Notre Dame Journal of Formal Logic
Volume 35, Number 4, Fall 1994

Some Results on Numerical Systems in λ-Calculus

BENEDETTO INTRIGILA

Abstract In this paper we study numeral systems in the $\lambda\beta\eta$-calculus. With one exception, we assume that all numerals have normal form. We study the independence of the conditions of adequacy of numeral systems. We find that, to a great extent, they are mutually independent. We then consider particular examples of numeral systems, some of which display paradoxical properties. One of these systems furnishes a counterexample to a conjecture of Böhm. Next, we turn to the approach of Curry, Hindley, and Seldin. We dwell with the general problem of obtaining their results with the additional requirement of nonconvertibility of numerals. In particular we solve a problem that they left open. Finally, we give the first example of an adequate unsolvable numeral system without a test for zero in the usual sense, thus solving a problem of Barendregt and Barendsen.

1 Introduction and summary In the λ-calculus numeral systems are, informally speaking, sets of terms suitable for playing the role of numerals in the representation of recursive functions by λ-terms. Formally a numeral system $\mathbf{d} = \mathbf{d}_0, \ldots, \mathbf{d}_n, \ldots$ is called *adequate* if all the recursive functions can be λ-defined with respect to $\mathbf{d}$ (see Definition 3.3).

In a classical approach, first proposed by Böhm and Gross [3] and further developed in Barendregt [1], Curry, Hindley, and Seldin [6], and Wadsworth [10], this question is reduced to the existence of four terms satisfying a few natural equations (which are exactly the functional counterpart of the first two Peano axioms of arithmetic). More formally, it can proved that if the successor, the predecessor and a test for zero can be λ-defined then the system is adequate (see Proposition 6.4.3 of [1]). The simple character of this test facilitated the understanding of the great complexity of numeral systems and related ones, see [10], and Rezus [7].

To simplify matters it is natural to require that numerals are terms in normal form. In this paper we adopt this requirement and we study (quasi)-numeral systems with numerals in normal form, (with the exception of Section 6).

Received November 1, 1990; revised August 1, 1994

Nevertheless, it is also possible to study numeral systems with numerals without normal form. This was first done by Böhm and Dezani, see [7]. Although these numerals do not have normal form it is generally possible to detect when the computation of a numerical result is completed. However complications arise for there exist adequate numeral systems without a test for zero in the usual sense, so that in this case the above-mentioned conditions are not necessary. Improving previous results of Barendregt and Barendsen, we are able to give an example of such a system in Section 6 (see Barendsen [2] for a different exposition). We also discuss how to refine in this context the Barendregt-Wadsworth thesis that "undefined means an unsolvable term."

In Section 3 of this paper we study the independence of the above mentioned conditions of adequacy. This problem has been extensively treated in Chapter 13 of [6]. In Section 3 we consider the independence questions in the general setting of (closed) quasi-numeral systems (in short QNS) see [7], i.e., infinite sets of pairwise non-convertible (closed) terms. We show that even in the restricted case of r.e. normal, closed QNS, the existence of successor, predecessor and test for zero are, with only one exception, completely independent of each other.

In fact the following table holds:

sucessor	predecessor	test for zero	r.e, QNS
Y	Y	Y	Y
Y	Y	N	Y
Y	N	Y	Y
Y	N	N	Y
N	Y	Y	N
N	Y	N	Y
N	N	Y	Y
N	N	N	Y

Table 1

where the first three columns describe the eight *a priori* possibilities (about the existence of successor, predecessor and test for zero) and the last one is set to Y or N according to whether there exists a r.e., normal, closed example. We also show that it is possible to find a normal, closed QNS of (N,Y,Y) "type" and that every such QNS must be not r.e.

