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Definability of Initial Segments

Saharon Shelah and Akito Tsuboi

Abstract In any nonstandard model of Peano arithmetic, the standard part is not first-order definable. But we show that in some model the standard part is definable as the unique solution of a formula $\varphi(P)$, where P is a unary predicate variable.

1 Introduction

Let T be a first-order theory formulated in the language $\mathcal{L}$ and P, P' new distinct relation symbols not in $\mathcal{L}$. Let $\varphi(P)$ be an $(\mathcal{L} \cup \{P\})$-sentence. Let us say that $\varphi(P)$ defines P implicitly in T if T proves $\varphi(P) \wedge \varphi(P') \to \forall x(P(x) \leftrightarrow P'(x))$. Beth's definability theorem states that if $\varphi(P)$ defines P implicitly in T then $P(x)$ is equivalent to an $\mathcal{L}$-formula.

However, if we consider implicit definability in a given model alone, the situation changes. For a more precise explanation, let us say that a subset A of a given model M of T is implicitly definable if there exists a sentence $\varphi(P)$ such that A is the unique set with $(M, A) \models \varphi(P)$. It is easy to find a structure in which two kinds of definability (implicit definability and first-order definability) are different. For example, let us consider the structure $M = (\mathbb{N} \cup \mathbb{Z}, <)$, where $<$ is a total order such that any element in the $\mathbb{Z}$-part is greater than any element in the $\mathbb{N}$-part. The $\mathbb{N}$-part is not first-order definable in M because the theory of M admits quantifier elimination after adding the constant 0 (the least element) and the successor function to the language. But the $\mathbb{N}$-part is implicitly definable in M because it is the unique nontrivial initial segment without a last element. On the other hand, for a given structure, we can easily find an elementary extension in which the two notions of implicit definability and first-order definability coincide.

In this paper, we shall consider implicit definability of the standard part $\{0, 1, \ldots\}$ in nonstandard models of Peano arithmetic (PA). It is clear that the standard part of a nonstandard model of PA is not first-order definable. As is stated above, there

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is a model in which every set defined implicitly is first-order definable. So we ask whether there is a nonstandard model of PA in which the standard part is implicitly definable.

In Section 2, we define a certain class of formulas and show that in any model of PA the standard part is not implicitly defined by such formulas.

Section 3 is the main section of the present paper; we shall construct a model of PA in which the standard part is implicitly defined. To construct such a model, first we assume a set theoretic hypothesis $\diamondsuit_{S_\lambda^{\lambda^+}}$, which is an assertion of the existence of a very general set. Then we shall eliminate the hypothesis using absoluteness for the existence of a model having a tree structure with a certain property.

In this paper $\mathcal{L}$ is a first-order countable language. $\mathcal{L}$-structures are denoted by $M, N, M_i, \ldots$. We do not distinguish a structure and its universe. $A, B, \ldots$ will be used for denoting subsets of some $\mathcal{L}$-structure. Finite tuples of elements from some $\mathcal{L}$-structure are denoted by $\bar{a}, \bar{b}, \ldots$. We simply write $A \subset M$ for expressing that A is a subset of the universe of M.

2 Undefinability Result

Let us first recall the definition of implicit definability.

Definition 2.1 Let M be an $\mathcal{L}$-structure. Let P be a unary second-order variable. A subset A of M is said to be *implicitly definable* in M if there is an $(\mathcal{L} \cup \{P\})$-sentence $\varphi(P)$ with parameters such that A is the unique solution to $\varphi(P)$, that is, $\{A\} = \{B \subset M : M \models \varphi(B)\}$.

In this section, $\mathcal{L}$ is the language $\{0, 1, +, \cdot, <\}$, and PA denotes the Peano arithmetic formulated in $\mathcal{L}$. We shall prove that the standard part is not implicitly definable in any model of PA by using a certain form of formulas. We fix a model M of PA and work on M.

