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Contractions of Closure Systems

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Abstract This essay shows that some recent work by George Weaver can be reformulated in an especially perspicuous way within the theory of closure systems. Closure theoretic generalizations of some theorems of Robert Goldblatt are presented. And, more generally, the relation between closure systems and the deducibility relations of Goldblatt is explored.

1 Introduction Pick some set $\mathfrak{D}$ to be our universe of discourse. Let $\wp(\mathfrak{D})$ be the power set of $\mathfrak{D}$. We adopt the convention that $\bigcap\varnothing = \mathfrak{D}$. If $C \subset \wp(\mathfrak{D})$, then we say that C is a *closure system* if and only if $\bigcap W \in C$ whenever $W \subset C$. (For an early discussion of closure systems, see Moore [6], pp. 59–60. For discussions from the point of view of lattice theory, universal algebra, and proof theory, respectively, see Birkhoff [1], Cohn [3], and Tarski [7].) In a typical proof theoretic application, the members of a closure system will be sets that are closed under certain inference rules. (In that case, $A \vdash \varphi$ just in case φ belongs to each member of C that contains A.) If additional inference rules are adopted, then fewer sets count as closed. So expansions of deductive systems correspond to *contractions* of closure systems. It turns out that some theorems in Weaver [9] can be recast as elementary results in the theory of such contractions.

We begin with a discussion of the relation between closure systems and the "deducibility relations" of Goldblatt [4] and Weaver [9].

Definition 1.1 If $\mathrm{Cl}:\wp(\mathfrak{D}) \to \wp(\mathfrak{D})$, then Cl is a *closure operator* just in case it satisfies the following postulates:

(K1) $\mathrm{Cl}(A) \subset \mathrm{Cl}(A \cup B)$;

(K2) $A \subset \mathrm{Cl}(A)$;

(K3) $\mathrm{Cl}(\mathrm{Cl}(A)) \subset \mathrm{Cl}(A)$.

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We say that a set A is *Cl-closed* just in case $\mathrm{Cl}(A) \subset A$. One can now prove that if Cl is a closure operator, then the Cl-closed subsets of $\mathfrak{D}$ form a closure system. Suppose, on the other hand, that C is a closure system. If $A \subset \mathfrak{D}$, we let $\mathrm{Cl}_c(A)=\bigcap\{B \in C : A \subset B\}$. Then Cl_c is a closure operator.

Definition 1.2 Now consider the following four postulates governing $\vdash$, $\neg$, and $\bot$; where $\vdash$ is a subset of $\wp(\mathfrak{D})\times\mathfrak{D}$, $\neg$ is a function on $\mathfrak{D}$ into $\mathfrak{D}$, and $\bot \in \mathfrak{D}$:

(D1) If $A \vdash x$, then $(A \cup B) \vdash x$;

(D2) If $x \in A$, then $A \vdash x$;

(D3) If $A \vdash y$ whenever $y \in B$, then $(A \cup B) \vdash x$ only if $A \vdash x$;

(D4) $A \vdash x$ if and only if $(A \cup \{\neg x\}) \vdash \bot$.

If $\vdash$, $\neg$, and $\bot$ satisfy D1, D2, D4, and all those instances of D3 in which B is finite, then $\vdash$ is a *deducibility relation* in the sense of Goldblatt and Weaver. (Weaver's formulation of D3 differs from Goldblatt's—though, as Weaver demonstrates, the two formulations are interderivable using D1 and D4. We follow Weaver here.) The relationship between closure systems and deducibility relations is somewhat obscure—unless, that is, we drop the requirement that B be finite in D3. We say that $\vdash$ is a *generalized deducibility relation* if and only if $\vdash$ satisfies D1, D2, and (the unrestricted version of) D3. Now suppose $x \in \mathrm{Cl}(A)$ if and only if $A \vdash x$. Then Cl is a closure operator if and only if $\vdash$ is a generalized deducibility relation. We can now see that, from the closure theoretic perspective, D1–D3 express the most natural notion of deducibility relation. Say that A is $\vdash$-closed just in case $\{x : A \vdash x\} \subset A$. If $\vdash$ is a generalized deducibility relation, then $\{A : A \text{ is } \vdash\text{-closed}\}$ is a closure system. And if C is a closure system, then $\{<A,x> : x \in \mathrm{Cl}_c(A)\}$ is a generalized deducibility relation.

