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Π^1_1 Wellfounded Relations

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Abstract If there is a good Δ^1_3 wellordering of the reals, then there is a Π^1_1 wellfounded relation for which the comparison relation is not projective.

1 Preface As much as possible in this paper I have tried to follow the notations and conventions of Jech [2]. That text also contains most basic results for this area. As seems to be traditional, I will call the topological space ω^ω "the reals." As usual, all the results proved below will hold for the true reals in virtue of standard coding arguments.

Section 2 provides an introduction to the problem and recalls various pertinent facts. Section 3 gives the main result, and Section 4 gives some further examples.

2 Introduction

Definition 2.1 Recall that for $R \subset \omega^\omega \times \omega^\omega$ which is wellfounded—that is to say, there is no $(x_i)_{i\in\omega} \subset \omega^\omega$ with $x_{i+1} R x_i$—we can form the ranking function which assigns to each $x \in \omega^\omega$ an ordinal α_x such that:

(i) yRx implies $\alpha_y < \alpha_x$;
(ii) this assignment is minimal—so that is $f : \omega^\omega \to \alpha$ some ordinal α satisfies (i), then for all $x \in \omega^\omega$, $f(x) \geq \alpha_x$.

For R a wellfounded relation, I will write $Rk_R(x)$ for the value of this ranking function at x. The *rank of* R will denote the strict supremum of $Rk_R(x)$ as x ranges over ω^ω, and will be written as $Rk(R)$. Again for R wellfounded, the *comparison relation for* R denotes $\{(x, y) \in \omega^\omega \times \omega^\omega : Rk_R(x) \leq Rk_R(y)\}$.

Fix a recursive bijection $\langle \cdot, \cdot \rangle : \omega \times \omega \cong \omega$. For $x \in \omega^\omega$, x can be viewed as modeling as structure with relation $\in^x$, where $n \in^x m$ if and only if $x(\langle n, m\rangle) = 0$. Here we can say that x codes this model, which I will denote by m^x. $x \in WO$ indicates that $(m^x, \in^x) \cong (\beta, \in)$ for some countable ordinal β.

We will also need a method of coding sequences of reals by a single element of ω^ω. Let $(\cdot, \ldots, \cdot) : (\omega^\omega)^{<\omega} \to \omega^\omega$ be a Δ^1_1 bijection.

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For a countable model $\mathbf{L}_\alpha[x]$, and S some Σ_2^1 set of reals, it will be convenient to use the expression $\mathbf{L}_\alpha[x] \models S(x)$ only for the situation that the model witnesses this by an appropriate embedding into the Shoenfield tree; if we do not take this precaution, it may not be upward absolute. For $\beta < \omega_1$, $x \in \omega^\omega$, and B a Π_1^1 set of reals, it is natural to speak of x being in B by stage β if the Kleene-Brouwer wellordering of ω induced by $x \in B$ has ordertype less than β; it will in fact be the case here that $\mathbf{L}_{\omega\cdot\beta}[x] \models B(x)$ in the strong sense above. A full discussion of the notions of Shoenfield tree and Kleene-Brouwer ordering can be found in [2].

Theorem 2.2 (Classical) *Let S be a wellfounded Σ_1^1 relation. Then there exists some $\beta < \omega_1^{ck}$ such that for all $x \in \omega^\omega$, $Rk_S(x) < \beta$. (See Moschovakis [4]; here $\beta < \omega_1^{ck}$ merely expresses that there is some recursive wellordering of ω of ordertype β).*

This is the best possible: given $\beta < \omega_1^{ck}$, there is certainly a Σ_1^1 wellfounded relation of rank β.

Corollary 2.3 *Let S be a Σ_1^1 wellfounded relation. Then the comparison relation for S is Δ_2^1.*

Proof: Notice that for $x \in \omega^\omega$, and $\beta < \omega_1^{ck}$, the statement that $Rk_S(x) > \beta$ is, uniformly $\Sigma_1^1(x)$ in any number coding a recursive wellordering of ordertype β; hence $Rk_S(x) \leq \beta$ is uniformily $\Pi_1^1(x)$. Hence $Rk_S(x_1) \leq Rk_S(x_2)$ if and only if for all $n \in \omega$ coding a recursive ordering γ_n on ω

$$\gamma_n \text{ illfounded} \vee Rk_S(x_1) \leq \gamma_n \vee Rk_S(x_2) > \gamma_n.$$

So this statement has the form

$$\forall n \in \omega(\Pi_2^0(n) \to (\Sigma_1^1(n) \vee \Pi_1^1(n, x_1) \vee \Sigma_1^1(n, x_2)).$$

But this is $\Delta_2^1(x_1, x_2)$.