Definition 2.2 An $(\mathcal{L} \cup \{P\})(M)$-formula $\varphi(\bar{y})$ (with parameters) will be called simple if it is equivalent (in M) to a prenex normal form

$$Q_1\bar{x}_1 \ldots Q_n\bar{x}_n[P(f(\bar{x}_1, \ldots, \bar{x}_n, \bar{y})) \to P(g(\bar{x}_1, \ldots, \bar{x}_n, \bar{y}))]$$

where Q_is are quantifiers and f and g are definable functions. If $Q_1 = \forall$ then φ will be called a simple Π_n-formula. Similarly, it is called a simple Σ_n-formula if $Q_1 = \exists$.

Remark 2.3 If P is an initial segment of M, then

1. $a_1 \in P \wedge a_2 \in P$ is equivalent to $\max\{a_1, a_2\} \in P$;
2. $a_1 \in P \vee a_2 \in P$ is equivalent to $\min\{a_1, a_2\} \in P$.

An $\mathcal{L}$-formula $\varphi(\bar{x})$ is equivalent to a formula of the form $P(f(\bar{x}))$, where f is a definable function such that $f(\bar{x}) = 0$ if $\varphi(\bar{x})$ holds and $f(\bar{x}) = a$ (a is a nonstandard element) otherwise. An initial segment I of an ordered structure will be called a *cut* if I does not have a last element. The statement that P is a cut is expressed by a simple Π_2-formula.

We shall prove that the standard part is not implicitly definable by a finite number of simple Π_2-formulas. In fact we can prove more.

Proposition 2.4 *Let I_0 be a cut of M with $I_0 < a$, that is, any element of I_0 is smaller than a. Let $\{\varphi_i(P) : i \leq n\}$ be a finite set of simple Π_2-sentences. If I_0 satisfies $\{\varphi_i(P) : i \leq n\}$, then there is another cut $I < a$ which also satisfies $\{\varphi_i(P) : i \leq n\}$.*

Let us say that a cut I is approximated by a decreasing ω-sequence if there is a definable function $f(x)$ with $I = \{a \in M : (\forall m \in \omega)\ a \leq f(m)\}$. Similarly, we say that I is approximated by an increasing ω-sequence if there is a definable function $g(x)$ with $I = \{a \in M : (\exists m \in \omega)\ a \leq g(m)\}$. Notice that no cut of M is approximated both by a decreasing ω-sequence and by an increasing ω-sequence. For a cut I with $I < a$, let $I^* = \{d : a - d \notin I\}$. I^* is a cut with $I^* < a$ and $I^{**} = I$. If I is approximated by a decreasing ω-sequence, then I^* is approximated by an increasing ω-sequence. For a sentence $\varphi(P)$, let $\varphi^*(P)$ denote the sentence obtained from $\varphi(P)$ by replacing all the occurrences of $P(*)$ by $\neg P(a - *)$. If a cut $I < a$ satisfies a simple Π_2-sentence $\varphi(P)$, then I^* satisfies $\varphi^*(P)$, which is also a simple Π_2-sentence. For a cut $I < a$, I satisfies $\varphi(P)$ if and only if I satisfies $\varphi^{**}(P)$ holds.

Proof of Proposition 2.4 For $i \leq n$, let $\varphi_i(P)$ have the form

$$\forall \bar{x} \exists \bar{y} [P(f_i(\bar{x}, \bar{y})) \to P(g_i(\bar{x}, \bar{y}))].$$

By the remark just after Proposition 2.4, we can assume that I_0 cannot be approximated by a decreasing ω-sequence. We shall show that there is an initial segment I with $I_0 \subsetneq I < a$ and $M \models \bigwedge_{i \leq n} \varphi_i(I)$. Since I_0 satisfies $\varphi_i(P)$, for each $b_0 \in M$ with $I_0 < b_0 < a$, we have $M \models \bigwedge_{i \leq n} \forall \bar{x} \exists \bar{y} [f_i(\bar{x}, \bar{y}) \in \omega \to g_i(\bar{x}, \bar{y}) \leq b_0]$. By overspill there is an element b_1 with $I_0 < b_1 < b_0$ such that

$$M \models \bigwedge_{i \leq n} \forall \bar{x} \exists \bar{y} [f_i(\bar{x}, \bar{y}) \leq b_1 \to g_i(\bar{x}, \bar{y}) \leq b_0].$$

By choosing maximum such $b_1 < b_0$, we may assume that $b_1 \in \mathrm{dcl}(\bar{a}, b_0)$, where $\bar{a}$ are parameters necessary for defining f_is and g_is. So we can choose an $\mathcal{L}(\bar{a})$-definable function, $h(x)$ such that

(i) $I_0 < b < a$ implies $I_0 < h(b) < b$ and
(ii) $M \models \bigwedge_{i \leq n} \forall x \exists y [f_i(\bar{x}, \bar{y}) \leq h(b) \to g_i(\bar{x}, \bar{y}) \leq b]$, for any nonstandard $b \in M$.

