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The Modal Logic of Agreement and Noncontingency

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Abstract The formula $\triangle$A (it is noncontingent whether A) is true at a point in a Kripke model just in case all points accessible to that point agree on the truth-value of A. We can think of $\triangle$-based modal logic as a special case of what we call the general modal logic of agreement, interpreted with the aid of models supporting a ternary relation, S, say, with OA (which we write instead of $\triangle$A to emphasize the generalization involved) true at a point w just in case for all points x, y, with $Swxy$, x and y agree on the truth-value of A. The noncontingency interpretation is the special case in which $Swxy$ if and only if Rwx and Rwy, where R is a traditional binary accessibility relation. Another application, related to work of Lewis and von Kutschera, allows us to think of OA as saying that A is entirely about a certain subject matter.

1 Introduction

We say that two valuations for a language—assignments of the truth-values T, F to its formula—*agree on* a formula if they both assign the same value to that formula. Humberstone [14] distinguishes two ways for a class of valuations $\mathcal{V}$ for a language to induce a consequence relation on that language, which we describe here in a slightly different notation. On the one hand, we have the consequence relation *inference-determined* by $\mathcal{V}$, denoted $\models_{\mathcal{V}}$, defined thus, where 'Γ' ranges over sets of formulas of the language, and 'A' over individual formulas:

$$\Gamma \models_{\mathcal{V}} \text{A if and only if for all } v \in \mathcal{V}, \text{ if } v(\text{C}) = \text{T for each C} \in \Gamma, \text{ then } v(\text{A}) = \text{T}.$$

On the other hand, we have the consequence relation *supervenience-determined* by $\mathcal{V}$, denoted $\Vdash_{\mathcal{V}}$, defined thus:

$$\Gamma \Vdash_{\mathcal{V}} \text{A if and only if for all } u, v \in \mathcal{V},$$
$$\text{if } u(\text{C}) = v(\text{C}) \text{ for each C} \in \Gamma, \text{ then } u(\text{A}) = v(\text{A}).$$

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The inference-determined consequence relation, in other words, is defined in terms of preservation of *truth* on arbitrarily selected valuations in the determining class, while the supervenience-determined consequence relation is defined in terms of preservation of *agreement* between arbitrarily selected pairs of valuations in the determining class. Sometimes the two will coincide, as with the case of the language with (for definiteness) countably many propositional variables (or sentence letters), $p_1, p_2, \ldots$ and a single binary connective $\leftrightarrow$. The class of valuations we are interested in for this language is the class of all valuations v which are "$\leftrightarrow$-Boolean" in the sense of respecting the familiar truth-table account of $\leftrightarrow$, that is, those v for which we have $v(\mathrm{A} \leftrightarrow \mathrm{B}) = \mathrm{T}$ if and only if $v(\mathrm{A}) = v(\mathrm{B})$ for all formulas A, B. For this choice of $\mathcal{V}$, it turns out that $\models_{\mathcal{V}} = \Vdash_{\mathcal{V}}$ ([14], Corollary 2.7), though in general the two relations will be quite different.[1] (When there is a standard truth-table account for a connective, we call it a *Boolean* connective. The Boolean connectives we assume to be present in the language are the binary $\leftrightarrow, \rightarrow, \wedge$, and $\vee$, singulary $\neg$, and nullary $\top$ and $\bot$; formulas are constructed with their aid in the usual fashion from a countable stock of propositional variables $p_1, p_2, \ldots, p_n, \ldots$—and we will generally write 'p' and 'q' for 'p_1' and 'p_2' in what follows.)[2]

The $\leftrightarrow$-Boolean valuations are those valuations which associate a particular binary truth-function with the connective $\leftrightarrow$ (namely, that mapping $\langle \mathrm{T}, \mathrm{T}\rangle$ and $\langle \mathrm{F}, \mathrm{F}\rangle$ to T, $\langle \mathrm{T}, \mathrm{F}\rangle$ and $\langle \mathrm{F}, \mathrm{T}\rangle$ to F), in the sense (of "associates") given by the following definition: A valuation v *associates* the n-ary truth-function f with the n-ary connective # just in case for all formulas $\mathrm{B}_1, \ldots, \mathrm{B}_n$ we have $v(\#(\mathrm{B}_1, \ldots, \mathrm{B}_n)) = f(v(\mathrm{B}_1), \ldots, v(\mathrm{B}_n))$. We say that the ($n$-ary) connective # is *truth-functional* with respect to $\mathcal{V}$ just in case there is some (n-ary) truth-function f such that for all $v \in \mathcal{V}$, v associates f with #. (Note the prefix form '$\exists f \forall v \in \mathcal{V}$', rather than '$\forall v \in \mathcal{V} \exists f$'. The latter gives the weaker notion of pseudo-truth-functionality with respect to $\mathcal{V}$ discussed in Humberstone [18] and cannot replace truth-functionality proper in the following claim.) If # is truth-functional with respect to $\mathcal{V}$, then this has a striking and simple effect on $\Vdash_{\mathcal{V}}$ in the shape of

(# Composition) $\quad \mathrm{B}_1, \ldots, \mathrm{B}_n \Vdash_{\mathcal{V}} \#(\mathrm{B}_1, \ldots, \mathrm{B}_n)$, for all $\mathrm{B}_1, \ldots, \mathrm{B}_n$.

For suppose that we have the truth-function f associated on every $v \in \mathcal{V}$ with #, and we have $u, v \in \mathcal{V}$ agreeing on $\mathrm{B}_1, \ldots, \mathrm{B}_n (u(\mathrm{B}_i) = v(\mathrm{B}_i)$, for $i = 1, \ldots, n$, that is). Then of course u and v must agree on $\#(B_1, \ldots, \mathrm{B}_n)$ since each must assign to this formula the result of applying the function f to the same n-tuple of arguments, the u, v-agreed truth-values of $\mathrm{B}_1, \ldots, \mathrm{B}_n$. We will not be further concerned (except in passing) with the supervenience-determined consequence relations, but will turn instead to a somewhat different apparatus for registering the agreement and disagreement between valuations on formulas, in terms of which, however, the above point about the principle of # Composition will emerge in a slightly different guise (namely, in the form of the axioms $(\mathrm{OComp})_\#$ in the following section).[3]

The new setting into which we intend to transpose the study of agreement is that of Kripke-style model theory for modal logics, in which we are typically given a set W of points (or "worlds") and a stipulation (notated by 'V' in Section 2 below, as well as at the end of this section) as to which of them are to verify which of the propositional variables, and some further apparatus—accessibility relations and so forth—varying from case to case, in terms of which truth at a point $x \in W$ for a formula A is defined. Representing the latter by the notation '$\mathcal{M} \models_x \mathrm{A}$', where $\mathcal{M}$ is the model concerned, we say that x, y *agree on* the formula A when either $\mathcal{M} \models_x$ A and $\mathcal{M} \models_y$ A or else $\mathcal{M} \not\models_x$ A and $\mathcal{M} \not\models_y$ A. This is not really a new use of the "agreement" terminology introduced above for valuations, since every pair $\langle \mathcal{M}, x\rangle$ with $x \in W$ gives rise to

$$\text{the valuation } v_x^{\mathcal{M}} \text{ defined by } v_x^{\mathcal{M}}(\mathrm{A}) = \mathrm{T} \text{ iff } \mathcal{M} \models_x \mathrm{A} \text{ (for all formulas A)},$$

and then the agreement (relative to $\mathcal{M}$) between points x and y on the formula A is just a matter of the valuations $v_x^{\mathcal{M}}$ and $v_y^{\mathcal{M}}$ agreeing on A in the sense of our opening sentence. By way of motivation for the change of setting, we say a few words about some of the content of von Kutschera [35], which leads directly to the discussion in our Section 2. In Section 3 we shall point out, among other things, the connections between that discussion and a traditional focus of occasional concern in the literature on modal logic: the subject of noncontingency. Since the noncontingency of a statement is a matter of all (accessible) worlds agreeing on its truth-value, it is not surprising that a modal approach to agreement should bear on that subject.

That aspect of von Kutschera's discussion in [35] we want to consider is his reaction to (some variations on) the *Fitch Derivation* (originating in Fitch [10], Theorem 4) of the conclusion that every truth is known from the apparently less implausible starting point that every truth is capable of being known. A somewhat informal version of this derivation runs as follows. Suppose that something, p, say, is true but not known to be true. Then we have as a truth $p \wedge \neg \mathrm{K}p$, so if everything true is capable of being known, we have (writing '$\Diamond$' for the possibility operator) $\Diamond\mathrm{K}(p \wedge \neg\mathrm{K}p)$. But this last formula is refutable in any mixed alethic-epistemic logic on minimal assumptions, since it will (given such assumptions) imply $\Diamond(\mathrm{K}p \wedge \mathrm{K}\neg\mathrm{K}p)$ and hence $\Diamond(\mathrm{K}p \wedge \neg\mathrm{K}p)$, where the expression in the scope of '$\Diamond$' is an explicit contradiction. There have been several philosophical reactions to this derivation, and in particular to the question of whether what it shows is that a "principle of knowability" to the effect that every truth is (logically) capable of being known is correctly captured in the form of the schema $\mathrm{A} \to \Diamond\mathrm{KA}$, as assumed here in deriving from it the unpalatable consequence that every truth is, in fact, known. (In the preceding derivation sketch, this schema is instantiated by taking A as '$p \wedge \neg\mathrm{K}p$'. One interesting rejection of this proposed formalization of the principle of knowability may be found in Edgington [8] (itself further discussed in, for example, Sorensen [33], pp. 124–29, where additional references to the literature may be found). It concentrates on the fact that the embedding of 'A' under '$\Diamond$K' in the consequent misleadingly directs us to consider A from the perspective of worlds different from the world at which we thought we were hypothesizing, with the antecedent, that A was true. Thus the principle requires amendment by the judicious insertion of occurrences of an "Actually" operator (written as '$\mathcal{A}$' in Section 4 below, where this is mentioned for other reasons).[4] A quite different proposal, essentially that to be found in [35], considers instead the imposition of a restriction on the schematic A: intuitively that we should not allow the substitution for A of (partially) epistemic statements such as that represented by the crucial '$p \wedge \neg\mathrm{K}p$' in the above derivation. Von Kutschera in fact considers similar problems for the notions of belief and true belief, rather than knowledge. Since we are using A, B, . . . as schematic letters for formulas here, let us write the belief-operator as 'K_0' rather than 'B'. In this notation, von Kutschera ([35], p. 104) considers two principles which have untoward consequences unless subjected to some restriction, calling them P1* and P2*:

P1* $$\Diamond \mathrm{A} \rightarrow \Diamond \mathrm{K}_0\mathrm{A}$$

P2* $$\mathrm{A} \rightarrow \Diamond(\mathrm{K}_0\mathrm{A} \wedge \mathrm{A})$$

It is the second of these which yields, by minor modifications of the Fitch derivation sketched above, to the conclusion that every truth is truly believed. (Von Kutschera does not actually allude to this derivation—and indeed makes no mention of Fitch—but it is clearly what he has in mind with the remark that "any logic is unacceptable in which this assumption implies omniscience"). P1* presents a somewhat different difficulty, since von Kutschera wants to consider doxastic logics in which the "self-confidence" principle $\mathrm{K}_0(\mathrm{K}_0\mathrm{A} \rightarrow \mathrm{A})$ is provable for all A. As he notes, putting '$\mathrm{K}_0 p \wedge \neg p$' for 'A' here makes trouble because the hypothesis that $\Diamond(\mathrm{K}_0 p \wedge \neg p)$ leads by P1* to the conclusion that $\Diamond\mathrm{K}_0(\mathrm{K}_0 p \wedge \neg p)$, clashing with the necessitation of corresponding instance of the self-confidence schema with this same substitution for 'A', given the additional (consistency) principle that $\mathrm{K}_0\mathrm{A}$ implies $\neg\mathrm{K}_0\neg\mathrm{A}$. Thus we can conclude that $\neg\Diamond(\mathrm{K}_0 p \wedge \neg p)$, most implausibly since consistency (and other aspects of rationality) together with what we are calling self-confidence should not preclude the possibility of false belief. The situation is somewhat different from that of the original Fitch derivation because in the latter case the critical subformula was $p \wedge \neg\mathrm{K}p$ (or $\neg\mathrm{K}p \wedge p$, as it may more conveniently be put for the present contrast) whereas in the current case it is $\mathrm{K}_0 p \wedge \neg p$.

The replacements for P1* and P2* which von Kutschera offers are subject to a further restriction expressed by the use of a new operator O, with the informal reading of OA as 'it is an objective (or nondoxastic) proposition that A':

P1 $$(\mathrm{OA} \wedge \Diamond\mathrm{A}) \rightarrow \Diamond\mathrm{K}_0\mathrm{A}$$

P2 $$(\mathrm{OA} \wedge \mathrm{A}) \rightarrow \Diamond(\mathrm{K}_0\mathrm{A} \wedge \mathrm{A}).$$

To interpret this language with the non-Boolean 1-ary operators K_0 and $\Diamond$ (or rather the necessity operator $\Box$, in terms of which we may take $\Diamond$ to be defined in the usual fashion) as well as the novel O, von Kutschera uses models $\langle W, \sim, S, V\rangle$ with W a nonempty set (the worlds), $\sim$ an equivalence relation (on W), S a binary relation (on W) satisfying certain conditions we need not go into, to make it suitable as the accessibility relation for the belief operator K_0 in a fairly strong doxastic logic,[5] and V assigning subsets of W to the propositional variables as the sets of worlds at which they are to be true, subject to the special condition that whenever for $w, x \in W$ we have $w \sim x$, we must have $w \in V(p_i)$ if and only if $x \in V(p_i)$. There is no special accessibility relation supplied for $\Box$, which is instead interpreted by universal quantification over W. Finally—and this is the aspect of the semantics which we take up below—for O a model deems OA true at any world just in case every pair of worlds standing in the relation $\sim$ agree on the truth-value of A.[6]

Unlike P1* and P2*, the schemes P1 and P2 can be added to the set of formulas valid (i.e., have only unfalsifiable instances) according to this semantics, without producing untoward consequences. (Von Kutschera is more interested in certain other extensions of his basic set of valid formulas, expressing the independence of belief and the objective world, and the tension between them and some supervenience theses. See [35] for details.) We will pursue the O-fragment of von Kutschera's language along more or less the above semantic lines, without the alethic and doxastic operators. The latter present no novelties since they just have familiar clauses in the definition of truth, just requiring the truth of the formula after the operators at a range of worlds, rather than, as for the case of 'O', agreement as to the truth of that formula at a range of pairs of worlds. We continue to use von Kutschera's 'O' notation as well, though preferring to think of it as neutrally mnemonic for 'operator (of current concern)' rather than for 'objective'. Our principal object in the following section will be the provision of a simple and illuminating axiomatic description of the set of valid formulas. (Von Kutschera's discussion is entirely model-theoretic.)