The statement of 2.3 is somewhat misleading, in that the proof shows the comparison relation to be a fairly simple Δ_2^1 set—in fact, it will be a countable Boolean combination of Π_1^1. On the Π_1^1 side we can obtain a bound for the possible ranks, and in the presence of sharps we can obtain a precise calculation.

Theorem 2.4 (Kunen-Martin) *For $x \in \omega^\omega$, the rank of every $\Pi_1^1(x)$ wellfounded relation will be less than $((\omega_1^{\mathbf{V}})^+)^{\mathbf{L}[\mathbf{x}]}$; if in fact every real has a sharp,*

$$sup\{Rk(R) : R \text{ is a wellfounded } \underset{\sim}{\Pi}_1^1 \text{ relation}\} = u_2.$$

(See [2] or [4].)

In fact, for $x, y \in \omega^\omega$ and R a $\Pi_1^1(y)$ wellfounded relation, $Rk_R(x)$ will be equal to some ordinal definable over $\mathbf{L}[y]$ from ω_1 and some countable ordinal α.

Here we might ask about analogs of 2.3 for Π_1^1 wellfounded relations. One answer is given by the following:

Theorem 2.5 (Harrington-Kechris) *Assume $\underset{\sim}{\Pi}_2^1$ determinacy. Let R be a Π_1^1 wellfounded relation. Then the comparison relation is Δ_4^1.*

Proof: The proof of this can be found in Harrington and Kechris [1].

I want to consider the other side of this problem. There are Π^1_1 wellfounded relations whose comparison relation is neither Σ^1_3 nor Π^1_3. If there exists a good Δ^1_3 wellordering of the reals, then there is a Π^1_1 wellfounded relation whose comparison relation is not $\underset{\sim}{\Sigma}^1_n$ for any $n \in \omega$. Hence, while ZFC puts a firm and reasonable bound on the comparison relation for Σ^1_1 wellfounded relations, at the level of Π^1_1 the possibilities are already wildly divergent. Assuming determinacy, the comparison is simple. On the other hand, if **V**=**L**, then the comparison relation is not even projective.

3 Good wellorderings The move in Theorem 2.5 from a Π^1_1 wellfounded relation to a Δ^1_4 comparison relation would seem like a big jump. However, there always exists a Π^1_1 wellfounded relation for which the comparison relation is not Σ^1_3 or Π^1_3.

Let $P \subset \omega^\omega \times \omega^\omega$ be a universal Σ^1_2 set. Define R as follows: the field of R consists of all $(0, x)$, $x \in \omega^\omega$, all $(1, x, y)$, $x, y \in \omega^\omega$, and all $(2, x, y, w)$ such that w codes some countable ordinal α, and $\mathbf{L}_\alpha[x, y] \models \neg P(x, y)$. For $(0, x)$, $(1, x, y)$, and $(2, x, y, w)$ in the field of R, set $(2, x, y, w)R(1, x, y)R(0, x)$. $Rk_R(0, x) = \omega_1 + 1$ if and only if $\exists y \neg P(x, y)$. We set $(2, x, y, w)R(2, x, y, w^*)$ if w and w^* code ordinals α and α^* respectively, with $\alpha < \alpha^*$. These cases are the only ones in which R holds between two elements.

Pictorially, a real, $(0, x)$, lies above some $(1, x, y)$ which stands above every single ordinal where it survives as a witness to $\exists y \neg P(x, y)$.

Definition 3.1 If there exists an onto function $f : \omega_1 \to \omega^\omega$ which is Δ^1_3 in the codes, then we say that r has a *good* Δ^1_3 *wellordering*.