In Section 4, we consider particular examples of numeral systems, some of which display paradoxical properties. One of these systems furnishes a counterexample to the following conjecture of Böhm [4]:

Claim 1.1 *Let* $\mathbf{d}$ *be an adequate numeral system with numerals in normal form. It is always possible to find a successor* $[s]_{\mathbf{d}}$*, a predecessor* $[p]_{\mathbf{d}}$ *and a test for zero* $[z]_{\mathbf{d}}$ *such that:*

1. $[s]_{\mathbf{d}}\ \mathbf{d}_n \gg \mathbf{d}_{n+1}$;
2. $[p]_{\mathbf{d}}\ \mathbf{d}_{n+1} \gg \mathbf{d}_n$;
3. $[z]_{\mathbf{d}}\ \mathbf{d}_0 \gg \mathbf{T}$, $\quad [z]_{\mathbf{d}}\mathbf{d}_{n+1} \gg \mathbf{F}$;

(where ≫ means strong normalization, i.e., every sequence of reductions is finite, see Section 2).

We exhibit as a counterexample a system **d** with numerals, successor, predecessor and a test for zero in normal form, such that for every choice of $[s]_{\mathbf{d}}$, $[p]_{\mathbf{d}}$ and $[z]_{\mathbf{d}}$ *none* of (i) – (iii) holds. Independently, Statman [9] found a different counterexample which, in a sense, furnishes a stronger negative result. Statman's NS has numerals in normal form, but it cannot have a test for zero in normal form.

These systems are, in a sense, paradoxical. In the numerals some information is coded which is completely useless from the computational point of view and has the unique function of creating infinite reduction processes. We show that another such system, consisting of λ**I**-terms in normal form, is an adequate numeral system in the λ**K**-calculus but not in the λ**I**-calculus. We also generalize some constructions of [7], showing that some natural sequences of terms turn out to be adequate numeral systems.

In Section 5 we come back to [6]. In this framework an abstract set of terms:

[0], [s][0], [s]([s][0])...

is considered, and abstract operations on this set satisfying some given equations are studied.

No hypothesis is made on previous terms, so they may turn out to be mutually convertible. We try to link this approach with the previous one (in term of QNS) and we also dwell with the general problem of obtaining the results of [6] with the additional requirement of the none convertibility of numerals. As far as successor and predecessor functions and test for zero are concerned we show that the corresponding results hold, but in [6] many other operations and functionals were considered: the recursion operator, the μ-operator, etc. We do not consider these operations here, with the exception of the μ-operator, and we show that it cannot be defined from successor and predecessor only, thus solving a problem left open in [6].

2 Preliminary remarks We work in the $\lambda\beta\eta$ calculus. We follow [1] for notation and terminology. However, especially with respect to *Böhm trees*, we often use informal arguments based on the well known informal picture ([1], 19.1.3) of the Böhm tree *BT*(N) of a term N with head normal form $\lambda x_1 \dots x_n.x_i \mathrm{M}_1 \dots \mathrm{M}_t$:

$$BT(\mathrm{N}) = \begin{array}{c} \lambda x_1 \dots x_n.x_i \\ / \setminus \\ BT(\mathrm{M}_1) \dots BT(\mathrm{M}_t) \end{array}$$

In this case we say that x_i is the *head variable* and $x_1 \dots x_n$ the *head abstraction variables* of the node. If N is as above, then the *grade* of N is t.

We shall often use, without explicitly mentioning them, the basic results on terms with head normal form (see Paragraph 8.3 of [1]). We use capital letters M, N, L,... for arbitrary terms. The lowercase letters x, y, z, and ξ, (possibly with indexes) denote variables. Other lowercase letters such as $k, n, m, p, r, t\dots$ denote natural numbers. We denote with $\mathcal{N}$ the set of natural numbers. The symbol "≡" between terms denotes identity modulo α-convertibility, whereas "=" denotes β-η-convertibility and

"$\neq$" denotes non-β-η-convertibility. We read X > Y as meaning that X reduces (multistep) to Y. A term X is in *normal form* if it cannot be reduced. Sometimes we shorten "normal form" to "nf." When Y is in normal form, we write X $\gg$ Y to mean that X strongly reduces to Y, i.e., X reduces to Y and every chain of reduction steps starting from X is finite. As usual, by *combinator* we mean a closed term. We shall often use the following combinators:

$$\mathbf{I} \equiv \lambda x_1.x_1;\ \mathbf{K} \equiv \lambda x_1 x_2.x_1;\ \mathbf{O} \equiv \lambda x_1 x_2.x_2.$$

In particular, we use **K** to represent the Boolean value "True" (also denoted by **T**) and **O** to represent the value "False" (also denoted by **F**). If M and N are terms, $\langle$ M, N $\rangle$ is $\lambda x.x$MN and, more generally, if $M_1 \ldots M_n$ are terms $\langle M_1 \ldots M_n \rangle$ is $\lambda x.xM_1 \ldots M_n$. Given a term M, M^n denotes the composition of M with itself n times.