As already remarked, our semantic treatment follows *more or less* the treatment von Kutschera gives: there are two respects of difference. In the first place, we will not impose the special condition on V that $w \sim x \Rightarrow (w \in V(p_i) \Leftrightarrow x \in V(p_i))$, because—however appropriate it may be for the philosophical agenda of [35]—this leads to an inconvenient failure of the set of valid formulas to be closed under uniform substitution (of arbitrary formulas for propositional variables): on von Kutschera's semantics, $p \to \mathrm{O}p$ comes out valid while substituting $\mathrm{K}_0 p$ for p at both occurrences gives an invalid formula, for instance.[7] The second respect in which for technical reasons it is appropriate not to follow precisely von Kutschera's example is over the equivalence relation $\sim$ in his models. The clause governing O says that OA is true at a world w just in case every pair of worlds standing in this relation to each other agree on A, and this truth-condition conspicuously makes no reference to the world w itself. An appropriately basic modal logic of agreement should allow the truth-value of OA at w to depend on w, leaving it open to us to consider strengthening the logic in such a way as to rule out, should the application demand it, the effects of this dependence: variation in the truth value of OA from world to world in a model. The situation is entirely parallel to the treatment of $\Box$ in alethic modal logic without the use of accessibility relations in the models. It is better to work with the general case to begin with (obtaining the smallest normal modal logic **K**, and then consider the effects of the special condition that the accessibility relation holds between every pair of worlds and so can be dispensed with, as it can be for the much stronger logic **S5**. (Note its absence from the semantics of [35] summarized above.) Similarly, we concentrate in Section 2 on the basic logic allowing world-to-world variation that corresponds to von Kutschera's relation, the relation $\sim$, and then ask later how to strengthen the logic to iron out this variation. To mark this difference, we will actually employ a different notation and write '$x \equiv_w y$' to mean that x and y are equivalent relative to w, invoking this relation for the determination of OA's truth value at the point w. In Section 3 we will look into the question of what difference it makes to the validation of formulas that, for a fixed w, this relation between x and y be an equivalence relation at all. Finally, as all this talk of formulas will have made clear, we are from now thinking of logics in the same way as most mainstream work in modal logic: as (certain) sets of formulas, rather than as (inference-determined) consequence relations. The reason for this is simply a desire for continuity with such work (including of course von Kutschera's). The reader who prefers to think in terms of consequence relations can easily recover suitable proof-systems, using sequent-to-sequent rules, from the axiomatic systems we describe. Its semantic characterization (analogous to Theorem 2.4 below) will then be as the consequence relation inference-determined by the class of all valuations $v_x^{\mathcal{M}}$ for $\mathcal{M}$ a model in the sense of our discussion below and x a point in that model.

In the terms with which this introduction opened, such an emphasis on the inference-determination (as opposed to supervenience-determination) of a logic by a class of valuations is not a departure from the theme of studying "agreement" from a logical point of view, though it represents the choice of a different locus for that study: now on the logical behavior of a sentence operator whose raison d'être is to invoke from within the object language itself the relation of agreement in truth-value.

2 The Basic Logic of Agreement and an Extension Thereof

With the preceding remarks as motivation, for the purposes of the present section a *model* $\mathcal{M}$ will be taken to be a structure $\langle W, \equiv, V\rangle$ in which W is a nonempty set, $\equiv$ is a function assigning to each $w \in W$ an equivalence relation $\equiv_w \subseteq W \times W$, and V is a function assigning to each propositional variable a subset of W. Truth at $w \in W$ in $\mathcal{M}$ for a formula A (notated '$\mathcal{M} \models_w$ A') is defined inductively in terms of the construction of the formula A:

$$\mathcal{M} \models_w p_i \text{ if and only if } w \in V(p_i),$$

$$\mathcal{M} \models_w \mathrm{A} \wedge \mathrm{B} \text{ if and only if } \mathcal{M} \models_w \mathrm{A} \text{ and } \mathcal{M} \models_w \mathrm{B},$$

and similarly for the other Boolean primitives, which for convenience we assume to include the binary $\to$ and also the nullary $\top$, $\bot$.

$$\mathcal{M} \models_w \mathrm{OA} \text{ if and only if for all } x, y \in W \text{ such that } x \equiv_w y \text{ we have } \mathcal{M} \models_x \mathrm{A} \text{ iff } \mathcal{M} \models_y \mathrm{A}.$$

A formula A is *valid* just in case for every model $\mathcal{M} = \langle W, \equiv, V\rangle$, for every $w \in W$, we have $\mathcal{M} \models_w$ A. (More refined terminology: abstracting from the details of what V does in a model, we have the notion of a *frame* $\langle W, \equiv\rangle$, on which a formula is said to be valid if it is true at every point in every model $\langle W, \equiv, V\rangle$ on that frame. The formulas valid tout court are then those which are valid on every frame.) We turn to the project of axiomatizing the valid formulas.

We offer two axiom schemes, one to supply classical propositional logic for the Boolean connectives, and one special O-axiom:

(TF) A for A any substitution-instance of a truth-functional tautology,

(OComp) $(\mathrm{OA}_1 \wedge \cdots \wedge \mathrm{OA}_n) \to \mathrm{O\#}(\mathrm{A}_1, \ldots, \mathrm{A}_n)$

for all formulas $A_1, \ldots, A_n$ and every primitive n-ary Boolean connective #. For the cases of #, =, $\wedge$, $\neg$, $\top$, $\bot$, in which n is respectively 2, 1, 0, 0, this metascheme instantiates to the schemes,

$(\mathrm{OComp})_\wedge$ $(\mathrm{OA}_1 \wedge \mathrm{OA}_2) \to \mathrm{O}(\mathrm{A}_1 \wedge \mathrm{A}_2)$

$(\mathrm{OComp})_\neg$ $\mathrm{OA} \to \mathrm{O}\neg\mathrm{A}$

$(\mathrm{OComp})_\top$ $\mathrm{O}\top$

$(\mathrm{OComp})_\bot$ $\mathrm{O}\bot$.

We also present two rules, one (Modus Ponens) for the truth-functional machinery, and the other a special rule for O, rendering it "congruential" (capable of supporting the replacement of provable equivalents):

(MP) From A $\to$ B and A to B

(OCong) From $A \to B$ and $B \to A$ to $OB \to OA$.

Note that in view of what we call (TFC) below, (OCong) could equivalently be formulated as licensing the transition from $A \leftrightarrow B$ to $OA \leftrightarrow OB$.

The smallest logic containing all instances of the axiom-schemes (TF) and (OComp) and closed under the rules (MP) and (OCong) we call **LO**, and to indicate that a formula A belongs to this logic we write $\vdash_{\mathbf{LO}} A$.[8] For the sake of establishing the soundness of this logic (the validity of all A for which $\vdash_{\mathbf{LO}} A$, that is), a result we incorporate into Theorem 2.4 below, we note that all instances of (TF) and (OComp) are valid—in the latter case by essentially the same reasoning as was deployed in Section 1 a propos of what we called the principle of *# Composition* (which is why we use the label 'OComp'). Note that the converse of $(\mathrm{OComp})_\wedge$ has, by contrast, invalid instances, showing that **LO** is not a normal modal logic (i.e., when 'O' is taken as $\Box$); in particular, the operator O is not monotone. (For 'normal' and 'monotone', not defined here—see, for example, Chellas and McKinney [3], or §8.2 of Chellas [2]—though the authors use "monotonic" instead of "monotone", a usage we prefer to avoid in case of confusion with the unrelated issue of monotonic versus nonmonotonic logics.) It is clear that (MP) preserves validity, since it preserves truth at an arbitrarily selected point in any model, while for (OCong) validity is preserved because, as we now show, this rule preserves, not truth at an arbitrary point in a model, but rather the property of being true at every point in a model (as with the rule of Necessitation in normal modal logics). Assume then that $A \to B$ and $B \to A$ are both true throughout a model $\mathcal{M} = \langle W, \equiv, V\rangle$, with a view to showing that $OB \to OA$ is likewise. If this last formula is not true throughout $\mathcal{M}$, we have some $w \in W$ with $\mathcal{M} \models_w OB$ while $\mathcal{M} \not\models_w OA$. From this last, we have $x, y \in W$ with $x \equiv_w y$ while, say (without loss of generality) $\mathcal{M} \models_x A$ but $\mathcal{M} \not\models_y A$. Since $A \to B$ and $B \to A$ are true throughout $\mathcal{M}$, A and B have the same truth values as each other at the points x and y, so $\mathcal{M} \models_x B$ and $\mathcal{M} \not\models_y B$, contradicting the fact that $\mathcal{M} \models_w OB$, since $x \equiv_w y$.

One might think that although for soundness purposes this argument is fine, in the interest of completeness (having **LO** prove all the valid formulas, that is), we should provide not only a rule licensing the passage to the conclusion $OB \to OA$ from the premise that $A \leftrightarrow B$, but another one to that same conclusion from the premise that $A \leftrightarrow \neg B$, since a corresponding argument will work in this case also: having to give opposite truth-values to A and B will work just as well to contradict the assumption that $\mathcal{M} \models_w OB$, since it will force $\mathcal{M} \models_y B$ and $\mathcal{M} \not\models_x B$. In fact, however, (OCong) does not need to be supplemented in this way since it takes us from $A \leftrightarrow \neg B$ to $OA \leftrightarrow O\neg B$ and we have $\vdash_{\mathbf{LO}} O\neg B \leftrightarrow OB$, in the backward direction by appeal to $(\mathrm{OComp})_\neg$, and in the forward direction by another appeal to $(\mathrm{OComp})_\neg$, giving $\vdash_{\mathbf{LO}} O\neg B \to O\neg\neg B$, and so by (OCong) with the equivalence of B and $\neg\neg B$ as its starting point, and truth-functional reasoning, we get the conclusion that $\vdash_{\mathbf{LO}} O\neg B \to OB$.

Note that by (TF) and (MP), when B is a truth-functional consequence of $A_1, \ldots, A_n$ then if $\vdash_{\mathbf{LO}} A_i$ (for each $i = 1, \ldots, n$) then $\vdash_{\mathbf{LO}} B$. We will appeal to this fact about truth-functional consequences by writing (TFC). This is an example of what falls under the heading (just used) of "truth-functional reasoning."

Unlike the various axioms $(\mathrm{OComp})_\#$ in which the implications take us from O-prefixed components to O-prefixed compounds, the principle governing negation just derived (i.e., $O\neg B \to OB$) goes in the reverse "decompositional" direction. Other examples of this phenomenon are worth noting at this point, since they require no more than the resources of **LO** but have in the past received attention (e.g., Humberstone [16], p. 218) only in the context of stronger logics—namely, modal noncontingency logics, the weakest of which will be displayed as a proper extension of **LO** in Section 3 below. We have, for example,

$$\vdash_{\mathbf{LO}} (O(A \leftrightarrow B) \wedge OA) \to O((A \leftrightarrow B) \leftrightarrow A)$$

by $(\mathrm{OComp})_{\leftrightarrow}$, and hence, by (OCong),

$$\vdash_{\mathbf{LO}} (O(A \leftrightarrow B) \wedge OA) \to OB,$$

which is decompositional in that the 'B' on the right is a component of one of the O-prefixed formulas on the left. Further, exporting the second conjunct and noting the symmetrical situation of A and B here, we get the following, the schema which often figures as an axiom in noncontingency-based modal logics (with 'O' usually written as 'Δ' in that case, our present point being that this fact only depends on the "agreement" aspect of the situation, and not the distinctively "noncontingency" aspect):

$$\vdash_{\mathbf{LO}} O(A \leftrightarrow B) \to (OA \leftrightarrow OB).$$

Other decompositional principles include the following, for which the 'A' in the consequent can be taken as obtained by (OCong) from the disjunction and from the conjunction, respectively, of the O-prefixed formulas in the antecedent:

$$\vdash_{\mathbf{LO}} (O(A \wedge B) \wedge O(A \wedge \neg B)) \to OA$$

$$\vdash_{\mathbf{LO}} (O(A \vee B) \wedge O(A \vee \neg B)) \to OA.$$

Slightly further afield are some interesting cases neither compositional nor decompositional:

$$\vdash_{\mathbf{LO}} (O(A \wedge B) \wedge O(A \leftrightarrow B)) \to O(A \vee B)$$

$$\vdash_{\mathbf{LO}} (O(A \vee B) \wedge O(A \leftrightarrow B)) \to O(A \wedge B)$$

$$\vdash_{\mathbf{LO}} (O(A \vee B) \wedge O(A \wedge B)) \to O(A \leftrightarrow B)$$

for which in all three instances the relevant case of $(\mathrm{OComp})_{\#}$ takes # as $\leftrightarrow$. Using this same choice of #, we have, similarly, the more fully decompositional

$$\vdash_{\mathbf{LO}} (O(A \wedge B) \wedge O(A \to B)) \to OA.$$

If one is wondering of a candidate principle of the 'conjunction of O-formulas implies a given O-formula' whether it is provable—for example, that last cited but with 'B' in place of 'A' in the consequent—a simple test is provided by the notion we baptized as '$\Vdash_{\mathcal{V}}$' in Section 1 (for $\mathcal{V}$ the class of Boolean valuations).[9] Replace the schematic letters by propositional variables, getting, in this case,

$$(O(p \wedge q) \wedge O(p \to q)) \to Oq,$$

and ask whether we have $p \wedge q,\ p \to q \Vdash_{\mathcal{V}} q$. A simple truth-table test shows that the answer is negative (in each of the last two lines of a conventionally set out four-line truth-table, the formulas on the left have the same value while the value of that on the right changes) from which it follows that the inset formula above is not **LO**-provable. The criterion we are using here is that for O-free formulas $A_1, \ldots, A_n$, B, we have $A_1, \ldots, A_n \Vdash_{\mathcal{V}} B$ if and only if $\vdash_{\mathbf{LO}} (OA_1 \wedge \cdots \wedge OA_n) \to OB$, the unobvious (if) half of which claim follows from the completeness theorem for **LO** given below (Theorem 2.4).