We shall now show that the employment of a closure theoretic notion of deducibility results in an immense simplification of some important proofs. In many applications, it also allows us to dispense with D4. Since this postulate guarantees that $\neg$ behaves like a classical negation, its elimination should allow us to extend our results to a variety of interesting deviant logics. (Note for example our remark below about cyclical negations.) At the very least, this change would remove a definite impediment to such applications. Our claim, then, is; (1) that many results obtained by Weaver and Goldblatt can be reformulated and proved for generalized deducibility relations, (2) that employment of closure theoretic methods greatly simplifies the proofs, and (3) that avoidance of a commitment to classical negation promises to widen the range of applicability of the results so proved.

Suppose $\mathfrak{f} : \wp(\mathfrak{D}) \to \wp(\mathfrak{D})$. If $\mathfrak{D}$ is a set of sentences and $A \subset \mathfrak{D}$, then we might say that each member of $\mathfrak{f}(A)$ is "derivable" from A by means of the "inference rule" $\mathfrak{f}$. If $B \subset \mathfrak{D}$, then we say that B is $\mathfrak{f}$-*closed* if and only if $\mathfrak{f}(A) \subset B$ whenever $A \subset B$. If $W \subset \wp(\mathfrak{D})$, then we say that W *respects* $\mathfrak{f}$ if and only if each member of W is $\mathfrak{f}$-closed. From now on, we assume that C is a closure system. Let $C_{\mathfrak{f}} = \{X \in C : X \text{ is } \mathfrak{f}\text{-closed}\}$. Then $C_{\mathfrak{f}}$ is (obviously) the largest subset of C that respects $\mathfrak{f}$. Furthermore:

Theorem 1.3 *$C_{\mathfrak{f}}$ is a closure system.*

Proof: If $A \subset \bigcap W$ and each member of W is $\mathfrak{f}$-closed, then $\mathfrak{f}(A) \subset \bigcap W$. So $\bigcap W$ is an $\mathfrak{f}$-closed member of C whenever $W \subset C_{\mathfrak{f}}$.

$C_{\mathfrak{f}}$ is the closure system we obtain when we incorporate the "inference rule" $\mathfrak{f}$ into a deductive system with closure topology C. As we shall soon see, $C_{\mathfrak{f}}$ is the closure theoretic counterpart of Weaver's $\vdash_{\mathrm{I}}$ (see [9], p. 454). We approach $\vdash_{\mathrm{I}}$ via a further discussion of closure operators. If Cl and Cl′ are closure operators, we stipulate that $\mathrm{Cl} \precsim \mathrm{Cl}'$ if and only if $\mathrm{Cl}(A) \subset \mathrm{Cl}'(A)$ whenever $A \subset \mathfrak{D}$. We say that Cl′ *extends* Cl just in case $\mathrm{Cl} \precsim \mathrm{Cl}'$. Suppose C and C' are the closure systems associated with the closure operators Cl and Cl′, respectively. Then it is fairly easy to show that $\mathrm{Cl} \precsim \mathrm{Cl}'$ if and only if $C' \subset C$, (see Ward [8], p. 194). From now on, we assume that Cl is a closure operator. Suppose, as above, that $\mathfrak{f} : \wp(\mathfrak{D}) \rightarrow \wp(\mathfrak{D})$. We say that Cl *respects* $\mathfrak{f}$ if and only if $\mathrm{Cl}(A)$ is $\mathfrak{f}$-closed whenever $A \subset \mathfrak{D}$. Notice that if Cl is the closure operator associated with the closure system C, then Cl respects $\mathfrak{D}$ if and only if C does.

