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Normal Forms in Combinatory Logic

PATRICIA JOHANN

Abstract Let R be a convergent term rewriting system, and let CR-equality on (simply typed) combinatory logic terms be the equality induced by $\beta\eta R$-equality on terms of the (simply typed) lambda calculus under any of the standard translations between these two frameworks for higher-order reasoning. We generalize the classical notion of strong reduction to a reduction relation which generates CR-equality and whose irreducibles are exactly the translates of long βR-normal forms. The classical notion of strong normal form in combinatory logic is also generalized, yielding yet another description of these translates. Their resulting tripartite characterization extends to the combined first-order algebraic and higher-order setting the classical combinatory logic descriptions of the translates of long β-normal forms in the lambda calculus. As a consequence, the translates of long βR-normal forms are easily seen to serve as canonical representatives for CR-equivalence classes of combinatory logic terms for nonempty, as well as for empty, R.

1 Introduction The interaction between higher-order and first-order algebraic reasoning has recently received much attention (see Breazu-Tannen [2], [3], and [4], and Dougherty [8]), particularly the situation in which the equational theory in question admits presentation as a convergent (confluent and terminating) term rewriting system R. Reasoning in theories combining rewriting with higher-order logic is typically described in terms of $\beta\eta R$-equality on the simply typed lambda calculus ($\mathcal{LC}$). But since simply typed combinatory logic ($C\mathcal{L}$) provides an algebraic formalization of higher-order reasoning, it is sometimes convenient to study $\beta\eta R$-equality on $\mathcal{LC}$ by examining the equality induced on $C\mathcal{L}$ under any of the standard effective translations between the terms of $\mathcal{LC}$ and $C\mathcal{L}$. We call this induced equality *extensional combinatory R-equality*, or *CR-equality* for short; in the special case when R is empty, we refer to *extensional combinatory equality* or *C-equality*.

The investigation of $\beta\eta$-equality is facilitated by the existence of a notion of reduction on $\mathcal{LC}$-terms which captures it precisely. Indeed, the fact that $\beta\eta$-reduction is convergent on $\mathcal{LC}$ guarantees that the $\beta\eta$-irreducibles comprise a class of canonical

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representatives for the $\beta\eta$-equivalence classes of $\mathcal{LC}$-terms. These "normal forms" provide a tool for proving the consistency of $\beta\eta$-equality on $\mathcal{LC}$, as well as for establishing other inequalities between $\mathcal{LC}$-terms, and computing the normal form for a $\mathcal{LC}$-term reflects the process of evaluating a function at an argument. Of course, $\beta\eta$-equality is generated by $\beta\eta^{-1}$-reduction as well, and so, as their name suggests, the class of long β-normal forms can also be taken as a class of canonical representatives for these purposes. Long β-normal forms have proven quite useful in the study of higher-order unification methods (see for example Huet [16] and Gallier and Snyder [10]).

Reflecting $\beta\eta$-equality, C-equality has been well-studied in the literature (see Curry and Feys [5], Hindley [11], Hindley and Lercher [13], Hindley and Seldin [15], Lercher [20] and [21], and Mezghiche [22]). Since long β-normal forms are fundamentally important in the investigation of $\beta\eta$-equality, it is natural to look for a reduction relation on $C\mathcal{L}$ capturing C-equality in hopes of characterizing their translates as the irreducibles with respect to this relation. Unfortunately, C-equality is not the equality generated by the axioms for weak equality, so that the weak irreducibles in $C\mathcal{L}$ do not correspond to the long β-normal forms in $\mathcal{LC}$. *Strong reduction* does, however, fit our specification exactly: if we say that a $C\mathcal{L}$-term is in *C-normal form* whenever it is the translate of a $\mathcal{LC}$-term in long β-normal form (equivalently, the translate of a $\mathcal{LC}$-term in $\beta\eta$-normal form, since the translations between $\mathcal{LC}$- and $C\mathcal{L}$-terms are blind to η-equality), then the irreducibles with respect to strong reduction are precisely the C-normal forms.

In the presence of rewriting, we may hope for an analogous result. We define here a notion of reduction on $C\mathcal{L}$-terms suitable for capturing CR-equality and demonstrate that the irreducibles with respect to this relation are precisely the translates of the long βR-normal forms (although not of the $\beta\eta R$-normal forms, since η-expansions can induce first-order algebraic redexes, and so the incorporation of rewriting into the higher-order paradigm requires that we restrict attention to translates of fully η-expanded terms, as discussed in more detail following Theorem 3.9 below). In fact, we show that three classical notions of normal form with respect to C-equality—namely, translates of long β-normal forms, strong irreducibles, and strong normal forms—can be extended to accommodate first-order algebraic rewriting in a natural way, and that when this is done, the resulting notions coincide to describe a class of canonical representatives with respect to CR-equality, just as their classical versions do for C-equality.

By analogy with the normal forms in $\mathcal{LC}$, we expect any canonical representatives of CR-equivalence classes to be irreducible with respect to the fundamental reduction relations on $C\mathcal{L}$ and to have the property that any $C\mathcal{L}$-term is CR-equal to exactly one such canonical representative. In addition, certain applications of higher-order equational logic may require that a further property of normal forms in $\mathcal{LC}$ persist under the transfer to $C\mathcal{L}$, namely that the class of representatives be closed under subterm extraction. Indeed, all three of these properties of CR-normal forms are required for the work which originally motivated the investigation reported here: in Dougherty and Johann [9], the observation that the class of CR-normal forms satisfies them if R is convergent is used to prove that, under that hypothesis, certain transformations for deciding CR-equality on $C\mathcal{L}$-terms can be "lifted" to transformations which are capable of enumerating complete sets of their $C R$-unifiers. In any case, it will follow easily from our tripartite characterization of $C R$-normal forms that:

- every $C R$-normal form is both weakly irreducible and R-irreducible,
- every subterm of a $C R$-normal form is also in $C R$-normal form, and
- every $C\mathcal{L}$-term is $C R$-equal to a unique $C R$-normal form.

2 Preliminaries We will assume familiarity with classical results about the lambda calculus and combinatory logic (as in, for example, [15]) and use the basic results on the combination of the simply typed lambda calculus and first-order algebraic rewriting. For definitions and notations regarding rewriting not given explicitly here, the reader is referred to Dershowitz and Jouannaud [7].

The *types* are formed by closing a set of *base types* under the operation $(\alpha_1 \to \alpha_2)$ for types α_1 and α_2. Fix an infinite set, *Vars*, of typed *variables* and an infinite set of typed *constants*. Certain constants, with associated arities, comprise the *signature* Σ over which our first-order algebraic terms will be defined. We assume that the constants include the symbols I, $\mathcal{K}$, and $\mathcal{S}$, given various types as usual. Although it is possible to postulate only the various $\mathcal{K}$ and $\mathcal{S}$, for technical reasons we will also need to take the various I as primitive—see the discussion following Lemma 3.4 below. An *atom* is either a variable or a constant; the typed $\mathcal{K}$, $\mathcal{S}$, and I are called *redex atoms*.

$\mathcal{L}C$ is the set of explicitly simply typed lambda terms over the atoms other than the redex atoms; $C\mathcal{L}$ is the set of explicitly simply typed combinatory logic terms over all atoms, including the redex atoms. We will never explicitly indicate the type of ($\mathcal{L}C$- or $C\mathcal{L}$-) terms unless it is necessary. By the *type-erasure* of a term T we will mean the untyped lambda calculus or combinatory logic term, as appropriate, obtained by disregarding all type information in T.

The set of variables appearing in a term T will be denoted $\mathit{Vars}(T)$. The combinatory abstraction operator, as defined, for example, on page 25 of [15], is written $[\cdot]$. We write $\equiv$ for syntactic equality between terms.

If $T \equiv hT_1 \ldots T_n$ is a term and h is an atom, then h is called the *head* of T. The class of *algebraic terms* contains all variables and all terms of the form $f T_1 \ldots T_k$ where $f \in \Sigma$ has arity k and T_i is algebraic for $i = 1, \ldots, k$.

A *substitution* is a finitely supported mapping from *Vars* to $\mathcal{L}C$ or $C\mathcal{L}$, as appropriate. A substitution σ induces a mapping on terms which, abusing notation, we will also denote by σ.

On $C\mathcal{L}$, *weak equality* is generated by *weak reduction*, denoted $\xrightarrow{w}$ and determined by the rules $Ix \longrightarrow x$, $\mathcal{K}xy \longrightarrow x$, and $\mathcal{S}xyz \longrightarrow xz(yz)$; *weak-$R$ reduction* ($\xrightarrow{wR}$) and *$\beta\eta R$-reduction* ($\xrightarrow{\beta\eta R}$) are the reduction relations generated by an algebraic term rewriting system R together with the rules for weak reduction or $\beta\eta$-reduction ($\xrightarrow{\beta\eta}$), respectively. For any notion of reduction $\xrightarrow{*}$, write $\xrightarrow{*}\!\!\!\!\rightarrow$ for the reflexive, transitive closure of $\xrightarrow{*}$ and $=_*$ for the symmetric closure of $\xrightarrow{*}\!\!\!\!\rightarrow$. If $\xrightarrow{*}\!\!\!\!\rightarrow$ is convergent (i.e., confluent and terminating), we may speak of the $*$-*normal form* of a term. In particular, we may refer to the w-*normal form* (usually called the *weak normal form*) of a $C\mathcal{L}$-term, and to the $\beta\eta$- or $\beta\eta^{-1}$-*normal form* (the latter usually called the *long β-normal form*) of a $\mathcal{LC}$-term. If R is convergent then so are $\xrightarrow{wR}\!\!\!\!\!\twoheadrightarrow$ and $\xrightarrow{\beta R}\!\!\!\!\!\twoheadrightarrow$ (see [2]), and so we may similarly speak of the *wR-normal form* or the *long βR-normal form* of a $\mathcal{LC}$- or $C\mathcal{L}$-term, as appropriate. We will write $l\beta nf(X)$ and $l\beta Rnf(X)$ for the long β-normal form and the long βR-normal form of the $\mathcal{LC}$-term X, respectively. Given any reduction relation, we of course have *irreducibles*, i.e., terms which are *irreducible* with respect to that relation, as well as *redexes*, i.e., terms which are *reducible*, with respect to it.