To make our way in the direction of that result, we need to begin with a generalized version of (OComp).

Lemma 2.1 *If* B *is a Boolean compound of formulas* $A_1, \ldots, A_n$, *then* $\vdash_{\mathbf{LO}} (OA_1 \wedge \cdots \wedge OA_n) \to OB$.

Proof The proof is by induction on the number of Boolean connectives used to construct B from $A_1, \ldots, A_n$, appealing to (OComp). □

We presume familiarity with the notions of consistency (with respect to a logic) and maximal consistency, and of the fact that every consistent set of formulas can be extended to a maximal consistent set (Lindenbaum's Lemma). Any standard text on modal logic—for example, [13], [2]— will supply these details.

The *canonical model* for **LO** is the model $\mathcal{M}_{\mathbf{LO}} = \langle W_{\mathbf{LO}}, \equiv_{\mathbf{LO}}, V_{\mathbf{LO}} \rangle$ in which $W_{\mathbf{LO}}$ is the set of all maximal **LO**-consistent sets of formulas, $V_{\mathbf{LO}}(p_i) = \{w \in W_{\mathbf{LO}} | p_i \in w\}$, and for all $w \in W_{\mathbf{LO}}$, $\equiv_{\mathbf{LO}_w}$ is that relation holding between $x, y \in W$ just in case for every formula $OC \in w$, we have $C \in x$ if and only if $C \in y$. Since this is clearly an equivalence relation $\mathcal{M}_{\mathbf{LO}}$ is a model, provided that $W_{\mathbf{LO}}$ is nonempty. But this last is equivalent to the claim that **LO** is consistent, which follows from the soundness of **LO**, already established. (Alternatively, consider a nonstandard truth-functional interpretation, in which O is interpreted as expressing the constant true 1-ary truth-function, on which all theorems of **LO** are easily seen to be truth-functional tautologies.) From now on, we will usually drop the subscript '**LO**', at least in the proofs (if not the initial formulation) of numbered results.

Given a set Γ of formulas, we define a *signing* of Γ to be any set of formulas whose elements comprise, for each $C \in \Gamma$, exactly one of $C, \neg C$. If the elements of Γ are enumerated as $C_1, \ldots, C_n, \ldots$ we can think of a signing of Γ as $s_1C_1, \ldots, s_nC_n, \ldots$ where each "sign" s_i is either positive (null) or negative—that is, s_iC_i is either C_i or $\neg C_i$. When the ordering of the formulas concerned is clear, we sometimes refer to s itself, conceived of as a function assigning s_i to i, as a signing of Γ.

Lemma 2.2 *For* $w \in W_{\mathbf{LO}}$ *with* $OA \notin w$, *suppose that* $\Gamma = \{C_1, \ldots, C_n, \ldots\}$ *is the set of formulas* C *for which* $OC \in w$. *Then there is some signing of* Γ *which is* **LO**-*consistent with* A *and also* **LO**-*consistent with* $\neg A$.

Proof We begin by showing that for all n, there is a signing of $\{C_1, \ldots, C_n, \ldots\}$ which is **LO**-consistent with A and also with $\neg A$. Suppose otherwise. Then for every signing s of this set we have either

$$\vdash (s_1C_1 \wedge \cdots \wedge s_nC_n) \to \neg A \tag{1}$$

or else

$$\vdash (s_1C_1 \wedge \cdots \wedge s_nC_n) \to A. \tag{2}$$

Let B be the disjunction of all those conjunctions $s_1C_1 \wedge \cdots \wedge s_nC_n$ falling under the description (2) here, and B′ be the disjunction of all those conjunctions $s_1C_1 \wedge \cdots \wedge s_nC_n$ falling under (1). Then by (TFC) we have

$$\vdash_{\mathbf{LO}} B \to A \tag{3}$$

and

$$\vdash_{\mathbf{LO}} \mathrm{B}' \to \neg\mathrm{A}. \tag{4}$$

But by (TF) we have $\vdash \mathrm{B} \vee \mathrm{B}'$, since this is the disjunction of all "state-descriptions" in the C_i. Rewriting this as

$$\vdash_{\mathbf{LO}} \neg\mathrm{B} \to \mathrm{B}' \tag{5}$$

makes it obvious, in view of (5), that

$$\vdash_{\mathbf{LO}} \neg\mathrm{B} \to \neg\mathrm{A} \tag{6}$$

and so

$$\vdash_{\mathbf{LO}} \mathrm{A} \to \mathrm{B} \tag{7}$$

giving us, in (3) and (7), the two premises for an application of (OCong) which yields

$$\vdash_{\mathbf{LO}} \mathrm{OB} \to \mathrm{OA}. \tag{8}$$

Now, recalling that B is a certain disjunction of conjunctions $s_1\mathrm{C}_1 \wedge \cdots \wedge s_n\mathrm{C}_n$ each of whose conjuncts is either C_i or $\neg\mathrm{C}_i$, and is thus a Boolean combination of formulas—the C_i—for each of which we have $\mathrm{OC}_i \in w$, we invoke Lemma 2.1 to conclude that $\mathrm{OB} \in w$, and hence, by (8), that $\mathrm{OA} \in w$, contradicting our initial supposition concerning OA. This contradiction establishes, then, that for all n, there is a signing of $\{\mathrm{C}_1, \ldots, \mathrm{C}_n\}$ which is **LO**-consistent with A and also with $\neg$A. To complete the proof, we must show that there is such a signing of the whole infinite set $\Gamma = \{\mathrm{C}_1, \ldots, \mathrm{C}_n \ldots\}$. (This set is clearly infinite because it has among its elements, for instance $\top, \mathrm{O}\top, \mathrm{OO}\top, \ldots$, each of these being a formula C for which $\mathrm{OC} \in w$.) But this conclusion follows from the finite version just established, by König's Lemma. Consider the infinite binary branching tree all of whose nodes except the origin are labeled with formulas, the labeling effected in the following way. At the first level (i.e., immediately dominated by the root) we have nodes labeled with the formulas C_1 in the one case and $\neg\mathrm{C}_1$ in the other, each in turn dominating nodes labeled, respectively, with C_2 and $\neg\mathrm{C}_2$, and so on. Each branch of this tree represents in the obvious way a signing of the set Γ. Now prune this tree by erasing any node such that the set of labels from the origin to that node is not both **LO**-consistent with A and also with $\neg$A, together with all descendants of that node. The resulting tree is finitary but still has infinitely many nodes, since, by the "finite version" established above, for each n there is a node with either C_n or $\neg\mathrm{C}_n$ as its label, there being a signing of $\{\mathrm{C}_1, \ldots, \mathrm{C}_n \ldots\}$ which is **LO**-consistent with A and also with $\neg$A. Thus by König's Lemma, the pruned tree contains at least one infinite branch, representing a signing of Γ which is **LO**-consistent with each of A, $\neg$A. □

Lemma 2.3 *For every formula* B, *and every* $w \in W_{\mathbf{LO}}$, $\mathcal{M}_{\mathbf{LO}} \models_w \mathrm{B}$ *if and only if* $\mathrm{B} \in w$.

Proof The proof follows the standard pattern by induction on the construction of B, so we deal only with the novel (inductive) case in which $\mathrm{B} = \mathrm{OA}$ for some formula A, with the inductive hypothesis assuring us that for all $u \in W_{\mathbf{LO}}$, $\mathcal{M}_{\mathbf{LO}} \models_u \mathrm{A}$ if and only if $\mathrm{A} \in u$. Let w be an arbitrary element of $W_{\mathbf{LO}}$. By the truth-definition, $\mathcal{M}_{\mathbf{LO}} \models_w \mathrm{OA}$ if and only if for all $x, y \in W$ such that $x \equiv_w y$ we have $\mathcal{M} \models_x \mathrm{A}$ if and only if $\mathcal{M} \models_y \mathrm{A}$. (Here we suppress the subscripted '**LO**'.) Thus by the inductive hypothesis and the definition (for the canonical model) of $\equiv$, what we have to establish is

$$\mathrm{OA} \in w \Leftrightarrow \forall x, y \in W[[(\forall \mathrm{C}(\mathrm{OC} \in w \Rightarrow (\mathrm{C} \in x \Leftrightarrow \mathrm{C} \in y))] \Rightarrow (A \in x \Leftrightarrow A \in y)].$$

The $\Rightarrow$ direction is immediate, instantiating the universally quantified variable 'C' here to the formula A itself. For the $\Leftarrow$ direction, suppose that $\mathrm{OA} \notin w$. We must find $x, y \in W$, which "agree" in respect of all formulas C for which $\mathrm{OC} \in w$ (i.e., $\mathrm{C} \in x$ if and only if $\mathrm{C} \in y$, for such C), yet do not agree on A. Lemma 2.2 tells us that there is some signing of the set $\Gamma = \{\mathrm{C}|\mathrm{OC} \in x\}$, Γ', say, which is **LO**-consistent with A and also **LO**-consistent with $\neg$A. Accordingly, let the desired x and y be maximal consistent extensions of the **LO**-consistent sets $\Gamma' \cup \{\mathrm{A}\}$ and $\Gamma' \cup \{\neg\mathrm{A}\}$, respectively. Then x and y disagree on A but agree on all the C for which $\mathrm{OC} \in w$, since for each such C, Γ' contains either C or else $\neg$C. □

Theorem 2.4 *For all formulas* A, $\vdash_{\mathbf{LO}}$ A *if and only if* A *is valid.*

Proof The "only if" direction (soundness) having already been established, we deal with the "if" direction (completeness), which follows directly from Lemma 2.3: if $\nvdash_{\mathbf{LO}}$ A, then $\{\neg\mathrm{A}\}$ is consistent and so can be extended some $w \in W_{\mathbf{LO}}$, which accordingly does not also have A as an element, whence by that lemma $\mathcal{M}_{\mathbf{LO}} \not\models_w$ A, showing that A is not valid. □

At the end of Section 1, a question was raised which we can now describe as the question of how to extend the basic logic **LO** to obtain a logic sound and complete with respect to the class not of all models but rather with respect to the class of all those models $\langle W, \equiv, V \rangle$ in which the equivalence relation assigning function $\equiv$ is a constant function, thereby returning to (suitable reducts—$\langle W, \sim, V \rangle$—of) the models figuring in von Kutschera's discussion in [35]. In other words, how can we strengthen our axiomatization of **LO** so that the provable formulas of the strengthened logic are precisely those that are *valid*$^+$ in the sense of being true at every $w \in W$ in any model $\langle W, \equiv, V \rangle$ satisfying the further condition that for all $u, w, x, y \in W$, $x \equiv_u y \Rightarrow x \equiv_w y$ (or equivalently, with '$\Leftrightarrow$' replacing '$\Rightarrow$' here).

One way of obtaining the desired extension of **LO** is to strengthen the rule (OCong) from – in its '$\leftrightarrow$ in the premise' formulation:

$$\text{(OCong)} \qquad \frac{\mathrm{A} \leftrightarrow \mathrm{B}}{\mathrm{OA} \rightarrow \mathrm{OB}}$$

to the following rule (or rules, if we count each choice of $m, n \geq 0$, as making for a different rule):

$$\text{(OCong)}^+ \quad \frac{((\mathrm{OD}_1 \wedge \cdots \wedge \mathrm{OD}_m) \wedge (\neg\mathrm{OE}_1 \wedge \cdots \wedge \neg\mathrm{OE}_n)) \rightarrow (\mathrm{A} \leftrightarrow \mathrm{B})}{((\mathrm{OD}_1 \wedge \cdots \wedge \mathrm{OD}_m) \wedge (\neg\mathrm{OE}_1 \wedge \cdots \wedge \neg\mathrm{OE}_n)) \rightarrow (\mathrm{OA} \rightarrow \mathrm{OB})}\,.$$

Let us denote by $\mathbf{LO}^+$ the logic axiomatized by (TF), (OComp), (OCong)$^+$, and (MP). Then we have the following theorem.

Theorem 2.5 *For all formulas* A, $\vdash_{\mathbf{LO}^+}$ A *if and only if* A *is valid*$^+$.

Proof The "only if" (soundness) direction just requires that (OCong)$^+$ preserves truth throughout a model with constant $\equiv$, the crucial point being that such constancy means that O-formulas have the same truth-values everywhere in the model. For the "if" (completeness) direction, we modify the earlier canonical model construction, beginning with Lemma 2.2, which should be replaced with this: For $w \in W_{\mathbf{LO}^+}$ with $\mathrm{OA} \notin w$, suppose that $\Gamma = \{\mathrm{C}_1, \ldots, \mathrm{C}_n, \ldots\}$ is the set of formulas C for which $\mathrm{OC} \in w$. Then, putting $\Delta = \{\mathrm{OD}|\mathrm{OD} \in w\} \cup \{\neg\mathrm{OE}|\mathrm{OE} \notin w\}$, there is some signing of Γ which is $\mathbf{LO}^+$-consistent with $\Delta \cup \{\mathrm{A}\}$ and also $\mathbf{LO}^+$-consistent with $\Delta \cup \{\neg\mathrm{A}\}$. This is established by using (OCong)$^+$ at the point at which (OCong) was used in the proof of Lemma 2.2. The effect of building in Δ to what, when the proof of Lemma 2.3 is adapted, is that when dealing with the hypothesis that $\mathrm{OA} \notin w$, we can find $x, y \in W$, which "agree" in respect of all formulas C for which $\mathrm{OC} \in w$ (i.e., $\mathrm{C} \in x$ if and only if $\mathrm{C} \in y$, for such C), yet do not agree on A, and which both contain precisely the same O-formulas as w. If we start with a formula B for which $\nvdash_{\mathbf{LO}^+}$ B, and let w be a maximal consistent extension of $\{\neg\mathrm{B}\}$, then we consider only the points from the full canonical model for $\mathbf{LO}^+$ which have precisely the same O-formulas in them as w, a restriction which by the reasoning just given, still gives the coincidence of truth and membership Lemma 2.3 (as adapted) speaks of, but now in a model for which $\equiv$ is a constant function (since all points in the submodel thus generated agree on all O-formulas), showing that the unprovable B is not valid$^+$. □

The new rule (OCong)$^+$, though very convenient for the sake of the above proof, looks rather cumbersome, and so some interest attaches to its replaceability by some simple axiom-scheme(s), to be taken alongside the axioms and rules—including the original (OCong)—used to axiomatize **LO**. The valid$^+$ principle OOA shows some promise in this regard.[10] It can be derived from the above basis for $\mathbf{LO}^+$ thus. We take an instance of (TF): $\mathrm{OA} \to (\mathrm{OA} \leftrightarrow \top)$ to which we apply (OCong)$^+$ with $m = 1$ (and $\mathrm{D}_1 = \mathrm{A}$), $n = 0$, getting the conclusion $\mathrm{OA} \to (\mathrm{OOA} \leftrightarrow \mathrm{O}\top)$, whence by (OComp)$_\top$ and (TFC), we get $\mathrm{OA} \to \mathrm{OOA}$. To conclude that $\vdash_{\mathbf{LO}^+}$ A by (TFC) it suffices to show that we can also prove $\neg\mathrm{OA} \to \mathrm{OOA}$. Again we start with a premise for (OCong)$^+$, this time with $m = 0, n = 1$ (and $\mathrm{E}_1 = \neg\mathrm{OA}$): $\neg\mathrm{OA} \to (\mathrm{OA} \leftrightarrow \bot)$, by (TF). The conclusion is then $\neg\mathrm{OA} \to (\mathrm{OOA} \leftrightarrow \mathrm{O}\bot)$, which by (TFC) and (OComp)$_\bot$ yields $\neg\mathrm{OA} \to \mathrm{OOA}$, as desired. However, the author does not know whether taking OOA as an axiom-scheme yields the derivability of (OCong)$^+$—this does not seem especially likely—or, more generally, whether $\mathbf{LO}^+$ can be presented as an axiomatic extension of **LO** by some finite set of similar principles (such as those quoted from Kuhn [22] and Zolin [37] for noncontingency versions of **K4** and **K5** in Section 4 below).