Theorem 3.2 *Assume that there is a good Δ^1_3 wellordering of the reals. Let P be Σ^1_{2n+1} formula. Then, uniformly in P, there is a Π^1_1 wellfounded relation R such that $Rk_R(z) = \omega_1^n$, in the sense of ordinal exponentiation, if z witnesses that P, and $Rk_R(z) < \omega_1^n$ otherwise.*

Proof: We prove by induction on n that the theorem holds for all $n \in \omega$. The example above indicates how this is proved in the case $n = 1$, even without the wellordering. So, assume $n > 1$. Fix $f : \omega_1 \to \omega^\omega$, an onto function which is Σ^1_3 in the codes, witnessing that there is a good Δ^1_3 wellordering of the reals. Suppose that P is of the form $\exists x \forall y \exists z Q(x, y, z)$, where Q is Π^1_{2n-2}. By inductive assumption, uniformly in (x, y) we have a Π^1_1 wellfounded relation $R_{(x,y)}$ such that $Rk_{R_{(x,y)}}(z) = \omega_1^{n-1}$ if $Q(x, y, z)$, and is less than ω_1^{n-1} otherwise. Now let the field of R consist of elements of the form $(0, x)$, where $x \in \omega^\omega$, and $(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, x_\beta)$ such that:

(i) $x_\beta \in WO$ codes an ordinal $\beta < \omega_1$ and $\zeta \leq \beta$;
(ii) $x \in \omega^\omega$ and each $y_\alpha, w_\alpha, z^{\bar\alpha}_\alpha \in \omega^\omega$;
(iii) each $z^{\bar\alpha}_\alpha$ is in the field of $R_{(x, y_{\bar\alpha})}$;
(iv) w_α attempts to witness that $f(\alpha) = y_\alpha$, and this attempt is not refuted by stage β; in other words, if $f(\gamma) = a$ is equivalent to $\exists w S(\gamma, w, a)$, where S is Π^1_2 in the codes, then $\mathbf{L}_\beta[w_\alpha, y_\alpha] \models S(\alpha, w_\alpha, y_\alpha)$.

The relation of R is then defined by its holding among objects in its field in just the following three cases:

(i) $(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, x_\beta) R(0, x)$ for all $x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, \zeta, x_\beta$ as above;

(ii) $(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta+1}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta+1}, x_\beta) R(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta+1}, ((\bar z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta+1}, x_{\bar\beta})$ if x_β codes a countable ordinal β such that for each $\alpha < \zeta, z^{\bar\alpha}_\alpha = \bar z^{\bar\alpha}_\alpha$ and for each $\bar\alpha < \zeta$ $z^{\bar\alpha}_\zeta R_{(x, y_{\bar\alpha})} \bar z^{\bar\alpha}_\zeta$ is seen to be true by stage β; that is, the Kleene-Brouwer ordering corresponding to this Π^1_1 fact is less than β;

(iii) $(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, x_\beta) R(1, x, (y_\alpha, w_\alpha)_{\alpha\in\bar\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\bar\zeta}, x_{\bar\beta})$ if $\zeta < \bar\zeta$.

Claim 3.3 *For any* $(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, x_\beta)$ *in the field of R,*

$$Rk_R(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, x_\beta) \le \omega_1^{n-1}\cdot\zeta.$$

Proof: The proof is by induction on ζ. At $\zeta = 1$ this follows by the assumption on $R_{(x,y_0)}$. At the successor step $\zeta = \bar\zeta + 1$, the inductive hypothesis implies that for all $\gamma < \zeta$, any $(1, x, (y_\alpha, w_\alpha)_{\alpha\in\gamma}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\gamma}, x_\beta)$ has rank at most $\omega_1^{n-1}\cdot\gamma$. Hence we set

$$g(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, \beta) = \omega_1^{n-1}\cdot\bar\zeta + 1 + \inf_{\bar\alpha\in\bar\zeta} Rk_{R_{(x,y_{\bar\alpha})}}(z^{\bar\alpha}_{\bar\zeta})$$

for any $(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, (z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, x_\beta)$ and for $\gamma < \zeta$ we set

$$(1, x, (y_\alpha, w_\alpha)_{\alpha\in\gamma}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\gamma}, x_\beta) = Rk_R(1, x, (y_\alpha, w_\alpha)_{\alpha\in\gamma}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\gamma}, x_\beta);$$

by clause (ii) of Definition 2.1 g witnesses that

$$Rk_R(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, x_\beta) \le \omega_1^{n-1}\cdot\zeta.$$

