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Representations for Small Relation Algebras

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Abstract There are eighteen isomorphism types of finite relation algebras with eight or fewer elements, and all of them are representable. We determine all the cardinalities of sets on which these algebras have representations.

1 Introduction We say that a relation algebra is *small* if it has no more than eight elements. A relation algebra is a Boolean algebra with additional operators, so every small relation algebra has cardinality 1, 2, 4, or 8. There are eighteen isomorphism types of small relation algebras. One of the types contains one-element algebras, thirteen of them contain simple algebras, and the remaining four contain direct products of simple relation algebras.

A simple or one-element relation algebra $\mathfrak{A}$ is *representable* if it is isomorphic to a subalgebra of $\mathfrak{Re}\,U$, for some set U, where $\mathfrak{Re}\,U = \left\langle Re\,U, \cup, \sim, |, {}^{-1}, Id_U \right\rangle$ is the relation algebra of all binary relations on the set U. A *representation of* $\mathfrak{A}$ *on* U is an isomorphism that embeds $\mathfrak{A}$ into $\mathfrak{Re}\,U$. Direct products of representable relation algebras are also representable. It has long been known that every simple small relation algebra is representable. Therefore all small relation algebras are representable.

Let $\mathfrak{G} = \left\langle G, \cdot, {}^{-1}, 1 \right\rangle$ be a group. The *complex algebra* of $\mathfrak{G}$ is the Boolean algebra of all subsets of G augmented with the binary operation ; defined by $X ; Y = \{xy : x \in X,\ y \in Y\}$ for all $X, Y \subseteq G$, the unary operation ${}^{-1}$, where $X^{-1} = \{x^{-1} : x \in X\}$, and the distinguished subset $\{1\}$. Using the fact that $\mathfrak{G}$ is isomorphic to a group of permutations of G, it is easy to show that the complex algebra of $\mathfrak{G}$ is representable.

Many of the results presented here are quite elementary and previously known, if not explicitly stated in print. For example, Lyndon noted (in [16], p. 307, footnote 13) that every small integral relation algebra is commutative and isomorphic to a subalgebra of the complex algebra of either the group of rational numbers under addition, or a cyclic group of order not exceeding thirteen. (See also McKenzie [19], p. 286, Section 5). This observation implies that every small relation algebra is representable, and that some special representations exist for these algebras. Therefore

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Lyndon knew that every small relation algebra is representable on a set of cardinality either ω or else not more than thirteen. Some explicit representations for simple small relation algebras were given in McKenzie's dissertation [18], pp. 38–40, and in Backer's seminar report [1], pp. 11–20. A simple 8-element relation algebra was missed in the latter survey, and an 8-element direct product of two simple relation algebras was omitted from the former. [1] contains a representation on infinitely many elements for the relation algebra of type #18 (defined below). A representation on 9 elements was found by Wostner and communicated to the second author in the early 1970s. This 9-element representation has undoubtedly been found independently. Comer [8] and others have accumulated many representations for many finite relation algebras. This work is still in progress. Not much of this work has been published, but see Comer [2], [3], [4], [5], [6], and [7], and Wostner [20]. We decided to undertake a survey of representations of small relation algebras with the following goal in mind: to determine all the cardinalities of sets on which small relation algebras have representations, and the cardinalities for which the representations are uniquely determined. The investigation has produced a few surprises.

The *spectrum* of a simple or one-element relation algebra $\mathfrak{A}$ is the set of cardinalities of sets on which there is a representation of $\mathfrak{A}$, so

$$\mathrm{spec}\,(\mathfrak{A}) = \{\kappa : \mathfrak{A} \in \mathsf{IS}\mathfrak{Re}\,U,\ |U| = \kappa\}\,.$$

There are different ways to extend the notion of spectrum to algebras that are not simple, but we do not consider such extensions here. In any case, the representations of an algebra that is not simple are completely determined by the representations of its simple homomorphic images.

In this paper we determine the spectra of all thirteen types of simple small relation algebras. We will also determine some cardinalities for which a given simple small relation algebra $\mathfrak{A}$ has a unique representation. If I and H are representations of $\mathfrak{A}$ on U and V, respectively, then I and H are *conjugate* if there is a bijection f from U to V such that, for every $x \in A$, $H(x) = f^{-1}|I(x)|f$. The representation I of $\mathfrak{A}$ on U is *unique* iff I is conjugate to every representation of $\mathfrak{A}$ on a set with the same cardinality as U. The representation I is *minimal* if there is no representation of $\mathfrak{A}$ on a smaller set, i.e., if H embeds $\mathfrak{A}$ into $\mathfrak{Re}\,V$ then $|V| \geq |U|$. We will see that the minimal representations of the thirteen simple small relation algebras are *all* unique. Let κ be any cardinal. The representation I is *κ-extendible* if there is a representation H of $\mathfrak{A}$ on a set V such that $U \subseteq V$, $|V \sim U| = \kappa$, and I is the restriction of H to U, i.e., $I(x) = H(x) \cap (U \times U)$ for every element x of $\mathfrak{A}$. A representation is *finitely extendible* if it is κ-extendible for some finite $\kappa > 0$. The representation I is *κ-redundant* if the restriction of I to $W \subseteq U$ is a representation of $\mathfrak{A}$ whenever $|U \sim W| = \kappa$.