By using recursion we can choose a definable function $l(x)$ with $l(m) = h^m(a)$ (the m-time application of h) for each $m \in \omega$. Now we put

$$I = \{d \in M : (\forall m \in \omega)\ d \leq l(m)\}.$$

Since $m < h(m)$ holds for any $m \in \omega$, by overspill, there is a nonstandard m^* such that $m^* < h(m^*)$. This shows that I is an initial segment different from I_0. Now we show the following.

Claim 2.5 *For all $i \leq n$ and for all $\bar{d} \in M$, there is $\bar{e} \in M$ such that*

$$f_i(\bar{d}, \bar{e}) \in I \ \to\ g_i(\bar{d}, \bar{e}) \in I.$$

Let $d \in M$ and $i \leq n$ be given. We can assume that $\forall y (f_i(\bar{d}, \bar{y}) \in I)$ holds in M. So by the definitions of I and l, for all $k \in \omega$, we have $M \models \forall y (f_i(\bar{d}, \bar{y}) \leq l(k))$. Hence, for some nonstandard $k^* \in M$ with $k^* \leq l(k^*)$, we have

$$M \models \forall \bar{y} (f_i(\bar{d}, \bar{y}) \leq l(k^*)).$$

On the other hand, by our choice of h and l, we can find $\bar{e}$ with

$$M \models f_i(\bar{d}, \bar{e}) \leq l(k^*) \to g_i(\bar{d}, \bar{e}) \leq l(k^* - 1).$$

Hence, for this $\bar{e}$, we have $g_i(\bar{d}, \bar{e}) \leq l(k^* - 1) \in I$. □

Corollary 2.6 *The standard part is not implicitly definable by a finite number of simple Σ_3-formulas.*

3 Definability Result

In this section we aim to prove the following theorem.

Theorem 3.1 *There is a model of* PA *in which the standard part is implicitly definable.*

Instead of proving the theorem, we prove a more general result, Theorem 3.5, from which Theorem 3.1 easily follows. For stating the result, we need some preparations.

We assume the countable language $\mathcal{L}$ contains a binary predicate symbol $<$, a constant symbol 0, and a unary function symbol S. We fix a complete countable $\mathcal{L}$-theory T with a partial definable function $F(x, y)$ such that the following sentences are members of T.

1. $<$ is a linear order with the first element 0;
2. For each x, $S(x)$ is the immediate successor of x with respect to $<$;
3. $\forall y_1, \dots, y_n \forall z_1, \dots, z_n \exists x (\bigwedge_{i \neq j} y_i \neq y_j \to \bigwedge_{i=1}^{n} F(x, y_i) = z_i)$ (for $n \in \omega$).

Remark 3.2 In PA, let $F(x, y) = z$ be a definable function such that the sequence coded by x has z as the yth element. It is easy to see that this F satisfies the third condition above. So, any completion of PA satisfies our requirements stated above.

Let P be a new unary predicate symbol not in $\mathcal{L}$. Throughout this section $\psi^*(P)$ is the conjunction of the following $(\mathcal{L} \cup \{P\})$-sentences:

1. P is a cut (nonempty proper initial segment closed under S), that is, $\neg(\forall x P(x)) \wedge P(0) \wedge \forall x \forall y (P(y) \wedge x < y \to P(x)) \wedge \forall x (P(x) \to P(S(x)))$;
2. for no x and z with $P(z)$ is $\{F(x, y) : y < z\} \cap P$ unbounded in P, that is, $\forall x \forall z [P(z) \to \exists w (P(w) \wedge \forall y (P(F(x, y)) \to F(x, y) < w))]$.