Theorem 1.4 *There is a $\precsim$-smallest closure operator that extends Cl and respects $\mathfrak{f}$.*

Proof: Let C be the closure system associated with Cl. By Theorem 1.3, we can let $\mathrm{Cl}_{\mathfrak{f}}$ be the closure operator associated with $C_{\mathfrak{f}}$. Then $\mathrm{Cl}_{\mathfrak{f}}$ respects $\mathfrak{f}$ because $C_{\mathfrak{f}}$ does. And $\mathrm{Cl}_{\mathfrak{f}}$ extends Cl because $C_{\mathfrak{f}} \subset C$. Suppose Cl′ is a closure operator that respects $\mathfrak{f}$ and extends Cl. Let C' be the associated closure system. Then C' respects $\mathfrak{f}$ and $C' \subset C$. So $C' \subset C_{\mathfrak{f}}$ and, hence, $\mathrm{Cl}_{\mathfrak{f}} \precsim \mathrm{Cl}'$.

Now suppose $\mathrm{I} \subset (\wp(\mathfrak{D}) \times \mathfrak{D})$. If $A \subset \mathfrak{D}$, we stipulate that $\mathfrak{f}(A) = \{x : \langle A, x\rangle \epsilon \mathrm{I}\}$. Let $\vdash$ be a generalized deducibility relation. And let Cl be the closure operator associated with $\vdash$. (That is, $A \vdash x$ if and only if $x \epsilon \mathrm{Cl}(A)$.) Then we say that $\vdash$ *respects* I just in case Cl respects $\mathfrak{f}$. (We depart here from the terminology of Goldblatt and Weaver. When we say that $\vdash$ respects I, they would say that each subset of $\mathfrak{D}$ respects I. For a set A respects I, in the sense of Goldblatt and Weaver, just in case $\mathrm{Cl}(A)$ is $\mathfrak{f}$-closed.) Note that $\vdash \subset \vdash'$ if and only if $\mathrm{Cl} \precsim \mathrm{Cl}'$—where Cl and Cl′ are the closure operators associated with the generalized deducibility relations $\vdash$ and $\vdash'$.

Theorem 1.5 *There is a smallest generalized deducibility relation that extends $\vdash$ and respects I. (See also Weaver [9], Lemma 2).*

Proof: Let Cl be the closure operator associated with $\vdash$. By 1.4, we can let $\mathrm{Cl}_{\mathfrak{f}}$ be the $\precsim$-smallest closure operator that extends Cl and respects $\mathfrak{f}$. If we stipulate that $A \vdash_{\mathrm{I}} x$ if and only if $x \epsilon \mathrm{Cl}_{\mathfrak{f}}(A)$, then $\vdash_{\mathrm{I}}$ is the smallest generalized deducibility relation that extends $\vdash$ and respects I.

The principal interest of Theorem 1.5 is not its guarantee that $\vdash_{\mathrm{I}}$ exists. Weaver already showed that in a more straightforward way. It is more significant that the proof of Theorem 1.5 indicates the closure system associated with $\vdash_{\mathrm{I}}$. If C is the closure system associated with $\vdash$ and $\mathfrak{f}$ is defined as above, then $C_{\mathfrak{f}}$ is the closure system associated with $\vdash_{\mathrm{I}}$. That is, $\{A : A \text{ is } \vdash_{\mathrm{I}}\text{-closed}\} = C_{\mathfrak{f}}$. And $A \vdash_{\mathrm{I}} x$ if and only if $x \epsilon \bigcap \{B \epsilon C_{\mathfrak{f}} : A \subset B\}$. (To see this, just recall that $\mathrm{Cl}_{\mathfrak{f}}$ is the closure operator associated with $C_{\mathfrak{f}}$, while $\vdash_{\mathrm{I}}$ is, by our construction, the generalized deducibility relation associated with $\mathrm{Cl}_{\mathfrak{f}}$.)