For γ a limit ordinal, it follows that

$$Rk_R(x, (y_\alpha, w_\alpha)_{\alpha\in\gamma}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\gamma}, x_\beta) \le \omega_1^{n-1}\cdot\gamma.$$

Claim 3.4 *For* $\alpha_0 < \zeta$ *and* w_{α_0} *failing to witness that* $f(\alpha_0) = y_{\alpha_0}$*, we have*

$$Rk_R(1, x, (y_\alpha, w_\alpha)_{\alpha\in\zeta}, ((z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\zeta}, x_\beta) < \omega_1^{n-1}\cdot(\alpha_0 + 1).$$

Proof: If w_{α_0} fails to witness, then there will be some ordinal β^* at which point it becomes apparent that it does not witness—that is,

$$\mathbf{L}_{\beta^*}[w_{\alpha_0}, y_{\alpha_0}] \models \neg S(\alpha_0, w_{\alpha_0}, y_{\alpha_0}).$$

But that means we can never go beyond this β^* until after we have dropped down to some $(1, x, (y_\alpha, w_\alpha)_{\alpha\in\alpha_0}, ((\bar z^{\bar\alpha}_\alpha)_{\bar\alpha\in\alpha})_{\alpha\in\alpha_0}, x_{\bar\beta})$, which must have rank no greater than $\omega_1^{n-1}\cdot\alpha_0$ by the previous claim. So, as long as we are above α_0, we are in effect looking at a $\utilde{\Delta}^1_1$ wellfounded relation, and when we are below α_0 we have a wellfounded relation of rank $\leq \omega_1^{n-1} \cdot \alpha_0$. But the rank of a wellfounded Borel relation is always strictly less than ω_1, and hence the total rank must be strictly less than $\omega_1^{n-1} \cdot (\alpha_0 + 1)$.

More formally, set $z_1^* R_\alpha z_2^*$ if $z_1^* R_{(x, y_\alpha)} z_2^*$ *and* this is seen to be the case by stage β^*. These are wellfounded Borel relations, and hence there is some γ which strictly bounds the rank of R_α for every α less than ζ. Now set

$$\begin{aligned} g(1, x, (y_\alpha, w_\alpha)_{\alpha \in \bar{\zeta}+1}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha} \in \alpha})_{\alpha \in \bar{\zeta}+1}, x_\beta) &= \\ & \omega_1^{n-1} \cdot \alpha_0 + 1 + \gamma \cdot \bar{\zeta} + 1 + \inf_{\alpha \in \bar{\zeta}} Rk_{R_\alpha}(z_{\bar{\zeta}}^\alpha) \end{aligned}$$

for $\bar{\zeta}$ greater than α_0; set

$$g(1, x, (y_\alpha, w_\alpha)_{\alpha \in \bar{\zeta}}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha} \in \alpha})_{\alpha \in \bar{\zeta}}, x_\beta) = \omega_1^{n-1} \cdot \alpha_0 + 1 + \gamma \cdot \bar{\zeta}$$

for $\bar{\zeta}$ a limit greater than α_0; and set

$$\begin{aligned} g(1, x, (y_\alpha, w_\alpha)_{\alpha \in \bar{\zeta}}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha}})_{\alpha \in \bar{\zeta}}, x_\beta) &= \\ & Rk_R(1, x, (y_\alpha, w_\alpha)_{\alpha \in \bar{\zeta}}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha} \in \alpha})_{\alpha \in \bar{\zeta}}, x_\beta) \end{aligned}$$

for $\bar{\zeta}$ less than or equal to α_0. So, as in the previous argument, g witnesses the appropriate bound on the rank function.