The subset $\{S^n(0) : n \in \omega\}$ of a model of T will be called the standard part of the model and denoted by $\mathbb{N}$. It is clear that $\mathbb{N}$ satisfies $\psi^*(P)$, that is, the sentence $\psi^*(P)$ holds in the $(\mathcal{L} \cup \{P\})$-structure $(M, \mathbb{N})$.

Definition 3.3 A model M of T will be called appropriate if the following two conditions are satisfied:

1. $M \neq \mathbb{N}$;
2. if $(M, I) \models \psi^*(P)$ then
   (a) $I = \mathbb{N}$ or
   (b) I is first-order definable with parameters.

Remark 3.4 In case that T is a completion of PA, part (b) of condition 2 in Definition 3.3 does not occur, because in any model of T no definable proper subset is closed under S.

Theorem 3.5 *There is an appropriate model of T.*

We shall prove Theorem 3.5 by a series of claims. For a period of time, we fix an uncountable cardinal λ with $\lambda = \lambda^{<\lambda}$. In our proof of the theorem we shall construct an appropriate model of cardinality λ^+ under a set-theoretic assumption, and later by eliminating this assumption, we get an appropriate model (of cardinality $\aleph_1$) in ZFC. We don't know whether the existence of a countable appropriate model can be shown in ZFC.

First we need some definition. The definition itself can be stated in a general context. However, we give the definition for a countable T. (Recall that our T is countable.)

Definition 3.6 Let M be a model of T and $\varphi(x, \bar{a})$ a formula with parameters from M. We say that $\varphi(x, \bar{a})$ is big (in M) if in some (any) $\aleph_1$-saturated model $N \succ M$ there is $A \subset N$ with $|A| \leq \aleph_0$ such that for any finite number of distinct elements $a_1, \ldots, a_n \in N \setminus A$, and any elements $b_1, \ldots, b_n \in N$, we have

$$N \models \exists x[\varphi(x, \bar{a}) \wedge \bigwedge_{i=1}^{n} F(x, a_i) = b_i].$$

Let us briefly recall the definition of *bigness* defined in Shelah [5]. Let $R \notin \mathcal{L}$ be a unary predicate symbol. A statement (or an infinitary $(\mathcal{L} \cup \{R\})$-sentence) $\Gamma(R)$ is called a notion of bigness for T, if any model M of T satisfies the following axioms, for all formulas $\varphi(x, \bar{y})$ and $\psi(x, \bar{y})$ (where $\Gamma(\varphi(x, \bar{y}))$ means that setting $R(x) = \varphi(x, \bar{y})$ [so $\bar{y}$ is a parameter] makes Γ true):

1. $\forall \bar{y}(\forall x(\varphi(x, \bar{y}) \rightarrow \psi(x, \bar{y})) \wedge \Gamma(\varphi(x, \bar{y})) \rightarrow \Gamma(\psi(\bar{x}, \bar{y})))$;
2. $\forall \bar{y}(\Gamma(\varphi(x, \bar{y}) \vee \psi(\bar{x}, \bar{y})) \rightarrow \Gamma(\varphi(x, \bar{y})) \vee \Gamma(\psi(x, \bar{y})))$;
3. $\forall \bar{y}(\Gamma(\varphi(x, \bar{y})) \rightarrow \exists^{\geq 2} x \varphi(\bar{x}, \bar{y}))$;
4. $\forall x \Gamma(x = x)$.