We now turn to analogues of theorems that Weaver proves with the help of a complicated transfinite construction. No such construction is necessary within closure theory. As promised, we also dispense with the classical negation operator of D4. Suppose that $W \subset \wp(\mathfrak{D})$. If $A \subset \mathfrak{D}$, then we say A is *W-consistent* if and only if some member of $W\backslash\{\mathfrak{D}\}$ contains *A*. In a proof theoretic application, the idea would be that *A* is contained in a deductively closed set distinct from the set of all sentences (that is, *A* is proof theoretically consistent). A *W*-consistent set is *maximally W-consistent* if and only if none of its proper supersets are *W*-consistent.

Theorem 1.6 *A set is $C_{\mathfrak{f}}$-consistent if and only if it is contained in a C-consistent, $\mathfrak{f}$-closed member of C. (See also Weaver [9], Lemma 7.)*

Proof: A set belongs to $C_{\mathfrak{f}}\backslash\{\mathfrak{D}\}$ if and only if it is an $\mathfrak{f}$-closed member of $C\backslash\{\mathfrak{D}\}$.

Theorem 1.7 *Each maximally W-consistent set belongs to W.*

Proof: Suppose *A* is maximally *W*-consistent. Let *B* be a member of $W\backslash\{\mathfrak{D}\}$ that contains *A*. Then *B* is *W*-consistent and, hence, *A*=*B*.

Theorem 1.8 *Each maximally C-consistent, $\mathfrak{f}$-closed set is also maximally $C_{\mathfrak{f}}$-consistent. (See Weaver [9], Theorem 5.1, right-to-left.)*

Proof: Suppose *A* is $\mathfrak{f}$-closed and maximally *C*-consistent. Then, by Theorems 1.6 and 1.7, A is $C_{\mathfrak{f}}$-consistent. Let *B* be a member of $C_{\mathfrak{f}}\backslash\{\mathfrak{D}\}$ that contains *A*. Then *B* is *C*-consistent (since $C_{\mathfrak{f}}\subset C$) and, hence, *A*=*B*. We conclude that no proper superset of *A* belongs to $C_{\mathfrak{f}}\backslash\{\mathfrak{D}\}$.

As our references to Weaver indicate, the preceding theorems are not entirely novel. We do not claim to be breaking completely new ground. We mean, rather, for the reader to contrast the triviality of our derivations with the Herculean labors expended by Weaver in his proofs of analogous results. This may help to explain our own preference for a closure theoretic approach to deducibility relations. Perhaps the simplifying power of closure theory is clearest in the case of Weaver's Lemma 6—which we now restate as a theorem about generalized deducibility relations. Suppose $\mathrm{I}\subset(\wp(\mathfrak{D})\times\mathfrak{D})$. As before let $\mathfrak{f}(A)=\{x : \langle A,x\rangle\in\mathrm{I}\}$ whenever $A\subset\mathfrak{D}$. Finally, let $\vdash$ be a generalized deducibility relation, let *C* be the closure system associated with $\vdash$, and let $\vdash_{\mathrm{I}}$ be the extension of $\vdash$ guaranteed by Theorem 1.5.

Theorem 1.9 *If A is maximally C-consistent and $\mathfrak{f}$-closed, then $A\vdash x$ if and only if $A\vdash_{I}x$.*

Proof: Suppose *A* is maximally *C*-consistent and $\mathfrak{f}$-closed. Then, by Theorem 1.7, $A\epsilon C_{\mathfrak{f}}$. Suppose $A\vdash_{\mathrm{I}}x$. Then $x\epsilon\bigcap\{B\epsilon C_{\mathfrak{f}} : A\subset B\}$ and, hence, $x\epsilon A$. So, by D2, $A\vdash x$. On the other hand, since $\vdash\subset\vdash_{\mathrm{I}}$, $A\vdash x$ only if $A\vdash_{\mathrm{I}}x$.