2 *Relation algebras and first-order theories* Let $\mathfrak{A}$ be a simple finite relation algebra with n atoms. Suppose $At\mathfrak{A} = \{a_0, \ldots, a_{n-1}\}$, and, for some $k < n$, $1' = a_0 + \cdots + a_k$. Let Ł be a first-order language with equality and binary relation symbols $\mathsf{R}_0, \ldots, \mathsf{R}_{n-1}$. Let $Th(\mathfrak{A})$ be the theory in Ł determined by $\mathfrak{A}$ as follows:

1. $\forall x \forall y(x\mathsf{R}_0 y \vee \ldots \vee x\mathsf{R}_{n-1} y)$ is in $Th(\mathfrak{A})$, and $\forall x \forall y(x\mathsf{R}_i y \to \neg x\mathsf{R}_j y)$ is in $Th(\mathfrak{A})$ whenever $i < j < n$,

2. $\forall x \forall y(x = y \leftrightarrow x\mathsf{R}_0 y \vee \ldots \vee x\mathsf{R}_k y)$ is in $Th(\mathfrak{A})$,
3. if $\breve{a}_i = a_j$ and $i < j < n$, then $\forall x \forall y(x\mathsf{R}_j y \leftrightarrow y\mathsf{R}_i x)$ is in $Th(\mathfrak{A})$,
4. if $i, j < n$, $0 < m$ and $a_i ; a_j = a_{k_0} + \cdots + a_{k_{m-1}}$, then $\forall x \forall y(\exists z(x\mathsf{R}_i z \wedge z\mathsf{R}_j y) \leftrightarrow x\mathsf{R}_{k_0} y \vee \ldots \vee x\mathsf{R}_{k_{m-1}} y)$ is in $Th(\mathfrak{A})$.

It is fairly easy to prove that if $\mathfrak{A}$ is a finite simple relation algebra, then $\mathfrak{A}$ is representable iff $Th(\mathfrak{A})$ has a model. In fact, if $\mathfrak{U} = \langle U, R_0, \ldots, R_{n-1}\rangle$ is a model of $Th(\mathfrak{A})$, where $R_0, \ldots, R_{n-1}$ are binary relations on U, then the function h, defined by $h(a_i) = R_i$ for all $i < n$, has a unique extension to an embedding H of $\mathfrak{A}$ into $\mathfrak{Re}\,U$. Conversely, if $\mathfrak{A}$ is representable, then there is an isomorphism H from $\mathfrak{A}$ into $\mathfrak{Re}\,U$ for some set U, and $\langle U, H(a_0), \ldots, H(a_{n-1})\rangle$ is a model of $Th(\mathfrak{A})$.

A similar construction is used in [18], pp. 108–109, and also in Jónsson [11]. McKenzie and Jónsson use a relation symbol for every element of the algebra. Their method therefore works even for infinite relation algebras, but for finite ones the resulting theory is exponentially larger than the one built here.

This connection between relation algebras and theories allows results about finite relation algebras to be translated into results about their first-order theories. The spectrum of a finite simple relation algebra $\mathfrak{A}$ is defined in such a way that it coincides with the (model-theoretic) spectrum of the associated theory $Th(\mathfrak{A})$. Thus $\mathfrak{A}$ has a unique representation on α points just in case the theory $Th(\mathfrak{A})$ is categorical in power α. Also, if spec$(\mathfrak{A})$ is infinite or contains an infinite cardinal, then $\{\kappa : \omega \leq \kappa\} \subseteq$ spec$(\mathfrak{A})$, by the Upward Löwenheim-Skolem-Tarski Theorem.

3 The eighteen types of small relation algebras Listed below are the eighteen types of small relation algebras. For each type we give a representative algebra, and state its spectrum if it is simple or trivial. If the type contains simple algebras, then the representative algebra is a subalgebra of $\mathfrak{Re}\,\kappa$ that is generated either by the empty set, or by a single relation on κ, where κ is the minimum cardinal in the spectrum. For any algebra $\mathfrak{C}$ and any subset X of the universe of $\mathfrak{C}$, we let $\mathfrak{Sg}^{(\mathfrak{C})}X$ be the subalgebra of $\mathfrak{C}$ that is generated by X.

The relations we use for the representative algebras are defined as follows. Let $\mathbb{Q}$ be the set of rational numbers, and let $L = \{\langle x, y\rangle : x, y \in \mathbb{Q},\ x < y\}$. Whenever $\kappa < \alpha < \omega$, let $\mathsf{P}^\alpha_\kappa = \{\langle \lambda, \mu\rangle : \lambda, \mu < \alpha,\ \mu - \lambda \equiv \kappa \pmod{\alpha}\}$ and $\mathsf{Q}^\alpha_\kappa = \mathsf{P}^\alpha_\kappa \cup (\mathsf{P}^\alpha_\kappa)^{-1}$. If, in addition, $\lambda < \beta < \omega$, then $\mathsf{Q}^{\alpha,\beta}_{\kappa,\lambda} = \left\{\langle\langle \mu, \nu\rangle, \langle \xi, \zeta\rangle\rangle : \langle \mu, \xi\rangle \in \mathsf{Q}^\alpha_\kappa,\ \langle \nu, \zeta\rangle \in \mathsf{Q}^\beta_\lambda\right\}$.

Suppose $\mathfrak{A}$ is a finite relation algebra. Then

$$C(\mathfrak{A}) = \left\{\langle a, b, c\rangle : a, b, c \in At\mathfrak{A},\ a ; b \geq c\right\}.$$

The triples in $C(\mathfrak{A})$ are called *cycles.* For any atoms a, b, c,

$$[a, b, c] = \left\{\langle a, b, c\rangle, \langle \breve{a}, c, b\rangle, \langle b, \breve{c}, \breve{a}\rangle, \langle \breve{b}, \breve{a}, \breve{c}\rangle, \langle \breve{c}, a, \breve{b}\rangle, \langle c, \breve{b}, a\rangle\right\}.$$

Then either $[a, b, c] \subseteq C(\mathfrak{A})$ or else $[a, b, c] \cap C(\mathfrak{A}) = \varnothing$. Thus $C(\mathfrak{A})$ is a union of sets of the form $[a, b, c]$, each of which may contain up to six different cycles. The cycles completely determine the operation ;, since $a ; b = \sum\{c : \langle a, b, c\rangle \in C(\mathfrak{A})\}$ for all atoms a, b.