Let $\mathcal{L} : C\mathcal{L} \to \mathcal{LC}$ and $\mathcal{H} : \mathcal{LC} \to C\mathcal{L}$ be the well-known translations between $\mathcal{LC}$ and $C\mathcal{L}$ defined as follows:

Let

- $\mathcal{L}(a) \equiv a$ when a is a non-redex atom,
- $\mathcal{L}(I) \equiv \lambda x.x$,
- $\mathcal{L}(\mathcal{K}) \equiv \lambda xy.x$,
- $\mathcal{L}(\mathcal{S}) \equiv \lambda xyz.xz(yz)$, and
- $\mathcal{L}(MN) \equiv \mathcal{L}(M)\mathcal{L}(N)$;

and

- $\mathcal{H}(a) \equiv a$ when a is an atom,
- $\mathcal{H}(XY) \equiv \mathcal{H}(X)\mathcal{H}(Y)$,
- $\mathcal{H}(\lambda x.Y) \equiv [x]\mathcal{H}(Y)$, where
  - $[x]M \equiv \mathcal{K}M$ when x does not appear in M,
  - $[x]x \equiv I$,
  - $[x](Mx) \equiv M$ when x does not appear in M, and
  - $[x](MN) \equiv \mathcal{S}([x]M)([x]N)$ otherwise.

These translations are such that $\mathcal{H}(\mathcal{L}(X)) \equiv X$ and $\mathcal{L}(\mathcal{H}(M)) =_{\beta\eta} M$. Note that although such translations allow passage between $\mathcal{LC}$- and *$C\mathcal{L}$-terms*, they are not translations of the respective *theories*, or even their higher-order parts, since weak equality is too coarse to reflect $\beta\eta$-equality. For instance, the terms $\mathcal{SK}$ and $\mathcal{K}I$ (of appropriate types) are distinct weak normal forms in $C\mathcal{L}$, but their translations are $\beta\eta$-equal $\mathcal{LC}$-terms.

Given a first-order algebraic theory which admits presentation as a convergent term rewriting system R, define *extensional combinatory R-equality*, abbreviated *CR-equality*, by $M =_{CR} N$ iff $\mathcal{L}(M) =_{\beta\eta R} \mathcal{L}(N)$; it follows that for any $\mathcal{LC}$-terms X and Y, $X =_{\beta\eta R} Y$ iff $\mathcal{H}(X) =_{CR} \mathcal{H}(Y)$. We omit the symbol R from the notation and terminology when R is empty.

Throughout this paper, we will assume that R is a convergent (first-order) term rewriting system defined on the set of algebraic terms over Σ.

3 A notion of normal form for $C\mathcal{L}$ This section is devoted to a characterization of a class of CR-normal forms which will be shown below to have the three properties we expect of any class of canonical representatives for the CR-equivalence classes in $C\mathcal{L}$, as discussed in the introduction.

3.1 Classical notions of normal form It will be helpful to have a brief review of normal forms in the pure higher-order calculi (without algebraic rewriting). We recall the basic results about three notions of combinatory normal form studied in the classical literature. These three notions coincide to yield a characterization of the class of C-normal forms, formally defined by:

Definition 3.1 A $C\mathcal{L}$-term M is in *C-normal form* if there exists a $\mathcal{L}C$-term X in long β-normal form such that $M \equiv \mathcal{H}(X)$.

It is well-known that C-equality can be generated by adding to the rules for weak reduction the extensionality rule: if Mx and Nx are C-equal and x does not appear free in M or N, infer that M and N are C-equal. The classical notion of *strong reduction* on $C\mathcal{L}$, introduced in [5] and shown there to generate C-equality, is a reflection of this observation.

Definition 3.2 *Strong reduction*, denoted $>_S$, is the reduction relation whose set S of rules is the smallest set containing the rules for weak reduction and closed under the inference rule

$$\text{If } M \equiv [x]P,\ N \equiv [x]Q,\ (P, Q) \in S, \text{ and } M \not\equiv N, \text{ then } (M, N) \in S.$$

We will say that M *strongly reduces* to N if $M >_S N$, and denote by $>>_S$ the reflexive, transitive closure of $>_S$. We say that a $C\mathcal{L}$-term M is *$>_S$-irreducible* if there is no term N such that $M >_S N$.

It was shown by Curry [5] that C-equality can be defined by adding a finite number of equations to the equations for weak equality, and so a natural question is whether or not strong reduction is also finitely axiomatizable. Hindley [11] has demonstrated that there can be no finite set of axioms, or even axiom schemes, which generate strong reduction when added to the rules for weak reduction. Indeed there is considerable difficulty even in recognizing the set of rules for strong reduction.

From the difficulty of identifying strong redexes we might infer that describing the class of terms which are strongly irreducible is an equally daunting task. But Lercher [21] has shown that it is possible to characterize them entirely by virtue of their structure—he shows that they are the terms described by the following definition.

Definition 3.3 A $C\mathcal{L}$-term M is in *strong normal form* provided either

- $M \equiv aM_1 \ldots M_k$ with M_i in strong normal form for $i = 1, \ldots, k$ and a a non-redex atom, or
- $M \equiv [x]N$ with N in strong normal form.

Note that the statements in Definition 3.3 hold for $\mathcal{L}C$-terms if "strong normal form" is replaced throughout by "long β-normal form" and combinatory abstraction is replaced by the usual lambda abstraction.

We have the following result, for whose proof the reader is referred to the sources cited in the previous discussion.

Theorem 3.4 *For a $C\mathcal{L}$-term M the following are equivalent:*

(1) M is in strong normal form;
(2) M is $>_S$-irreducible;

(3) M is in C-normal form.

Proof: The equivalence of (1) and (2) is the combination of results from [21] and either [13] or Section 6F in [5], while the equivalence of (1) and (3) is straightforward (see Exercise 9.16 in [15]).

Note that if I is defined as $\mathcal{SKK}$ instead of being taken as primitive, then the $>_S$-irreducibles are *not* precisely the strong normal forms: in this case, I is clearly in strong normal form, being $[x]x$, but since $\mathcal{SK} >_S \mathcal{K}I$, we have

$$I >_S \mathcal{K}I\mathcal{K} >_S \mathcal{K}(\mathcal{K}I\mathcal{K})\mathcal{K} >_S \ldots$$

The class of C-normal forms indeed represents the C-equivalence classes in $C\mathcal{L}$:

Theorem 3.5 *The class of C-normal forms is such that:*

(1) Every term in C-normal form is weakly irreducible;

(2) Every subterm of a term in C-normal form is also in C-normal form;

(3) Every $C\mathcal{L}$-term is C-equal to a unique term in C-normal form.

Proof: (1) If M is in C-normal form, then M is $>_S$-irreducible. But then M is weakly irreducible since strong reduction contains the rules for weak reduction.

(2) If M is in C-normal form, then M is $>_S$-irreducible. But any subterm of a $>_S$-irreducible term is again $>_S$-irreducible, and is therefore in C-normal form. Thus any subterm of M is itself in C-normal form.

(3) Suppose $M =_C P$ and $M =_C Q$ for two C-normal forms P and Q. Then there exist long β-normal forms X and Y such that $P \equiv \mathcal{H}(X)$ and $Q \equiv \mathcal{H}(Y)$. Then $\mathcal{L}(M) =_{\beta\eta} \mathcal{LH}(X)$ and $\mathcal{L}(M) =_{\beta\eta} \mathcal{LH}(Y)$, and so $X =_{\beta\eta} \mathcal{LH}(X) =_{\beta\eta} \mathcal{L}(M) =_{\beta\eta} \mathcal{LH}(Y) =_{\beta\eta} Y$. Since $\beta\eta$-reduction is convergent, we must have $X \equiv Y$ and so $P \equiv \mathcal{H}(X) \equiv \mathcal{H}(Y) \equiv Q$.

3.2 Normal forms in the presence of algebraic rewriting We define extensions of the three classical notions of normal form on $C\mathcal{L}$ presented in the last section, each of which is seen to satisfy one of the properties we require of CR-normal forms. Theorem 3.4 guarantees that the classical notions coincide and Theorem 3.5 shows that they give a class of canonical representatives with respect to C-equality; the content of the next section is that our extensions behave analogously in the presence of algebraic rewriting.

Recall that the class of terms in $C\mathcal{L}$ in which we are interested is given by:

Definition 3.6 A $C\mathcal{L}$-term M is in CR-*normal form* if there exists a $\mathcal{L}C$-term X in long βR-normal form such that $M \equiv \mathcal{H}X$.

Since every $\mathcal{L}C$-term is $\beta\eta R$-equivalent to a unique long βR-normal form, it follows that every $C\mathcal{L}$-term is CR-equal to a unique CR-normal form. The proof is by analogy with that of the third part of Theorem 3.5.

We extend in a straightforward manner the notion of strong reduction to incorporate reduction via a convergent term rewriting system R, and thereby arrive at a notion of reduction which generates CR-equality.

Definition 3.7 *R-strong reduction*, denoted $>_{SR}$, is the reduction relation whose set S_R of rules is the smallest set containing the rules for wR-reduction and closed under the inference rule

$$\text{if } M \equiv [x]P,\ N \equiv [x]Q,\ (P, Q) \in S_R, \text{ and } M \not\equiv N, \text{ then } (M, N) \in S_R.$$

As above, we will say that M *R-strongly reduces* to N if $M >_{SR} N$, and denote by $>>_{SR}$ the reflexive, transitive closure of $>_{SR}$. We say that a $C\mathcal{L}$-term M is *$>_{SR}$-irreducible* if there is no term N such that $M >_{SR} N$. That the equality generated by $>_{SR}$ is exactly CR-equality is straightforward.