Claim 3.5 *If* $\forall z \neg Q(x, y_{\alpha_0}, z)$ *then*

$$Rk_R(1, x, (y_\alpha, w_\alpha)_{\alpha \in \zeta}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha} \in \alpha})_{\alpha \in \zeta}, \beta) < \omega_1^{n-1} \cdot (\alpha_0 + 1).$$

Proof: This follows by the assumption on $R_{(x, y_{\alpha_0})}$. If $\forall z \neg Q(x, y_{\alpha_0}, z)$, then we are permanently constrained above α_0. Suppose that for a given condition

$$(1, x, (y_\alpha, w_\alpha)_{\alpha \in \zeta}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha} \in \alpha})_{\alpha \in \zeta}, x_\beta)$$

we have

$$\sup_{\alpha \in \zeta}(Rk_{R_{(x, y_{\alpha_0})}}(z_\alpha^{\alpha_0})) = \gamma < \omega_1^{n-1},$$

then above α_0 we have, in effect, a wellfounded relation of size at most $\gamma \cdot (\zeta + 1) < \omega_1^{n-1}$ by the assumption on $R_{(x, y_\alpha)}$ and by adjusting the argument from the first claim. Below α_0 we have a wellfounded relation of rank at most $\omega_1^{n-1} \cdot \alpha_0$. Hence, the total rank is less than $\omega_1^{n-1} \cdot (\alpha_0 + 1)$.

More formally, set

$$\begin{aligned} g(1, x, (y_\alpha, w_\alpha)_{\alpha \in \gamma}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha} \in \alpha})_{\alpha \in \gamma}, x_\beta) &= \\ & \omega_1^{n-1} \cdot \alpha_0 + 1 + \Sigma_{\alpha \in \bar{\zeta}}(Rk_{R_{(x, y_{\alpha_0})}}(z_\alpha^{\alpha_0}) + 1) \end{aligned}$$

for $\bar{\zeta}$ greater than α_0, and set

$$\begin{aligned} g(1, x, (y_\alpha, w_\alpha)_{\alpha \in \gamma}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha} \in \alpha})_{\alpha \in \gamma}, x_\beta) &= \\ & Rk_R(1, x, (y_\alpha, w_\alpha)_{\alpha \in \gamma}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha} \in \alpha})_{\alpha \in \gamma}, x_\beta) \end{aligned}$$

pfor $\bar{\zeta}$ less than α_0. As before, g witnesses the bound.

Claim 3.6 $\exists y \forall z \neg Q(x, y, z) \Rightarrow Rk_R(0, x) < \omega_1^n$.

Proof: If $f(\alpha_0) = y$ with $\forall z \neg Q(x, y, z)$, then the previous two claims established that $Rk_R(0, x) \leq \omega_1^{n-1} \cdot (\alpha_0 + 1)$.

Claim 3.7 $\forall y \exists z Q(x, y, z) \Rightarrow Rk_R(0, x) = \omega_1^n$.

Proof: Let $f(\alpha) = y_\alpha$ be witnessed by w_α. Let z_α be such that $Q(x, y_\alpha, z_\alpha)$. Set $z_\alpha^{\bar{\alpha}} = z_\alpha$ for each $\bar{\alpha} < \alpha$. It suffices to show that each $(1, x, (y_\alpha, w_\alpha)_{\alpha \in \zeta}, ((z_\alpha^{\bar{\alpha}})_{\bar{\alpha} \in \alpha})_{\alpha \in \zeta}, x_\beta)$ has rank at least $\omega_1^{n-1} \cdot \zeta$, but this follows by the assumption on the $R_{(x, y_\alpha)}$ and by induction on ζ.

But these last two claims are exactly as required.

Corollary 3.8 *If there is a good Δ^1_3 wellordering of the reals, then there is Π^1_1 wellfounded relation for which the comparison relation is nonprojective.*

Proof: Using that the last argument produced the wellfounded relations uniformly, we can join them together to obtain a wellfounded relation whose comparison relation is not Σ^1_n for any $n \in \omega$. But by just relativizing the above construction to every real, and joining all those relations together, we can obtain a relation whose comparison relation is not $\Sigma^1_n(y)$ for any $n \in \omega$ and $y \in \omega^\omega$.