For each isomorphism type we describe a typical algebra by listing its atoms and cycles. Let a be an atom. We say that a is an identity atom if $a \leq 1'$, a is symmetric if

$\breve{a} = a$, and a is antisymmetric if a and $\breve{a}$ are disjoint. We first list the identity atoms, then the symmetric atoms, and finally the pairs of antisymmetric atoms. We use e_0, e_1, etc., to denote identity atoms, unless 1' is itself an atom. Pairs of antisymmetric atoms are denoted by a, $\breve{a}$, and b, $\breve{b}$, etc., while symmetric atoms are denoted by just a, b, etc., since they coincide with their converses. Thus 1' and $\breve{}$ are determined by the list of atoms.

#1 Representative: $\mathfrak{Re}\,0$. Spectrum: $\{0\}$.
No atoms. No cycles.

#2 Representative: $\mathfrak{Re}\,1$. Spectrum: $\{1\}$.
Atom: 1'. Cycle: [1', 1', 1'].

#3 Representative: $\mathfrak{Re}\,1 \times \mathfrak{Re}\,1$.
Atoms: e_0, e_1. Cycles: $[e_0, e_0, e_0]$, $[e_1, e_1, e_1]$.

#4 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}2)}\varnothing$. Spectrum: $\{2\}$.
Atoms: 1', 0'. Cycles: [1', 1', 1'], [1', 0', 0'].

#5 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}3)}\varnothing$. Spectrum: $\{\kappa : 3 \le \kappa\}$.
Atoms: 1', 0'. Cycles: [1', 1', 1'], [1', 0', 0'], [0', 0', 0'].

#6 Representative: $\mathfrak{Re}\,1 \times \mathfrak{Re}\,1 \times \mathfrak{Re}\,1$.
Atoms: e_0, e_1, e_2. Cycles: $[e_0, e_0, e_0]$, $[e_1, e_1, e_1]$, $[e_2, e_2, e_2]$.

#7 Representative: $\mathfrak{Re}\,1 \times \mathfrak{Sg}^{(\mathfrak{Re}2)}\varnothing$.
Atoms: e_0, e_1, 0'. Cycles: $[e_0, e_0, e_0]$, $[e_1, e_1, e_1]$, $[e_1$, 0', 0'].

#8 Representative: $\mathfrak{Re}\,1 \times \mathfrak{Sg}^{(\mathfrak{Re}3)}\varnothing$.
Atoms: e_0, e_1, 0'. Cycles: $[e_0, e_0, e_0]$, $[e_1, e_1, e_1]$, $[e_1$, 0', 0'], [0', 0', 0'].

#9 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}3)}\{\mathsf{P}^3_1\}$. Spectrum: $\{3\}$.
Atoms: 1', a, $\breve{a}$. Cycles: [1', 1', 1'], [1', a, a], [a, 1', a], [a, a, $\breve{a}$].

#10 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}\mathbb{Q})}\{L\}$. Spectrum: $\{\kappa : \omega \le \kappa\}$.
Atoms: 1', a, $\breve{a}$. Cycles: [1', 1', 1'], [1', a, a], [a, 1', a], [a, a, a].

#11 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}7)}\{\mathsf{P}^7_1 \cup \mathsf{P}^7_2 \cup \mathsf{P}^7_4\}$. Spectrum: $\{7\} \cup \{\kappa : 9 \le \kappa\}$.
Atoms: 1', a, $\breve{a}$. Cycles: [1', 1', 1'], [1', a, a], [a, 1', a], [a, a, $\breve{a}$], [a, a, a].

#12 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}4)}\{\mathsf{Q}^4_1\}$. Spectrum: $\{4\}$.
Atoms: 1', a, b. Cycles: [1', 1', 1'], [1', a, a], [1', b, b], [a, b, b].

#13 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}6)}\{\mathsf{Q}^6_2\}$. Spectrum: $\{\kappa : 6 \le \kappa\}$.
Atoms: 1', a, b. Cycles: [1', 1', 1'], [1', a, a], [1', b, b], [a, b, b], [a, a, a].

#14 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}6)}\{\mathsf{Q}^6_3\}$. Spectrum: $\{2\kappa : 3 \le \kappa\}$.
Atoms: 1', a, b. Cycles: [1', 1', 1'], [1', a, a], [1', b, b], [a, b, b], [b, b, b].

#15 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}9)}\{\mathsf{Q}^9_3\}$. Spectrum: $\{\kappa : 9 \le \kappa\}$.
Atoms: 1', a, b. Cycles: [1', 1', 1'], [1', a, a], [1', b, b], [a, b, b], [a, a, a], [b, b, b].

#16 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}5)}\{\mathsf{Q}^5_1\}$. Spectrum: $\{5\}$.
Atoms: 1', a, b. Cycles: [1', 1', 1'], [1', a, a], [1', b, b], [a, b, b], [a, a, b].

#17 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}8)}\{\mathsf{Q}^8_1 \cup \mathsf{Q}^8_4\}$. Spectrum: $\{\kappa : 8 \le \kappa\}$.
Atoms: 1', a, b. Cycles: [1', 1', 1'], [1', a, a], [1', b, b], [a, b, b], [a, a, b], [b, b, b].

#18 Representative: $\mathfrak{Sg}^{(\mathfrak{Re}(3\times 3))}\{Q^{3,3}_{1,1}\}$. Spectrum: $\{\kappa : 9 \leq \kappa\}$.
Atoms: 1', a, b. Cycles: [1', 1', 1'], [1', a, a], [1', b, b], [a, b, b], [a, a, b], [a, a, a], [b, b, b].

The action of ; on pairs of atoms can be deciphered from the list of cycles, but it is rather tedious to do so. The tables for the products of atoms in the algebras described above are therefore given here explicitly. Many properties of the algebras can be more easily recognized from these tables.