Now let $\Gamma(\varphi)$ be the statement 'φ is big' in the sense of Definition 3.6. Then this Γ satisfies the above four axioms: It is easy to see that our Γ satisfies Axioms 1, 3, and 4. So let us prove Axiom 2. Suppose that neither φ nor ψ is big. Let M be a model of T and $N \succ M$ be $\aleph_1$-saturated. Let A be a subset of N of cardinality $\leq \aleph_0$. Since φ is not big, A cannot witness the definition of bigness, so there are a finite number of elements $a_1, \ldots, a_n \in N \setminus A$ with no repetition and $b_1, \ldots, b_n \in N$ such that $N \models \forall x[\bigwedge_{i \leq n} F(x, a_i) = b_i \rightarrow \neg\varphi(x)]$. Since ψ is not big, $A' = A \cup \{a_1, \ldots, a_n\}$ cannot witness the definition of bigness, hence there are $a_{n+1}, \ldots, a_m \in N \setminus (A \cup \{a_1, \ldots, a_n\})$ with no repetition and $b_{n+1}, \ldots, b_m \in N$ such that $N \models \forall x[\bigwedge_{n+1 \leq i \leq m} F(x, a_i) = b_i \rightarrow \neg\psi(x)]$. So $N \models \forall x[\bigwedge_{i \leq m} F(x, a_i) = b_i \rightarrow \neg(\varphi(x) \vee \psi(x))]$. Since A was chosen arbitrarily, this shows that $\varphi \vee \psi$ is not big.

We introduce some terminology. A Dedekind cut of M of cofinality (μ_1, μ_2) is a pair (C_1, C_2) such that

(i) $M = C_1 \cup C_2$,
(ii) $\forall x \in C_1 \forall y \in C_2 [x <^M y]$,
(iii) the cofinality of C_1 with respect to $<$ is μ_1, and
(iv) the coinitiality of C_2 (i.e., the cofinality of C_2 with respect to the reverse ordering) is μ_2.

Let $S_\lambda^{\lambda^+} = \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$. From now on, until the end of Claim 3.8, we assume $\diamondsuit_{S_\lambda^{\lambda^+}} + \diamondsuit_\lambda$, where $\lambda = \lambda^{<\lambda}$.

By [5], we have the following claim. (Recall that our λ is uncountable. Even if $\lambda = \aleph_0$, a similar (but not identical) statement holds. For more details, see Shelah [6].)

Claim 3.7 (Under $\diamondsuit_{S^{\lambda^+}_{\lambda}} + \diamondsuit_\lambda$) *There is a model M of T such that the condition*

(a) *if (C_1, C_2) is a Dedekind cut of M of cofinality (λ^+, λ^+) then C_1 is a subset of M definable with parameters*

holds, and which is the union of a continuous elementary chain $\langle M_i : i < \lambda^+\rangle$ of models of T such that for some sequence $\langle a_i : i < \lambda^+\rangle$ of elements $a_i \in M_{i+1} \setminus M_i$,

(b) $|M_i| = \lambda$*;*
(c) *M_i is saturated unless i is a limit ordinal with* $\mathrm{cf}(i) < \lambda$*;*
(d) $\mathrm{tp}_{M_{i+1}}(a_i/M_i)$ *is big, that is, each formula in it is big;*
(e) *$M_i \subset \{F^{M_{i+1}}(a_i, c) : M_i \models c < b\}$ if $b \in M_i \setminus \mathbb{N}$.*

Now we expand the language $\mathcal{L}$ by adding new binary predicate symbols. Let $\mathcal{L}^* = \mathcal{L} \cup \{E_1, E_2, <_{\mathrm{lev}}, <_{\mathrm{tr}}\}$. We expand the $\mathcal{L}$-structure M obtained in Claim 3.7 to an $\mathcal{L}^*$-structure M^* by the following interpretation. For $a \in M$, let $i(a) = \min\{i < \lambda^+ : a \in M_{i+1}\}$.

1. $E_1^{M^*} = \{(a, b) : i(a) = i(b)\}$;
2. $E_2^{M^*} = \{(a, b) : i(a) = i(b)$ and $M \models (c < a \equiv c < b)$ for every $c \in M_{i(a)}\}$; in other words, $(a, b) \in E_2^{M^*}$ iff a and b determine the same Dedekind cut of $M_{i(a)} (= M_{i(b)})$;
3. $<_{\mathrm{lev}}^{M^*} = \{(a, b) : i(a) < i(b)\}$;
4. $<_{\mathrm{tr}}^{M^*} = \{(a, b) : i(a) < i(b)$ and $M \models (c < a \equiv c < b)$ for every $c \in M_{i(a)}\}$.