Like strong reduction, R-strong reduction is not suitable as a tool for normal form computations—the nonfinite axiomatizability in the classical setting is inherited by this richer reduction relation. But of course, every subterm of a $>_{SR}$-irreducible term is also $>_{SR}$-irreducible, and it will turn out that the $>_{SR}$-irreducibles are precisely the CR-normal forms.

We will see in Theorem 3.9 that an R-enriched variation on the classical notion of strong normal form provides a third characterization of CR-normal forms. The proof that terms in this class are wR-irreducible is by direct analogy with the classical case (see Lemma 3.12 below).

Definition 3.8 A $C\mathcal{L}$-term M is in *R-strong normal form* provided either

- $M \equiv aM_1 \ldots M_k$ with M_i in R-strong normal form for $i = 1, \ldots, k$, M in R-normal form, a a non-redex atom, and $arity(a) = k$ if $a \in \Sigma$, or
- $M \equiv [x]N$ with N in R-strong normal form.

The restrictions to R-normal form and on $arity(a)$ in the first clause of Definition 3.8 are discussed immediately following the statement of Theorem 3.9. For the remainder of this section we will assume the validity of the following theorem, whose proof comprises the next section.

Theorem 3.9 *For a $C\mathcal{L}$-term, the following are equivalent:*

(1) M is in R-strong normal form;
(2) M is $>_{SR}$-irreducible;
(3) M is in CR-normal form.

Note that if the CR-normal forms and R-strong normal forms are to coincide, we must have that $arity(a) = k$ when $a \in \Sigma$ in the first clause of Definition 3.8. Otherwise, $X \equiv \lambda y_1 \ldots y_n.aX_1 \ldots X_k$ is not in long βR-normal form, and we would have to consider a term $X' \equiv \lambda y_1 \ldots y_n z_1 \ldots z_m.aX_1 \ldots X_k l\beta nf(z_1) \ldots l\beta nf(z_m)$, where $k+m$ is the arity of a, to potentially circumvent this difficulty. But X' is not necessarily in R-normal form, since introduction of the arguments $l\beta nf(z_i)$ could introduce an R-redex.

For example, if we have the rule $fx \longrightarrow a$ and do not insist that arity$(a) = k$ in the definition of R-strong normal form, then the term f is in R-strong normal form. But there exists no X in long βR-normal form such that $\mathcal{H}X \equiv f$, since the only candidates for X are f itself and $\lambda x.\,fx$, and so in this case the equivalence between the first and third conditions in Theorem 3.9 can fail. Insisting that $arity(a) = k$ in the definition of R-strong normal form precludes f from satisfying that definition: f

is not in long β-normal form, and if $f \equiv [x]Y$, then $Y \equiv fx$, which is not in R-normal form and so not in R-strong normal form. So f is not in R-strong normal form under the definition which is sensitive to arity concerns, and, as we will see, the equivalence of the first and third conditions in Theorem 3.9 is restored.

Of course, such care is not required when algebraic rewriting is not permitted, and it is therefore easy to see that if $f \in \Sigma$ but f is not at the head of the left-hand side of any rule in R, then $f \equiv [x_1 \ldots x_k] f x_1 \ldots x_k$ is always in CR-normal form.

We have the following analogue of Theorem 3.5.

Theorem 3.10 *The class of CR-normal forms is such that:*

(1) Every term in CR-normal form is wR-irreducible;

(2) Every subterm of a term in CR-normal form is also in CR-normal form;

(3) Every $C\mathcal{L}$-term is CR-equal to a unique term in CR-normal form.

Proof: That CR-normal forms are wR-irreducible follows from their characterization as those terms which are $>_{SR}$-irreducible, together with the fact that R-strong reduction contains the rules for weak-R reduction. Since the class of $>_{SR}$-irreducibles is also closed under subterm formation and the class of CR-normal forms provides a unique canonical representation for every CR-equivalence class (as discussed immediately following Definition 3.6), the class of CR-normal forms must have these properties.

It is also possible to see directly that each R-strong normal form is R- and weakly irreducible, a fact of which we will make much use in Section 4. The proof requires a simple lemma adapted from [21].

Lemma 3.11 *Let $M \equiv [x]N$. Then M is wR-irreducible iff every wR-redex of N contains x. Moreover, if U is a wR-redex of M, then U is a subterm of N.*

Proof: By induction on N with cases corresponding to the clauses in the abstraction algorithm.

If $x \notin Vars(N)$, then $M \equiv \mathcal{K}N$. Any wR-redex of N must be a wR-redex of M and vice-versa.

If $N \equiv x$, then $M \equiv I$. Then both M and N are wR-irreducible.

If $N \equiv Px$ with $x \notin Vars(P)$, then $M \equiv P$. If M is wR-irreducible, then the only possible wR-redex of N is N itself, which contains x. Conversely, if every wR-redex of N contains x, then $P \equiv M$ is wR-irreducible. Clearly every wR-redex of M is also a subterm of N.

If $N \equiv PQ$ with $Q \not\equiv x$ and $x \in Vars(PQ)$, then $M \equiv \mathcal{S}([x]P)([x]Q)$. If M is wR-irreducible, then $[x]P$ and $[x]Q$ are. By the induction hypothesis, then, every wR-redex of P or Q contains x. The only other possible wR-redex of N is N itself which contains x. Conversely, if every wR-redex of N contains x, then every wR-redex of P or Q contains x. By the induction hypothesis, this implies that $[x]P$ and $[x]Q$ are wR-irreducible. But then M is also. If U is a wR-redex of M, then it is contained in $[x]P$ or $[x]Q$. By the induction hypothesis, U is a subterm of P or Q, and therefore of N.

In particular, if M is wR-irreducible, then so is $[x]M$.

Lemma 3.12 *If M is in R-strong normal form, then M is wR-irreducible.*

Proof: The proof is by induction on M with cases corresponding to the clauses in Definition 3.8.

If $M \equiv aM_1 \ldots M_k$, then M in R-strong normal form implies that M_i is also in R-strong normal form for $i = 1, \ldots, k$, and M is in R-normal form. By the induction hypothesis, M_i is wR-irreducible for $i = 1, \ldots, k$. Since M is in R-normal form and there can be no head weak redex in M, M is wR-irreducible.

If $M \equiv [x]N$ is in R-strong normal form, then N is in R-strong normal form. By the induction hypothesis, N is wR-irreducible, so by Lemma 3.11, M must be also.

4 Equivalence of normal forms In this section we prove Theorem 3.9; our proof is by direct analogy with the classical case. We begin by showing that the CR-normal forms are precisely the R-strong normal forms.

The observation that R-redexes in C-normal forms are translates of R-redexes in $\mathcal{L}C$ will be useful in seeing that CR-normal forms are in R-strong normal form.

Lemma 4.1 *If X is in long β-normal form and $\mathcal{H}X \equiv \theta S$ for S algebraic, then $X \equiv \theta' S$ where $\theta'(x) \equiv l\beta nf(\mathcal{L}\theta(x))$ for every $x \in Vars$.*

Proof: By induction on S.

Lemma 4.2 *If M is in CR-normal form, then M is in R-strong normal form.*

Proof: If M is in CR-normal form, then $M \equiv \mathcal{H}X$ for some long βR-normal form X. The proof is by induction on X.

If $X \equiv aX_1 \ldots X_k$, then $\mathcal{H}X \equiv a\mathcal{H}X_1 \ldots \mathcal{H}X_k$, with X_i in long βR-normal form for $i = 1, \ldots, k$, so by the induction hypothesis, $\mathcal{H}X_i$ is in R-strong normal form for $i = 1, \ldots, k$. Since X is in long βR-normal form, arity$(a) = k$ if $a \in \Sigma$. Finally, $\mathcal{H}X$ is in R-normal form since otherwise, $\mathcal{H}X \equiv \theta S$ for some θ and some algebraic S. Then by Lemma 4.1, $X \equiv \theta' S$ where $\theta'(x) \equiv l\beta Rnf(\mathcal{L}\theta(x))$ for all $x \in Vars$. But this contradicts the assumption that X is in R-normal form.

If $X \equiv \lambda x.Y$, then Y is in long βR-normal form, so by the induction hypothesis, $\mathcal{H}Y$ is in R-strong normal form. Then $\mathcal{H}(\lambda x.Y) \equiv [x]\mathcal{H}Y$ is in R-strong normal form.

The converse is not difficult:

Lemma 4.3 *If M is in R-strong normal form, then M is in CR-normal form.*

Proof: By induction on M with cases corresponding to the clauses in Definition 3.8.

If $M \equiv aM_1 \ldots M_k$ for $a \in \Sigma$, then M_i is in R-strong normal form so that there exists a long βR-normal form X_i such that $\mathcal{H}X_i \equiv M_i$ for $i = 1, \ldots, k$. Consider $X \equiv aX_1 \ldots X_k$. Clearly $\mathcal{H}X \equiv M$ and X is in long β-normal form. If X is not in R-normal form, then $aX_1 \ldots X_k \equiv \theta S$ for some left-hand side S of a rule in R. But then $M \equiv \mathcal{H}X \equiv \mathcal{H}(\theta S) \equiv (\mathcal{H} \circ \theta)S$, contradicting the fact that M is in R-normal form.

If $M \equiv aM_1 \ldots M_k$ for $a \in Vars$, then M_i is in R-strong normal form so that there exists a long βR-normal form X_i such that $\mathcal{H}X_i \equiv M_i$ for $i = 1, \ldots, k$. Consider $X \equiv \lambda z_1 \ldots z_m.aX_1 \ldots X_k l\beta nf(z_1) \ldots l\beta nf(z_m)$, where $aX_1 \ldots X_k l\beta nf(z_1) \ldots l\beta nf(z_m)$ is of base type. Clearly $\mathcal{H}X \equiv M$, and the fact that $aX_1 \ldots X_k l\beta nf(z_1) \ldots l\beta nf(z_m)$ is in long βR-normal form implies that X is as well.