3 Variations on the Semantics and a Noncontingency Extension of the Basic Logic

We want to consider another extension of **LO** in this section in the interest of bringing the notion of noncontingency into the range of our study of agreement. (Whenever we speak of extensions of **LO**, we mean extensions closed under (MP) and (OCong).) But we begin somewhere else, by drawing attention to two possible variations on the notion of model and on the clause for 'O' in the definition of truth. Let us consider the latter first. In Section 2, we defined truth at a point in a model with the aid of the following clause for 'O':

$$\mathcal{M} \models_w \mathrm{OA} \text{ if and only if for all } x, y \in W \text{ such that } x \equiv_w y \text{ we have } \mathcal{M} \models_x \mathrm{A} \Leftrightarrow \mathcal{M} \models_y \mathrm{A}.$$

Now consider what we shall call the $\Rightarrow$-variant:

$$\mathcal{M} \models_w \text{OA if and only if for all } x, y \in W \text{ such that } x \equiv_w y \text{ we have } \mathcal{M} \models_x \text{A} \Rightarrow \mathcal{M} \models_y \text{A}.$$

Since $\equiv_w$ is an equivalence relation, the two clauses are equivalent. In more detail, if OA is true at w in $\mathcal{M}$ in the sense of the first definition then since we may weaken the '$\Leftrightarrow$' to a '$\Rightarrow$', OA is true at w in $\mathcal{M}$ in the sense of the second definition. Conversely, suppose that OA is true at w in $\mathcal{M}$ in the sense of the second, $\Rightarrow$-employing definition. This gives us that if $x \equiv_w y$ then $\mathcal{M} \models_x$ A $\Rightarrow$ $\mathcal{M} \models_y$ A, and also, interchanging variables, that if $y \equiv_w x$ then $\mathcal{M} \models_y$ A $\Rightarrow$ $\mathcal{M} \models_x$ A. But since $\equiv_w$ is an equivalence relation, $x \equiv_w y$ implies $y \equiv_w x$, so if $x \equiv_w y$ then we have not only $\mathcal{M} \models_x$ A $\Rightarrow$ $\mathcal{M} \models_y$ A, but also $\mathcal{M} \models_y$ A $\Rightarrow$ $\mathcal{M} \models_x$ A; in other words, we have $\mathcal{M} \models_x$ A $\Leftrightarrow$ $\mathcal{M} \models_y$ A, and OA is true at w in the sense of the first definition. We have labored this very obvious point because we shall presently need to allude to the particular feature of the original semantics which gives rise to the fact that there is no difference between the '$\Leftrightarrow$' form of the clause for O and the '$\Rightarrow$' variant: specifically this is because, being an equivalence relation, $\equiv_w$ is *symmetric*.

Now consider what we may call *generalized models* $\langle W, S, V\rangle$ in which W and V are as before and S (no relation to the S of [35] mentioned in Section 1) assigns to each $w \in W$ any binary relation on W, rather than specifically an equivalence relation. We write '$S_w xy$' to say that x and y stand in the relation which is the value of S for the argument w. Of course, we can equally well regard S here an arbitrary ternary relation on W. But we continue to subscript the first relatum, to emphasize its different status (as the point of evaluation for the O-formula concerned) and for continuity with the '$x \equiv_w y$' notation. The clause for O in the truth-definition just replaces $\equiv$ with S:

$$\mathcal{M} \models_w \text{OA if and only if for all } x, y \in W \text{ such that } S_w xy \text{ we have } \mathcal{M} \models_x \text{A} \Leftrightarrow \mathcal{M} \models_y \text{A}.$$

Call the formulas which are true at every point in one of these generalized models, with the above clause in place for O, valid in the generalized sense.

Proposition 3.1 *The following are equivalent for any formula* A*:*

1. A *is valid in the generalized sense,*
2. A *is valid,*
3. $\vdash_{\mathbf{LO}}$ A.

The equivalence of (2) and (3) here is already the content of Theorem 2.4. We can bring (1) into the fold either via (3) or via (2). Taking the first route, we note that a soundness proof for **LO** in terms of the generalized models presents no difficulties: at no point was the fact that for a given w, $\equiv_w$ was an equivalence relation, actually relied upon in establishing soundness with respect to the original models. And of course the same canonical model completeness proof works to show that when $S_w xy$ is defined exactly as $x \equiv_w y$, except that for the proof in this case we do not need to allude to the fact—which still is a fact—that this relation (for any given w) is an equivalence relation. Taking now, instead, the second route, let us explore the relation between (1) and (2) in purely semantic terms, without bringing in (3) with its reference to the axiomatically presented logic **LO**. We isolate the right-hand sides of the original clause and the new generalized clause as (∗) and (∗∗):

(∗) $$\forall x, y \in W(x \equiv_w y \Rightarrow (\mathcal{M} \models_x \mathrm{A}) \Leftrightarrow \mathcal{M} \models_y \mathrm{A}))$$

(∗∗) $$\forall x, y \in W(S_w xy \Rightarrow (\mathcal{M} \models_x \mathrm{A} \Leftrightarrow \mathcal{M} \models_y \mathrm{A})).$$

Abstracting from the present context, it is easy to see that whenever R_1, R_2 are binary relations on a set U, with R_2 an equivalence relation, then $R_1 \subseteq R_2$ if and only if $R_1^{\mathrm{eq}} \subseteq R_2$, where R_1^{eq} is the smallest equivalence relation including R_1. (Of course this is a special case of a more general phenomenon, another case of which would be that on the hypothesis that R_2 is, say, some transitive relation, rather than specifically an equivalence relation, then $R_1 \subseteq R_2$ if and only if $R_1^{\mathrm{tr}} \subseteq R_2$, where R_1^{tr} is the smallest transitive relation including—alias the transitive closure of—R_1.) Applying this fact to our current concerns, we note that the relation (playing here the role of R_2) for a given formula A and model $\mathcal{M}$ holding between x and y when $\mathcal{M} \models_x \mathrm{A} \Leftrightarrow \mathcal{M} \models_y \mathrm{A}$, the relation of "agreeing on" A, is an equivalence relation, we see that claim made by (∗∗), that the binary relation S_w is included in this relation, is itself equivalent to the claim that S_w^{eq} is included in this relation. Thus (∗∗) is equivalent to (∗) for a suitable choice of $\equiv$, and there is nothing to choose between the notions of validity and validity in the generalized sense.

We have considered two minor variants on the semantics of Section 2. First we considered replacing the original '$\Leftrightarrow$' form of the clause for O with the '$\Rightarrow$', and noted that this made no difference to the class of valid formulas. Then we considered another variant, which consisted in replacing the original notion of model (truth at a point of which was defined using the '$\Leftrightarrow$' form of the clause for O) with a more general notion in which an arbitrary binary relations S_w replaced the equivalence relations $\equiv_w$, and again found that the alteration had no impact on validity. However, if both changes are made at once, there is a dramatic effect on the class of valid formulas, as is predictable from the fact that our demonstration of the equivalence of the '$\Rightarrow$' and '$\Leftrightarrow$' clauses for O conspicuously exploited the symmetry of the relations $\equiv_w$. If we employ the '$\Rightarrow$' style clause in the setting of generalized models and say

$$\mathcal{M} \models_w \mathrm{OA} \text{ if and only if for all } x, y \in W \text{ such that } S_w xy, \text{ we have } \mathcal{M} \models_x \mathrm{A} \Rightarrow \mathcal{M} \models_y \mathrm{A},$$

then we lose the validity of various cases of the (OComp) scheme, most obviously (OComp)$_\neg$ as we see from a consideration of its instance $\mathrm{O}p \to \mathrm{O}\neg p$. We can falsify this at a point w in a generalized model $\mathcal{M} = \langle W, S, V \rangle$ with the above clause in place for O, by having for some $w \in W$, $\mathcal{M} \models_w \mathrm{O}p$ while $\mathcal{M} \not\models_w \mathrm{O}\neg p$, as the latter requires there to be $x, y \in W$ for which $S_w xy$, with $\mathcal{M} \models_x \neg p$ and $\mathcal{M} \not\models_y \neg p$, so that $\mathcal{M} \not\models_x p$ and $\mathcal{M} \models_y p$. In the semantics with the '$\Leftrightarrow$' clause for O, this would cause trouble because x and y would then disagree on p, making it impossible after all that $\mathcal{M} \models_w \mathrm{O}p$. But in the present context, there is no such trouble, since from the hypothesis that $S_w xy$, all that is required by the supposition that $\mathcal{M} \models_w \mathrm{O}p$, is that if p is true at x, it must be true at y; we are not given that if p is false at x, it must be false at y. To put it another way: we are not given that if p is true at y, it must be true at x. For that we should require, not the hypothesis that $S_w xy$, but rather, that $S_w yx$, something we are in no position to conclude since we have not required the relations S_w to be symmetric.

As the reasoning just gone through makes evident, a proof-system for the set of formulas valid when the '$\Rightarrow$' variant truth-definition is used with generalized models requires a weakening of (OComp), specifically by deleting the cases $(\text{OComp})_\#$ for # which are not (in an obvious sense) monotone #; which means, for the Boolean connectives # with which we have been working, that we lose not only the case of $\# = \neg$, but also those of $\rightarrow$ (nonmonotone in the first position) and $\leftrightarrow$ (not monotone in either position). Into the question of whether the **LO** proof-system thus trimmed provides a complete axiomatization of the formulas valid in the sense just isolated, the author has made no investigations. Though the question has considerable interest, it is somewhat removed from the topic of agreement, which requires a clause at least equivalent to the '$\Leftrightarrow$' form. (The '$\Rightarrow$' form is instead close to the relation of 'persistence' or 'heredity' familiar from the Kripke semantics for intuitionistic logic, with the requirement that the truth of a formula A persists on passage from a point x to an accessible point y.) Returning to agreement proper, then, we pass to the special case of this idea: the notion of noncontingency. We shall need to use the generalized models $\langle W, S, V\rangle$ of our recent discussion in order to treat this topic, rather than the $\langle W, \equiv, V\rangle$ models of Section 2, for a reason which will become clear shortly (immediately after $(***)$ below, in fact).

We recall (see [16], [22], and references) that in terms of a Kripke model for normal modal logic $\langle W, R, V\rangle$ with R a binary relation (accessibility) on W, the noncontingency operator $\triangle$ is interpreted by a clause in the definition of truth to the effect that $\triangle$A is true at $w \in W$ for such a model just in case either A is true at all y for which Rxy or else false at all x for which Rxy. Thus if sufficiently many conditions on R have been imposed for universal quantification over R-accessible to amount to (some intuitive notion of) *necessity* at a given point, this amounts to saying that $\triangle$A is true at w just in case either A is necessary at x, or A is impossible at x. We are not concerned with such additional conditions here—such as reflexivity[11] —but rather with the fact that this account of $\triangle$ can be formulated in terms of agreement in an obvious way: $\triangle$A is true at w just in case all points in $R(w) = \{x \in W | Rwx\}$ *agree* on the formula A. Accordingly, for continuity with the foregoing discussion, we can write 'OA' rather than '$\triangle$A', and ask how **LO** might need to be strengthened to reflect this special noncontingency interpretation of 'O'. Begin by considering how to convert a model $\langle W, R, V\rangle$ into one of our generalized models $\langle W, S, V\rangle$ with the same W and V, in such a way that $(**)$ above amounts to the noncontingency of A at $w \in W$. (i.e., A's truth throughout $R(w)$ or A's falsity throughout $R(w)$). The appropriate S is of course given by

$$S_w xy \Leftrightarrow Rwx \ \&\ Rwy$$

since plugging this in for S in $(**)$ gives

$$(***) \qquad \forall x, y \in W((Rwx \ \&\ Rwy) \Rightarrow (M \models_x \text{A} \Leftrightarrow M \models_y \text{A})).$$

(Note that the relations S_w as defined here are not in general reflexive, which is why for this part of the discussion we have dropped the '$\equiv_w$' and moved to generalized models.) Since **LO** is sound and complete with respect to the class of generalized models and the structures arising from Kripke models with binary accessibility relations via the above definition of S are generalized models, **LO** is sound with respect to this special class and there arises the question of whether **LO** is also complete for this class. A negative answer means that S's being equivalent to a definiens of the above form is sufficient to validate some special formulas not true throughout arbitrary generalized models.

