

The Complexity of Revision

GIAN ALDO ANTONELLI

Abstract In this paper we show that the Gupta-Belnap systems $\mathbf{S}^\#$ and \mathbf{S}^* are Π_2^1 . Since Kremer has independently established that they are Π_2^1 -hard, this completely settles the problem of their complexity. The above-mentioned upper bound is established through a reduction to countable revision sequences that is inspired by, and makes use of a construction of McGee.

1 Introduction In his [1], Kremer provides an answer to a question raised by Gupta and Belnap in their [2]:

Is a [sound and] complete calculus for $\mathbf{S}^\#$ possible? If not, what is the complexity of the theorems of $\mathbf{S}^\#$ relative to that of \mathcal{D} ? (p. 185)

Kremer settles the first part of the question and provides a partial answer to the second one by showing, *first*, that the set of (Gödel numbers of) arithmetical truths is reducible to the set of (Gödel numbers of) $\mathbf{S}^\#$ or \mathbf{S}^* validities, which is therefore *at least* Δ_1^1 . This part is established by showing that for any arithmetical sentence φ one can uniformly find a sentence ψ that is a theorem of $\mathbf{S}^\#$ (or \mathbf{S}^*) if and only if φ is true in the standard model of arithmetic. This fact is *then* used to characterize all Π_1^1 subsets of the natural numbers. This allows one to represent, in $\mathbf{S}^\#$ (or \mathbf{S}^*), the relation $D \models_i \varphi$, (whose definition is given below). Since this relation is Π_2^1 -complete, it follows that the complexity of $\mathbf{S}^\#$ (or \mathbf{S}^*) is at least Π_2^1 .

This paper shows that this lower bound is optimal, in that the (Gödel numbers of) (\mathbf{S}^* - or) $\mathbf{S}^\#$ -validities can be written in Π_2^1 form.

2 The Lower Bound In this section, for the sake of completeness, we are going briefly to rehearse the argument given by Kremer establishing a Π_2^1 lower bound for the complexity of $\mathbf{S}^\#$ and \mathbf{S}^* .

We first define \models_i , the notion of validity mentioned above. Fix a countable first-order language \mathcal{L} , and let \mathcal{L}^+ be obtained by expanding \mathcal{L} by a new predicate constant G . Let D comprise the definition $G \stackrel{\text{def}}{\iff} A(G)$, where $A(G)$ is a formula in which G occurs positively. Then we say that $D \models_i \varphi$ if and only if for every model

Received September 30, 1993; revised January 4, 1994

\mathcal{M} , the sentence φ is true in \mathcal{M} when G is interpreted over the least fixed point of the inductive definition D over \mathcal{M} . This can be easily extended to the case in which the language is expanded by finitely many new predicate constants and D comprises finitely many definitions.

The set of (codes of) sentences φ such that $D \models_i \varphi$ is Π_2^1 -complete, in the sense that for any D it can be written in Π_2^1 form, and it is Π_2^1 -hard for at least some D 's.¹ We first show that this set can be written in Π_2^1 -form. Observe that “ Y is a fixed point” can be expressed arithmetically:

$$\forall x(x \in Y \leftrightarrow A(x, Y)).$$

However, the property of being the *least* fixed point is Π_1^1 :

$$\forall Z(\forall x(x \in Z \leftrightarrow A(x, Z)) \Rightarrow Z \subseteq Y).$$

Moreover, it is easy to convince oneself that the following properties are all Δ_1^1 (in X and Y) once the syntax of \mathcal{L} has been arithmetized (in fact, the first two properties below can be expressed arithmetically in X and Y):

- X is a model for \mathcal{L} ;
- X assigns Y as the interpretation of G ;
- φ is true in X .

Therefore, the following sentence is indeed Π_2^1 : “For all X , for all Y : if X is a model for \mathcal{L} and Y is the least fixed point of $A(G)$ over X and X assigns Y to G , then φ is true in X .”