If $M \equiv [x]N$ for N in R-strong normal form, then by the induction hypothesis, there exists a Y such that $\mathcal{H}Y \equiv N$ and Y is in long βR-normal form. Consider $X \equiv \lambda x.Y$. Then $\mathcal{H}X \equiv [x]\mathcal{H}Y \equiv [x]N \equiv M$ and X is in long βR-normal form since Y is.

Having established the equivalence of the conditions in the first and third clauses in Theorem 3.9, we now turn our attention to showing the equivalence of the first and second, i.e., we prove that for any $C\mathcal{L}$-term M, M is $>_{SR}$-irreducible iff M is in R-strong normal form. We begin by proving that every $>_{SR}$-irreducible term is in R-strong normal form. The next lemma and corollary exhibit a simple relation between reduction in $\mathcal{L}C$ and in $C\mathcal{L}$.

Lemma 4.4 *For any $\mathcal{L}C$-terms X and Y:*

- *if $X \xrightarrow{R} Y$, then $\mathcal{H}X >_{SR} \mathcal{H}Y$, and*
- *if $X \xrightarrow{\beta} Y$, then $\mathcal{H}X >>_{SR} \mathcal{H}Y$.*

Proof: Both statements are proved by induction on X.

Corollary 4.5 *If $X \xrightarrow{\beta\eta\eta^{-1}R} Y$, then $\mathcal{H}X >>_{SR} \mathcal{H}Y$.*

Proof: By Lemma 4.4, if $X \xrightarrow{\beta} Y$, then $\mathcal{H}X >>_{SR} \mathcal{H}Y$, and if $X \xrightarrow{R} Y$, then $\mathcal{H}X >_{SR} \mathcal{H}Y$. If $X \xrightarrow{\eta\eta^{-1}} Y$, then $\mathcal{H}X \equiv \mathcal{H}Y$.

We can now see that:

Theorem 4.6 *If M is $>_{SR}$-irreducible, then M is in R-strong normal form.*

Proof: Since $\mathcal{L}M \overset{\beta\eta^{-1}R}{\twoheadrightarrow} l\beta Rnf(\mathcal{L}M)$, $M \equiv \mathcal{H}\mathcal{L}M >>_{SR} \mathcal{H}(l\beta Rnf(\mathcal{L}M)) \equiv N$. But since M is $>_{SR}$-irreducible, we must have $M \equiv N$. Then M is in CR-normal form since N is, and by Lemma 4.2 is therefore in R-strong normal form.

To prove the converse, we first prove this result for untyped combinatory logic terms, (called *ob*s), and then immediately infer that it holds for $C\mathcal{L}$-terms since typing is irrelevant to structural and reduction properties of terms. The proof of the converse given below relies heavily on the fact that obs are untyped—which simply allows us to avoid many of the purely technical difficulties which would arise in a similar treatment of simply typed $C\mathcal{L}$-terms—while the proof just presented uses the typing of $C\mathcal{L}$-terms. Indeed, any reasonably simple proof that a term which is $>_{SR}$-irreducible is in R-strong normal form seems to require the use of types. It may, however, be possible to extend the notion of *normal reduction* from [5] to a suitable notion involving R, prove an analogue of Curry's characterization of $>_S$-irreducibles as termini of normal reductions, and then use a justification as in [21] to see that R-strong normal forms and $>_{SR}$-irreducibles are identical in untyped combinatory logic. But since the main thrust of this section is that Curry's normal form theorem is preserved for simply typed combinatory logic—insuring that we have a suitable notion of CR-normal form there—we do not hesitate to restrict our attention to typed systems whenever possible.

We begin the proof of

Theorem 4.7 *Every $C\mathcal{L}$-term M in R-strong normal form is $>_{SR}$-irreducible.*

The following definitions and alphabetic conventions will be assumed.

Definition 4.8 The set of *ob*s consists of all possible type-erasures of $\mathcal{CL}$-terms.

In the remainder of this paper, the letters u, v, w, x, y, z, etc., will denote the type-erasures of variables, while L, M, N, P, Q, and their subscripted versions will denote arbitrary obs. We will abuse terminology and henceforth refer to the type-erasure of a variable as a variable. As usual, application will be assumed to be left-associative.

Of course, the notions of $>_{SR}$-reduction and R-strong normal forms can be extended to obs in an obvious fashion (since symbols from Σ have fixed arities). As suggested above, our proof of Theorem 4.7 then involves proving its analogue for obs.

We would like to prove that every ob M which is in R-strong normal form is $>_{SR}$-irreducible by induction on M, with cases according to the analogue of Definition 3.8 for obs. But this requires the ability to characterize $>_{SR}$-redexes, and the manner in which the reduction relation $>_{SR}$ is defined makes such a characterization especially difficult. To remedy this situation, we define a new axiomatic relation $>^R$ on obs (Definition 4.11) whose redexes can be characterized with considerably less difficulty. We then establish in Lemmas 4.12 through 4.15 some facts about the interaction between combinatory abstraction and substitution on obs which are used to prove that $>>_{SR}$ and the reduction relation induced by $>^R$ are equivalent (Theorem 4.20). En route to proving this equivalence we ascertain a result far more important for our purposes, namely that if M is an $>^R$-irreducible ob then it is $>_{SR}$-irreducible. Together with the observation that any ob in R-strong normal form is necessarily also $>^R$-irreducible (Theorem 4.21), this result allows us to conclude that any ob in R-strong normal form is also $>_{SR}$-irreducible, as desired. The proof of Theorem 4.21 is by induction on obs; it uses the properties of $>^R$-redexes established in Lemmas 4.24 through 4.26 and some straightforward facts about weak reduction (Lemmas 4.27 and 4.28).

The reader familiar with the classical proof in [13] will observe that the axiom schemes for $>^R$ do not include a reflexivity axiom but otherwise comprise an extension of those for Hindley's ([11]) relation $\succ$ by the rules of the algebraic reduction relation R. The added complications in the proof here arise as consequences of the possibility of reduction according to the rules in R.

Definition 4.9 Given an infinite set of *metavariables* disjoint from the set of (all type-erasures of) variables, the set of *ob-schemes* is defined inductively:

- every metavariable is an ob-scheme,
- $\mathcal{K}$, $\mathcal{S}$, and I are ob-schemes,
- for every (type-erasure of a) variable there is a distinct symbol which is an ob-scheme, and
- if U and V are ob-schemes, then so is (UV).

We assume the same conventions regarding application for ob-schemes that we observe for obs. We denote metavariables by A, B, C and their subscripted versions; we will sometimes write $A_1, \ldots, A_k$ for the first k metavariables. In what follows U, V, W and their subscripted versions stand for arbitrary ob-schemes.

Identifying variables with the ob-scheme symbols denoting them gives a "denotation" for each ob. Thus, every ob-scheme without metavariables denotes a unique ob, each ob is denoted by precisely one such scheme, and we may identify an ob with the ob-scheme without metavariables by which it is denoted.

Metavariables are intended, of course, to denote arbitrary obs in ob-schemes, and interpreting the metavariables in an ob-scheme by obs defines an interpretation of the entire ob-scheme. While such an interpretation is, strictly speaking, another ob-scheme, under the convention of the last paragraph, we will adopt the point of view that an interpretation of the metavariables in an ob-scheme yields an ob.

Of course, it is possible to do away entirely with metavariables and ob-schemes and the distinctions they induce. But the alternatives are either using protected sets of variables, or requiring a set of variables disjoint from those used in term formation from which all axioms would be constructed, and prefacing many of the results here with restrictions on the variables (both are essentially the same, although the latter is what might be done in an implementation). In the interest of clarity, and to preserve the parity between the results here and those for the classical calculi, we use metavariables in the style of Hindley.

Note that we can extend the usual definition of the combinatory abstraction algorithm to ob-schemes containing metavariables by abstracting over ob-schemes with respect to the (symbols corresponding to) variables. Write $U[A_1 := M_1]\ldots[A_k := M_k]$ for the result of simultaneously replacing every occurrence of the metavariable A_i by the ob-scheme M_i, $i = 1, \ldots, k$.

Definition 4.10 If $>$ denotes a binary relation between ob-schemes, then a *sentence scheme* is an expression $U > V$. A *sentence* is a sentence scheme containing no metavariables.

Let a set of sentence schemes called *axiom schemes* be given. If $U > V$ is an axiom scheme and $\{A_1, \ldots, A_k\}$ includes all the metavariables in this scheme, and if $M_1, \ldots, M_k$ are any obs, then $U[A_1 := M_1]\ldots[A_k := M_k] > V[A_1 := M_1]\ldots[A_k := M_k]$ is an *instance* of the axiom scheme, called an *axiom*.

For the definition of the relation $>^R$ we need a final piece of notation. Let x be a variable, and let A be a metavariable. Define $A^{x1} \equiv A$, $A^{x2} \equiv Ax$, and $A^{x3} \equiv x$. Let U be an ob-scheme with metavariables from the set $\{A_1, \ldots, A_k\}$. For any $i_1, \ldots, i_k$ with each i_j taking on values from the set $\{1, 2, 3\}$, define $U_{x,i_1\ldots i_k}$ to be $U[A_1 := A_1^{xi_1}]\ldots[A_k := A_k^{xi_k}]$.

Observe that being able to replace a metavariable A by the ob-schemes Ax and x in an ob-scheme U requires that obs and ob-schemes are untyped. For notational convenience, we will write A^i for A^{xi} and $U_{i_1\ldots i_k}$ in place of $U_{x,i_1\ldots i_k}$ when the variable x is discernible from the context.

