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A Note on Generic Projective Planes

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Abstract Hrushovski constructed an ω-categorical stable pseudoplane which refuted Lachlan's conjecture. In this note, we show that an ω-categorical projective plane cannot be constructed by "the Hrushovski method."

1 Introduction

An infinite *projective plane* is a structure with two sorts, points and lines, together with an incidence relation satisfying the following:

1. on any line there are infinitely many points, and through any point there are infinitely many lines;
2. any two lines intersect in exactly one point, and through any two points there is exactly one line.

A typical example is a projective plane over an algebraically closed field which is $\aleph_1$-categorical but not ω-categorical. Our concern is the following well-known problem (for reference, see Baldwin [1]; Cameron [4], p. 133; Hodges [6], p. 344).

Problem 1.1 Is there an ω-categorical projective plane?

There is a *Hrushovski class* K_α for each $\alpha > 0$ (see Section 3). In [7] Hrushovski constructed the generic models for some subclass K of K_α that is an ω-categorical pseudoplane, which refuted Lachlan's conjecture. (A *pseudoplane* is a structure satisfying condition (2) above with "exactly one" replaced by "finitely many" in both places.) His pseudoplane is not a counterexample of the above problem because it has two points that no line passes through. On the other hand, "the Hrushovski method" is being applied to construct new ones.

For example, Baldwin constructed the generic model for some subclass K of $K_{\frac{1}{2}}$ that is almost strongly minimal non-Desarguesian projective plane, contradicting a conjecture of Zilber (Baldwin [2]). However no one can construct an ω-categorical projective plane using a reasonably general interpretation of the Hrushovski method.

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Let K^* be the subclass of K_α consisting of all graphs which have *no squares* (see Example 3.4). Baldwin and Shi showed that the generic model for the subclass K^* cannot be an ω-categorical projective plane using Marker's lemma (see Baldwin and Shi [3], Section 6). We extend this to show the same result for *any* amalgamation class K contained in K_α. That is, our goal is to show the following theorem: There are no δ-generic ω-categorical projective planes. (The Hrushovski construction finds a theory from a pair $(K, \leq)$. In all known examples of this construction the strong submodel relation $\leq$ is derived from a predimension δ. We show in this paper that natural predimension functions do not yield projective planes. Our final question asks whether some strong submodel relation not imposed by a δ could yield a projective plane.)

2 Generic Structures

Let $\mathcal{L}$ be a finite relational language and K a class of finite $\mathcal{L}$-structures closed under isomorphism and substructures. For any $A, B \in K$ with $A \subset B$ let $A \leq B$ be a reflexive and transitive relation which is invariant under isomorphism. Consider the following set of axioms:

(A1) $A \subset B \subset C \in K$ and $A \leq C$ implies $A \leq B$;
(A2) $\varnothing \leq A$ for any $A \in K$;
(A3) $A \leq B \in K$ and $X \subset B$ implies $A \cap X \leq X$;
(A4) There are no infinite chains $A_1 \subset A_2 \subset \cdots$ such that, for each $i < \omega$, $A_i \in K$, $A_i \not\leq A_{i+1}$ and any proper nonempty subset X of $A_{i+1} - A_i$ satisfies $A_i \leq A_i X$.

For an infinite $\mathcal{L}$-structure M satisfying $A \in K$ for any finite $A \subset M$, define $A \leq M$ if $A \leq B$ for all finite B with $A \subset B \subset M$.

Remark 2.1 Let M satisfy $A \in K$ for all finite $A \subset M$. By (A1)–(A4), for a finite $B \subset M$ there is a unique smallest superset B^* of B with $B^* \leq M$. Such a B^* is called the *closure* of B in M (in symbol, $\mathrm{cl}_M(B)$).

Definition 2.2 Let $(K, \leq)$ satisfy (A1)–(A4). A structure M is said to be $(K, \leq)$-*generic*

1. if A is a finite substructure of M then $A \in K$,
2. if $A \leq M$ and $A \leq B \in K$ then there is an A-embedding $f : B \to M$ with $f(B) \leq M$. (An A-embedding is an embedding fixing A pointwise.)

Whenever we consider a $(K, \leq)$-generic structure, $(K, \leq)$ is supposed to satisfy the above conditions (A1)–(A4). However, even if $(K, \leq)$ satisfies (A1)–(A4), then a $(K, \leq)$-generic structure does not necessarily exist.

Definition 2.3 $(K, \leq)$ is said to have the *amalgamation property* if for any $A \leq B \in K$ and $A \leq C \in K$ there is $D \in K$ such that $f(B) \leq D$ and $g(C) \leq D$ for some A-embeddings $f : B \to D$ and $g : C \to D$.

Fact 2.4 ([3], [8]) If $(K, \leq)$ has the amalgamation property, then there exists a unique $(K, \leq)$-generic structure.