The relation $<_{\mathrm{tr}}$ defines a preorder on M^* and induces a tree structure on the E_2-equivalence classes. This tree structure $(M^*/E_2, <_{\mathrm{tr}})$ is a definable object of $M^{*\mathrm{eq}}$. (We do not use a new symbol for the order induced by $<_{\mathrm{tr}}$.) Similarly $<_{\mathrm{lev}}$ induces a linear order on the E_1-equivalence classes. Let R be the definable function which maps a_{E_2} to a_{E_1}. R is considered as a rank function which assigns a level to each node of the tree. Then $\langle <_{\mathrm{tr}}, <_{\mathrm{lev}}, R\rangle$ is an *$\mathcal{L}^*$-tree* in the sense of Shelah [4]. A subset B of M^*/E_2 will be called a *branch* of the tree if

(i) it is linearly ordered by $<_{\mathrm{tr}}$,
(ii) $a_{E_2} \in B$ and $b \leq_{\mathrm{tr}} a$ imply $b_{E_2} \in B$, and
(iii) the set $\{R(a_{E_2}) : a_{E_2} \in B\}$ of all levels in B is unbounded in M^*/E_1.

Claim 3.8 *Every branch of the tree $(M^*/E_2, <_{\mathrm{tr}}, <_{\mathrm{lev}}, R)$ is definable in M^*.*

Proof Let B be a branch of the tree $(M^*/E_2, <_{\mathrm{tr}}, <_{\mathrm{lev}}, R)$. We show that B is definable in M^*. Let I be the $<$-initial segment determined by B, that is,

$$I = \{a \in M^* : M^* \models (\forall b_{E_2} \in B)(\exists c_{E_2} \in B)[b_{E_2} <_{\mathrm{tr}} c_{E_2} \wedge a < c]\}.$$

It is easy to see that I and B are interdefinable in M^*. In fact, we have $b_{E_2} \in B$ if and only if there exist $c \in I$ and $d \in M^* \setminus I$ such that

1. b_{E_2} intersects the interval $[c, d]$,
2. if $b_{E_2} \subset I$ then any other b'_{E_2} with $[c, d] \cap I \cap b'_{E_2} \neq \varnothing$ has a strictly larger level than b_{E_2}, and
3. if $b_{E_2} \subset M^* \setminus I$ then any other b'_{E_2} with $[c, d] \cap (M^* \setminus I) \cap b'_{E_2} \neq \varnothing$ has a strictly larger level than b_{E_2}.

If the cofinality of $(I, M^* \setminus I)$ is (λ^+, λ^+), then I is definable in M by property (a) of Claim 3.7, so B is definable in M^*. So we may assume that the cofinality is not (λ^+, λ^+).

First suppose that $\mathrm{cf}(I) \leq \lambda$. Then we can choose a set $\{a_i : i < \lambda\}$ which is cofinal in I. Choose $j < \lambda^+$ with $\mathrm{cf}(j) = \lambda$ and $\{a_i : i < \lambda\} \subset M_j$. If $M_j \setminus I$ is bounded from below in $M^* \setminus I$, say by $d \in M^* \setminus I$, then I is defined in M^* by the formula $\exists y[x < y < d \wedge y <_{\mathrm{lev}} e]$, where e is an element from $M_{j+1} \setminus M_j$. So we may assume that there is a set $\{a_i' : i < \lambda\} \subset M_j \setminus I$ which is coinitial in $M^* \setminus I$. (We shall derive a contradiction from this.) Let $b_{E_2} \in B$ with $b \notin M_j$. Since the other case can be treated similarly, we can assume that $b \in I$. Then b_{E_2} is included in some interval $[0, a_i]$. By the definition of I, there is $c_{E_2} \in B$ such that $b_{E_2} <_{\mathrm{lev}} c_{E_2}$ and $a_i < c$. But then b and c determine different Dedekind cuts of M_j, hence b and c are not comparable with respect to $<_{\mathrm{tr}}$. This contradicts our assumption that B is a branch.