Definition 4.11 Let $>^R$ be defined by the following axiom schemes:

(1) $\mathcal{S}A_1A_2A_3 >^R A_1A_3(A_2A_3)$,
(2) $\mathcal{K}A_1A_2 >^R A_1$,
(3) $IA_1 >^R A_1$,
(4) $\mathcal{S}(\mathcal{K}A_1)I >^R A_1$,
(5) $\mathcal{S}(\mathcal{K}A_1)(\mathcal{K}A_2) >^R \mathcal{K}A_1A_2$,

(6) $\mathcal{S}(\mathcal{K}I) >^R I$,

(7) if $U \xrightarrow{R} V$, then $U >^R V$,

(8) if $U >^R V$ is an axiom scheme other than 3 or 6, and if $A_1, \ldots, A_k$ are all the metavariables occurring in U, then for all variables x,

$$[x]U_{i_1\ldots i_k} >^R [x]V_{i_1\ldots i_k}$$

is an axiom scheme for $>^R$ unless $i_j = 1$ for all j such that A_j occurs in U, or $[x]U_{i_1\ldots i_k} \equiv [x]V_{i_1\ldots i_k}$.

If $U >^R V$, then the metavariables in U are the only metavariables which may occur in V, and likewise for the variables in U and V. The notations A^{i_j}, $i_j \in \{1, 2, 3\}$, correspond to the three cases which might arise when forming an axiom from an axiom scheme: $i_j = 1$ corresponds to interpreting a metavariable by an ob which does not contain x; $i_j = 2$ corresponds to interpreting a metavariable by an ob of the form Mx for some M with x not appearing in M; and $i_j = 3$ accommodates the remaining cases. This distinction will be important in Lemma 4.16.

The situation when $i_1 = \ldots = i_k = 1$ corresponds to interpreting the metavariables of U and V by obs containing no occurrence of x. In this case we already have

$$[x]U_{i_1\ldots i_k} \equiv \mathcal{K}U_{i_1\ldots i_k} >^R \mathcal{K}V_{i_1\ldots i_k} \equiv [x]V_{i_1\ldots i_k},$$

so clearly this should not be added as a "new" axiom scheme. Similarly, if the axiom scheme 3 is used as $U >^R V$ in scheme 8, then we have one of two cases:

- x is not in $U_{i_1\ldots i_k}$, so that $i_1 = 1$ and $[x]U_{i_1\ldots i_k} \equiv [x]IA_1 \equiv \mathcal{K}(IA_1) >^R \mathcal{K}A_1 \equiv [x]A_1 \equiv [x]V_{i_1\ldots i_k}$.
- x appears in $U_{i_1\ldots i_k}$. Then either $i_1 = 3$ so that $U_{i_1,\ldots,i_k} \equiv [x]Ix \equiv I \equiv [x]x \equiv [x]A_1^{i_1} \equiv [x]V_{i_1\ldots i_k}$, or $i_1 = 2$ so that $[x]U_{i_1,\ldots,i_k} \equiv [x]IA_1^{i_1} \equiv \mathcal{S}([x]I)([x]A_1^{i_1}) \equiv \mathcal{S}(\mathcal{K}I)([x]A_1^{i_1}) >^R I([x]A_1^{i_1}) >^R [x]A_1^{i_1} \equiv [x]V_{i_1\ldots i_k}$.

In both cases, we derive no more information from the application of scheme 8. Likewise, no additional information is obtained by using axiom scheme 6 in axiom 8. But observe that if we use, for example, the axiom scheme 4 in scheme 8, and if $i_1 = 3$, then

$$[x]U_{i_1\ldots i_k} \equiv [x](\mathcal{S}(\mathcal{K}A)I)_{i_1\ldots i_k} \equiv [x]\mathcal{S}(\mathcal{K}x)I >^R [x]x \equiv [x]V_{i_1\ldots i_k}$$

by scheme 8, but this is not derivable from axiom schemes 1 through 7, since

$$\begin{aligned} [x]\mathcal{S}(\mathcal{K}x)I &\equiv \mathcal{S}([x]\mathcal{S}(\mathcal{K}x))([x]I) \\ &\equiv \mathcal{S}(\mathcal{S}([x]\mathcal{S})([x]\mathcal{K}x))(\mathcal{K}I) \\ &\equiv \mathcal{S}(\mathcal{S}(\mathcal{K}\mathcal{S})\mathcal{K})(\mathcal{K}I) \end{aligned}$$

is in normal form with respect to those axiom schemes. Since we indeed have derived "new" information, this information is encoded as a new axiom scheme generated from the ones that came "before" it. We can think of axiom scheme 8 as recording lemma schemata.

Interpretation of metavariables can be accomplished sequentially and in any order:

Lemma 4.12 *If $A \not\equiv B$, then for all obs M and N, and ob-schemes U,*

$$U[A := M][B := N] \equiv U[B := N][A := M].$$

Proof: Just observe that the metavariable A cannot occur in the ob N, and that B cannot occur in M.

The next two lemmas examine interpretations of abstractions and abstractions of interpretations, providing insight into the structure of the axioms of $>^R$.

Lemma 4.13 *Let $\{A_1, \ldots, A_k\}$ contain the metavariables occurring in the ob-scheme U. Then for any obs $M_1, \ldots, M_k$,*

$$([x]U_{i_1 \ldots i_k})[A_1 := M_1] \ldots [A_k := M_k] \equiv [x](U[A_1 := M_1^{i_1}] \ldots [A_k := M_k^{i_k}])$$

provided x does not appear in U, $M_1, \ldots, M_k$.

Proof: By induction on $U_{i_1 \ldots i_k}$.

• If x does not appear in $U_{i_1 \ldots i_k}$, then x also does not appear in U and $i_1 = \ldots = i_k = 1$, so that

$$\begin{aligned}
([x]U_{i_1 \ldots i_k})[A_1 := M_1] \ldots [A_k := M_k] &\equiv (\mathcal{K}U_{i_1 \ldots i_k})[A_1 := M_1] \ldots [A_k := M_k] \\
&\equiv \mathcal{K}(U_{i_1 \ldots i_k}[A_1 := M_1] \ldots [A_k := M_k]) \\
&\equiv \mathcal{K}(U[A_1 := M_1] \ldots [A_k := M_k]) \\
&\equiv \mathcal{K}(U[A_1 := M_1^{i_1}] \ldots [A_k := M_k^{i_k}]) \\
&\equiv [x](U[A_1 := M_1^{i_1}] \ldots [A_k := M_k^{i_k}]).
\end{aligned}$$

• If $U_{i_1 \ldots i_k} \equiv x$, then $U \equiv A_j$ for some j such that $i_j = 3$, so that

$$\begin{aligned}
([x]U_{i_1 \ldots i_k})[A_1 := M_1] \ldots [A_k := M_k] &\equiv ([x]x)[A_1 := M_1] \ldots [A_k := M_k] \\
&\equiv [x]x \\
&\equiv [x]M_j^{i_j} \\
&\equiv [x](U[A_1 := M_1^{i_1}] \ldots [A_k := M_k^{i_k}]).
\end{aligned}$$

• If $U_{i_1 \ldots i_k} \equiv W_{i_1 \ldots i_k} x$ with x not appearing in $W_{i_1 \ldots x_k}$, then $U \equiv WA_j$ for some j such that $i_j = 3$ and A_j is not in W, and moreover, $i_n = 1$ for all n such that A_n is in W. Then

$$\begin{aligned}
([x]U_{i_1 \ldots i_k})[A_1 := M_1] \ldots [A_k := M_k] &\equiv ([x]W_{i_1 \ldots i_k} x)[A_1 := M_1] \ldots [A_k := M_k] \\
&\equiv ([x](WA_j)_{i_1 \ldots i_k})[A_1 := M_1] \ldots [A_k := M_k] \\
&\equiv [x]((WA_j)[A_1 := M_1^{i_1}] \ldots [A_k := M_k^{i_k}]) \\
&\equiv [x](U[A_1 := M_1^{i_1}] \ldots [A_k := M_k^{i_k}]).
\end{aligned}$$

• If $U_{i_1 \ldots i_k} \equiv (VW)_{i_1 \ldots i_k}$, with none of the above holding, then

$$\begin{aligned}
[x](U_{i_1 \ldots i_k})[A_1 := M_1] \ldots [A_k := M_k] &\equiv [x]((VW)_{i_1 \ldots i_k}[A_1 := M_1] \ldots [A_k := M_k]) \\
&\equiv [x]((VW)[A_1 := M_1^{i_1}] \ldots [A_k := M_k^{i_k}]) \\
&\equiv [x](U[A_1 := M_1^{i_1}] \ldots [A_k := M_k^{i_k}]).
\end{aligned}$$

Lemma 4.14 *For any ob M and variable x, define $M^{x+} \equiv ([x]M)x$ if x appears in M but $x \not\equiv M$, and $M^{x+} \equiv M$ otherwise. Then for any ob M, metavariable A, and ob-scheme U,*

$$[x](U[A := M^{x+}]) \equiv [x](U[A := M]).$$

Proof: By induction on U, under the assumption that $M \not\equiv x$ and x appears in M (otherwise there is nothing to prove).

Corollary 4.15 *For any ob-scheme U, distinct metavariables $A_1, \ldots, A_k$, and obs $M_1, \ldots, M_k$,*

$$[x](U[A_1 := M_1^{x+}] \ldots [A_k := M_k^{x+}]) \equiv [x](U[A_1 := M_1] \ldots [A_k := M_k]).$$

Proof: The proof uses Lemma 4.14 k times in conjunction with Lemma 4.12.

The notation M^{x+} reflects the internal structure of the terms obtained when the metavariable A in U is instantiated, which must be taken into account when applying the abstraction algorithm. In what follows, we will write M^+ for M^{x+} when x is clear from the context.