It only requires a bit more work to show that for any Π_2^1 set S there is a system of definitions D such that S is many-one reducible to $\{\varphi : D \models_i \varphi\}$. Let S be Π_2^1 . This means that for an arithmetical predicate R and a set X of natural numbers, we have

$$i \notin S \iff \forall Y \neg R(i, X, Y).$$

Let $\mathcal{N}[X]$ be the standard model of arithmetic expanded by X . We know that a set is $\Pi_1^1[X]$ if and only if it is many-one reducible to a set P that is inductively definable over $\mathcal{N}[X]$. So let Γ_X inductively define such a P for

$$\{i : \forall Y \neg R(i, X, Y)\}.$$

If $\overline{\Gamma_X}$ is the least fixed point of Γ_X over $\mathcal{N}[X]$, then there is a total recursive function f such that

$$i \notin S \iff f(i) \in \overline{\Gamma_X}.$$

As shown in Kremer [1], let F_1 be the inductive definition fixing the extension of a new predicate constant G_1 to be isomorphic to the natural numbers, and let F_2 be the inductive definition fixing the extension of a new predicate constant G_2 to be the isomorphic image of $\overline{\Gamma_X}$ in (the extension of) G_1 . For each natural number i let A^i be the sentence “ $\forall x G_1(x) \rightarrow \neg G_2(i)$.” Then we have

$$i \in \overline{\Gamma_X} \iff F_1, F_2 \not\models_i A^i[G_1, G_2].$$

Hence, as promised,

$$i \in S \iff F_1, F_2 \models_i A^{f(i)}[G_1, G_2].$$

It only remains to observe that the systems \mathbf{S}^* and $\mathbf{S}^\#$ can mimick any inductive definition over \mathcal{N} , as shown in [1] (to which the reader is referred for the details). The idea is first to mimick the inductive definition fixing the extension of a predicate P to be isomorphic to the natural numbers, and then mimick inductive definitions over P .

3 Reduction to Countable Sequences In this section, we are going to show that in a sense we can dispense with revision sequences of length unbounded in \mathcal{O} , the collection of all ordinals, and only look at revision sequences of countable length. In order to show this, we will crucially employ ideas from a theorem due to McGee (see his [4], p. 135, and also Gupta & Belnap [2], p. 176). We first restrict our attention to the system \mathbf{S}^* , and deal with $\mathbf{S}^\#$ later.

Definition 3.1 We say that φ is valid in \mathbf{S}^* if and only if φ is true in every model $\mathcal{M} + h$, where h is reflexive in some revision sequence for $\delta_{\mathcal{M}, D}$.²

Theorem 3.2 Let h be a reflexive hypothesis of a revision sequence \mathcal{S} for $\delta_{\mathcal{M}, D}$, and let $\kappa = \max(|\mathcal{M}|, \aleph_0)$. Then h is α -reflexive for some $\alpha < \kappa^+$, i.e., $|\alpha| = \kappa$.

The import of this theorem, due to McGee (see [2], p. 176), is that if we want to find out whether a sentence φ is validated by some model $\mathcal{M} + h$, where h is recurring for \mathcal{M} , we only need to look at revision sequences of length bounded by κ . The theorem can indeed be strengthened (as shown in [4]) by taking κ to be cardinality of the language (which in our case is assumed to be countable). Our main purpose in this section is to establish the following result.

Theorem 3.3 A sentence φ is valid in \mathbf{S}^* if and only for any hypothesis h that is reflexive in a countable revision sequence \mathcal{S} relative to a countable model \mathcal{M} , φ is true in $\mathcal{M} + h$.

Proof. For the “only if” direction of the theorem it suffices to show that if φ is false in a countable model $\mathcal{M} + h$, where h is reflexive in a countable revision sequence, it is not valid in \mathbf{S}^* . We use a construction due to McGee (see [4], p. 135).