3 Quasi-numeral systems

3.1 Definitions and notation We recall some definitions from Paragraph 6 of [1], and from [7].

Definition 3.1 A *quasi-numeral system* (QNS) **d** is an infinite sequence of terms:

$$\mathbf{d} = \mathbf{d}_0, \ldots \mathbf{d}_n \ldots$$

such that for i $\neq$ j, $\mathbf{d}_i \neq \mathbf{d}_j$. A QNS is *normal* if each $\mathbf{d}_n$ is in normal form. A QNS is *closed* if each $\mathbf{d}_n$ is closed.

In the remainder of this paper (with the exception of Section 6) we only consider normal closed QNSs so that, *par abus de langage*, QNS always means normal closed QNS. We denote QNSs with letters **c**, **d**, **q**.

Definition 3.2 A QNS **d**:

- has a *successor* if there exists a term $[s]_{\mathbf{d}}$ such that $[s]_{\mathbf{d}}\mathbf{d}_n = \mathbf{d}_{n+1}$;
- has a *predecessor* if there exists a term $[p]_{\mathbf{d}}$ such that $[p]_{\mathbf{d}}\mathbf{d}_{n+1} = \mathbf{d}_n$;
- has a *test for zero* if there exists a term $[z]_{\mathbf{d}}$ such that $[z]_{\mathbf{d}}\mathbf{d}_0 = \mathbf{T}$ and $[z]_{\mathbf{d}}\mathbf{d}_{n+1} = \mathbf{F}$.

If the QNS **d** is clear from the context, we drop the subscript and simply write [s], etc.

Definition 3.3 A QNS **d**:

- is a *numeral system in the strong sense* (NS) if it has a successor, a predecessor, and a test for zero;
- is an *adequate numeral system* if for every partial recursive function f with value $f(n_1, \ldots, n_m)$ there is a term F such that:

$$\begin{aligned} \mathrm{F}\mathbf{d}_{n_1} \ldots \mathbf{d}_{n_m} &= \mathbf{d}_{f(n_1 \ldots n_m)} \text{ if } f \text{ is defined for } n_1 \ldots n_m; \\ &= \text{an unsolvable term otherwise.} \end{aligned}$$

Remark 3.4 It is well known that every NS is an adequate numeral system (see Proposition 6.4.3 of [1]). Conversely an adequate numeral system with numerals in normal form is a NS. However this holds true only for normal QNSs, see Section 6.

Remark 3.5 An example of a classical NS is the Church NS: $\underline{n} = \lambda yx.y^n x$.

In the following, we shall denote with $\underline{n}$ the nth element of Church system, and with $[\underline{s}]$, $[\underline{p}]$ and $[\underline{z}]$ respectively a (standard) successor, predecessor and test for zero of the Church NS (see Chapter 13A of [6]).

3.2 Independence of adequacy conditions A natural question to ask is whether the three requirements in the definition of NS (the existence of successor, predecessor and test for zero) are mutually independent. To answer this question we observe that a well known result of Barendregt and Wadsworth (see [10] or 6.8.21 of [1]) establishes that the QNS **d** defined as follows:

$$\mathbf{d}_n = \mathbf{K}^n\mathbf{I}$$

has successor and predecessor but not test for zero. Looking for a QNS without successor, we have first to make another distinction between different QNS.

Definition 3.6 A QNS **d** is *recursively enumerable* (r.e.) if $\{\mathbf{d}_n\}_{n\in\mathcal{N}}$ is a r.e. set of terms.