If $M >^R N$ is an axiom, then $[x]M$ admits an $>^R$-reduction:

Lemma 4.16 *If $M >^R N$ is an axiom, then $[x]M >>^R [x]N$ for all variables x.*

Proof: For some axiom scheme $U >^R V$ and obs $M_1, \ldots, M_k$,

$$M \equiv U[A_1 := M_1] \ldots [A_k := M_k]$$

and

$$N \equiv V[A_1 := M_1] \ldots [A_k := M_k].$$

If M_j does not contain x, define $i_j = 1$ and $M_{j0} \equiv M_j$. If M_j contains x but is distinct from x, define $i_j = 2$ and $M_{j0} \equiv [x]M_j$. If $M_j \equiv x$, define $i_j = 3$ and $M_{j0} \equiv \mathcal{S}$. Then if we write $M_{j0}^{i_j}$ for $(M_{j0})^{i_j}$, $M_{j0}^{i_j} \equiv M_j^+$ as defined in Lemma 4.14. We have

$$\begin{array}{rcl} [x]M & \equiv & [x](U[A_1 := M_1] \ldots [A_k := M_k]) \\ & \equiv & [x](U[A_1 := M_1^+] \ldots [A_k := M_k^+]) \\ & \equiv & [x](U[A_1 := M_{10}^{i_1}] \ldots [A_k := M_{k0}^{i_k}]) \\ & \equiv & ([x]U_{i_1 \ldots i_k})[A_1 := M_{10}] \ldots [A_k := M_{k0}], \end{array}$$

and so by a similar result for N, we see that if $[x]U_{i_1\ldots i_k} \not\equiv [x]V_{i_1\ldots i_k}$ then clearly the sentence $[x]M >^R [x]N$ must be an instance of the sentence scheme $[x]U_{i_1\ldots i_k} >^R [x]V_{i_1\ldots i_k}$. If $[x]U_{i_1\ldots i_k} \equiv [x]V_{i_1\ldots i_k}$, then clearly $[x]M >>^R [x]N$ holds, and otherwise $[x]U_{i_1\ldots i_k} >^R [x]V_{i_1\ldots i_k}$ is an axiom scheme in all but the following cases:

- When $i_1 = \ldots = i_k = 1$. In this case x does not occur in $M_1, \ldots, M_k$, so $[x]M \equiv \mathcal{K}M >^R \mathcal{K}N \equiv [x]N$.
- When $U >^R V$ is the scheme 6. Then x is in neither M nor N, so the conclusion follows as in the second case above.
- When $U >^R V$ is the scheme 3. Then $M \equiv IP$ and $N \equiv P$. If $P \equiv x$, then $[x]M \equiv I \equiv [x]x$. If x does not appear in P, then neither M nor N contains x and we again proceed as in the second case above. If x appears in P but $x \not\equiv P$, then $[x]M \equiv [x]IP \equiv \mathcal{S}([x]I)([x]P) \equiv \mathcal{S}(\mathcal{K}I)([x]P) >^R I([x]P) >^R [x]P \equiv [x]N$, and the lemma is proved.

The above lemma is the key to proving that $>>^R$ is equivalent to $>>_{SR}$ for obs.

Lemma 4.17 *If $M >>_{SR} N$, then $M >>^R N$, and if there is a nontrivial step in the first derivation then there is a nontrivial step in the second.*

Proof: Since the rules for $\mathcal{K}$, $\mathcal{S}$, I and R-reduction are part of the definition of $>^R$, it suffices to prove that for any variable x and obs M and N, $M >^R N$ implies that $[x]M >>^R [x]N$; there must always be at least one nontrivial step in the derivation $[x]M >>^R [x]N$ if there is one in the derivation $[x]M >>_{SR} [x]N$ since if $[x]M >_{SR} [x]N$, then $[x]M \not\equiv [x]N$ by Definition 3.7 and $>^R$ is not reflexive. We induct on the deduction of $M >^R N$, remembering that abstraction does not necessarily preserve the structure of obs.

- If $M >^R N$ is an axiom, then by Lemma 4.16, $[x]M >>^R [x]N$.
- If $M >^R N$ is deduced because we know $M \equiv LP$, $N \equiv LQ$, and $P >^R Q$, then the proof of this induction step is broken into clauses according to the evaluation of $[x]M$ and $[x]N$.
  - If x does not appear in M, then x does not appear in N, so $[x]M \equiv \mathcal{K}M >^R \mathcal{K}N \equiv [x]N$.
  - If $x \equiv P$ and x does not appear in L, then $x >^R Q$. But then $Q \equiv x$ since x can be shown to be $>_R$-irreducible (directly). Therefore, $M \equiv Lx$ and $N \equiv Lx$, so that $M >^R N$ is impossible.
  - If x appears in M but $P \not\equiv x$, then either
    - x does not appear in LQ, so that $[x]M \equiv [x]LP \equiv \mathcal{S}([x]L)([x]P) \equiv \mathcal{S}(\mathcal{K}L)([x]P) >^R \mathcal{S}(\mathcal{K}L)([x]Q) \equiv \mathcal{S}(\mathcal{K}L)(\mathcal{K}Q) >^R \mathcal{K}(LQ) \equiv [x]N$.
    - $Q \equiv x$ and x is not in L, so that $[x]M \equiv [x]LP >^R \mathcal{S}(\mathcal{K}L)([x]Q) \equiv \mathcal{S}(\mathcal{K}L)I >^R L \equiv [x]N$.
    - $[x]M \equiv [x]LP \equiv \mathcal{S}([x]L)([x]P) >>^R \mathcal{S}([x]L)([x]Q) \equiv [x]LQ$.
- If $M >^R N$ is deduced because $M \equiv PL$, $N \equiv QL$, and $P >^R Q$, then the proof is similar to the above case.

Corollary 4.18 *If M is $>^R$-irreducible, then it is $>_{SR}$-irreducible.*

It is easy to get a converse to Lemma 4.17, and so we prove the equivalence of the reduction relations generated by $>^R$ and $>_{SR}$.

Lemma 4.19 *If $M >>^R N$, then $M >>_{SR} N$, and if there is a nontrivial step in the first derivation then there is a nontrivial step in the second.*

Proof: It is enough to show that $M >>_{SR} N$ for every axiom $M >^R N$; there must always be at least one nontrivial step in the derivation $M >>_{SR} N$ if there is one in $M >>^R N$ since if $M >^R N$ then $M \not\equiv N$ by Definition 4.11 and since $>_{SR}$ is not reflexive. For instances of schemes 1 through 7, this is easy. So suppose that the result has been proved for every instance of an axiom scheme $U >^R V$, where $M >^R N$ is an instance of $[x]U_{i_1 \ldots i_k} >^R [x]V_{i_1 \ldots i_k}$. By Lemma 4.13, $M \equiv [x]M^*$ and $N \equiv [x]N^*$ for some instance $M^* >^R N^*$ of $U >^R V$. Since $M^* >^R N^*$, the induction hypothesis gives $M^* >>_{SR} N^*$. Then $M \equiv [x]M^* >>_{SR} [x]N^* \equiv N$ by the definition of $>_{SR}$.

Combining Lemmas 4.17 and 4.19, we have:

Theorem 4.20 *$M >>_{SR} N$ iff $M >>^{R} N$.*

In order to prove Theorem 4.7, in light of Corollary 4.18 it suffices to see that:

Theorem 4.21 *If M is in R-strong normal form, then M is $>^{R}$-irreducible.*

The proof of Theorem 4.21 requires some preliminary notions and results.

Definition 4.22 An ob-scheme U is a *redex scheme* if it is the left-hand side of some axiom scheme. A redex scheme U is *basic* if it is the left-hand side of one of the first seven axiom schemes in Definition 4.11. A redex scheme is *based on* axiom scheme 7 if it is $[x]U_{i_1 \ldots i_k}$ where either U is the left-hand side of axiom scheme 7 or U itself based on axiom scheme 7. An ob M is a *redex* if it is an instance of a redex scheme.

Lemma 4.23 *Let U be a redex scheme. Then $head(U)$ is not a metavariable.*

Proof: This is clearly true if U is basic. Otherwise, $U \equiv [x]V_{i_1 \ldots i_k}$ and the result is obtained by induction on V.

For any ob or ob-scheme U, let $n(U)$ be the number of occurrences of non-variable symbols in U. As expected, abstraction increases n:

Lemma 4.24 *If $U \equiv [x_1 \ldots x_k]V$ for some variables $x_1, \ldots, x_k$, then $n(U) \geq n(V)$.*

Proof: The proof is by induction on k, with cases on the clauses in the definition of the abstraction algorithm in case $k = 1$.

An analysis of redex schemes will facilitate our investigation of $>^{R}$-reduction.

Lemma 4.25 *Let U be a redex scheme.*

(1) If U is not based on scheme 7, then U contains at most one occurrence of each metavariable.

(2) If U is not basic, then U is of one of the forms $\mathcal{S}U_1U_2$ or $\mathcal{S}U_1$ with $n(U_1) > 0$, or else U is of the form $fU_1 \ldots U_k$, with $arity(f) > k$ and $f \in \Sigma$.

Proof:

(1) This is clear for basic redexes from schemes 1 through 6. Moreover, the application of axiom scheme 8 does not introduce new metavariables.

(2) If U is not basic, then $U \equiv [x]V_{i_1 \ldots i_k}$ for some redex scheme V, and $i_1 = \ldots = i_k = 1$ is impossible. Thus x appears in $V_{i_1 \ldots i_k}$ (otherwise $V_{i_1 \ldots i_k}$ is identically x and so $V \equiv A_j$ for some j such that $i_j = 3$, but this is impossible by virtue of Lemma 4.23).

If V is basic and V is an instance of axiom schemes 1, 4, and 5, then the result clearly holds. If V is an instance of axiom scheme 2, then U is not a redex scheme unless $V \equiv \mathcal{K}A_1A_2$ and either $x \not\equiv A_{2,i_1 \ldots i_k}$ or x appears in $\mathcal{K}A_{1,i_1 \ldots i_k}$. But then U is indeed of the form $\mathcal{S}U_1U_2$ with $n(U_1) > 0$. If V is as in scheme 7, then either U is of the form $\mathcal{S}U_1U_2$ with $n(U_1) > 0$, or U is of the form $fU_1 \ldots U_{n-1}$, where $n \leq arity(f)$.