So assume the antecedent, and let \mathcal{S} be a countable revision sequence in which h is reflexive. Let β be the length of \mathcal{S} . Since any ordinal can be written in the form $(\beta \cdot \gamma) + \delta$, with $\delta < \beta$, we can define a revision sequence \mathcal{T} by setting for any ordinal γ ,

$$\mathcal{T}_{(\beta \cdot \gamma) + \delta} = \mathcal{S}_\delta.$$

Now h is reflexive in \mathcal{T} , which shows that φ is not valid in \mathbf{S}^* .

Now for the “if” direction. Let φ be a sentence, and assume that φ is true in $\mathcal{N} + h$, where h is any hypothesis that is reflexive in some countable revision sequence relative to a countable model \mathcal{N} . Let \mathcal{M} be any model, \mathcal{S} a revision sequence for \mathcal{M} , and h a hypothesis that is reflexive in \mathcal{S} . We need to show that φ is true in $\mathcal{M} + h$.

Assume for contradiction that φ is false in $\mathcal{M} + h$. We will show that there is a countable model \mathcal{M}' , a countable revision sequence \mathcal{S}' for \mathcal{M}' and a hypothesis h' that is reflexive for \mathcal{S}' such that φ is false in $\mathcal{M}' + h'$.

By the downward Löwenheim-Skolem, let $\mathcal{M}' + h'$ be a countable elementary submodel of $\mathcal{M} + h$ (where $h = \mathcal{S}_0$). Observe that also \mathcal{M}' is an elementary submodel of \mathcal{M} (relative to the original language \mathcal{L}). We define a new revision sequence \mathcal{S}' by

setting $\mathcal{S}'_0 = h'$, and $\mathcal{S}'_{\alpha+1} = \delta_{\mathcal{M}', D}(\mathcal{S}'_\alpha)$. In order to define \mathcal{S}' at limit stages, for limit ordinals λ (and in particular for limit ordinals $< \omega_1$) we set:

$$\mathcal{S}'_\lambda = \{d \in \mathcal{M}' : d \in \mathcal{S}_\lambda\}.$$

This definition makes sense since $\mathcal{M}' \subseteq \mathcal{M}$.

We need to establish (i) that \mathcal{S}' is a revision sequence; (ii) that h' is recurring for \mathcal{S}' ; and (iii) that φ is false in $\mathcal{M}' + h'$. In order to do so, we will first prove the following auxiliary results.

Lemma 3.4 $\mathcal{M}' + \mathcal{S}'_\alpha$ is an elementary submodel of $\mathcal{M} + \mathcal{S}_\alpha$, for each ordinal α , and in particular for $\alpha < \omega_1$.

Proof. We proceed by induction on α . We already know that $\mathcal{M}' \prec \mathcal{M}$. The lemma holds by definition for $\alpha = 0$. We now deal with the case $\alpha + 1$. We first argue that $\mathcal{S}'_{\alpha+1}$ and $\mathcal{S}_{\alpha+1}$ agree on \mathcal{M} :

$$\begin{aligned} d \in \mathcal{S}'_{\alpha+1} &\iff d \in \delta_{\mathcal{M}', D}(\mathcal{S}'_\alpha) \\ &\iff \mathcal{M}' + \mathcal{S}'_\alpha, d \models A(G) \\ &\iff \mathcal{M} + \mathcal{S}_\alpha, d \models A(G) \\ &\iff d \in \delta_{\mathcal{M}, D}(\mathcal{S}_\alpha) \\ &\iff d \in \mathcal{S}'_{\alpha+1}. \end{aligned}$$

In particular, this shows that $\mathcal{M}' + \mathcal{S}'_{\alpha+1}$ is a submodel of $\mathcal{M} + \mathcal{S}_{\alpha+1}$. An easy induction on the complexity of formulas $\varphi(x_1, \dots, x_n)$ (which is left to the reader) now shows that for d_1, \dots, d_n in \mathcal{M}' ,

$$\mathcal{M}' + \mathcal{S}'_{\alpha+1} \models \varphi[d_1, \dots, d_n] \iff \mathcal{M} + \mathcal{S}_{\alpha+1} \models \varphi[d_1, \dots, d_n],$$