Weaver's Lemma 6 is a sophisticated result in the theory of deducibility relations. Its proof could scarcely be described as trivial. So it is striking that the corresponding result for *generalized* deducibility relations (our Theorem 1.9) is quite elementary. It is all the more striking when one recalls that the move to generalized deducibility relations involves not just the strengthening of Weaver's D3, but the *omission* of D4—that is, the omission of a classical negation operator. This allows the theory of generalized deducibility relations (or, as we prefer to emphasize, the theory of closure systems) to accommodate a common form of logical deviance (i.e., the employment of a non-classical negation).

Clearly, a theorem that characterizes every deducibility relation also characterizes every closure system with a classical negation. And, in some cases, classical negation is indispensable. (That is, some theorems about deducibility relations are true only of those generalized deducibility relations that satisfy D4.) On the other hand, we have just given examples of properties of deducibility relations that apply to all closure systems—even those in which no classical negation is definable. In these cases, neither classical negation nor any expressive capacity similar to classical negation is required. It is natural to wonder whether there are logically significant properties of deducibility relations that, while not applying to all closure systems, do apply to closure systems that fall somewhere between closure-systems-in-general and closure-systems-with-a- classical-negation. Properties of just this sort are explored in Goldblatt [4]—as we shall see after some preliminary definitions and theorems.

Suppose, as above, that A and B are both subsets of $\mathfrak{D}$. Then we say that B *W-decides $\mathfrak{f}$ at A* if and only if either $\mathfrak{f}(A) \subset B$ or $(A \cup B)$ is not W-consistent. B *W-decides $\mathfrak{f}$* (simpliciter) if and only if B W-decides $\mathfrak{f}$ at each subset of $\mathfrak{D}$.

Theorem 1.10 *Each W-consistent set that W-decides $\mathfrak{f}$ is $\mathfrak{f}$-closed.*

Proof: If B W-decides $\mathfrak{f}$ and $A \subset B$, then either $\mathfrak{f}(A) \subset B$ or B is not W-consistent.

We say that W has *Lindenbaum's property* if and only if each W-consistent subset of $\mathfrak{D}$ is contained in a maximally W-consistent subset of $\mathfrak{D}$. A subset of $\mathfrak{D}$ is *finitely W-consistent* if and only if each of its finite subsets is W-consistent. W is *compact* if and only if each finitely W-consistent subset of $\mathfrak{D}$ is W-consistent. (It is easy to see that W-consistent sets are always finitely W-consistent.) It may seem perverse to employ the model theoretic term "compact" in a setting that invites proof theoretic applications of closure systems. Note, however, that closure systems are not intrinsically proof theoretic structures. Sets closed under a model theoretic consequence relation typically form a closure system—and, in such applications, our definition supplies "compact" with its usual model theoretic sense. Furthermore, we are motivated here by a conception of compactness that is common in lattice theory. Let C be, as usual, a closure system. Say that W is a *cover* of A just in case $A \subset \mathrm{Cl}_c(\bigcup W)$. A finite cover, then, is a cover with only finitely many members. Lattice theorists (following the practice in real number analysis) say that a subset of $\mathfrak{D}$ is *compact* if and only if each of its covers contains a finite cover (see Birkhoff [1], p. 186). We find our notion of compactness suggestive because one can show that C is compact (in our sense) if and only if $\mathfrak{D}$ is compact (in the lattice theoretic sense).