Such a negative answer may be gleaned from a perusal of the very elegant axiomatization in [22] of the basic system of noncontingency logic $\mathbf{K}^{\triangle}$. This logic comprises those formulas true at every point in every model when $\triangle$A is taken to be true at a point w if and only if condition ($***$) is satisfied. As already explained, we shall write 'OA' rather than '$\triangle$A' in order to display the logic as an extension of **LO** (reverting to the '$\triangle$' notation later, after we have finished with our commentary on how the details of this transition from **LO** to $\mathbf{K}^{\triangle}$ work). In this notation, and with other minor cosmetic alterations, Kuhn's axiomatization of $\mathbf{K}^{\triangle}$ extends our axiomatization by one further scheme (his A3 on p. 231 of [22])—which we take the liberty of referring to as an axiom, though of course it is really schematic for all those axioms we obtain on substituting particular formulas for the 'A', 'B', 'C':

Kuhn's Axiom $\quad \mathrm{OA} \to (\mathrm{O}(\mathrm{A} \vee \mathrm{B}) \vee \mathrm{O}(\neg\mathrm{A} \vee \mathrm{C}))$.

While invalid (i.e., having some invalid instances) on the generalized models semantics for 'O', or equivalently, on the semantics with models as in Section 2, this is valid when attention is restricted to models—call them *noncontingency models*—with S defined by the inset equivalence above in terms of some binary relation R. (Informally: the antecedent then says that A is noncontingent, in which case it is either necessary, making the first disjunct of the consequent true, or else impossible, making the second disjunct true.) Thus cutting down from the class of generalized models to the class of noncontingency models does indeed properly extend the class of valid formulas and calls for a stronger logic than **LO**. Another scheme which is interdeducible with Kuhn's Axiom, given the deductive apparatus of **LO** (as axiomatized in Section 2) is the following $\neg$-free principle, whose validity for the class of noncontingency models is evident—in fact the same justification informally sketched applies here too:

$$\mathrm{OA} \to (\mathrm{O}(\mathrm{A} \vee \mathrm{B}) \vee \mathrm{O}(\mathrm{A} \wedge \mathrm{C})).$$

A further alternative, especially easily seen to be equivalent to Kuhn's Axiom via the above variant, would be to employ a rule which weakens what would be an incorrect claim that noncontingency is monotone (as embodied in a rule allowing passage from $\mathrm{B} \to \mathrm{C}$ to $\mathrm{OB} \to \mathrm{OC}$) as well as an incorrect claim that noncontingency is antitone (embodied in a rule allowing passage from $\mathrm{A} \to \mathrm{B}$ to $\mathrm{OB} \to \mathrm{OA}$): namely, the rule

$$\frac{\mathrm{A} \to \mathrm{B} \qquad \mathrm{B} \to \mathrm{C}}{\mathrm{OB} \to (\mathrm{OA} \vee \mathrm{OC})} \quad .$$

It is always regarded as an improvement in simplicity when such rules are shown to be replaceable without loss of deductive power by axioms, however. (Indeed [22] provides just such a simplification in replacing a cumbersome set of rules from [16], in which the rule just formulated appears as (2.9) on p. 218.) Another dimension of simplicity in the formulation of axioms is the number of variables or (with axiom schemes) the number of distinct schematic letters. Kuhn's Axiom can itself be simplified in this regard, because the 'B' and 'C' can in fact be identified. Here we write them as 'D' to avoid confusion:

$$\mathrm{OA} \to (\mathrm{O}(\mathrm{A} \vee \mathrm{D}) \vee \mathrm{O}(\neg\mathrm{A} \vee \mathrm{D})).$$

Obviously we obtain this form from Kuhn's Axiom by putting 'D' for 'B' and 'C'. To obtain Kuhn's Axiom from the simplified form, we note that classical propositional logic allows us the following claim: for any formulas A, B, and C, there is a formula D with the property that (1) $A \vee$ D is logically equivalent to A $\vee$ B and (2) $\neg$A $\vee$ D is logically equivalent to $\neg$A $\vee$ C. The desired D can be written most perspicuously (though not most concisely) as ($\neg$A $\rightarrow$ B) $\wedge$ (A $\rightarrow$ C). This is accordingly the substitution to make for D to obtain (with the aid of (OCong), since we are making this replacement within the scope of 'O') Kuhn's Axiom in its original form from the above two-letter form. We shall stick with Kuhn's Axiom in its original form, since that is the form most immediately employed in the proof given in [22] (see also Section 4 below) that **LO** with this added axiom is sound and complete with respect to the class of noncontingency models, so what we want to do here is to analyze the semantic effect of the addition against the background of our discussion to this point.[12]

The author originally hoped that this analysis could take a particularly simple form: it would be shown that a certain first-order condition—roughly, the conjunction of symmetry with a condition we call [$\wedge$] below—would be guaranteed to be satisfied by the canonical model's relations S_w (defined as the canonical $\equiv_w$ were in Section 2) for any logic extending **LO** with Kuhn's Axiom, and which condition was necessary and sufficient for there to exist a binary relation R for which (∗∗∗) above held. This initial thought turned out to be overly optimistic, since there appears to be no way to force the canonical accessibility relation(s) to satisfy [$\wedge$], and a slightly more complicated condition, which we shall call [$\wedge$]$^{\neq}$, is needed instead. With this alteration, the strategy just sketched does indeed work, as we shall see after supplying some explanation for this rather cryptic summary.

The explanation calls for a detour through some aspects of the general theory of binary relations. The following points may be found in Humberstone [15] and references cited therein.[13] Let U be any set and suppose $T \subseteq U \times U$. We call T $\wedge$-*representable* if there exist $X, Y \subseteq U$ such that for all $x, y \in U$, Txy if and only if $x \in X$ and $y \in Y$. Since the definiens here quantifies over subsets of U, we can think of it as a second-order condition. Consider now the following condition, written by contrast in the first-order language[14] of the relational structure $\langle U, T\rangle$:

$$[\wedge] \qquad \forall x, y, u, v((Txy \wedge Tuv) \rightarrow Txv).$$

Note that this condition strengthens (considerably!) the condition of transitivity for R, the latter resulting when '$y = u$' is added as a further conjunct to the antecedent. It turns out that T is $\wedge$-representable if and only if it satisfies the condition [$\wedge$]. Similar first-order conditions can be found which are equivalent to the second-order condition of $\wedge$-representability when the "and" in "Txy if and only if $x \in X$ and $y \in Y$" is replaced by other (informal) Boolean connectives, so that we have a similar treatment for $\vee$-representable relations, and so on. Note that we can formulate the definition of $\wedge$-representability in more succinct (but still second-order) terms by saying that there exist $X, Y \subseteq U$ with $T = X \times Y$, for which reason $\wedge$-representable relations are usually called *rectangular* in the literature. (Visualize a graphical depiction of T.) We use the vocabulary of $\wedge$-representability, instead of that more conventional terminology to emphasize the analogy with, for example, $\vee$-representability. We shall have no need of the latter variants here, however. In fact, what we need is a further special case of $\wedge$-representability, called sameness-representability in Section 4 of [15]. It is the special case in which we require $X = Y$. That is, with T, U as above, T is *sameness-representable* if and only if there is some $X \subseteq U$ such that for all $x, y \in U$, Txy if and only if $x \in X$ and $y \in X$. In the reference just cited, it is observed[15] that T is sameness-representable just in case T is symmetric and satisfies [$\wedge$]. Taken thus in conjunction with symmetry [$\wedge$] could be replaced by several variants, for instance those resulting from replacing the 'Txv' in the consequent by any of Txu, Tux, Tyv, Tvy. Suitably choosing from among these, we could even drop the reference to symmetry altogether; this is the case with the last variant, for instance—a relation T is symmetric and satisfies [$\wedge$] if and only if T satisfies $\forall x, y, u, v((Txy \wedge Tuv) \to Tvy)$. While this gives an even more economical first-order characterization of sameness-representability, we will stick with the *symmetry* + [$\wedge$] formulation here. In fact, a much simpler first-order characterization—at least if simplicity is measured by the number of bound variables involved—of sameness-representability can be given, as is more or less explicitly noted on p. 251 of Williamson [36], namely, the following:

$$\forall x, y(Txy \leftrightarrow (Txx \wedge Tyy)).$$

However, we prefer to focus on (the combination of symmetry with) [$\wedge$] as above, because it is a restricted version ([$\wedge$]$^{\neq}$) of that condition that we shall need below.

Applying all this to our discussion of the relations S_w from our semantic apparatus for **LO**, we can now describe the initial hope mentioned above. Take one such relation for an element w of the canonical model for $\mathbf{K}^{\triangle}$, understood as the extension (still required to be closed under (MP) and (OCong)) by Kuhn's Axiom. The idea was that we show that the effect of the latter axiom is to force such a relation S_w to satisfy [$\wedge$], and hence, since it is already symmetric by definition (since we envisage the same definition as for the canonical $\equiv_w$ from Section 2), the above characterization of sameness-representability, with S_w in the role of T, gives us the conclusion that S_w is sameness-representable. We can thus find a subset X of $W(= U$ in the preceding discussion) with S_wxy if and only if x and y both belong to X. Accordingly, define a binary accessibility relation on W by stipulating, for one $w \in W$ at a time, that $R(w) = X$ for the relevant choice of X. This then guarantees that S_wxy if and only if Rwx & Rwy and thus that we have on our hands a noncontingency model. As already remarked, however, there is no reason to expect the canonical model to supply relations S_w satisfying [$\wedge$], and we cannot proceed quite so straightforwardly. Instead, we attend to a related condition:

[$\wedge$]$^{\neq}$ $\quad \forall x, y, u, v[((Txy \wedge x \neq y) \wedge (Tuv \wedge u \neq v)) \to Txv]$.

Proposition 3.2 *Let $\langle W, S, V\rangle$ be the canonical model for any consistent extension of $\mathbf{K}^{\triangle}$, with S_w defined to hold between $x, y \in W$ just in case for all* OA $\in w$, A $\in x$ *if and only if* A $\in y$. *Then the relations S_w satisfy* [$\wedge$]$^{\neq}$, *when taken as T.*

Proof Suppose we have $w, x, y, u, v \in W$ with S_wxy, $x \neq y$, S_wuv, $u \neq v$. We must show that S_wxv. Suppose the latter is not the case. Then for some OA $\in w$, we have A $\in x$ and A $\notin v$ or vice versa: but if we have the "vice versa" case for a given A, we have the original case for its negation, the result of attaching O to which is also an element of w—by (OComp)$_{\neg}$—so it suffices to deal with the case described. Also, since OA $\in w$ and S_wxy, we have A $\in y$ (as A $\in x$), and since OA $\in w$ and S_wuv, we have A $\notin u$ (as A $\notin v$). Further, since $x \notin y$, there is some formula C with C $\in x$ and C $\notin y$. (Again, if for a given C we have the reverse distribution, take its negation.) Likewise, since $u \notin v$, there is some B $\in u$ with B $\notin v$. Taking stock of all this: we have A, C $\in x$, A $\in y$, C $\notin y$, which means that x and y disagree on $\neg$A $\vee$ C (in x, not in y); and we also have A $\notin u$, B $\in u$, A $\notin v$, B $\notin v$, which means that u and v disagree on A $\vee$ B (in u, not in v). We now have a contradiction with Kuhn's Axiom, since OA $\in w$, so by that axiom, we must have O(A $\vee$ B) $\in w$ or O($\neg$A $\vee$ C) $\in w$, contradicting the disagreements just noted since we supposedly also have $S_w uv$ and $S_w xy$. □

We need now to recall the understanding of S_w mentioned above (before $(***)$) in terms of which the noncontingency of A at w amounts to the agreement between all S_w-related pairs x, y in respect of A, which for ease of reference we now label (α), and a propos of which we remarked that this cannot in general be expected to be reflexive[16]—since that would imply that $R(w) = W$:

$$(\alpha) \qquad S_w xy \Leftrightarrow Rwx \;\&\; Rwy$$

Now in Proposition 3.2, our effort to extract some structural information about a canonical S relation from the provability of Kuhn's Axiom, we used the same definition as for the canonical $\equiv$ of Section 2, meaning that the relations S_w were equivalence relations, and hence, in particular, reflexive. Fortunately, however, for the purpose of showing that Kuhn's Axiom turns **LO** into a complete logic for noncontingency models—or more accurately, since this was already shown in [22]—for the purpose of analyzing the role of that axiom against the background of the more agreement models of our discussion, we do not need to show how to supply our canonical model(s) with an R satisfying (α). (Cf. the discussion following $(*)$ and $(**)$ above.) It will be sufficient to find a binary relation R for which (β) and (γ) are equivalent for the canonical model $\mathcal{M} = \langle W, S, V\rangle$ of any consistent extension of $\mathbf{K}^{\triangle}$, understood as **LO** + Kuhn's Axiom, for arbitrary $w \in W$ and any formula A:

$$(\beta) \qquad \forall x, y \in W(S_w xy \Rightarrow (\mathcal{M} \models_x \mathrm{A} \Leftrightarrow \mathcal{M} \models_y \mathrm{A}));$$

$$(\gamma) \qquad \forall x, y \in W((Rwx \;\&\; Rwy) \Rightarrow (\mathcal{M} \models_x \mathrm{A} \Leftrightarrow \mathcal{M} \models_y \mathrm{A})).$$

Accordingly, consider the following way of defining $R \subseteq W \times W$, quantifiers ranging over W:

$$(\delta) \qquad \forall w, u(Rwu \Leftrightarrow \exists v \neq u. S_w uv).$$

Then we have the following.

Proposition 3.3 *If $\mathcal{M} = \langle W, S, V\rangle$ is the canonical model for $\mathbf{K}^{\triangle}$ or any consistent extension thereof, then for the binary relation R on W defined by (δ), we have (β) and (γ) equivalent for any formula* A*, any $w \in W$.*

Proof $(\beta) \Rightarrow (\gamma)$: Assume (β), and that for $x, y \in W$ we have Rwx and Rwy, meaning, by (δ), that $S_w xx'$ for some $x' \neq x$ and $S_w yy'$ for some $y' \neq y$, in which case, since S_w is symmetric, $S_w y'y$. By Proposition 3.2, S_w satisfies the condition $[\wedge]^{\neq}$, so $S_w xy$, and thus by (β), $\mathcal{M} \models_x$ A $\Leftrightarrow \mathcal{M} \models_y$ A, establishing (γ).