# 2	1'
1'	1'

#3	e_0	e_1
e_0	e_0	0
e_1	0	e_1

#4	1'	0'
1'	1'	0'
0'	0'	1'

#5	1'	0'
1'	1'	0'
0'	0'	1

#6	e_0	e_1	e_2
e_0	e_0	0	0
e_1	0	e_1	0
e_2	0	0	e_2

#7	e_0	e_1	0'
e_0	e_0	0	0
e_1	0	e_1	0'
0'	0	0'	e_1

#8	e_0	e_1	0'
e_0	e_0	0	0
e_1	0	e_1	0'
0'	0	0'	$e_1 + 0'$

#9	1'	a	$\breve{a}$
1'	1'	a	$\breve{a}$
a	a	$\breve{a}$	1'
$\breve{a}$	$\breve{a}$	1'	a

#10	1'	a	$\breve{a}$
1'	1'	a	$\breve{a}$
a	a	a	1
$\breve{a}$	$\breve{a}$	1	a

#11	1'	a	$\breve{a}$
1'	1'	a	$\breve{a}$
a	a	0'	1
$\breve{a}$	$\breve{a}$	1	0'

#12	1'	a	b
1'	1'	a	b
a	a	1'	b
b	b	b	$1'+a$

#13	1'	a	b
1'	1'	a	b
a	a	$1'+a$	b
b	b	b	$1'+a$

#14	1'	a	b
1'	1'	a	b
a	a	1'	b
b	b	b	1

#15	1'	a	b
1'	1'	a	b
a	a	$1'+a$	b
b	b	b	1

#16	1'	a	b
1'	1'	a	b
a	a	$1'+b$	0'
b	b	0'	$1'+a$

#17	1'	a	b
1'	1'	a	b
a	a	$1'+b$	0'
b	b	0'	1

#18	1’	a	b
1’	1’	a	b
a	a	1	0’
b	b	0’	1

4 Minimal Representations In this section we give representations for each of the thirteen types of simple small relation algebras, and prove that these representations are minimal and unique. Among the eighteen types of small relation algebras, those that are simple are the ones of type #2, #4, #5, and #9 – 18. To describe the representations, it is necessary to specify just the images of the atoms. This is done in the following tables, except for type #18, where the image of b is $(3 \times 3) \sim Id_{3\times 3} \sim \mathsf{Q}^{3,3}_{1,1}$, a relation for which we have introduced no convenient designation.

	1’	0’
#2	Id_1	$\varnothing$
#4	Id_2	Di_2
#5	Id_3	Di_3

	1’	a	$\breve{a}$
#9	Id_3	P^3_1	P^3_2
#10	$Id_{\mathbb{Q}}$	L	L^{-1}
#11	Id_7	$\mathsf{P}^7_1 \cup \mathsf{P}^7_2 \cup \mathsf{P}^7_4$	$\mathsf{P}^7_6 \cup \mathsf{P}^7_5 \cup \mathsf{P}^7_3$

	1’	a	b
#12	Id_4	Q^4_2	Q^4_1
#13	Id_6	Q^6_2	$\mathsf{Q}^6_1 \cup \mathsf{Q}^6_3$
#14	Id_6	Q^6_3	$\mathsf{Q}^6_1 \cup \mathsf{Q}^6_2$
#15	Id_9	Q^9_3	$\mathsf{Q}^9_1 \cup \mathsf{Q}^9_2 \cup \mathsf{Q}^9_4$
#16	Id_5	Q^5_1	Q^5_2
#17	Id_8	$\mathsf{Q}^8_1 \cup \mathsf{Q}^8_4$	$\mathsf{Q}^8_2 \cup \mathsf{Q}^8_3$
#18	$Id_{3\times 3}$	$\mathsf{Q}^{3,3}_{1,1}$	

From results in Jónsson and Tarski [14] we can draw some conclusions concerning some of the small algebras. Recall that an element x is functional if $\breve{x} ; x \leq 1’$. The algebras in which every atom is a functional element are those of type #2, #3, #4, #6, #7, and #9. These algebras are all representable by Theorem 4.29 of [14]. Among these algebras, the ones of type #2, #4, and #9 are integral. It follows from Theorem 5.11 of [14] that #2, #4, and #9 are isomorphic to the complex algebras of certain groups. By Theorem 4.32 of [14], a simple relation algebra in which 1 is the join of m functional elements is representable on a set with at most m elements. It follows that the integers 1, 2, and 3 belong to the spectra of #2, #4, and #9, respectively. Among the small algebras that are simple, the ones in which $0’ ; 0’ \leq 1’$ are #2 and #4. By Theorem 4.33 of [14] these algebras are both representable on sets containing at most two elements. The only simple small algebra in which 0' ; 0' $= 0$ is #2. By Theorem 4.35 (i) of [14], algebra #2 is representable on a set containing exactly one element. The only simple small algebra in which 0' ; 0' $=$ 1' is #4. By Theorem 4.35 (ii) of [14], algebra #4 is representable on a set containing exactly two elements. By Theorem 4.35 (iii) of [14], if $\mathfrak{A}$ is a simple representable relation algebra in which 0' ; 0' $= 1$, then neither 1 nor 2 is in the spectrum of $\mathfrak{A}$. These observations serve to prove several parts of the following theorem. (The theorem is so elementary that it must be regarded as well known by everyone who has studied small relation algebras.)

Theorem 4.1 *Assume that $\mathfrak{A}$ is a small simple relation algebra, and that I is a representation of $\mathfrak{A}$ on U.*

(i) If $\mathfrak{A}$ is #2 then $|U| = 1$, I is unique, and spec $(\mathfrak{A}) = \{1\}$.
(ii) If $\mathfrak{A}$ is #4 then $|U| = 2$, I is unique, and spec $(\mathfrak{A}) = \{2\}$.
(iii) If $\mathfrak{A}$ is #5 then $|U| \geq 3$, I is unique, and spec $(\mathfrak{A}) = \{\kappa : \kappa \geq 3\}$.
(iv) If $\mathfrak{A}$ is #9 then $|U| = 3$, I is unique, and spec $(\mathfrak{A}) = \{3\}$.
(v) If $\mathfrak{A}$ is #10 then $|U| \geq \omega$, I is unique if $|U| = \omega$, and spec $(\mathfrak{A}) = \{\kappa : \kappa \geq \omega\}$.