Our treatment of the term "compact" is essentially the same as Goldblatt's use of the term "finitary" (see Goldblatt [4], p. 37). Weaver, on the other hand, uses "finitary" in much the same way as Birkhoff (see Birkhoff [1], p. 185.) A closure system C is *finitary*, in this sense, just in case $x \in \mathrm{Cl}_c(A)$ for some finite subset A of B whenever $x \in \mathrm{Cl}_c(B)$. As it turns out, the finitary deducibility relations are precisely the compact ones. In the case of closure systems, the connection between the two notions is only slightly more complicated. Each compact closure system with a classical negation is finitary. And if some finite subset of $\mathfrak{D}$ is not C-consistent, then C is finitary only if C is compact. (Note, incidentally, that Cleave uses "compact" in the way that Birkhoff and Weaver use "finitary"! See his [2], p. 76.) We turn now to a fundamental result concerning compact sets.

Theorem 1.11 *Each compact subset of $\wp(\mathfrak{D})$ has Lindenbaum's property.*

Proof: If V consists of finitely W-consistent subsets of $\mathfrak{D}$ linearly ordered by containment, then $\bigcup V$ is finitely W-consistent. So, by Zorn's Lemma, if A is finitely W-consistent, then there is a largest finitely W-consistent set that contains A. The theorem follows since, in a compact subset of $\wp(\mathfrak{D})$, there is no distinction between *W*-consistent and finitely *W*-consistent sets (see also Goldblatt [4], p. 37).

We are now in a position to discuss two logically significant types of closure system with features similar to those provided by classical negation. We say that *W* is *expressive* if and only if, for each *A* in $W \backslash \{\mathfrak{D}\}$ and each *B* that properly contains *A*, there is a subset *D* of $\mathfrak{D}$ such that $(A \cup D)$ is *W*-consistent, but $(B \cup D)$ is not (that is, *A* has a consistent extension that is incompatible with *B*). *W* is *strongly expressive* if and only if, for each *A* and *B* as above, there is a finite *D* of the indicated sort. Note that each compact, expressive subset of $\wp(\mathfrak{D})$ is strongly expressive. It is also easy to prove that each closure system with a classical negation is strongly expressive. The converse, however, is false. (To confirm this, consider the closure system whose members are the filters of the lattice M_5.) So expressiveness is a generalization of the classical notion of negation.

Expressiveness has a substantial claim on our interest. It is the dual of a notion that has received attention (albeit scant) from lattice theorists, (see Maeda & Maeda [5], Lemma 7.2.) More importantly, it figures prominently in some powerful results in the general theory of closure systems. (One can prove, for example, that each *C*-consistent member of a compact, expressive closure system *C* is the intersection of maximally *C*-consistent sets.) Finally, some significant deviant logics (logics with "cyclical negations," for example) have expressive formalizations that preserve designated values. So it is not frivolous to extend some of Goldblatt's results to expressive or strongly expressive closure systems.

Theorem 1.12 *If C is strongly expressive, B is C-consistent,* $\mathfrak{f}(A)$ *is finite, and* $A \subset Cl_c(B)$ *only if* $\mathfrak{f}(A) \subset Cl_c(B)$*, then B has a C-consistent, finite extension that C-decides* $\mathfrak{f}$ *at A. (See Goldblatt [4], Lemma 1.4).*

Proof: Assume the hypotheses of the theorem. Recall that $\mathrm{Cl}_c(A) = \bigcap\{B \epsilon C : A \subset B\}$. (That is, $\mathrm{Cl}_c(A)$ is the smallest member of *C* that contains *A*.) If $A \subset \mathrm{Cl}_c(B)$, then $(B \cup \mathfrak{f}(A))$ is a *C*-consistent, finite extension of *B* that *C*-decides $\mathfrak{f}$ at *A*. Suppose $\mathrm{Cl}_c(B)$ does not contain *A*. If $\mathrm{Cl}_c(A \cup B)$ is not *C*-consistent, then neither is $(A \cup B)$ and, hence, *B* itself *C*-decides $\mathfrak{f}$ at *A*. Suppose, on the other hand, that $\mathrm{Cl}_c(A \cup B)$ is *C*-consistent. Then we can pick a finite *D* such that $(\mathrm{Cl}_c(B) \cup D)$ is *C*-consistent, but $(\mathrm{Cl}_c(A \cup B) \cup D)$ is not. It follows that $(B \cup D)$ is *C*-consistent, but $(A \cup (B \cup D))$ is not. So $(B \cup D)$ is a *C*-consistent, finite extension of *B* that *C*-decides $\mathfrak{f}$ at *A*.