Second suppose that the coinitiality of $M^* \setminus I$ is $\leq \lambda$ and that the cofinality of I is λ^+. As in the first case, we can choose $j < \lambda^+$ such that $M_j \setminus I$ is coinitial in $M^* \setminus I$. Choose $d \in I$ which bounds $I \cap M_j$ from above and an element $e \in M_{j+1} \setminus M_j$. Then I is defined by the formula $\forall y[d < y \wedge y <_{\mathrm{lev}} e \rightarrow x < y]$. Lastly the case where the cofinality of $(I, M^* \setminus I)$ is (μ_1, μ_2) with $\mu_1, \mu_2 \leq \lambda$ is impossible by the definition of branch. $\square$

Let T^* be the $\mathcal{L}^*$-theory of M^*. Under the hypothesis of Claims 3.7 and 3.8 (i.e., $\diamondsuit_{S_\lambda^{\lambda^+}}$ and so on), we have proven the existence of $M^* \models T^*$ having a tree with the property stated in Claim 3.8. So, for example, we have such a model M^* in the constructible universe L, as our hypothesis holds there. Now we extend the structure L. Let P be the forcing notion $Levy(\aleph_0, \lambda)$, and $G \subset P$ generic over V. In the generic extension $L[G]$ (the Lévy collapse), we have $\lambda = \aleph_0$ and $\lambda^+ = \aleph_1$. This extension does not add branches to the tree as a branch has length λ^+. We can now apply the absoluteness (e.g., Theorem 6 in [4]) and get such a model without using the hypothesis. Moreover, as T^* is countable, we can assume that relevant properties of M^* expressed by one $\mathcal{L}^*_{\omega_1\omega}(Q)$-sentence are also possessed by such models. (Q is the quantifier which expresses "there are uncountably many".) Thus in ZFC we can show the following claim.

Claim 3.9 *There is a model $N^* \models T^*$ of cardinality $\aleph_1$ that satisfies*

1. *the tree $(N^*/E_2, <_{\mathrm{tr}})$ has no undefinable branch;*
2. *the set N^*/E_1 of levels has cardinality $\aleph_1$, but for each $b_{E_1} \in N^*/E_1$, $\{c_{E_1} : c_{E_1} <_{\mathrm{lev}} b_{E_1}\}$ is countable;*
3. *if I is a definable subset of N^* with the Dedekind cut $(I, N^* \setminus I)$ of cofinality $(\aleph_1, \aleph_1)$, then I is definable in N;*
4. *the clause (e) of Claim 3.7, namely, for each level d_{E_1} there is $a \in N^*$ such that if $b \in N^* \setminus \mathbb{N}$ and $b \leq_{\mathrm{lev}} d$ then $\{F(a, c) : c < b\}$ includes $\{c \in N^* : c \leq_{\mathrm{lev}} d\}$.*

Claim 3.10 *Let N^* be a model of T^* with the properties stated in Claim 3.9. Then the reduct N of N^* to the language $\mathcal{L}$ is ψ^*-appropriate.*

Proof Toward a contradiction, we assume that there is an undefinable (in the sense of N) subset $I \subset N$ with $(N, I) \models \psi^*(P)$ and $I \neq \mathbb{N}$. We show that the cofinality of $(I, N^* \setminus I)$ is $(\aleph_1, \aleph_1)$. Suppose that this is not the case. First assume that the cofinality of $(I, <)$ is less than $\aleph_1$. As $(N^*/E_1, <_{\mathrm{lev}})$ has the cofinality $\aleph_1$, there is d_{E_1} such that $\{c \in I : c \leq_{\mathrm{lev}} d\}$ is unbounded in I. Since $I \neq \mathbb{N}$, we can choose $b \in I \setminus \mathbb{N}$ with $b \leq_{\mathrm{lev}} d$. By condition (4) of Claim 3.9, there is $a \in N^*$ such that $\{F(a, c) : c < b\}$ includes $\{c \in I : c \leq_{\mathrm{lev}} d\}$. So $\{F(a, c) : c < b\} \cap I$ is unbounded in I. This contradicts the last clause in the definition of ψ^*. Second, assume that the coinitiality of $N^* \setminus I$ is less than $\aleph_1$. For a similar reason as in the first case, we can find d_{E_1} such that $\{c \in N^* \setminus I : c \leq_{\mathrm{lev}} d\}$ is unbounded from below in $N^* \setminus I$. Also we can choose $a \in N^*$ and $b \in I$ such that $\{F(a, c) : c < b\}$ includes $\{c \in N^* \setminus I : c \leq_{\mathrm{lev}} d\}$. If $I \cap \{F(a, c) : c < b\}$ were bounded (from above) say by $e \in I$, then I would be definable in N by the $\mathcal{L}$-formula