Note that the table for type #9 is actually the multiplication table for a three-element group. According to the table, $\breve{a} ; a =$ 1', $a ; \breve{a} =$ 1', and $a + \breve{a} =$ 0'. These equations imply that $I(a)$ is a permutation of U that is disjoint from its inverse, such that $I(a) \cup I(a)^{-1} = Di_U$. Such a permutation can only be a cyclic permutation of a 3-element set. Part (iv) is actually a special case of the easily proved fact that the complex algebra of a group of order n has a one-element spectrum, namely $\{n\}$. This fact can also be easily generalized.

Concerning part (v), note that if $\mathfrak{A}$ is type #10, then the theory $Th(\mathfrak{A})$ states that $I(a)$ is a dense linear ordering of U without endpoints. Also, the relation algebra generated by any dense linear ordering without endpoints is of type #10. The theory of dense linear orderings without endpoints has spectrum $\{\kappa : \kappa \geq \omega\}$ and is categorical in power ω, so the same is true of $\mathfrak{A}$.

The next theorem contains one of the surprises of our investigation. The representation of algebra #11 on seven elements is well known, as is its uniqueness, although this has not yet been explicitly stated in print. From the evidence of the spectra of the other small algebras it would be natural to suspect that the spectrum of type #11 would be $\{n : n \geq 7\}$, but it turns out that 8 is missing. This was discovered first by a computer search.

Theorem 4.2 *If $\mathfrak{A}$ is algebra #11 and I is a representation of $\mathfrak{A}$ on U, then $|U| \geq 7$, I is unique if $|U| = 7$, and spec $(\mathfrak{A}) = \{7\} \cup \{\kappa : \kappa \geq 9\}$.*

Proof: Let $A = I(a)$. We will use the following proposition several times.

(1) Assume $u \in U$, and either $X = \{x : uAx\}$ or $X = \{x : xAu\}$. For every $x \in X$ there are $y, z \in X$ such that $yAxAz$, u, x, y, z are distinct, and thus $|X| \geq 3$.

To prove (1), suppose first that $X = \{x : uAx\}$. From 1' $\leq a ; \breve{a}$ we get $\langle u, u\rangle \in Id_U = I(1') \subseteq I(a ; \breve{a}) = A|A^{-1}$, so $X \neq \varnothing$. Let $x \in X$. From $a \leq a ; a \cdot a ; \breve{a}$ we get $\langle u, x\rangle \in A = I(a) \subseteq I(a ; a \cdot a ; \breve{a}) = A|A \cap A|A^{-1}$, so there are $y, z \in U$ such that uAy, yAx, uAz, and $zA^{-1}x$. Therefore $y, z \in X$, $yAxAz$, and u, x, y, z are distinct since $a \cdot 1' = 0 = a \cdot \breve{a}$. If $X = \{x : x A u\}$, then the same conclusions follow from $\breve{a} \leq \breve{a} ; \breve{a} \cdot \breve{a} ; a$.

It follows immediately from (1) that $|U| \geq 7$.

Suppose $|U| = 7$. Choose $u \in U$. Let $X = \{x : x A u\}$ and $Y = \{y : u A y\}$. Since $|U| = 7$, (1) implies that $|X| = |Y| = 3$ and the restrictions of A to X and to Y are 3-cycles. Hence there are $x_1, x_2, x_3, y_1, y_2, y_3$ such that $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, $x_1 A x_2$, $x_2 A x_3$, $x_3 A x_1$, $y_1 A y_2$, $y_2 A y_3$, and $y_3 A y_1$. By (1), x_1 must have exactly one other A-image, which must be in Y. So we may also assume $x_1 A y_1$, $y_2 A x_1$, and $y_3 A x_1$. By (1), the restrictions of A to $\{z : x_1 A z\} = \{u, x_2, y_1\}$ and to $\{z : z A x_1\} = \{x_3, y_2, y_3\}$ must be 3-cycles, so we also get $y_1 A x_2$, $y_3 A x_3$, and $x_3 A y_2$. Now y_2 must have one more A-image, hence $y_2 A x_2$, and then x_2 must have one more, so $x_2 A y_3$, and finally, y_1 must have one more A-image, which must be x_3. This completely determines A, so the representation is unique when $|U| = 7$.

Assume $|U| = 8$. We will derive a contradiction. By (1), every $u \in U$ has either three or four A-images. If every $u \in U$ has exactly three A-images, then $|A| = 3 \cdot 8 = 24$, but $|A| = 28$, so we get some $u \in U$ with exactly four A-images, say y_1, y_2, y_3, and y_4. Let $Y = \{y_1, y_2, y_3, y_4\}$ and $X = \{x_1, x_2, x_3\} = \{z : z A u\}$. By (1) we may assume $x_1 A x_2$, $x_2 A x_3$, and $x_3 A x_1$. Also by (1), every $y \in Y$ has an A-image in Y, and is the A-image of something in Y. Up to isomorphism there is just one restriction of A to Y that has these properties, and we may therefore assume $y_1 A y_2$, $y_2 A y_3$, $y_3 A y_4$, $y_4 A y_1$, $y_1 A y_3$, and $y_2 A y_4$. Since $\langle y_2, y_3 \rangle \in I(0') = A|A$, there is some $z \in U$ such that $y_2 A z$ and $z A y_3$. Then $z \in X$, so we may assume $y_2 A x_1$ and $x_1 A y_3$. Then $\langle y_2, x_2 \rangle \in I(0') = A^{-1}|A$, so there must be some z such that $z A y_2$ and $z A x_2$, but the only possibility for z is y_1, so $y_1 A x_2$. There is some z such that $y_4 A z$ and $x_2 A z$, but the only possibility is x_3, so $y_4 A x_3$. Finally, there must be some z such that $x_3 A z$ and $y_3 A z$, but there is no possibility for z, so we have a contradiction. Thus $8 \notin \operatorname{spec}(\mathfrak{A})$.