$(\gamma) \Rightarrow (\beta)$: Assume (γ) and that for $x, y \in W$ we have $S_w xy$. Distinguish two cases. First, $x = y$. In that case certainly $\mathcal{M} \models_x$ A $\Leftrightarrow \mathcal{M} \models_y$ A. Next, $x \neq y$. In that case, since $S_w xy$ and $x \neq y$, Rwx by (δ), and since S_w is symmetric $S_w yx$ and again by (δ), Rwy. So by (γ), $\mathcal{M} \models_x$ A $\Leftrightarrow \mathcal{M} \models_y$ A. Thus in either case $\mathcal{M} \models_x$ A $\Leftrightarrow \mathcal{M} \models_y$ A, establishing (β). □

We have now explained the way Kuhn's Axiom succeeds in completing **LO**, our general modal logic of agreement, to the stronger logic required for the noncontingency interpretation of 'O' via its securing the satisfaction of the condition $[\wedge]^{\neq}$ for the canonical models of logics containing all its instances. Any consistent formula of such a logic is true at some point in its canonical model, whose relation S can be used to define a binary relation R, as Proposition 3.3 assures us, in such a way that any formula OA is true at point w not only if and only if (β) is satisfied (as for all models) but also if and only if (γ) is, that is, in the manner of a noncontingency model.

As a footnote to this discussion, we may extract from it the following general observation in the style of 'a binary relation T is $\wedge$-representable if and only if T satisfies $[\wedge]$', 'T is sameness-representable if and only if T is symmetric and satisfies $[\wedge]$', and the like, with the aid of the following notation. For any $T \subseteq U \times U$, we denote by T^{ref} (the reflexive closure of T) the smallest reflexive relation extending T. By contrast with the operations $(\cdot)^{\text{eq}}$ and $(\cdot)^{\text{tr}}$ considered earlier, $(\cdot)^{\text{ref}}$ has a very simple explicit description: $T^{\text{ref}} = T \cup \{\langle u, u\rangle | u \in U\}$. Now the observation to be extracted can be put thus: $T^{\text{ref}} = T_0^{\text{ref}}$ for some sameness-representable T_0 if and only if T is symmetric and satisfies $[\wedge]^{\neq}$. The "only if" direction here is routine. For the "if" direction, assume T is symmetric and satisfies $[\wedge]^{\neq}$, and define $X \subseteq U$ as $\{u \in U | \exists v \in U.\ u \neq v \ \&\ Tuv\}$, and show that taking T_0 as $X \times X$, we have $T_0^{\text{ref}} = T^{\text{ref}}$.

4 More on Noncontingency

An interesting further issue is raised by our progress toward the conclusion summarized in the second to last paragraph of Section 3. Suppose we started with a traditional Kripke model $\langle W, R_1, V\rangle$ with a binary accessibility relation R_1 and a truth-definition setting the truth of OA, or $\triangle$A, as we shall now write it for conformity with the usual notation for noncontingency, at a point w equal to the satisfaction of the condition ($***$). Suppose, next, that we introduced a ternary relation S by means of the natural definition (α) above (with R_1 for R), and finally, that we employed (δ) to define a binary relation R_2 (where (δ) speaks of R). Would we be back where we started? Would, that is, R_2 be the original relation R_1?

In general, the answer to the question just raised is negative: R_2 will be a proper subrelation of R_1, though as we shall see, this difference makes no difference to the point-by-point truth-values of formulas in the language whose sole non-Boolean connective is $\triangle$, as we pass between the models $\langle W, R_1, V\rangle$ and $\langle W, R_2, V\rangle$. To see why the answer is negative, unpack the suggested definition a la (δ) of R_2 by replacing references to S in terms of R_1 a la (α); this gives the following equivalence, in which the replaceability of the 'R_2' on the left by 'R_1' would amount to an affirmative answer to our question:

$$\forall w, u(R_2wu \Leftrightarrow \exists v \neq u(R_1wu \ \&\ R_1wv)).$$

If we replace 'R_2' on the left with 'R_1' we get something unobjectionable in its $\Leftarrow$ direction, but far from guaranteed to be true in its $\Rightarrow$ direction: just because w bears R_2 to u, why should it follow that w bears R_2 to something other than u as well? Plainly, the equivalence inset above tells us that u is an R_2-successor of w (i.e., R_2wu) if and only if u is one of at least two R_1-successors of w, and thus R_2 coincides with R_1 only in the case in which for all w, $|R_1(w)| \neq 1$. For the general case, the passage from R_1 to R_2 is as described here in Section 4 of [16]—several themes from which are recalled here for current purposes—in terms of "R_1-reduction," though here we use a more explicit terminology: Given a Kripke model $\mathcal{M}_1 = \langle W, R_1, V\rangle$ we say that $\mathcal{M}_2 = \langle W, R_2, V\rangle$ is obtained from $\mathcal{M}_1$ by *severance of sole successors* just in case

$$R_2 = R_1 \setminus \{\langle w, x\rangle | R_1(w) = \{x\}\}.$$

Note that we do not exclude the case in which $x = w$ here. Note also that the change from $\mathcal{M}_1$ to $\mathcal{M}_2$ here described can be regarded as a change from one frame to another (since V is not affected).

If we had '$\Box$' in the language, interpreted as usual, then the transition from $\mathcal{M}_1$ to $\mathcal{M}_2$ would make a difference to which formulas were true at points w for which this transition severs the accessibility connection from w to its sole successor, since in $\mathcal{M}_1$ a formula $\Box$A is true at w just in case A is true at x, whereas in $\mathcal{M}_2$ all formulas $\Box$A are true regardless of what A is. (In particular, then, $\Box\bot$ is bound to change from being false at w in $\mathcal{M}_1$ to be being true at w in $\mathcal{M}_2$.) By contrast, in the case in which the only non-Boolean primitive is $\triangle$, there is no difference over $\triangle$-formulas between successorless points and points with a single successor: both such points verify all such formulas, since the only way for $\triangle$B to be false at a point is for it to have successors disagreeing on B—which requires at least two successors. This reasoning gives the heart of the inductive case for $\triangle$ in a proof of the following (= Lemma 4.1(i) in [16]), by induction on the construction of A, the remainder of which proof the reader is invited to supply.

Theorem 4.1 *If $\mathcal{M}_2$ is obtained from $\mathcal{M}_1$ by severance of sole successors, then we have $\mathcal{M}_1 \models_w$ A if and only if $\mathcal{M}_2 \models_w$ A, for all $w \in W$, all formulas A (with $\triangle$ as the only non-Boolean connective).*

The following is evident even from our informal remarks above; it tells us that $\Box$, with its customary interpretation, is not definable in terms of the Boolean connectives and $\triangle$ in $\mathbf{K}^{\triangle}$. (Think of A(p) below as a candidate definiens for $\Box p$. More information on the definability of $\Box$ in terms of $\triangle$ may be found in Creswell [4].)

Corollary 4.2 *There is no formula A(p) of the language of $\mathbf{K}^{\triangle}$ with the property that for all models $\mathcal{M} = \langle W, R, V\rangle$ and all $w \in W$, $\mathcal{M} \models_w$ A(p) if and only if for all $x \in R(w)$, $\mathcal{M} \models_x p$.*

Proof Suppose, for a contradiction, that A(p) is a formula of the kind claimed here not to exist. With $W = \{u\}$, $R_1 = \{\langle u, u\rangle\}$, $R_2 = \varnothing$, and $V(p_i) = \varnothing$ for all i, the models $\mathcal{M}_1 = \langle W, R_1, V\rangle$ and $\mathcal{M}_2 = \langle W, R_2, V\rangle$ are as in Theorem 4.1, which therefore implies $\mathcal{M}_1 \models_u$ A(p) $\Leftrightarrow$ $\mathcal{M}_2 \models_u$ A(p), contradicting the fact that $\mathcal{M}_1 \not\models_u$ A(p) and $\mathcal{M}_2 \models_u$ A(p), the former being given by A(p)'s requiring for its at u truth throughout $R_1(u) = \{u\} \not\subseteq V(p)$, and the latter by the fact that $R_2(u) = \varnothing$. $\Box$

In [16] an easy corollary of Theorem 4.1 was stressed, showing the resemblance between severance of sole successors and other operations on frames, such as taking disjoint unions, generated subframes, and p-morphic images in standard $\Box$-based modal logic (results which remain intact in the present setting): for a class $\mathbb{C}$ of frames to be modally definable, $\mathbb{C}$ and its complement must be closed under the transition from $\langle W, R_1\rangle$ to $\langle W, R_2\rangle$ by severance of sole successors.[17] As noted in

[16], several familiar classes of frames modally definable in the $\Box$-language can be shown to be undefinable in the $\triangle$-language by appeal to this fact. These include the classes of reflexive frames, which are not closed under severance of sole successors, and the classes of transitive frames and symmetric frames, whose complements are not closed under this operation. To them we can readily add the class of serial frames and the class of euclidean frames. Much more information on modal definability in the $\triangle$-language can be found in [37]; a similar study of the more general agreement operator O of **LO** would be equally welcome. (In that setting the anomalies raised by sole successors here are raised by the situation in which for a given w, we only have $Swxy$—or $x \equiv_w y$—when $x = y$: no O-formula can then be false at w.) The other side of the coin from these facts of undefinability consists of corresponding facts of multiple determination, where (as usual) a logic is *determined* by a class of frames when its theorems coincide with the formulas valid (in the sense there defined) on every frame in the class. Many distinct classes of frames can determine the same normal modal logic in the usual language with $\Box$, but the feature we are now highlighting concerns the fact that classes of frames determining different $\Box$-based modal logics can, because of the expressive weakness just alluded to, determine the same $\triangle$-based logic. An example related to the severance of sole successors issue was given in [16], p. 225: In fact for any ($\Box$-based) **S** determined by a class of frames in which for every frame element w, $|R(w)| \leq 1$ we have (using the notation of note 12) $\mathbf{S}^{\triangle} = \mathbf{K}^{\triangle} + \triangle\mathrm{A}$ (i.e., the extension of $\mathbf{K}^{\triangle}$ by the schema $\triangle$A); thus the Verum system $\mathbf{KVer} = \mathbf{K} + \Box\bot$, the logic $\mathbf{K} + \Diamond\mathrm{A} \leftrightarrow \Box\mathrm{A}$ ('**KD!**' in Chellas's nomenclature—see [2]), and their intersection $\mathbf{K} + \Diamond\mathrm{A} \to \Box\mathrm{A}$ ('$\mathbf{KD}_c$'), as well as $\mathbf{KT}_c$ and **KT!** (= $\mathbf{K} + \mathrm{A} \to \Box\mathrm{A}$ and $\mathbf{K} + \mathrm{A} \leftrightarrow \Box\mathrm{A}$, respectively) all have the same noncontingency fragment. (As mentioned in [16], this is the sole Post-complete extension of $\mathbf{K}^{\triangle}$—or of NC as it was there called. The interested reader will find such themes from [16] taken further in [37].)

Before we turn temporarily to a slightly different language, one allowing propositional quantifiers in terms of which a workable surrogate $\Box$ can be defined (using $\triangle$), we mention a further aspect of the passage from **S** to $\mathbf{S}^{\triangle}$ inspired by the inclusion of $\mathbf{KD}_c$ on the above list, which as mentioned, is the intersection of two other logics on the list, neither of which is included in the other: $\mathbf{K} + \Box\bot$ and **KD!**. As is well known,[18] this guarantees that the logic $\mathbf{KD}_c$ is Halldén-incomplete, that is, proves for some A, B, the formula A $\vee$ B without proving A or proving B, even when A and B have no propositional variables in common. For an example in the case of $\mathbf{KD}_c$, take $\mathrm{A} = \Box p$ and $\mathrm{B} = \Diamond q \leftrightarrow \Box q$. It is an immediate consequence of the definitions that it is Halldén-completeness (rather than Halldén-incompleteness) that is passed from **S** to $\mathbf{S}^{\triangle}$, however, so the fact that $\mathbf{K}^{\triangle} + \triangle\mathrm{A}$ is $(\mathbf{K}+\Box\bot)^{\triangle}$ as well as being $\mathbf{KD}_c^{\triangle}$ shows that this noncontingency logic is Halldén-complete. It would be interesting to know if this is also the case for $\mathbf{K}^{\triangle}$ itself, in view of the Halldén-incompleteness of **K**. The best known witnessing disjunctions in the latter case are formulas such as $\Box p \vee \Diamond(q \vee \neg q)$ and $\Box(p \wedge \neg p) \vee \Diamond(q \vee \neg q)$, the latter being a reformulation of the variable-free $\Box\bot \vee \Diamond\top$ designed to exhibit variable-disjoint disjuncts: it is not clear how to say anything analogous in the $\triangle$-language. To clinch matters, as in the case of $\mathbf{K}^{\triangle} + \triangle\mathrm{A}$ just reviewed, it would suffice to show that $\mathbf{K}^{\triangle}$ is also $\mathbf{S}^{\triangle}$ for some Halldén-complete **S**.

We turn now to the promised temporary change of language. The languages we have been considering do not allow propositional quantification, but it is worth noting that if that is added to our expressive resources, then we can come quite close to defining necessity in the resulting second-order propositional $\mathbf{K}^{\triangle}$.[19] The formulation we have in mind is inspired by Kuhn's Axiom, repeated here for convenience, though now with '$\triangle$' in place of the 'O' of our earlier discussion (especially as we shall have more to say about it below):

Kuhn's Axiom $\quad \triangle A \to (\triangle(A \vee B) \vee \triangle(\neg A \vee C))$.

Recall that if $\triangle A$ is true at w because points in $R(w)$ verify A—that is, informally speaking, because A is necessary at w—then the first disjunct of the consequent is true at w, for any formula B. This gives the forward direction of what we have in mind as close to a suitable definition of $\Box$, and we shall instead take to define a new operator $\Box^{-}$, so that we can compare its upshot with $\Box$.

Definition 4.3 ($\Box^{-}$) $\quad \Box^{-} A \leftrightarrow \forall q \triangle(A \vee q)$.

In Definition 4.3 we require 'q' to be a variable not occurring in A, so if A does contain $q (= p_2)$, just replace q by p_{k+1} where all variables in A are among $\{p_1, \ldots, p_k\}$.

Proposition 4.4 *Let $\mathcal{M} = \langle W, R, V\rangle$ be a model interpreting the language of* $\mathbf{K}^{\triangle}$ *with $\Box$ added (understood as usual), and w any element of W with $|R(w)| \neq 1$. Then $\mathcal{M} \models_w \Box$ A if and only if $\mathcal{M} \models_w \Box^{-}$ A.*

Proof (only if) This direction holds without requiring the condition on w, since if $\mathcal{M} \models_w \Box$A then all elements of $R(w)$ verify A and hence $A \vee q$, independently of $V(q)$; thus however V is altered to V' like V on all propositional variables other than q, $\langle W, R, V'\rangle = \mathcal{M}' \models_w \Box(A \vee q)$ and thus $\mathcal{M}' \models_w \triangle(A \vee q)$, so $\mathcal{M} \models_w \forall q \triangle (A \vee q)$, that is, $\mathcal{M} \models_w \Box^{-}$A.