If V is not basic, then it is of the form $\mathcal{S}V_1V_2$, $\mathcal{S}V_1$ with $n(V_1) > 0$, or $fV_1 \ldots V_m$ for $arity(f) > m$, so that $U \equiv [x]V_{i_1 \ldots i_k}$ is

- $\mathcal{S}([x]\mathcal{S}V_1')([x]V_2')$ or $\mathcal{S}V_1'$ if $V \equiv \mathcal{S}V_1V_2$, and in either case the first argument to the outermost occurrence of $\mathcal{S}$ contains at least one occurrence of a nonvariable symbol since V_1 does.
- $\mathcal{S}(\mathcal{K}\mathcal{S})([x]V_1')$ if $V \equiv \mathcal{S}V_1$, since $V_1' \not\equiv x$. Clearly $n(\mathcal{K}\mathcal{S}) > 0$.
- $fV_1' \ldots V_{m-1}'$ or $\mathcal{S}([x]fV_1' \ldots V_{m-1}')([x]V_m')$ if $V \equiv fV_1 \ldots V_m$, and in the latter case, $n([x]fV_1' \ldots V_{m-1}') \geq n(fV_1' \ldots V_{m-1}') > 0$.

The properties of weak reduction will be important in obtaining our proof that R-strong normal forms are $>^R$-irreducible. Our first observation is that redex schemes other than those for weak reduction are not weakly reducible.

Lemma 4.26 *Let U be a redex scheme. If U is not basic, then U is weakly irreducible.*

Proof: If U is not basic, then $U \equiv [x]V_{i_1 \ldots i_k}$ and x appears in $V_{i_1 \ldots i_k}$ for some $>^R$-redex V. If V is basic, then $V_{i_1 \ldots i_k}$ is of the form V_1V_2 with x appearing in V_1V_2 and V_1 and V_2 weakly irreducible. By Lemma 3.11, then $[x]V_1$ and $[x]V_2$ are weakly irreducible, and therefore $U \equiv [x]V_{i_1 \ldots i_k}$ is as well. If V is not basic, then V is weakly irreducible by the induction hypothesis, and therefore so is $V_{i_1 \ldots i_k}$. Again by Lemma 3.11, $U \equiv [x]V_{i_1 \ldots i_k}$ is also weakly irreducible.

The following fact is an easy consequence of confluence for weak reduction.

Lemma 4.27 *If V is weakly irreducible, $U \equiv [x]V$, and $Ux \overset{w}{\twoheadrightarrow} W$, then $W \overset{w}{\twoheadrightarrow} V$.*

Interpretations of weakly irreducible ob-schemes admit weak reductions only in the interpretation part of the term:

Lemma 4.28 *Suppose U is weakly irreducible, U contains at most one occurrence of each metavariable, and either U is itself a metavariable or else does not have a metavariable at the head. If $U[A_1 := M_1] \ldots [A_k := M_k] \overset{w}{\twoheadrightarrow} V$, then $V \equiv U[A_1 := M_1^0] \ldots [A_k := M_k^0]$ for some $M_1^0, \ldots, M_k^0$ such that $M_i \overset{w}{\twoheadrightarrow} M_i^0$ for all i.*

Proof: By induction on U. If U contains no occurrences of metavariables $A_1, \ldots, A_k$, then $U[A_1 := M_1] \ldots [A_k := M_k] \equiv V$, so we may take $M_i^0 \equiv M_i$ for $i = 1, \ldots, k$.

- If $U \equiv A_j$ for some j, then $U[A_1 := M_1] \ldots [A_k := M_k] \equiv M_j \overset{w}{\twoheadrightarrow} V$. Let $V \equiv M_j^0$ and $M_i^0 \equiv M_i$ for $i \neq j$. Then $V \equiv U[A_1 := M_1^0] \ldots [A_k := M_k^0]$ and $M_i \overset{w}{\twoheadrightarrow} M_i^0$ for all i.
- If $U \equiv TW$ for some T and W, then $U[A_1 := M_1] \ldots [A_k := M_k] \equiv T[A_1 := M_1] \ldots [A_k := M_k]W[A_1 := M_1] \ldots [A_k := M_k] \overset{w}{\twoheadrightarrow} V$. Since U is weakly irreducible and the rules for weak reduction are shallow, if $T[A_1 := M_1] \ldots [A_k := M_k]W[A_1 := M_1] \ldots [A_k := M_k]$ contains any weak redexes, it must be because the M_i's contain some. Thus we must have $V \equiv T[A_1 := M_1^0] \ldots [A_k := M_k^0]W[A_1 := M_1^0] \ldots [A_k := M_k^0]$ with $M_i \overset{w}{\twoheadrightarrow} M_i^0$ for all i, i.e., $V \equiv U[A_1 := M_1^0] \ldots [A_k := M_k^0]$ with $M_i \overset{w}{\twoheadrightarrow} M_i^0$ for $i = 1, \ldots, k$.

We are now ready to prove that an ob in R-strong normal form is $>_{SR}$-irreducible by proving Theorem 4.21.

Proof: By induction on the definition of R-strong normal form.

- If $M \equiv x$, then M contains no $>^R$-redex by Lemma 4.23.
- If $M \equiv xM_1 \dots M_k$ with $M_1, \dots, M_k$ in R-strong normal form, then by the induction hypothesis, each M_i contains no $>^R$-redex. Thus the only possible $>^R$-redex in M is M itself, which is impossible (this uses part 2 of Lemma 4.25).
- If $M \equiv fM_1 \dots M_k$ with $M_1, \dots, M_k$ in R-strong normal form, then by the induction hypothesis, each M_i contains no $>^R$-redex. Thus the only possible $>^R$-redex is M itself. Since M is in R-strong normal form, $arity(f) = k$, and if M is a $>^R$-redex it must be by virtue of axiom scheme 7. But then $M \equiv \sigma U$ for U an R-redex scheme, contradicting M in R-normal form, and therefore M in R-strong normal form.
- If $M \equiv [x]N$ with N in R-strong normal form, then by the induction hypothesis, N contains no $>^R$-redexes. The remainder of the proof is by induction on N, noting that, in particular, N is weakly irreducible since it is $>^R$-irreducible.

  1. If $N \equiv x$, then $M \equiv I$ which is not a $>^R$-redex by part 2 of Lemma 4.25.
  2. If x does not appear in N, then $M \equiv \mathcal{K}N$. By part 2 of Lemma 4.25, neither $M \equiv \mathcal{K}N$ nor $\mathcal{K}$ itself are $>^R$-redexes, so the only possible redexes of M are those of N. But N is $>^R$-irreducible.
  3. If $N \equiv Mx$, then since N contains no $>^R$-redexes, clearly X contains none.
  4. If $N \equiv N_1N_2$, then $M \equiv [x]N_1N_2 \equiv \mathcal{S}([x]N_1)([x]N_2) \equiv \mathcal{S}M_1M_2$ where $M_i \equiv [x]N_i$ for $i = 1, 2$. Moreover, since N is $>^R$-irreducible, so are N_1 and N_2. But then $[x]N_1$ and $[x]N_2$ must also be $>^R$-irreducible by the induction hypothesis on N, so the only possible $>^R$-redexes in M are $\mathcal{S}M_1$ and M itself by part 2 of Lemma 4.25. We consider the two cases separately.

     (a) Suppose $\mathcal{S}M_1$ is a redex. Then $\mathcal{S}M_1$ is an instance of a redex scheme $\mathcal{S}U$. If $\mathcal{S}U$ is basic, then $\mathcal{S}U \equiv \mathcal{S}(\mathcal{K}I)$, so that $\mathcal{K}I \equiv [x]N_1$. Then $N_1 \equiv I$ contradicting the hypothesis that $N \equiv N_1N_2 \equiv IN_2$ is weakly irreducible. So $\mathcal{S}U \equiv [x]V_{i_1 \dots i_k}$ for some redex scheme V. But then V cannot be basic, and cannot be $fV_1 \dots V_n$ with $n < arity(f)$ or $\mathcal{S}V_1$, either (by inspection). By Lemma 4.25, part 2, the only other possibility is that $V \equiv \mathcal{S}V_1V_2$. But then $\mathcal{S}U \equiv [x]V_{i_1 \dots i_k} \equiv [x]\mathcal{S}V_{1i_1..i_k}V_{2i_1..i_k}$ implies $V_{2i_1 \dots i_k} \equiv x$, so that $[x]V_{i_1 \dots i_k} \equiv [x]\mathcal{S}Ux$. Finally, $V \equiv \mathcal{S}UA$ with A not appearing in U, and with the index corresponding to A being 3, and hence $M \equiv \mathcal{S}M_1M_2$ is an instance of $([x]V_{i_1 \dots i_k})M_2$ $\equiv ([x]\mathcal{S}Ux)M_2 \equiv \mathcal{S}UM_2$, and so is an instance of the redex scheme $\mathcal{S}UA \equiv V$. So M itself is a $>^R$-redex if $\mathcal{S}M_1$ is, and therefore this case may be included in the next.

     (b) Suppose $M \equiv \mathcal{S}M_1M_2$ is a redex, i.e., a substitution instance of the redex scheme $\mathcal{S}U_1U_2$. If $\mathcal{S}U_1U_2$ is basic, then $U_1 \equiv \mathcal{K}A$, so that $M_1 \equiv \mathcal{K}P \equiv [x]N_1$. This implies $N_1 \equiv P$ and x does not appear in P.