We get a representation on 9 points by setting $I(a) = \mathsf{P}^9_1 \cup \mathsf{P}^9_2 \cup \mathsf{P}^9_4 \cup \mathsf{P}^9_6$, so $9 \in \operatorname{spec}(\mathfrak{A})$. Let $\Gamma_0 = \{1, 3, 4, 5, 9\}$, and, for every $\alpha < \omega$, let $\Gamma_{\alpha+1} = \{1\} \cup \{2\alpha + 12 - \kappa : \kappa \in \Gamma_\alpha\}$. Set $I_\alpha(a) = \bigcup \left\{ \mathsf{P}^{2\alpha+11}_\kappa : \kappa \in \Gamma_\alpha \right\}$. Straightforward calculations show that I_α determines a 1-redundant representation on $2\alpha + 11$ points, so by deleting any point we also get a representation of $\mathfrak{A}$ on $2\alpha + 10$ points. Thus $\{\kappa : 10 \leq \kappa\} \subseteq \operatorname{spec}(\mathfrak{A})$.

For our next theorem we need the following simple lemma, concerning the small relation algebras in which $1' + a$ is an equivalence element. An element x is an equivalence element if $x ; x = x$ and $\breve{x} = x$. Parts (ii) and (iii) of the lemma contain observations that are also made (in slightly different terminology) in the remarks following Definition 9.1 on page 41 of Jónsson [12].

Lemma 4.3 *If $\mathfrak{A}$ is type #12, #13, #14, or #15, and $E = I(a + 1')$, then*

- *(i) E is an equivalence relation on U,*
- *(ii) each E-class has exactly two elements iff $[a, a, a]$ is not a cycle (i.e., $\mathfrak{A}$ is #12 or #14),*
- *(iii) each E-class has three or more elements iff $[a, a, a]$ is a cycle (i.e., $\mathfrak{A}$ is #13 or #15),*
- *(iv) there are exactly two E-classes iff $[b, b, b]$ is not a cycle (i.e., $\mathfrak{A}$ is #12 or #13),*

(v) there are three or more E-classes iff $[b, b, b]$ *is a cycle (i.e.,* $\mathfrak{A}$ *is #14 or #15).*

In Theorem 10.2 of [12], Jónsson notes that there are four isomorphism types of simple relation algebras generated by an equivalence element e such that $1' < e$. These types are #12, #13, #14, and #15. Jónsson mentions only that they are representable on sets with nine or fewer elements. Of course, the exact cardinalities of their minimal representations, and the uniqueness of their minimal representations have been clear to many who have studied small relation algebras. We summarize these facts in the next theorem, together with our observations concerning spectra and extendibility of representations.

Theorem 4.4 *Assume that* $\mathfrak{A}$ *is a small simple relation algebra, and that* I *is a representation of* $\mathfrak{A}$ *on* U.

(i) If $\mathfrak{A}$ *is #12 then* $|U| = 4$*, I is unique, and spec* $(\mathfrak{A}) = \{4\}$.

(ii) If $\mathfrak{A}$ *is #13 then* $|U| \geq 6$*, I is 1-extendible, I is unique iff* $|U| = 6$ *or* $U = 7$*, and spec* $(\mathfrak{A}) = \{\kappa : \kappa \geq 6\}$.

(iii) If $\mathfrak{A}$ *is #14 then* $|U| \geq 6$*,* $|U|$ *is even, I is unique, I is 2-extendible, and spec*$(\mathfrak{A})$ $= \{2\kappa : \kappa \geq 3\}$.

(iv) If $\mathfrak{A}$ *is #15 then* $|U| \geq 9$*, I is 1-extendible, I is unique iff* $|U| = 9$ *or* $|U| = 10$*, and spec* $(\mathfrak{A}) = \{\kappa : \kappa \geq 9\}$.

Proof Proof of (i): By Lemma 4.3, there are exactly two $I(a + 1')$-classes with exactly two elements each.

[*Proof of (ii)*] By Lemma 4.3, $I(a + 1')$ is an equivalence relation with exactly two classes, each of which contains three or more elements. Therefore $|U| \geq 6$, and any representation may be extended by one element by increasing the size of one or the other equivalence class. Note that there are n nonisomorphic representations of type #13 on $2n + 4$ and $2n + 5$ elements. So I is unique iff $|U|$ is 6 or 7. There are representations over every cardinality greater than 6 since every representation is 1-extendible.

[*Proof of (iii)*] By Lemma 4.3, there are at least three $I(a + 1')$-classes with exactly two elements each. Therefore $|U| \geq 6$ and $|U|$ must be even. Clearly I is unique. To obtain a 2-extension, just add one more equivalence class to the representation. By repeatedly forming 2-extensions, we get representations of $\mathfrak{A}$ for all even finite cardinalities larger than 6, and all infinite cardinalities.

[*Proof of (iv)*] By Lemma 4.3, $I(a + 1')$ is an equivalence relation with three or more classes, each of which contains three or more elements. Therefore $|U| \geq 9$, and any representation may be extended by one element by increasing the size of one of the equivalence classes. The number of nonisomorphic representations is not so easy to compute, but it is easy to see that uniqueness occurs just in case U has just nine or ten elements.

The next result is quite well known (see, for example, Maddux [17], pp. 369–70, or Comer [2].)

Theorem 4.5 *If* $\mathfrak{A}$ *is #16 and* I *is a representation of* $\mathfrak{A}$ *on* U*, then* $|U| = 5$*, I is unique, and spec* $(\mathfrak{A}) = \{5\}$.

Proof: Let $A = I(a)$ and $B = I(b)$. Note A and B form a partition of Di_U with $A^{-1} = A$, $B^{-1} = B$, and $A|A \cap A = B|B \cap B = \varnothing$. We may therefore apply Theorem 1 of Greenwood and Gleason [10] to conclude that $|U| \le 5$. (The theorem appeared as a question in the William Lowell Putnam Mathematical Competition of March 1953. "Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them are drawn, and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color.") Since $0 \neq a \le (b \cdot a\,;a); (b \cdot a\,;a)$, there are $u_0, u_1, u_2, u_3, u_4 \in U$ such that $u_0\,Au_1$, $u_0\,Bu_3$, $u_3\,Bu_1$, $u_0\,Au_4$, $u_4\,Au_3$, $u_1\,Au_2$, and $u_3\,Au_2$. Thus $|U| = 5$. Finally, u_0Bu_2, $u_2\,Bu_4$, and $u_4\,Bu_1$ since $A|A \cap A = \varnothing$, so I is unique.