Lemma 3.7 *A QNS with successor is r.e.*

Proof: We use Church thesis. Let [s] be a successor for the QNS **d**. To have an effective enumeration of the elements $\mathbf{d}_n$ of **d**, we start with $\mathbf{d}_0$ and observe that for each n, $[\mathrm{s}]^n\mathbf{d}_0$ reduces to the unique normal form $\mathbf{d}_n$.

Theorem 3.8 *There exists a QNS without successor but with predecessor and test for zero.*

Proof: Let **Q** be a non r.e. set of Church numerals such that $\underline{0} \in \mathbf{Q}$. We set:

$$\begin{array}{lcl} \mathbf{q}_n & = & \text{the } n\text{th element of } \mathbf{Q} \text{ in the increasing order,} \\ \mathbf{d}_0 & = & \langle \underline{0}, \underline{0} \rangle \\ \mathbf{d}_{n+1} & = & \langle \mathbf{q}_{n+1}, \mathbf{d}_n \rangle. \end{array}$$

Obviously, **d** is a QNS with predecessor $[\mathrm{p}]\mathbf{d}_{n+1} = \mathbf{d}_{n+1}\mathbf{O}$, and test for zero $[\mathrm{z}]\mathbf{d}_n = [\underline{z}](\mathbf{d}_n\mathbf{K})$. By Lemma 3.7 **d** does not have a successor.

In the following section, we show that is impossible to find a r.e. QNS with the properties of Theorem 3.8. In order to establish the independence of adequacy requirements, it remains to prove that there exist QNSs with successor and test for zero but without predecessor.

Theorem 3.9 *There exists a QNS with successor and test for zero, but without predecessor.*

Proof: Let $\mathbf{d}_0 = \langle \mathbf{I},\mathbf{T} \rangle$, then $\mathbf{d}_{n+1} = \langle \mathbf{K}^{n+1}\mathbf{I},\mathbf{F} \rangle$. Obviously **d** has a successor and a test for zero. Assume [p] to be a predecessor for **d**. Let X be of the form $\mathbf{K}^n\mathbf{I}$, then we have:

$$\begin{array}{rcl} [\mathrm{p}]\langle \mathbf{K}\mathrm{X}, \mathbf{F}\rangle\mathbf{O} & = & \mathbf{T} \;\; \text{if} \;\; \mathrm{X} = \mathbf{I}, \text{ and} \\ & = & \mathbf{F} \;\; \text{if} \;\; \mathrm{X} \neq \mathbf{I}. \end{array}$$

But then $\lambda x.[\mathrm{p}]\langle \mathbf{K}x,\mathbf{F}\rangle\mathbf{O}$ would be a test for zero for the QNS $\mathbf{q}$ with $\mathbf{q}_n = \mathbf{K}^n\mathbf{I}$, which is impossible.

3.3 Further results on independence We show that even in the restricted case of r.e. QNSs the existence of successor, predecessor and test for zero are, with only one exception, completely independent of each other. In fact Table 1 holds.

Now the line (Y,Y,Y) of this table corresponds to NS. The lines (Y,Y,N) and (Y,N,Y) were treated above, and we shall consider line (N,Y,Y) in the following section. We consider the other lines in Table 1 below.

Theorem 3.10 *There exists a r.e. QNS without any successor, predecessor or test for zero. (Case (N, N, N)).*

Proof: Let $\mathbf{d}_n = \mathbf{K}^{n+1}\underline{n+1}$. We claim that $\mathbf{d}$ is the required example. Our method of proof is almost the same as that of [10]. For brevity we consider only the successor, the other cases being similar.

Assume that $[\mathrm{s}] = \lambda x_1 \ldots x_k.\xi \mathrm{M}_1 \ldots \mathrm{M}_p$ is a successor for $\mathbf{d}$. If $\xi \neq x_1$, then $[\mathrm{s}]\mathbf{d}_m$ cannot have, when reduced to nf, more than $k-1$ head abstractions for every m, which is impossible. If $\xi = x_1$, then for $m > p$ we have:

$$[\mathrm{s}]\mathbf{d}_m = \lambda x_2 \ldots x_k.\mathbf{K}^{m+1-p}\underline{m+1} = \mathbf{K}^{m+k-p}\underline{m+1}.$$

On the other hand:

$$\mathbf{d}_{m+1} = \mathbf{K}^{m+2}\underline{m+2},$$

and hence we have a contradiction.