(if) Take $w \in W$ with $|R(w)| \neq 1$ and $\mathcal{M} \not\models_w \Box$A, with a view to showing that $\mathcal{M} \not\models_w \Box^{-}$A. If $|R(w)| = 0$, we cannot have $\mathcal{M} \not\models_w \Box$A, so it suffices to consider the case of $|R(w)| \geq 2$. As $\mathcal{M} \not\models_w \Box$A, there is some $x \in R(w)$ with $\mathcal{M} \not\models_x$ A, and since $|R(w)| \geq 2$ we can find $y \in R(w)$ with $y \neq x$. Let V' differ from V at most on q (and hence not at all on the variables in A) in any way that puts $y \in V'(q), x \notin V'(q)$. For the resulting model $\mathcal{M}' = \langle W, R, V'\rangle$, we have R-successors of w (namely, x, y) differing on $A \vee q$, since $\mathcal{M}' \not\models_x A \vee q$ while $\mathcal{M}' \models_y A \vee q$. Thus $\mathcal{M}' \not\models_w \triangle(A \vee q$, so $\mathcal{M} \not\models_w \forall q \triangle (A \vee q)$, that is, $\mathcal{M} \not\models_w \Box^{-}$A. $\Box$

Since Kripke models whose accessibility relations satisfy $|R(w)| = 1$ for all model elements w are called *functional* in the literature (the relation R being a functional relation—or, not to put too fine a point on it, a function—in this case), we might call models in which $|R(w)| = 1$ for *no* element w, *antifunctional* models. Proposition 4.4 tells us that our necessity surrogate $\Box^{-}$ behaves exactly like the real thing ($\Box$) in any antifunctional model. Although we shall have no more to say about second-order propositional modal logic and $\Box^{-}$ in what follows, some observations on this antifunctionality property are in order.

In the first place, the models $\langle W, R_2, V\rangle$ obtained from models $\langle W, R_1, V\rangle$ by severance of sole successors are obviously antifunctional. Secondly, the Kripke models $\langle W, R, V\rangle$ arising from the canonical models $\langle W, S, V\rangle$ of Propositions 3.2

and 3.3 by means of the definition (δ) of R in terms of S are all antifunctional, since if Rwu then, by (δ), there is a $v \in W$ which is distinct from u and for which $S_w uv$, in which case, by the symmetry of S_w and (δ) again, Rwv. So if w bears R to something, it bears R to something else as well. One might wonder whether every w bears R to something. It turns out to be possible to have $R(w) = \varnothing$, namely, if and only if for every formula A, we have $\triangle$A $\in w$.[20] The same is true of the canonical model constructed in [22]. Kuhn there uses an ingenious but straightforward definition of the canonical accessibility relation R for the canonical model for the basic system $\mathbf{K}^{\triangle}$ of noncontingency logic. For a maximal consistent set w, we define $\lambda(w) = \{\text{A} | \triangle (\text{A} \vee \text{B}) \in w$ for all formulas B$\}$. This is a way of simulating, in the absence of '$\Box$', the set of formulas which are 'necessary according to w' (cf. our '$\Box^{-}$' above).[21] Accordingly we then define the canonical accessibility relation R by: $Rwx \Leftrightarrow \lambda(w) \subseteq x$. $R(w)$'s being empty then amounts to $\lambda(w)$'s being inconsistent.[22] If for every formula A, $\triangle$A $\in w$, then for every formula A, for every formula B, $\triangle$(A $\vee$ B) $\in w$—not because, case by case, $\triangle$(A $\vee$ B) follows from $\triangle$A (which is of course not so), but because A $\vee$ B is itself a formula which could be chosen instead of the original A, our having said "for *every* formula A, $\triangle$A $\in w$." Thus every formula belongs to $\lambda(w)$, so $R(w) = \varnothing$. Conversely, if $R(w) = \varnothing$, then $\lambda(w)$ is inconsistent, and so for all A, B, $\triangle$(A $\vee$ B) $\in w$, and thus (e.g., taking B as A), for all A, $\triangle$A $\in w$.

We can extend the above reasoning to show that Kuhn's canonical model for $\mathbf{K}^{\triangle}$ is antifunctional. Suppose for maximal consistent w, x, we have $R(w) = x$. Then for every A $\in x$, A $\in \lambda(w)$, so for all formulas B, $\triangle$(A $\vee$ B) $\in w$ and as before, this implies that $\triangle$A $\in w$. Now for each formula A, either A $\in x$, or else $\neg$A $\in x$, so for each formula A, either $\triangle$A $\in w$ or $\triangle\neg$A $\in w$. But $\triangle\neg$A $\in w$ implies $\triangle$A $\in w$ (since as noted in Section 2, O$\neg$A provably implies OA even in **LO**). Thus for every formula A, $\triangle$A $\in w$, and in that case by the reasoning of the preceding paragraph, $R(w) = \varnothing \neq \{x\}$ after all.

The same definition of the canonical accessibility relation yields completeness results for $\mathbf{K4}^{\triangle}$ and $\mathbf{K5}^{\triangle}$ with respect to the classes of transitive and of euclidean frames, respectively, when the following axiom-schemes are added to $\mathbf{K}^{\triangle}$:

$$\triangle\text{A} \to \triangle(\triangle\text{A} \vee \text{B}) \quad \text{and} \quad \neg \triangle \text{A} \to \triangle(\neg \triangle \text{A} \vee \text{B})$$

as is shown in [22] and [37], respectively. (The latter work also gives first-order characterizations of the classes of frames these schemes modally define—which, as we have already noted, are certainly not the classes of transitive and euclidean frames.) The author does not know if the same definition for the canonical R works to show that this relation is reflexive for $\mathbf{KT}^{\triangle} = \mathbf{K}^{\triangle}+$

$$(\triangle(\text{A} \vee \text{B}) \wedge \triangle\text{A}) \to (\text{A} \vee \triangle\text{B})$$

though in this case we can use the standard definition for canonical accessibility in normal modal logic because of the definability of $\Box$A as $\triangle$A $\wedge$ A (i.e., put $Rwu \Leftrightarrow \{\text{A} | \triangle \text{A} \wedge \text{A} \in w\} \subseteq u$). (The following fact would appear to bear on this question. We can also characterize $\mathbf{KT}^{\triangle}$ as **LO**+

$$(\triangle\text{A} \wedge \text{A}) \to \triangle(\text{A} \vee \text{B}).$$

In other words, this schema is **LO**-interderivable with the combination of the preceding schema and Kuhn's Axiom.) Apart from combining these various ingredients to obtain $\mathbf{S5}^{\triangle}$, we have in any case another axiomatic route available, by replacing

(OCong) by (OCong)$^+$ in the above axiomatization of $\mathbf{K}^{\triangle}$ (i.e., add Kuhn's Axiom not to **LO** but to $\mathbf{LO}^+$). We conclude our discussion with some variants on the theme of noncontingency, returning to the 'O' notation to avoid confusion with noncontingency proper.

Suppose that for a Kripke model $\mathcal{M} = \langle W, R, V\rangle$ we used the following clause in the truth-definition:

$$\mathcal{M} \models_w \text{OA if and only if for all } y \in W \text{ such that } Rwy, \mathcal{M} \models_w \text{A iff } \mathcal{M} \models_y \text{A}.$$

The truth of OA at w requires then the agreement of all of w's successors with w on A. This implies that all of w's successors agree with each other on A (i.e., the truth of $\triangle$A at w) but is in general stronger if $w \notin R(w)$.[23] In terms of the generalized models of Section 3 with their ternary relations S, this amounts to setting $S_w xy \Leftrightarrow (x = w \,\&\, Rxy)$. A sound and complete axiomatization is obtained by replacing Kuhn's Axiom with the following variant:

$$\triangle\text{A} \to ((\text{A} \wedge \triangle(\text{A} \vee \text{B})) \vee (\neg\text{A} \wedge \triangle(\neg\text{A} \vee \text{C}))).^{24}$$

Soundness is clear and completeness follows by an easy adaptation of Kuhn's argument on p. 232 of [22]. (For this adaptation, define $\lambda'(w) = \{\text{A}|\text{A} \wedge \text{O}(\text{A} \vee \text{B}) \in w$ for all B$\}$, then putting $Rwu \Leftrightarrow \lambda'(w) \subseteq u$.)

Because of the fact that in terms of our generalized models with S, for the last case we have $S_w xy$ if and only if $x = w$ and Rxy, the effect of the universal quantifier on x in our general clause, repeated here for convenience,

$$\mathcal{M} \models_w \text{OA if and only if for all } x, y \in W \text{ such that } S_w xy,$$

$$\text{we have } \mathcal{M} \models_x \text{A} \Leftrightarrow \mathcal{M} \models_y \text{A},$$

is nullified: the only candidate for x is w itself. There are some reasonably well-motivated further variants in which not only the quantifier on x but also that on y is similarly nullified. Consider, for example, the models $\langle W, w^*, V\rangle$ with $w^* \in W$ sometimes used to interpret an "actually"operator—thinking of the distinguished element w^* as the actual world of the model. (If we have $\Box$ in the language we can supplement these models with an accessibility relation satisfying some reasonable conditions, or simply interpret it—for an **S5** treatment of necessity—as quantifying over the whole of W.) Writing this operator as '$\mathcal{A}$', we take $\mathcal{A}$A as true at an arbitrary $w \in W$ relative to such a model just in case A itself is true at w^*. (See Chapter 9 of Davies [5] for further details, applications, and references.) Now consider the following clause governing our agreement operator O in this setting, where $\mathcal{M} = \langle W, w^*, V\rangle$:

$$\mathcal{M} \models_w \text{OA iff } \mathcal{M} \models_w \text{A iff } \mathcal{M} \models_{w^*} \text{A}.$$

In terms of the earlier general clause for O formulated with the aid of S, what we have done with this now wholly $\forall$-free condition amounts to taking $S_w xy \Leftrightarrow (x = w \,\&\, y = w^*)$. OA means 'Things stand as they do in the actual world insofar as the truth-value of A is concerned'; an alternative gloss on 'OA' that works well for many applications (especially indirect speech and propositional attitude embeddings) is simply 'whether A': see Lewis [24] for further details, presented in the framework of two-dimensional modal logic. Notice that not only is O definable in the object language in terms of the actuality operator—since our truth-definition validates OA $\leftrightarrow$ (A $\leftrightarrow$ $\mathcal{A}$A)—but also, rearranging this biconditional, we could have started with O and defined $\mathcal{A}$, by means of the valid $\mathcal{A}$A $\leftrightarrow$ (A $\leftrightarrow$ OA). This allows one to mimic a completeness proof for an axiomatization of the valid formulas in the language of **LO** by translating a complete axiomatization of the appropriate actuality logic. A related subvariation of this second variant on noncontingency arises when there is an accessibility relation which is functional, in which context we have already seen standard noncontingency ($\triangle$) is a dull affair, everything being noncontingent everywhere. Rather than writing $R(w) = \{x\}$, let us write $f(w) = x$ for this case, thinking of the models as of the form $\langle W, f, V\rangle$ with $f : W \longrightarrow W$. Then we can put

$$\mathcal{M} \models_w \text{OA if and only if } \mathcal{M} \models_w \text{A iff } \mathcal{M} \models_{f(w)} \text{A}.$$

In terms of the general clause, we have now put $S_w xy \Leftrightarrow (x = w \;\&\; y = f(w))$. If we thought of W as a set of discretely ordered moments of time and of $f(x)$ as the immediate successor of moment x, then OA says that there is no change over whether or not A between now and the next moment. (This is not really a "subvariation" but a "supervariation", since plainly the w^* example is just the special case in which f is a constant function.)

Whatever its own interest might be, we have described the first of the above two variants on noncontingency (the one with $S_w xy \Leftrightarrow (x = w \;\&\; Rxy)$) in order to notice an analogous variant on von Kutschera's treatment of O described in Section 1. Here again the models had a binary relation $\sim$, and since this is all that concerns us we may take such models to be triples $\langle W, \sim, V\rangle$; again, while [35] imposes a special condition on the interaction between V and $\sim$, we ignored this, and concentrated only on the restriction that $\sim$ should be an equivalence relation. We axiomatized the complete logic for this semantics as the system $\mathbf{LO}^+$ in Section 2. Theorem 2.5 showed that system to be determined by the class of frames $\langle W, \equiv, V\rangle$, with $\equiv$ an equivalence-relation assigning function and assigning the same equivalence relation to each $w \in W$: thus we could simply treat these models as having the type of von Kutschera's models. We will use the '$\equiv$' rather than the '$\sim$' notation. Then the first of the following two clauses gives the semantic treatment of 'O' with which we became familiar in Section 2, while the second is related to it in exactly as that contemplated above for 'O' is related to noncontingency. So as to compare the different notions, we write the operator as 'O′' in the second case; we take $\mathcal{M}$ in both cases to be of the type $\langle W, \equiv, V\rangle$, $\equiv$ an equivalence relation on W:

$$\mathcal{M} \models_w \text{OA if and only if for all } x, y \in W \text{ such that } x \equiv y$$
$$\text{we have } \mathcal{M} \models_x \text{A iff } \mathcal{M} \models_y \text{A;}$$

$$\mathcal{M} \models_w \text{O}'\text{A if and only if for all } y \in W \text{ such that } w \equiv y$$
$$\text{we have } \mathcal{M} \models_w \text{A iff } \mathcal{M} \models_y \text{A.}$$

The first of these is the von Kutschera clause for O. We suggested at the end of Section 1 that it was an inappropriate starting point for investigating the logical issues because it made the truth of OA at w independent of the choice of w, the latter not putting in any appearance after the 'if and only if'. Our way of fixing this in Section 2 was to change '$x \equiv y$' to '$x \equiv_w y$,' giving the choice of w a chance of mattering by having it affect which equivalence relation was at issue. (As we saw in Section 3 this added element of generality was especially useful when '$x \equiv_w y$' was replaced by '$S_w xy$' so we could stop insisting that the binary relation left after fixing w was an equivalence relation, and thereby subsume the case in which this relation was that holding between R-successors x and y of w for a traditional accessibility relation R.)