as required. We now deal with the limit case. Let λ be a limit ordinal. By definition of \mathcal{S}' we already know that $\mathcal{M}' + \mathcal{S}'_\lambda$ is a submodel of $\mathcal{M} + \mathcal{S}_\lambda$, when λ is a limit ordinal. In order to show that it is an elementary submodel we need to verify the condition on $\varphi(x_1, \dots, x_n)$ above. Proceeding by induction on the complexity of φ , the only non-trivial case is for the new constant G , which is interpreted on \mathcal{S}'_λ . But then we only need to notice that by definition of \mathcal{S}' , as required, \mathcal{S}'_λ is the restriction of \mathcal{S}_λ to the universe of \mathcal{M}' .

Lemma 3.5 \mathcal{S}' is a revision sequence.

Proof. We need to show that for each limit ordinal λ , \mathcal{S}'_λ coheres with $\{\mathcal{S}'_\alpha : \alpha < \lambda\}$, in the sense that if some $d \in \mathcal{M}'$ is stably in \mathcal{S}'_α as α approaches λ , then it is in \mathcal{S}'_λ .

Using the previous lemma we have:

$$\begin{aligned} d \in \mathcal{S}'_\lambda &\iff d \in \mathcal{S}_\lambda \\ &\iff d \in \bigcup_{\alpha < \lambda} \bigcap_{\alpha < \beta < \lambda} \mathcal{S}_\beta \\ &\iff d \in \bigcup_{\alpha < \lambda} \bigcap_{\alpha < \beta < \lambda} \mathcal{S}'_\beta, \end{aligned}$$

which establishes the lemma.

Now we go back to the proof of theorem 3.3. By lemma 3.4, we know that each S'_α is the restriction of S_α to the universe of \mathcal{M}' . It follows that if $h = S_0$ is reflexive for \mathcal{S} , then $h' = S'_0$ is reflexive for \mathcal{M}' . But \mathcal{M}' is countable, so by McGee's theorem h' is reflexive for some countable sequence \mathcal{T} for \mathcal{M}' . Moreover, $\mathcal{M}' + h'$ is elementarily equivalent to $\mathcal{M} + h$, so that φ is false in $\mathcal{M}' + h'$. This completes the proof of the theorem.

In order to see that this section establishes a reduction to countable sequences also for $\mathbf{S}^\#$, recall the definition:

Definition 3.6 We say that φ is valid in $\mathbf{S}^\#$ if and only if for all hypotheses h that are reflexive in some revision sequence for $\delta_{\mathcal{M},D}$, there is n such that for all $p \geq n$, φ is true the model

$$\mathcal{M} + \underbrace{\delta_{\mathcal{M},D} \cdots \delta_{\mathcal{M},D}}_{p \text{ times}}(h).$$

We now notice that the construction of the theorem can be easily modified to apply to $\mathbf{S}^\#$. Let \mathcal{M} , \mathcal{M}' , h and h' be as in the theorem. Then, as we already know, $\mathcal{M}' + \delta_{\mathcal{M}',D}(S'_\alpha)$ is an elementary submodel of $\mathcal{M} + \delta_{\mathcal{M},D}(S_\alpha)$. This shows that for every n there is $p \geq n$ such that φ is false in the model

$$\mathcal{M} + \underbrace{\delta_{\mathcal{M},D} \cdots \delta_{\mathcal{M},D}}_{p \text{ times}}(h),$$

as required.

4 The Upper Bound Given the reduction of the previous section, we now show how to write \mathbf{S}^* in Π_2^1 -form. This will be accomplished by writing out formally a long series of definitions and by some simple quantifier manipulations. Leaving much of the work to the reader, we will point out how this can be accomplished.