We say that $\mathfrak{f}$ is *countable* if and only if $\mathfrak{f}$ assigns a finite set to each subset of $\mathfrak{D}$ and assigns $\varnothing$ to all but countably many subsets of $\mathfrak{D}$.

Theorem 1.13 *If C is compact and expressive, B is C-consistent,* $\mathfrak{f}$ *is countable, and* $Cl_c(D)$ *is* $\mathfrak{f}$*-closed whenever D is a finite extension of B, then B has a maximally* $C_{\mathfrak{f}}$*-consistent extension.*

Proof: Assume the hypotheses of the theorem. Let $\{A_n : n \epsilon \omega\}$ be an enumeration of the sets to which $\mathfrak{f}$ assigns a nonempty subset of $\mathfrak{D}$. Furthermore, let $B_0 = B$; let B_{n+1} be a *C*-consistent, finite extension of B_n that *C*-decides $\mathfrak{f}$ at A_n (as guaranteed by Theorem 1.12); and let $B_\omega = \bigcup\{B_n : n \epsilon \omega\}$. Then B_ω is a *C*-consistent extension of *B* that *C*-decides $\mathfrak{f}$. By Theorem 1.11, we can let B^* be a maximally *C*-consistent extension of *B* that *C*-decides $\mathfrak{f}$. Then, by Theorems 1.8 and 1.10, B^* is maximally $C_{\mathfrak{f}}$-consistent. (See also Goldblatt, [4], pp. 38–39).

For an indication of how Theorem 1.13 might be applied within proof theory, see Weaver's discussion of his Corollary 2 in his [9], p. 454. Weaver's deducibility relation $\vdash^1$ is finitary and, hence, is a generalized deducibility relation. So it is routine to give a closure theoretic reformulation of his remarks. Furthermore, if C^1 is the closure system associated with $\vdash^1$, then C^1 is both compact and expressive. So our Theorem 1.13 can do the work of Weaver's Corollary 1.

Since Goldblatt and Weaver only require that their deducibility relations satisfy those instances of D3 in which *B* is finite, their approach does have a particular kind of generality that the closure theoretic approach lacks. Nonetheless, under a wide variety of conceptions of proof, deductively closed sets form closure systems. Say, for example, that $\langle \pi, \leq \rangle$ is an $\mathfrak{f}$-proof of *x* from *A* if and only if;

(I) π is a subset of $\mathfrak{D}$ partially ordered by $\leq$;

(II) given any *y* in $\mathfrak{D}$, $y \epsilon \pi$ if and only if $y \leq x$;

(III) given any *y* in π, $\{z \epsilon \pi : z \leq y\}$ is well-ordered by $\leq$;

(IV) given any *y* in π, either $y \epsilon A$ or $y \epsilon \mathfrak{f}(D)$ for some subset *D* of $\{z \epsilon \pi : z < y\}$.

According to this account, a proof of conclusion *x* from premise-set *A* is a tree whose greatest element is *x* and all of whose elements are members of *A* or are derivable from prior elements by inference rule $\mathfrak{f}$. Now say that $A \vdash x$ just in case there is an $\mathfrak{f}$-proof of *x* from *A*. Then it is an amusing exercise to confirm that $\vdash$ is a generalized deducibility relation. So $\{A : A \text{is} \vdash -\text{closed}\}$ is a closure system. The above notion of proof (and, hence, the closure theoretic approach itself) can accommodate all sorts of curious features. To cite just one, $\mathfrak{f}(D)$ might be non-empty only when *D* is infinite. And postulates (I) through (IV) themselves express just one of the expansive conceptions of proof that can be explored closure theoretically.

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