Lemma 4.6 *If $\mathfrak{A}$ is #17 or #18 then for every $x \in U$, $|\{y : \langle x, y\rangle \in I(a)\}| \ge 3$ and $|\{y : \langle x, y\rangle \in I(b)\}| \ge 4$.*

Proof: Let $A = I(a)$ and $B = I(b)$. Choose u_1 so that $x\,Au_1$. By the type of $\mathfrak{A}$, $a \le a\,;(b \cdot b\,; b)$, so there are $u_2, u_3 \in U$ such that $x\,Au_2$, $u_1\,Bu_2$, $u_2\,Bu_3$, $u_3\,Bu_1$, and $|\{x, u_1, u_2, u_3\}| = 4$. If $x\,Au_3$ then $u_1, u_2, u_3 \in \{y : \langle x, y\rangle \in I(a)\}$, so assume $x\,Bu_3$. Since $b \le a\,; a$ there is some $u_4 \in U$ such that $x\,Au_4$ and $u_4\,Au_3$. Note that $u_4 \neq u_1, u_2$ since $u_1\,Bu_3$ and $u_2\,Bu_3$. So in either case we get $|\{y : \langle x, y\rangle \in I(a)\}| \ge 3$.

Next choose $u_1 \in U$ so that $x\,Bu_1$. The type of $\mathfrak{A}$ is such that $b \le b\,;(b \cdot a\,;(b \cdot b\,;(b \cdot a\,;b)))$, so there are $u_2, u_3, u_4, u_5 \in U$ such that $x\,Bu_2$, $u_1\,Bu_2$, $x\,Bu_3$, $u_2\,Au_3$, $x\,Bu_4$, $u_3\,Bu_4$, $x\,Bu_5$, and $u_4\,Au_5$. Clearly $u_1 \neq u_2 \neq u_3 \neq u_4 \neq u_5$, $u_1 \neq u_3$, $u_2 \neq u_4$, and $u_3 \neq u_5$. Notice also that either $u_1 \neq u_4$ or $u_2 \neq u_5$. Hence $|\{u_1, u_2, u_3, u_4, u_5\}| \ge 4$.

Theorem 4.7 *If $\mathfrak{A}$ is #17 then $|U| \ge 8$, if $|U| = 8$ then I is unique, and spec $(\mathfrak{A}) = \{\kappa : \kappa \ge 8\}$.*

Proof: It follows from Lemma 4.6 that $|U| \ge 8$.

Suppose $|U| = 8$. Let $A = I(a)$ and $B = I(b)$. There are no A-triangles since $A|A \cap A = \varnothing$. Since $a \le (b \cdot a\,;a); (b \cdot a\,;a)$, there are $u_0, u_1, u_2, u_3, u_4 \in U$ such that $u_0\,Au_1$, $u_0\,Bu_2$, $u_2\,Bu_1$, $u_0\,Au_3$, $u_3\,Au_2$, $u_2\,Au_4$, and $u_4\,Au_1$. It follows that u_0, u_1, u_2, u_3, u_4 are distinct. Since there are no A-triangles, we also get $u_0\,Bu_4$, $u_4\,Bu_3$, and $u_3\,Bu_1$. Let $X = \{u_0, u_1, u_2, u_3, u_4\}$. Notice that I is now completely determined on X, and the restrictions of A and B to X are 5-cycles.

Let $Y = U \sim X = \{u_5, u_6, u_7\}$. By Lemma 4.6, every $x \in U$ has exactly four B-images and exactly three A-images. For every $y \in Y$, the three A-images of y cannot all be in X, since there are no A-triangles, but every three-element subset of X contains a pair in A. So every $y \in Y$ has an A-image in Y. Since Y does not contain an A-triangle, there is essentially only one way this can happen, namely $u_5\,Au_6$, $u_5\,Au_7$, and $u_6\,Bu_7$.

Now u_5 has one more A-image, which must be in X. By the symmetry of A and B on X we may assume that $u_5\,Au_0$. Next, u_1 has exactly one other A-image besides u_0 and u_4. It cannot be u_5 since u_5 already has three A-images. Hence either $u_1\,Au_6$ or $u_1\,Au_7$. But u_6 and u_7 are still interchangeable, so we may assume $u_1\,Au_6$. The remaining A-image of u_4 must be in Y, cannot be u_5 since u_5 has three A-images, and cannot be u_6 since otherwise u_1, u_4, and u_6 would form an A-triangle. Hence

$u_4\,A\,u_7$. So far, u_7 has two A-images. The remaining one cannot be u_0, u_1, or u_2, since otherwise there would be an A-triangle. Hence $u_3\,A\,u_7$.

At this point every element of U has three A-images except u_2 and u_6, each of which has only two A-images. It follows that $u_2\,A\,u_6$. Now all elements of U have three A-images, hence B must be all pairs of distinct elements of U that are not in A. Thus A and B have been completely determined, and the representation is unique. Incidentally, if f is the mapping from U to 8 that takes u_0, u_1, u_2, u_3, u_4, u_5, u_6, and u_7 to 0, 1, 6, 7, 2, 4, 5, and 3, respectively, then $\mathrm{Q}^8_1 \cup \mathrm{Q}^8_4 = f^{-1}|A|f$.