$$\varphi(x, a, b, e) \overset{\mathrm{def}}{\equiv} \forall z[(e < z \wedge \exists y(y < b \wedge z = F(a, y))) \rightarrow x < z],$$

contradicting our assumption that I is not definable. So $I \cap \{F(a, c) : c < b\}$ is not bounded in I. Again this contradicts the last clause in the definition of ψ^*. So we have proven that the cofinality of $(I, N^* \setminus I)$ is $(\aleph_1, \aleph_1)$.

As in the proof of Claim 3.8, we shall define a set $\{(b_i)_{E_2} : i < \aleph_1\}$ and definable intervals $J_i \subset N^*$ $(i < \aleph_1)$ such that for each $i < \aleph_1$,

1. J_is are decreasing,
2. $b_i \in J_i$, $J_i \cap I \neq \varnothing$, $J_i \cap (N^* \setminus I) \neq \varnothing$,
3. there is no element $d \in J_i$ with $d <_{\mathrm{lev}} b_i$.

Suppose that we have chosen d_js and J_js for all $j < i$. Since the cofinality of I and the coinitiality of $N^* \setminus I$ are both $\aleph_1$, $\bigcap_{j<i} J_i$ intersects both I and $N^* \setminus I$. Choose $b \in \bigcap_{j<i} J_i \cap I$ and $c \in \bigcap_{j<i} J_i \cap (N^* \setminus I)$. Then we put $J_i = \{e \in N^* : N^* \models b < e < d\}$. Choose $b_i \in J_i$ of the minimum level. (Such b_i exists and $(b_i)_{E_2}$ is unique, because every nonempty definable subset of N^*/E_1 has the minimum element with respect to $<_{\mathrm{lev}}$. If there are two such elements, they are distinguished by elements of lower levels, contradicting the minimality.) We claim that $\{(b_i)_{E_2} : i < \aleph_1\}$ determines a branch $B = \{c_{E_2} : c_{E_2} \leq_{\mathrm{tr}} (b_i)_{E_2} \text{ for some } i\}$. For this it is sufficient to show that the b_is are linearly ordered by $\leq_{\mathrm{tr}}$. Let $i \leq i' < \aleph_1$. Then both b_i and $b_{i'}$ are members of the interval J_i. Suppose that b_i and $b_{i'}$ are not comparable with respect to $\leq_{\mathrm{tr}}$. They determine different Dedekind cuts of the elements of lower levels. So there is an element $c \in J_i$ with $c <_{\mathrm{lev}} b_i$. This contradicts our choice of $b_i \in J_i$. By our assumption (the first condition in Claim 3.9), the branch $B = \{(b_i)_{E_2} : i < \aleph_1\}$ is definable in N^*. It is easy to see that I and B are interdefinable in N^*. So I is also definable in N^*, hence I is definable in N by condition (3) in Claim 3.9. This contradicts our assumption that I is undefinable in N. □

Remark 3.11 Theorem 3.5 is a rather general statement. However, there are several related results concerning models of PA. The following are pointed out by the referee. Our model constructed in the proof of Theorem 3.5 has the property that the standard part $\mathbb{N}$ is the only semi-regular cut. (See Kirby and Paris [2] for the definition of semi-regularity.) Such property is also possessed by the models constructed in Theorem 3.14 of Kaufmann and Schmerl [1] (under $\diamondsuit$) and Theorem 2.1 of Schmerl [3] (under $\diamondsuit_{\lambda^+}$).

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Einstein Institute of Mathematics
The Hebrew University of Jerusalem
ISRAEL
and
Department of Mathematics
Hill Center-Busch Campus
Rutgers, The State University of New Jersey
Piscataway NJ 08854-8019
shelah@math.huji.ac.il

Institute of Mathematics
University of Tsukuba
Tsukuba Ibaraki 305-8571
JAPAN
tsuboi@sakura.cc.tsukuba.ac.jp