But the second clause above, written for ‘O$'$’, reintroduces the feature of dependence on w in a different way: not by making which binary relation is at issue depend on w, but much more straightforwardly, by taking w as the world with which $\equiv$-related worlds must agree on A.

We have stopped talking in abstract terms about “points” in our models, and started talking about them as worlds, to get back to the motivation we extracted from [35] for pursuing the modal logic of agreement in the first place. Von Kutschera was working with the idea—an idea which had been impressively elaborated and applied by Lewis in [25], [26]—that subject matters be identified with (perhaps only certain “natural”) equivalence relations on the set of possible worlds, a statement (or declarative sentence) being *entirely about* a given subject matters when any worlds standing in the equivalence relation in question agreed on the sentence in question.[25] In von Kutschera’s discussion, we fix on a thinker and consider the equivalence relation of being alike with respect to everything with the possible exception of thinker’s mental states. One of Lewis’s favored examples in [25] concerns the seventeenth-century statements being entirely about that subject matter when any worlds whose seventeenth-century parts are duplicates (are exactly alike, qualitatively, that is) agree on the sentence. Another, showing that not all subject matters are thus “part-based,” the subject matter of how many stars there are, worlds standing in the corresponding equivalence relation when there are the same number of stars in each, and a statement entirely about this subject matter is one on which any two worlds alike in respect of that number agree.[26] Now, fixing on a subject matter and its associated equivalence relation $\equiv$, the clause for ‘O’ above says that OA is true (at any world) just in case A is entirely about the subject matter in question. With the clause for ‘O$'$’ on the other hand, interpreted with the same equivalence relation $\equiv$, OA’s truth at a world amounts to what is called in Humberstone [17] A’s being *settled* in w on the basis of the subject matter concerned. For example, the statement that either there were some carpenters alive in 1650 or there would be in 1750 is not entirely about the seventeenth century, since there are worlds which are seventeenth-century-alike which differ as to its truth-value (say, because in one there are no carpenters before 1740 but there are from then on, and in the other there are no carpenters before 1760 but there are from then on). But this disjunction is settled—and settled as true—in the actual world on the basis of facts about the seventeenth century, since any world whose seventeenth-century duplicates that of the actual world (with all of its 1650 carpenters) agrees with the actual world on the truth-value of the statement. This, then, is a simple example in which, taking out disjunction as A and taking $\equiv$ as the relation of having matching seventeenth centuries and the actual world as w, OA is false in w while O$'$A is true in w. Note that while in general OA does not follow, as this example shows, from O$'$A, there is an entailment in the opposite direction. In fact, if we had in the same language a necessity operator understood in terms of universal quantification over the whole of W, then we could define OA as $\Box$O$'$A. Though we conclude our discussion here and do not consider any such extensions of the language here, one other dimension of expressiveness that might be explored would include the simultaneous several operators playing the roles of O and O$'$, one pair for each subject matter, together with binary functors on the operators to correspond to meets and joins in the lattice of subject matters (cf. the program-combining operations of propositional dynamic logic)—so that various logical relations whose description is usually confined to the metalanguage (e.g., in Humberstone [21]) would find expression in the object language itself.

Notes

1. What lies behind this coincidence is the fact that the relation for a given formula A between valuations u and v when $u(\mathrm{A}) = v(\mathrm{A})$ is dual to the relation for a given valuation v between formulas A and B when $v(\mathrm{A}) = v(\mathrm{B})$ in terms of a certain Galois duality discussed in Humberstone [20]. (These relations are of course the agreement-on-A relation and the condition for $\mathrm{A} \leftrightarrow \mathrm{B}$ to be true on a $\leftrightarrow$-Boolean valuation v.)

2. As metalinguistic analogues of $\leftrightarrow$, $\rightarrow$, $\wedge$, we sometimes use $\Leftrightarrow$, $\Rightarrow$, & , respectively.

3. More information on supervenience-determination may be found in Humberstone [19], [20].

4. There is a very serious technical problem with this suggestion (if it is understood against the background of the standard possible worlds semantics of the operators concerned), as explained in [32]; Rabinowicz and Segerberg there offer a revised semantics to fix the problem and compare their solution with one proposed by Lindström (then unpublished, but now available in Lindström [27]).

5. **KD45**, to be precise.

6. At several points in this summary of von Kutschera's discussion, we have assimilated his notation to ours in various ways. As already noted, he writes 'B' where we write 'K_0'; he writes 'N', 'M', for our '$\Box$', '$\rightarrow$'; and he uses the V part of the model to assign truth-values to all formulas rather than just the propositional variables, as in our summary. (This departure foreshadows our use of '$\models$' in the following section for the general truth-relation.)

7. A similar failure of closure under uniform substitution holds for the epistemic-doxastic logic of Halpern [11], where it is again the result of an objective/subjective distinction which puts all the propositional variables on the objective side: $\mathrm{K}p \rightarrow \mathrm{K}_0 p$ is valid on Halpern's semantics, as is the result of substituting any objective (i.e., K, K_0-free) formula for p, though—as he is keen to have be the case—the substitution-instance $\mathrm{K}\neg\mathrm{K}p \rightarrow \mathrm{K}_0\neg\mathrm{K}p$ is not. (Of course, we are using our own temporary notation here. [11] has 'B' for 'K_0'.)

8. '**LO**' is just mnemonic for 'the Logic of O'. There is little danger of confusion with other similarly named systems in the literature such as the system **LO** of Ono [29], [30], etc.

9. Note that the notion of a Boolean valuation still makes sense for a language such as that of **LO**, not all of whose connectives are Boolean: it simply places no constraints on a formula with a non-Boolean main connective.

10. This is inspired by the use made in Montgomery and Routley [28] of $\triangle\triangle\mathrm{A}$ as a schema—one of several alternatives considered in this regard—for extending the noncontingency formulation of **KT** to a noncontingency formulation of **S5 (KT5)**. We discuss some aspects of the logic of '$\triangle$' (or of 'O' as it behaves in what we call noncontingency models)

in Section 3. Unaware of [28], Demri provides his own noncontingency treatment of **S5** in [6], using semantic methods to show that it deserves this description (whereas Montgomery and Routley employ purely syntactic arguments). A more elaborate version of a similar idea, with an intended epistemic application, can be found in Demri [7]. Both [6] and [7] make the incorrect claim that 'KA' in Hintikka [12] is interpreted as meaning 'It is known whether A' (whereas in fact Hintikka reads this as 'It is known that A'—the epistemic analogue of necessity, in other words, rather than of noncontingency). This misconception may have arisen from the fact that this nonstandard interpretation of 'K' is to be found in work by Orłowska cited in [6] and [7]. Readers interested in [28] will find other papers on noncontingency formulations of several modal logics in their subsequent papers in the same journal, in Volumes 11 (1968) and 12 (1969). Additional historical information: Prior ([31], p. 313) cites an unpublished axiomatization, dated 1959, of (non)contingency-based **S5** by Lemmon and Gjertsen.

11. If attention is restricted to models with reflexive accessibility relations, then not only is $\triangle$ definable in terms of the (familiarly interpreted) $\Box$ and the Boolean connectives, by putting $\triangle A = \Box A \vee \Box\neg A$, but also—and this is where we need the reflexivity—conversely, we can put $\Box A = \triangle A \wedge A$.

12. Some misprints in [22] may slow readers down. Though Kuhn undoubtedly intends to call his basic logic for noncontingency '$\mathbf{K}\triangle$' (replacing the label '**NC**' from [16], associated there with a particular axiomatization which differs from Kuhn's), this label does not actually appear in [22], being misprinted as '$\mathbf{K4}\triangle$' on p. 231 when the system is introduced, as well as at four later occurrences on that page, and as '**K**' on another occasion there. Of the five occurrences of '$\mathbf{K4}\triangle$' on p. 233, the first and third should also be '$\mathbf{K}\triangle$'. The general convention is clear enough though: where **S** is a $\Box$-based modal logic, $\mathbf{S}\triangle$ is to be its $\triangle+$ Boolean connectives fragment, thinking of $\triangle$ as having been introduced into **S** by the definition of $\triangle A$ as $\Box A \vee \Box\neg A$. We shall follow instead the conventions of [37] and write '$\mathbf{S}^{\triangle}$' in place of '$\mathbf{S}\triangle$'.

13. The cited source uses 'R' rather than 'T' as a general variable for binary relations, avoided here since we have associated 'R' with the binary accessibility relations of Kripke models (and when we come to apply these ideas to the semantics, the relations T will correspond instead to the binary relations S_w); there are some other notational changes also.

14. Though for simplicity we make no notational distinction between the relation R and the relation symbol with this relation as its extension in the structure $\langle U, R\rangle$.

15. See Proposition 7(i) of [15].

16. It is for this reason that we use the 'S' notation, and generalized models, rather than the '$\equiv$' notation of the models of Section 2, even though in our canonical generalized models, the S-relations, or more accurately the relations S_w for w an element of such a model, are indeed equivalence relations.

17. We are using the "frame" terminology parenthetically introduced after the truth-definition at the start of Section 2 (though now understood to apply to frames as abstracted from Kripke models rather than the models there in play) and say that a set of formulas Σ *modally defines* a class $\mathbb{C}$ of frames when for every frame $\langle W, R\rangle$, we have $\langle W, R\rangle \models A$ for all $A \in \Sigma$ if and only if $\langle W, R\rangle$; here we have written '$\langle W, R\rangle \models A$'

for 'A is valid on the frame $\langle W, R\rangle$'. (If Σ consists of a single formula A, we say that A modally defines $\mathbb{C}$ in this case.) A class $\mathbb{C}$ of frames for which there is some Σ modally defining $\mathbb{C}$ is called *modally definable*. (See van Benthem [34] for these concepts and their properties in the $\Box$-based setting.) Theorem 4.1 and Corollary 4.2 just stated can be strengthened and have been formulated as above for easy visualizability. The key is that it doesn't matter whether a point has no successors or exactly one, so we can add or delete sole successors as we please, rather than having to do so uniformly. Thus the general situation for Theorem 4.1 is that $\mathcal{M}_1 = \langle W, R_1, V\rangle$ and $\mathcal{M}_2 = \langle W, R_2, V\rangle$ stand in the following relation: for all $w \in W$, if $|R_1(w)| \geq 2$ or $|R_2(w)| \geq 2$, then $R_1(w) = R_2(w)$. This allows for considerably more leeway in "rewiring" elements w with $|R_i(w)| < 2$ than Theorem 4.1, but still suffices for the inductive case of $\triangle$ in the proof.

18. The connection between Halldén-completeness and this intersection characterization may be found in Lemmon [23]; it can also be found as Theorem 15.22 in Chagrov and Zakharyaschev [1] together with further information and references on the topic.

19. The π^+ semantics from §2.1 of Fine [9] will do for our purposes here, which allows every subset of W as a proposition. (All we actually exploit in the proof of the 'if' half of Proposition 4.4 below, however, is the assumption that for any pair of elements there is a proposition containing one and not the other.)

20. If some $\triangle$A $\notin w$ then by Lemmas 2.2, 2.3—where we were writing 'O' rather than '$\triangle$'—there are two points agreeing on all C for which $\triangle$C $\in w$ but disagreeing on A, so if we call them x and y we have $S_w xy$('$x \equiv_w y$' we wrote there) with $x \neq y$; thus each of x, y, belongs to $R(w)$ as defined by (δ), making $R(w)$ nonempty. Conversely if $R(w)$ is nonempty then there are distinct x, y for which, canonically, $S_w xy$, so there must be some formula in x but not in y, say B, in which case $\triangle$B $\notin w$.

21. The 'λ' in [22] is, as in [16], mnemonic for 'labeling' where a labeling of (maximal consistent) w is what in Section 2 above we call a signing of the set $\{C| \triangle \text{C} \in w\}$. Whereas in [16], there is a complicated argument for the existence of a suitable labeling for each w, in [22] we have the simple explicit definition just given. Kuhn appeals to what we are calling Kuhn's Axiom to show that $\lambda(w)$ is indeed, in the sense just defined, a labeling of w: see [22], p. 232, Property P3. (The axiom as formulated gives this result very directly, whereas for the "two schematic letters" version mentioned in Section 3 above, this consequence would be far from evident and extracting it would amount to deriving the "three schematic letters" form used by Kuhn.) Kuhn's way of explicitly specifying $\lambda(w)$ avoids the appeal to König's Lemma (or a version thereof called the "Word Lemma") in the completeness proof in [16]. It would be interesting to know if the appeal we make to König's Lemma at the end of Lemma 2.2 above is also avoidable.

22. Inconsistency can be understood here, not in the weak sense of having every formula as a consequence, but of actually containing every formula as an element, because $\lambda(w)$ is actually closed under (**LO**)-consequences, that is, if $(\text{A}_1 \wedge \cdots \wedge \text{A}_n) \to \text{B}$ is provable in **LO**, and each $\text{A}_i \in \lambda(w)$, then B $\in \lambda(w)$. (See Properties 1, 3, 4, on p. 232 of [22].) The definition of the canonical accessibility relations R in [37] is essentially the same as Kuhn's. Instead of the definition just given of λ, that in [37] amounts to taking $\lambda(w)$ as $\{\text{A}| \triangle (\text{B} \to \text{A}) \in w$ for all formulas B$\}$, thus replacing the B in Kuhn's $\triangle(\text{A} \vee \text{B})$ formulation by its negation. While this is a different set of formulas from Kuhn's $\lambda(w)$, the one is included in any set closed under consequences if and only if the other is, so the same canonical R emerges from both definitions. (The '$\lambda(w)$' is not used in [37].)

23. The distinction arises only for nonreflexive R, of course. Another way of describing the effect of the above clause is to say that for OA to be true at w, we require that all points in $R(w) \cup \{w\}$ agree on A.

24. We can drop the first conjunct of either disjunct in the consequent here, provided we leave the first conjunct of the other intact. But the present form is easiest for conducting the completeness argument sketched immediately below.

25. The fact that what we call $(\text{OComp})_{\#}$ holds for the various Boolean # is already explicit with Lewis's "Compositional Condition" in [25], when 'OA' is interpreted as the claim that A is entirely about a given subject matter. (See [17], p. 123 and note 6 for a slightly more accurate statement of this point.)

26. A fuller discussion of part-based versus other subject matters may be found in [21].

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