\otimes$.

36. Uzquiano [15] proves that second-order Zermelo set theory with the axiom of regularity has models where second-order foundation fails. This provides us with another way in which *NewerV* is stronger than Zermelo set theory, since *NewerV* implies the equivalence of foundation and regularity.

37. That is, $(\exists x)(\mathrm{Set}(x) \wedge (\forall y)(y \in_N x \leftrightarrow y = x))$.

38. Note that Ω is a member of the basis but not an urelement.

39. It is worth noting that the following principle of *Urelement-Basis*,

    Ur-Base: $(\forall x)(BASE(x) \leftrightarrow UR(x))$,

    provides us with the following:

    Fact: *SOAP* + *NewerV* + *NewV* + *Ur-Base* implies the axiom of foundation.

40. I am not arguing either for the claim that replacement is unneeded in mathematics (the fact that we cannot prove that every Borel game is determined without replacement rules this out) nor for the claim that powerset is necessary to reconstruct modern mathematics (the fact that many mathematicians, either out of constructivist scruples or mathematical curiosity, have formulated interesting versions of analysis and other theories that do not depend on uncountable infinities rules this out). The point is merely that, given the situation as it stands now, if the neologicist can have a set theory with one or the other but not both of these principles, choosing powerset over replacement seems well motivated given that powerset allows for elegant and natural reconstructions of the continuum and other central mathematical structures while there are comparably fewer constructions and (currently identified) results that depend on replacement.

41. Of course, there are abstraction principles, such as the "distractions" found in Shapiro and Weir [13] and Weir [16] that provide all of ZFC and more. For example, we can define '$Bad(P)$' as 'P is the size of an inaccessible' (a notion definable in second-order logic) and then consider:

    $$(\forall P)(\forall Q)[\mathrm{EXT}(P) = \mathrm{EXT}(Q) \leftrightarrow ((\forall x)(Px \leftrightarrow Qx) \vee (Bad(P) \wedge Bad(Q)))].$$

    This principle will give us all of ZFCU. The point, however, is that *SOAP* + *NewerV* + *NewV* is likely the best candidate for a theory based on abstraction principles that defines extensions in terms of conditions that are well motivated and can be justified independently of an extensive prior knowledge of set theory.

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Arché: The AHRB Centre for the Philosophy
of Logic, Language, Mathematics, and Mind
Department of Logic and Metaphysics
University of St Andrews
St Andrews Fife KY16 9AL
SCOTLAND
rtc1@st-andrews.ac.uk