The assumption of the existence of a good Δ^1_3 wellordering of the reals is not overly restrictive. **L** has a good Δ^1_2 wellordering of the reals, and so certainly it has a good Δ^1_3 wellordering. It follows from early work by Silver that the existence of a measurable cardinal is compatible with a good Δ^1_3 wellordering of the reals (see [4]).

4 Examples

Example 4.1 (The Solovay model) Suppose κ is inacessible over **L**, then in $\mathbf{L}^{Coll(\omega, <\kappa)}$ there is a wellfounded Π^1_1 relation whose comparison relation is not projective. This is a slight variant on the argument in Theorem 3.2 using that the first order theory of $HC \cap \mathbf{L}$ is not projective in this model, but the initial segments of the form $\mathbf{L}_\alpha$, some $\alpha < \kappa$, can be easily enumerated in ordertype ω_1; whereas the argument of Theorem 3.2 tried to grab hold of a good guess at an initial segment of some good wellordering, and we showed that only the good guesses could have high rank, the variant here consists, in part, of attempts to enumerate the countable levels of **L** and witness some fact about the theory of $\mathbf{L}[x]$, for various x.

If, on the other hand, we begin with $\mathbf{L}[\mu] \models \kappa$ measurable, then the comparison relation for a Π^1_1 wellfounded relation in $\mathbf{L}[\mu]^{Coll(\omega, <\kappa)}$ is always Δ^1_4. Fix δ some big ordinal such that $\mathbf{L}_\delta[\mu]$ satisfies a large fragment of ZFC. Work in the generic extension. For $x \in \omega^\omega$, R a wellfounded Π^1_1 relation, $\alpha \in \kappa$, and τ a Skolem term over **L**, it will be the case that $Rk_R(x) > \tau(\alpha, \kappa)$ if and only if for some

$$(X[x], \kappa, \mu, \alpha) \prec (\mathbf{L}[\mu][x], \kappa, \mu, \alpha),$$

with $\alpha \subset X$, and for transitive

$$(M[x], \bar{\kappa}, \bar{\mu}, \alpha) \cong (X[x], \kappa, \mu, \alpha)$$

we have

$$M[x]^{Coll(\omega,<\bar{\kappa})} \models Rk_R(x) > \tau(\alpha,\kappa).$$

But the existence of such an iterable $M[x]$ will be a $\Sigma^1_3(x,\alpha)$ fact.

In fact in this model $\mathbf{L}[\mu]^{Coll(\omega,<\kappa)}$ we obtain that there is a real y such that $\mathbf{L}[x,y]^{Coll(\omega,<\kappa)}$ correctly calculates $Rk_R(x)$. Indeed, this observation is not restricted to Π^1_1 relations. What we really needed was that the rank is shorter than $\underset{\sim}{\delta}^1_2$. More generally one can argue that if κ is measurable in $\mathbf{V}$, and $G \subset$Coll$(\omega,<\kappa)$ is $\mathbf{V}$-generic, then for any Σ^1_3 wellfounded relation, R, in $\mathbf{V}[G]$ with rank less than the $\underset{\sim}{\delta}^1_2$ of $\mathbf{V}[G]$, there will exist a real y in $\mathbf{V}[G]$ such that $\mathbf{L}[y]^{Coll(\omega,<\kappa)}$ correctly calculates Rk_R.

At this point we might wonder why the argument that shows there to be a Π^1_1 wellfounded relation with a violently complicated comparison in $\mathbf{L}^{Coll(\omega,<\kappa)}$ fails in $\mathbf{L}[\mu]^{Coll(\omega,<\kappa)}$. The problem is not that the natural wellordering of this later model is too complicated; the natural ordering is still only Δ^1_3 and hence would seem to provide no obstruction to the technique of Theorem 3.2. The problem is that the natural enumeration of $\mathbf{L}[\mu]\cap\mathbf{V}_\kappa$ does not have ordertype κ. Indeed, one can prove that in this model there is no onto $f:\omega_1\to\mathbf{L}[\mu]\cap\mathbf{V}_\kappa$ which is Δ^1_3 in the codes.