3 δ-Generic Graphs

Let $\mathcal{L}$ consist of three relations P, Q, R, where P, Q are unary relations and R is a nonreflexive and symmetric binary relation. Let α be a real number. Then

1. for a finite $\mathcal{L}$-structure A, $\delta_\alpha(A) = |A| - \alpha|R^A|$, where $R^A = \{\{a,b\} \mid R(a,b), a, b \in A\}$;
2. $K_\alpha = \{A : A$ is a finite $\mathcal{L}$-structure, $\forall B \subset A[\delta_\alpha(B) \geq 0]\}$;
3. for $A \subset B \in K_\alpha$, $A \leq B$ is defined by $\delta_\alpha(XA) \geq \delta_\alpha(A)$ for any $X \subset B - A$.

Remark 3.1 It is easily checked that $(K_\alpha, \leq)$ satisfies (A1)–(A3), and moreover that if α is rational then $(K_\alpha, \leq)$ satisfies (A4).

Fact 3.2 ([3], [8]) Let α be a positive rational number. Let $K \subset K_\alpha$. If a $(K, \leq)$-generic structure M is saturated, then $\mathrm{Th}(M)$ is ω-stable.

Definition 3.3 We say that an $\mathcal{L}$-structure M is *δ-generic* if M is $(K, \leq)$-generic for some α and $K \subset K_\alpha$.

For any elements e, a, b of a graph we say a pair (e, ab) is *special* if $R(e,a) \wedge R(e,b)$ holds.

Example 3.4 It is not difficult to construct a δ-generic projective plane: Let $K \subset K_{\frac{1}{2}}$ be the subclass of all graphs in $K_{\frac{1}{2}}$ which have *no squares*, that is, there are no distinct $a, b, c, d \in A$ with $R(a,b)$, $R(b,c)$, $R(c,d)$, and $R(d,a)$ for any $A \in K$. We claim that $(K, \leq)$ has the amalgamation property.

Proof Let $A \leq B \in K$ and $A \leq C \in K$. Take maximal $B_0 \leq B$ and $C_0 \leq C$ with $B_0 \cong_A C_0$. We can assume that $B_0 = C_0 (= A_0$ say) and also that there are no relations between $B - A_0$ and $C - A_0$. Let $D = B \cup C$. Then it is easily checked that $B, C \leq D \in K$. By maximality of A_0, D has no squares, and so $D \in K_{\frac{1}{2}}$. Hence $(K, \leq)$ has the amalgamation property. □

By Fact 2.4, there exists the $(K, \leq)$-generic structure M. To see that M is a projective plane, it is enough to show that

1. for any $a \in M$ there are infinitely many $b \in M$ with $R(a,b)$;
2. for any distinct $a, b \in M$ with the same sort there is a unique $e \in M$ with (e, ab) special.

Since M has no squares, it satisfies condition (2). We show that M satisfies condition (1). Take any $a \in M$ and let $A = \mathrm{cl}_M(a)$. For any $n < \omega$ let $B_n = \{b_1, \ldots, b_n, a\}$ be a bipartite graph such that for any $i \leq n$, b_i is related to a but there is no relation between b_i and any element of A. It is easily checked that $A \leq AB_n \in K$. By genericity of M, B_n is embedded into M over A. Hence M satisfies (1).

4 Proof of Theorem

In this section, we assume, for the sake of contradiction, the following.

Assumption 4.1 K is a subclass of K_α which has the amalgamation property, and M is a $(K, \leq)$-generic projective plane.

A bipartite graph B is said to be *connected* if for any distinct $a, b \in B$ there exist $b_1, b_2, \ldots, b_n (= b) \in B$ with $R(a, b_1)$, $R(b_1, b_2)$, $\ldots$, $R(b_{n-1}, b_n)$.

Remark 4.2 Let $A (\in K_\alpha)$ be a finite bipartite graph with no loops, that is, for each $n > 2$ there do not exist distinct $a_1, a_2, \ldots, a_n \in A$ with $R(a_1, a_2)$, $R(a_2, a_3)$,$\ldots$, $R(a_{n-1}, a_n)$, and $R(a_n, a_1)$. Then we have $A \in K$.

Proof Take any $a_0 \in A$. Let C_0 be a connected component of a_0 in A. As A has no loops, C_0 can be regarded as a tree with height$(a_0) = 0$. Since M is a projective plane, we can inductively construct $C_0^* \subset M$ with $C_0^* \cong C_0$. Take any $a_1 \in A - C_0$. Let C_1 be a connected component of a_1. In the same way, we have $C_1^* \subset M$ with $C_0^* C_1^* \cong C_0 C_1$. Iterating this process, we have $A^* \subset M$ with $A^* \cong A$. Hence $A \in K$. □

To simplify our notation, we denote δ_α by δ, and $\delta(AB) - \delta(B)$ by $\delta(A/B)$.

Lemma 4.3 $\frac{1}{3} < \alpha \leq \frac{1}{2}$.

Proof Suppose by way of contradiction that $\alpha \leq \frac{1}{3}$ or $\frac{1}{2} < \alpha$.