Let α be a function from pairs of natural numbers into $\{0, 1\}$. It is easy to see that “ α codes a linear ordering” can be written out arithmetically in α . However, as is well known, the condition on well-orderings cannot be so expressed, as it requires a second-order universal quantifier in an essential way. We need to say that there are no infinite descending sequences; writing “ $x <_\alpha y$ ” for “ x precedes y in the well-ordering (coded by) α .”

$$\forall f[\forall n(f(n+1) <_\alpha f(n)) \rightarrow \exists n(f(n) = f(n+1))].$$

Let us write $W(\alpha)$ for “ α is a well-ordering of the natural numbers,” i.e., “ α is a countable ordinal.” (For this and other recursion-theoretic details see for instance [3].)

Likewise, we can code a model by means of a function over the natural numbers satisfying certain conditions, and in turn we can code a countable sequence of models by means of such a function satisfying certain other conditions. These restrictions can be expressed arithmetically in the corresponding second-order variables. We will write, e.g., $\forall \mathcal{M}$ or $\exists \mathcal{S}$ as second-order quantifiers with the above-mentioned restrictions.

We introduce predicates $\text{Rev}(\mathcal{M}, \mathcal{S}, D, \alpha)$ and $\text{Ref}(h, \mathcal{S}, \alpha)$ meaning, respectively, “ \mathcal{S} is a revision sequence for $\delta_{\mathcal{M},D}$ of length α ” and “ h is reflexive for \mathcal{S} .” Predicates Rev and Ref can be written arithmetically in their parameters. (We need

α as an explicit parameter in **Ref** because we want to say $\mathcal{S}_0 = h$ and for some $\beta < \alpha$, also $\mathcal{S}_\beta = h$.)

Finally, using the reduction to countable sequences given in the previous section, we see that “ φ is valid for **S***” can be written in Π_2^1 form as follows:

$$\forall \mathcal{M} \forall h \forall \mathcal{S} \forall \alpha [W(\alpha) \wedge \mathbf{Rev}(\mathcal{M}, \mathcal{S}, D, \alpha) \wedge \mathbf{Ref}(h, \mathcal{S}, \alpha) \implies \mathcal{M} + h \models \varphi].$$

Finally, it is again easy to see how to modify the above argument for **S[#]**. Since the “jump” $\delta_{\mathcal{M}, D}$ can be expressed in Δ_1^1 form, we can write “ φ is valid in **S[#]**” in Π_2^1 form by replacing the consequent $\mathcal{M} + h \models \varphi$ of the above sentence by

$$\exists n (\forall p \geq n) \mathcal{M} + \delta_{\mathcal{M}, D}^p(h) \models \varphi,$$

where $\delta_{\mathcal{M}, D}^p(h)$ is $\underbrace{\delta_{\mathcal{M}, D} \cdots \delta_{\mathcal{M}, D}}_{p \text{ times}}(h)$.

Acknowledgment The result reported upon in this paper was also established, independently and about at the same time, by Vann McGee, who generously declined joint authorship, but deserves equal credit. The author is grateful to the Department of Philosophy at the University of Pittsburgh for their hospitality at the time the present project was started.

NOTES

1. Kremer [1] mentions that a proof for this was sketched by Yiannis Moschovakis in private correspondence.
2. The original definition of [2] is formulated with respect to *recurring* hypotheses, i.e., hypotheses that occur cofinally in some revision sequence. By theorem **5C.13**, p. 174 of [2], these are the same as the reflexive hypotheses.

REFERENCES

- [1] Kremer, P., “The Gupta-Belnap systems **S[#]** and **S*** are not Axiomatisable,” *The Notre Dame Journal of Formal Logic*, vol. 34 (1993), pp. 583–596.
- [2] Belnap, N., and A. Gupta, *The Revision Theory of Truth*, MIT Press, Cambridge, MA, 1993.
- [3] Hinman, P., *Recursion-Theoretic Hierarchies*, Springer-Verlag, Berlin, 1978.
- [4] McGee, V., *Truth, Vagueness, and Paradox: An Essay on the Logic of Truth*, Hackett, Indianapolis, 1991.

Department of Philosophy
Yale University
New Haven, CT 06520–8306
e-mail: antonelli-aldo@cs.yale.edu