Example 4.2 (Precipitous ideals) It is now known to be possible to have a precipitous ideal on ω_1 along with a good Δ^1_3 wellordering of the reals. This follows from as yet unpublished work of Shelah, Martin, and Steel: Shelah for obtaining the precipitous ideal with a forcing notion that adds no reals; Martin and Steel for showing compatibility of his initial assumptions with the existence of such a wellordering. So, as a consequence of Theorem 3.2, the existence of a precipitous ideal on ω_1 is not sufficient to guarantee that the comparison relation for Π^1_1 wellfounded relations is simple.

However, in a wide range of situations, a precipitous ideal will give a simple comparison relation. For instance, if the ideal arises as a result of Levy collapsing a measurable cardinal κ in $\mathbf{V}$, then as in Example 4.1 the comparison will be Δ^1_4; this follows by essentially the same argument.

Example 4.3 (Changing ranks but not Σ^1_3 truth) It is possible to change the rank of a wellfounded Π^1_1 relation without changing ω_1 or introducing any new Σ^1_3 truths. Recall from Jensen and Solovay [3] that if δ is inacessible over $\mathbf{L}$, but not Mahlo, there is a δ-c.c. notion of forcing which introduces a real x such that $\mathbf{L}[x]\models\omega_1=\delta$. Suppose now that κ is Mahlo over $\mathbf{L}$, and it is the least such. A downward Lowenheim-Skolem argument and an application of Shoenfield absoluteness shows that there must exist an inacessible δ such that if $G\subset Coll(\omega,<\delta)$ is generic, then for all $P\in L_\kappa$, and $H\subset P$ which are $\mathbf{L}[G]$ generic,

$$\mathbf{L}[G]\prec_{\Sigma^1_3}\mathbf{L}[G][H],$$

that is to say, no new Σ^1_3 truths in parameters from $\mathbf{L}[G]$ are introduced by the further forcing. Now let x be a real as above, so that $\delta=\omega_1^{\mathbf{L}[x]}$. Let $G\subset Coll(\omega,<\delta)$ be $\mathbf{L}[x]$ generic. It follows from the factor lemma for forcing that x is generic over $\mathbf{L}[G]$, and that $\mathbf{L}[G]$ and $\mathbf{L}[G, x]$ both calculate the same value for ω_1—namely δ. It follows by assumption on δ that

$$\mathbf{L}[G] \prec_{\Sigma^1_3} \mathbf{L}[G, x],$$

so to complete the example we need only find a Π^1_1 wellfounded relation which has greater rank in $\mathbf{L}[G, x]$ than $\mathbf{L}[G]$.

Now for y, w_0, and w_1 all reals, w_0 and w_1 both reals coding wellorderings of ordertypes $\beta_0 < \beta_1$ respectively, and for $(a_\beta)_{\beta\in\alpha_1}$ a sequence of reals, with $z_1 \in \omega^\omega$ coding this sequence and $z_0 \in \omega^\omega$ coding $(a_\beta)_{\beta\in\alpha_0}$, where $\alpha_0 < \alpha_1$, $\alpha_0 < \beta_0$, and $\alpha_1 < \beta_1$ set

$$(y, w_0, z_0) R (y, w_1, z_1) R (0, y)$$

if and only if

$$(a_\beta)_{\beta\in\alpha_1} \subset \mathbf{L}_{\beta_1}[y],$$

and

$$(a_\beta)_{\beta\in\alpha_0} \subset \mathbf{L}_{\beta_0}[y].$$

This relation will have the property that $Rk_R(0, y) = \omega_1$ if and only if $\mathbf{L}[y]$ has uncountably many reals. So

$$(Rk(R))^{\mathbf{L}[G]} = \omega_1,\ (Rk(R))^{\mathbf{L}[G][x]} = \omega_1 + 1.$$

Here x will be generic over $\mathbf{L}[G]$ for a notion of forcing that is c.c.c. in $\mathbf{L}[G]$. Hence, we have a change of ranks induced by a c.c.c. notion of forcing that introduces no new Σ^1_3 truths.

A slightly more complicated version of this example, using the same type of argument as in Theorem 3.2, will show that is possible to have models where there is a wellfounded Π^1_1 relation whose rank can change after forcing without affecting ω_1 or introducing new Σ^1_4 truths.

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