Case 1 ($\alpha \leq \frac{1}{3}$) Let $abcd$ be an $\mathcal{L}$-structure with the relations $R(d, a)$, $R(d, b)$, $R(d, c)$. By Remark 4.2, we have $abcd \in K$. By $\alpha \leq \frac{1}{3}$, we have $\delta(d/abc) \geq 0$, and so $abc \leq abcd$. By genericity we can assume that $abc \leq abcd \leq M$. Take e with $R(e, a)$, $R(e, b)$, $\neg R(e, c)$. Again, by (4.2), $abce \in K$. By genericity, there is abc-embedding f with $abcf(e) \leq M$. Since $e \not\cong_{abc} d$, we have $f(e) \neq d$. This contradicts axioms of a projective plane.

Case 2 ($\frac{1}{2} < \alpha$) By genericity there are distinct $a, b \in M$ (with the same sort) such that $ab \leq M$. Then (e, ab) is not special for any $e \in M$. (In fact, if there is $e \in M$ with (e, ab) special, then $\delta(e/ab) = 1 - 2\alpha < 0$: a contradiction.) But this contradicts axioms of a projective plane. □

Definition 4.4 A special pair (e, ab) is called *small* if

1. $ab \leq eab$;
2. for any disjoint $A, B \in K$, if $ab \in A$, $e \in B$, $A \leq AB \in K$, $\delta(e/A) = \delta(e/ab)$, then $\delta(e/A) \leq \delta(B/A)$.

Remark 4.5 By Definition 4.4(1), it is clear that if a special pair is small then $\alpha \leq \frac{1}{2}$, and moreover that if $\alpha = \frac{1}{2}$ then a special pair is small.

Lemma 4.6 *For each $n \geq 2$, if $\frac{n-1}{2n-1} < \alpha \leq \frac{n}{2n+1}$ then there is a special pair that is not small.*

Proof Let $a_1 b_1 a_2 b_2, \ldots, a_n b_n cd$ be a finite $\mathcal{L}$-structure with the relations $R(a_1, c)$, $R(a_n, d)$, $\{R(a_i, b_i)\}_{i=1,\ldots,n}$, and $\{R(a_i, a_{i+1})\}_{i=1,\ldots,n-1}$. Let $A = \{a_i\}_{i=1,\ldots,n}$ and $B = \{b_i\}_{i=1,\ldots,n}$. By Remark 4.2, we have $ABcd \in K$.

Claim 4.7 $Bcd \leq ABcd$.

Proof Take any $X \subset A$. It is easily seen that if $X \neq A$ then $\delta(X/Bcd) \geq |X| - 2|X|\alpha$. So, by $\alpha \leq \frac{n}{2n+1} \leq \frac{1}{2}$ we have $\delta(X/Bcd) \geq 0$. If $X = A$ then $\delta(X/Bcd) = n - (2n+1)\alpha \geq n - (2n+1)\frac{n}{2n+1} = 0$. Hence $Bcd \leq ABcd$. □

Claim 4.8 $\delta(a_1/Bcd) > \delta(A/Bcd)$.

Proof $\delta(A/Bcd) - \delta(a_1/Bcd) = (n-1) - (2n-1)\alpha < (n-1) - (2n-1)\frac{n-1}{2n-1} = 0$. □

By Claims 4.7 and 4.8, special pair $(a_1, b_1 c)$ is not small. This completes the proof of Lemma 4.6. □

Lemma 4.9 *Every special pair is small.*

Proof Assume the contrary. Let (e, ab) be a nonsmall special pair. Then there are disjoint A, B such that $ab \in A$, $e \in B$, $A \leq AB \in K$, $Ae \not\leq AB$, $\delta(e/ab) = \delta(e/A)$. By genericity of M, we can assume that $AB \leq M$. In particular, $Ae \not\leq M$. On the other hand, take e^* such that (e^*, ab) is special and $e^* \cong_A e$. By genericity we can assume that $e^*A \leq M$. Since $eA \not\leq M$ and $e^*A \leq M$, we have $e^* \neq e$. This contradicts axioms of a projective plane. □

Lemma 4.10 $\alpha = \frac{1}{2}$.

Proof By Lemma 4.9, a special pair is small. So, by Lemma 4.6, we have

$$\alpha \notin \bigcup_{n=2}^{\infty} \left(\frac{n-1}{2n-1}, \frac{n}{2n+1} \right] = \left(\frac{1}{3}, \frac{1}{2} \right).$$

On the other hand, by Lemma 4.3, $\alpha \in \left(\frac{1}{3}, \frac{1}{2} \right]$. Hence we have $\alpha = \frac{1}{2}$. □

Theorem 4.11 *There are no δ-generic ω-categorical projective planes.*

Proof Assume there is a δ-generic ω-categorical projective plane M. Thus M is $(K, \leq)$-generic for some α and $K \subset K_\alpha$. Since M is ω-categorical, it is saturated. On the other hand, by Lemma 4.10, we have $\alpha = \frac{1}{2}$. By Fact 3.2, $\mathrm{Th}(M)$ is ω-stable. This contradicts the well-known fact that there are no ω-categorical ω-stable pseudoplanes (Cherlin et al. [5]). □

Question 4.12 *Are there no $(K, \leq)$-generic ω-categorical projective planes ?*

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