Now, either $U_2 \equiv I$ or $U_2 \equiv \mathcal{K}B$. If $U_2 \equiv I$, then $M_2 \equiv I$, so $N_2 \equiv x$ (since N_2 is weakly irreducible). But then $M \equiv \mathcal{S}M_1M_2 \equiv \mathcal{S}(\mathcal{K}P)I$ on one hand, and $M \equiv [x]N_1N_2 \equiv [x]Px \equiv P$ on the other. Clearly this is impossible. In the second case, $M_2 \equiv \mathcal{K}Q$ and $N_2 \equiv Q$ for x not appearing in N_2. But then x does not appear in N_1N_2, which implies that x is not in N, so that $M \equiv [x]N \equiv \mathcal{K}N_1$, contrary to $M \equiv \mathcal{S}M_1M_2$. So $\mathcal{S}U_1U_2$ cannot be basic. Therefore, $\mathcal{S}U_1U_2 \equiv [x]U_{i_1\ldots i_k}$ for a redex scheme U, so that M is an instance $([x]U_{i_1\ldots i_k})[A_1 := M_j]\ldots[A_n := M_n]$ of the axiom scheme $[x]U_{i_1\ldots i_k}$. Since $M \equiv [x]N$, we know M does not contain x, and so x does not appear in M_j for all j. By Lemma 4.13, $M \equiv [x](U[A_1 := M_1^{i_1}]\ldots[A_n := M_n^{i_n}]) \equiv [x]V$. Moreover, we must have $V \equiv V_1V_2$ and $M \equiv \mathcal{S}([x]V_1)([x]V_2)$ since otherwise $\mathcal{S}M_1M_2 \equiv [x]V$ implies that $V \equiv \mathcal{S}M_1M_2x$. But then $V \equiv U[A_1 := M_1^{i_1}]\ldots[A_k := M_k^{i_k}]$ for the redex scheme U, so that $U \equiv \mathcal{S}A_1A_2A_3$ with i_1 and i_2 equal to 1 and $i_3 = 3$ (there are no other redex schemes in which $\mathcal{S}$ takes three arguments). Thus $V \equiv \mathcal{S}M_1M_2x$ and $\mathcal{S}U_1U_2 \equiv \mathcal{S}A_1A_2$, which is not a $>^R$-redex scheme. Therefore, $M \equiv [x]N \equiv [x]V$, and by Lemma 4.27 the facts that $Mx \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow V$ and N is weakly irreducible together imply $V \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow N$. Similarly, $M_i \equiv [x]N_i \equiv [x]V_i$ implies $V_i \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow N_i$ for $i = 1, 2$. There are three cases:

- If U is weakly irreducible, and U is linear in its metavariables, then Lemma 4.28 implies that $N \equiv U[A_1 := M_1^{i_1 0}]\ldots[A_k := M_k^{i_k 0}]$ with $M_j^{i_j} \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow M_j^{i_j 0}$ for all j. This contradicts the fact that N is not a redex.
- If U is weakly irreducible, and U is not linear in its metavariables, then it must be an instance of the left-hand side of an axiom scheme based on scheme 7. We have $U[A_1 := M_1^{i_1}]\ldots[A_k := M_k^{i_k}] \equiv V \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow N$, so all the weak reductions must take place inside the terms $M_j^{i_j}$ or terms obtained from them by weak reductions. Since N is not an instance of U (this would contradict the fact that N is not a redex), there exists an A_j such that one occurrence of A_j is replaced by $M_j^{i_j\prime}$ in N and another is replaced by $M_j^{i_j\prime\prime}$ in N. But then since weak reduction is Church-Rosser, there exists an L_j which is a common reduct of $M_j^{i_j\prime}$ and $M_j^{i_j\prime\prime}$, and in the reductions $M_j^{i_j\prime} \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow L$ and $M_j^{i_j\prime\prime} \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow L$, we are assuming that there is at least one weak reduction done. So $N \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow U[A_1 := L_1]\ldots[A_k := L_k]$ with at least one step used here, contradicting the assumption that N is weakly irreducible.
- If U is weakly reducible, then $U \equiv \mathcal{S}A_1A_2A_3$ or $U \equiv \mathcal{K}A_1A_2$ or $U \equiv IA_1$, so that either $V \equiv \mathcal{S}M_1^{i_1}M_2^{i_2}M_3^{i_3}$ or $V \equiv \mathcal{K}M_1^{i_1}M_2^{i_2}$ or $V \equiv IM_1^{i_1}$. But since $V_1 \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow N_1$, either $\mathcal{S}M_1^{i_1}M_2^{i_2} \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow N_1$ and hence $N_1 \equiv \mathcal{S}N'N''$, or $\mathcal{K}M_1^{i_1} \xrightarrow{w}\!\!\!\!\!\twoheadrightarrow N_1$ and hence $N_1 \equiv \mathcal{K}N'$, or

$I \xrightarrow{w}\!\!\!\!\!\rightarrow N_1$, so that $N_1 \equiv I$. In each case, $N \equiv N_1 N_2$ contains a weak redex, contradicting the weak irreducibility of N. This final contradiction completes the proof.

Given a $C\mathcal{L}$-term M in R-strong normal form, we may disregard its type information, i.e., we may consider its type-erasure, and conclude via our restated proposition that the resulting ob, which is also in R-strong normal form, is $>^R$-irreducible. This ob is therefore also $>_{SR}$-irreducible, and so M must be as well.

This completes the proof of Theorem 4.7, and hence of Theorem 3.9.

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REFERENCES

[1] Barendregt, H., *The Lambda Calculus, Its Syntax and Semantics*, North-Holland, Amsterdam, 1984. Zbl 0551.03007 MR 86a:03012

[2] Breazu-Tannen V., "Combining algebra and higher-order types," pp. 82–90 in *Proceedings of the Third Annual Symposium on Logic in Computer Science*, 1988. 1, 2

[3] Breazu-Tannen, V., and J. Gallier, "Polymorphic rewriting conserves algebraic strong normalization," *Theoretical Computer Science*, vol. 83 (1991), pp. 3–28. Zbl 0745.68065 1

[4] Breazu-Tannen, V., and J. Gallier, "Polymorphic rewriting conserves algebraic confluence," *Information and Computation*, vol. 114 (1994), pp. 1–29. Zbl 0820.68059 MR 95i:68066 1

[5] Curry, H., and R. Feys, *Combinatory Logic, Vol. I*, North-Holland, Amsterdam, 1958. Zbl 0081.24104 MR 20:817 1, 3.1, 3.1, 3.1, 4

[6] Curry, H., J. Hindley and J. Seldin, *Combinatory Logic, Vol. II*, North-Holland, Amsterdam, 1972. Zbl 0242.02029

[7] Dershowitz, N., and J.-P. Jouannaud, "Rewrite systems," pp. 243–320 in *Handbook of Theoretical Computer Science B: Formal Methods and Semantics*, edited by J. van Leeuwen, North-Holland, Amsterdam, 1990. Zbl 0900.68283 MR 1127191 2

[8] Dougherty, D., "Adding algebraic rewriting to the untyped lambda calculus," pp. 37–48 in *Proceedings of the Fourth International Conference on Rewriting Techniques and Applications*, Springer-Verlag LNCS 488, 1991. Also appears in *Information and Computation*, vol. 101, 102 (1992), pp. 251–267. Zbl 0769.68058 MR 94g:68060 1

[9] Dougherty, D., and P. Johann, "A combinatory logic approach to higher-order E-unification," pp. 79–93 in *Proceedings of the Eleventh International Conference on Automated Deduction*, Springer-Verlag LNAI 607, 1992. Expanded version appears in *Theoretical Computer Science*, vol. 139 (1995), pp. 207-242. Zbl 0874.68271 MR 96d:68113 1

[10] Gallier, J., and W. Snyder, "Higher-order unification revisited: complete sets of transformations," *Journal of Symbolic Computation*, vol. 8 (1989), pp. 101–140. Zbl 0682.03034 MR 90j:03025 1

[11] Hindley, R., "Axioms for strong reduction in combinatory logic," *Journal of Symbolic Logic*, vol. 32 (1967), pp. 224–236. Zbl 0153.00603 MR 35:5320 1, 3.1, 4

[12] Hindley, R., "Combinatory reductions and lambda reductions compared," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 23 (1977), pp. 169–180. Zbl 0361.02035 MR 58:5080

[13] Hindley, R., and B. Lercher, "A short proof of Curry's normal form theorem," pp. 808–810 in *Proceedings of the American Mathematical Society*, vol. 24, 1970. Zbl 0195.02101 MR 41:67 1, 3.1, 4

[14] Hindley, J., B. Lercher, and J. Seldin, *Introduction to Combinatory Logic*, Cambridge University Press, Cambridge, 1972. Zbl 0269.02005 MR 49:25

[15] Hindley, J., and J. Seldin. *Introduction to Combinators and λ-Calculus*, Cambridge University Press, Cambridge, 1986. Zbl 0614.03014 MR 88j:03009 1, 2, 2, 3.1

[16] Huet, G., "A unification algorithm for typed λ-calculus," *Theoretical Computer Science*, vol. 1 (1975), pp. 27–57. Zbl 0337.68027 1

[17] Johann, P., *Complete sets of transformations for unification problems*, Dissertation, Wesleyan University, 1991.

[18] Johann, P., "Normal forms in combinatory logic," Technical Report, Wesleyan University, 1992. Zbl 0830.03005 MR 96d:03025

[19] Klop, J., "Combinatory reduction systems," *Mathematical Center Tracts* 129, Amsterdam, 1980. Zbl 0466.03006 MR 83e:03026

[20] Lercher, B., "The decidability of Hindley's axioms for strong reduction," *Journal of Symbolic Logic*, vol. 32 (1967), pp. 237–239. Zbl 0153.00701 MR 35:5321 1

[21] Lercher, B., "Strong reduction and normal forms in combinatory logic," *Journal of Symbolic Logic*, vol. 32 (1967), pp. 213–223. MR 36:2505 1, 3.1, 3.1, 3.2, 4

[22] Mezghiche, M., "On pseudo-$c\beta$ normal form in combinatory logic," *Theoretical Computer Science*, vol. 66 (1989), pp. 323–331. Zbl 0673.03010 MR 91e:03018 1

[23] Réty, P., "Improving basic narrowing techniques," pp. 228-241 in *Proceedings of the Second International Conference on Rewriting Techniques and Applications*, 1987. Zbl 0638.68101 MR 903675

Department of Computer Science and Engineering
Oregon Graduate Institute
P.O. Box 91000
Portland, OR 97291-1000