The unique representation of #17 on eight elements happens to be finitely extendible but not 1-extendible. We will construct explicit represions on all cardinalities of 9 or more. Let α be any cardinal, and let $U = \{u_\kappa : \kappa < \alpha\} \cup \{v_\kappa : \kappa < \alpha\} \cup \{w_\kappa : \kappa < \alpha\} \cup \{x, y, z\}$, so that $|U| = 3\alpha + 3$. Let $I(a) = R \cup R^{-1}$ and $I(b) = (U \times U) \sim (I(a) \cup Id_U)$, where R is the relation on U that contains the following pairs:

- $\langle x, y\rangle$, $\langle y, z\rangle$,
- $\langle x, u_\kappa\rangle$, $\kappa < \alpha$,
- $\langle y, v_\kappa\rangle$, $\kappa < \alpha$,
- $\langle z, w_\kappa\rangle$, $\kappa < \alpha$,
- $\langle u_\kappa, v_\lambda\rangle$, $\kappa < \lambda < \alpha$,
- $\langle u_\kappa, w_\kappa\rangle$, $\kappa < \alpha$,
- $\langle v_\kappa, w_\kappa\rangle$, $\kappa < \alpha$.

Then I yields a representation of $\mathfrak{A}$ on $3\alpha + 3$ points whenever $\alpha \geq 2$. Furthermore, if $\alpha \geq 3$, then the restriction of I to $U \sim \{x\}$ is a representation of $\mathfrak{A}$ on $3\alpha + 2$ points, and the restriction of I to $U \sim \{x, y\}$ a representation of $\mathfrak{A}$ on $3\alpha + 1$ points. It is interesting to note that if $\alpha = 3$ then the restriction of A to $U \sim \{x, y\}$ is the Peterson graph ([9], pp. 186–7).

Theorem 4.8 *If $\mathfrak{A}$ is #18 then $|U| \geq 9$, if $|U| = 9$ then I is unique, and spec $(\mathfrak{A}) = \{\kappa : \kappa \geq 9\}$.*

Proof: Let $u \in U$. By Lemma 4.6, u has at least four B-images. There is an automorphism of $\mathfrak{A}$ that interchanges a and b, so by Lemma 4.6 it also follows that u has at least four A-images. Hence $|U| \geq 9$.

Assume $|U| = 9$. Let $X = \{x_0, x_1, x_2, x_3\}$ be the set of A-images of u, and let $Y = \{y_0, y_1, y_2, y_3\}$ be the set of B-images of u. Since $a \leq a\,;a$, every $x \in X$ has an A-image in X, so x has at most two B-images in X. Therefore every $x \in X$ has at least two B-images in Y, and $|B \cap (X \times Y)| \geq 8$. Similarly, $|A \cap (X \times Y)| \geq 8$. But $|X \times Y| = 16$, so $|B \cap (X \times Y)| = 8 = |A \cap (X \times Y)|$, every $x \in X$ has one A-image in X, two B-images in X, and two B-images in Y, and every $y \in Y$ has one B-image in Y, two A-images in Y, and two A-images in X. It follows that $|A \cap X^2| = 2 = |B \cap Y^2|$, $|B \cap X^2| = 4 = |A \cap Y^2|$, $A \cap X^2$ and $B \cap Y^2$ are transpositions, and $B \cap X^2$ and $A \cap Y^2$ are 4-cycles. For every pair $\langle x, x'\rangle \in B \cap X^2$, there is some y such that $x\,B\,y$ and $y\,B\,x'$. There is no B-triangle in $\{u\} \cup X$, so $y \in Y$. No two such y's are the same, since otherwise some element of Y would have more than four B-images. We may therefore assume that $x_1\,B\,x_2$, $x_2\,B\,x_3$, $x_3\,B\,x_4$, $x_4\,B\,x_1$, $x_1\,A\,x_3$, $x_2\,A\,x_4$, $x_1\,B\,y_1$, $x_2\,B\,y_1$, $x_2\,B\,y_2$, $x_3\,B\,y_2$, $x_3\,B\,y_3$, $x_4\,B\,y_3$, $x_4\,B\,y_4$, and $x_1\,B\,y_4$. Suppose $y_1\,A\,y_3$. Then there must be some $z \in U \sim \{y_1, y_3\}$ such that $y_1\,A\,z$ and $z\,A\,y_3$, since $a \leq a; a$. But every $z \in U \sim \{y_1, y_3\}$ is B-related to either y_1 or y_3. Hence $y_1\,B\,y_3$, and, similarly, $y_2\,B\,y_4$. We therefore have $y_1\,A\,y_2$, $y_2\,A\,y_3$, $y_3\,A\,y_4$, and $y_4\,A\,y_1$, so A and B are now completely determined.

Let $\alpha > 2$. Let $U = \{u_\kappa : \kappa < \alpha\} \cup \{v_\kappa : \kappa < \alpha\} \cup \{w_\kappa : \kappa < \alpha\} \cup \{x, y\}$. Let R be the relation on U that contains the following pairs:

- $\langle u_\kappa, u_\lambda\rangle$, $\kappa < \lambda < \alpha$,
- $\langle v_\kappa, v_\lambda\rangle$, $\kappa < \lambda < \alpha$,
- $\langle w_\kappa, w_\lambda\rangle$, $\kappa < \lambda < \alpha$,
- $\langle u_\kappa, v_\kappa\rangle$, $\langle u_\kappa, w_\kappa\rangle$, $\langle v_\kappa, w_\kappa\rangle$, $\kappa < \alpha$,
- $\langle x, w_0\rangle$, $\langle x, v_0\rangle$, $\langle x, v_1\rangle$, $\langle x, u_0\rangle$,
- $\langle y, w_1\rangle$, $\langle y, w_2\rangle$, $\langle y, v_0\rangle$, $\langle y, v_1\rangle$.

Then I yields a representation of $\mathfrak{A}$ on $3\alpha + 2$ points, the restriction of I to $U \sim \{y\}$ is a representation of $\mathfrak{A}$ on $3\alpha + 1$ points, and the restriction of I to $U \sim \{x, y\}$ a representation of $\mathfrak{A}$ on 3α points. Thus spec $(\mathfrak{A}) = \{\kappa : \kappa \geq 9\}$.

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