Theorem 3.11 *There exists a r.e. QNS with only a test for zero. (Case (N, N, Y)).*

Proof: Let $\mathbf{d}_n$ be defined as in the proof of Theorem 3.10. We set:

$$\mathbf{q}_0 = \langle \mathbf{d}_0,\mathbf{T} \rangle \qquad \mathbf{q}_{n+1} = \langle \mathbf{d}_{n+1},\mathbf{F} \rangle.$$

Obviously, $\mathbf{q} = \mathbf{q}_0,\ldots, \mathbf{q}_n,\ldots$ is a r.e. QNS with test for zero. On the other hand, $\mathbf{q}$ cannot have a successor $[\mathrm{s}]_{\mathbf{q}}$ for otherwise we could find a successor $[\mathrm{s}]_{\mathbf{c}}$ for $\mathbf{c} = \mathbf{d}-\{\mathbf{d}_0\}$, setting:

$$[\mathrm{s}]_{\mathbf{c}} = \lambda x.[\mathrm{s}]_{\mathbf{q}}\langle x, \mathbf{F} \rangle\mathbf{K}.$$

However $\mathbf{c}$ cannot have a successor (see the proof of Theorem 3.10). To prove that a predecessor cannot exist, we invoke Corollary 4.7 of the next section. In fact, $\mathbf{q}$ being r.e., a test for zero together with the existence of a predecessor would imply the existence of a successor.

Theorem 3.12 *There exists a r.e. QNS with only a successor. (Case (Y, N, N)).*

Proof: We set:

$$\mathbf{d}_0 = \mathbf{K}$$
$$\mathbf{d}_{n+1} = \mathbf{d}_n \circ \mathbf{d}_n,$$

where $\circ$ represents infixed composition. Obviously $\mathbf{d} = \mathbf{d}_0, \ldots,\mathbf{d}_n, \ldots$ is a r.e. QNS with a successor. By a simple induction, we find that:

$\mathbf{d}_n = \lambda y x_1 \ldots x_{2^n}.y.$

Now suppose [p] is a predecessor for **d**. We can assume [p] = $\lambda z_1 \ldots z_r.z_i \mathrm{M}_1 \ldots \mathrm{M}_m$. We note that i must be 1, for otherwise [p] $\mathbf{d}_{n+1}$ = $\mathbf{d}_n$ could not have unbounded many head abstractions. It follows, for $2^{n+1} > m$, [p]$\mathbf{d}_{n+1} = \lambda z_2 \ldots z_r x_1 \ldots x_s.(\mathrm{M}_1)\star$; where $(\mathrm{M}_1)\star$ = M[$\mathbf{d}_{n+1}/z_1$] and $s = (2^{n+1} - m + 1)$. Now observe that the variables $z_1 \ldots z_s$ do not appear, after renaming, in $(\mathrm{M}_1)\star$, so they cannot be eliminated by η-reduction.

It follows that [p] $\mathbf{d}_{n+1}$—when reduced to nf—has no less than s irreducible head abstractions, but $s > 2^n + 1$ (beginning from a suitable n_0) and this is impossible. For a test for zero, the proof is similar.

Theorem 3.13 *There exists a r.e. QNS with only the predecessor. (Case (N, Y, N)).*

Proof: We set:

$\mathbf{d}_n = \langle\, \mathbf{K}^{2^n}\mathbf{I}, \ldots, \mathbf{K}^{2^i}\mathbf{I}, \ldots, \mathbf{KI}\, \rangle.$

where $0 \leq i \leq n$. Obviously **d** = $\mathbf{d}_0, \ldots, \mathbf{d}_n, \ldots$ is a r.e. QNS with a predecessor [p] $= \lambda x.x \circ \mathbf{K}$. Now, suppose that [s] is a successor for **d**. We can assume [s] = $\lambda x_1 \ldots x_r.x_i \mathrm{M}_1 \ldots \mathrm{M}_t$. We note that i must be 1, for otherwise [s] $\mathbf{d}_n$ = $\mathbf{d}_{n+1}$ could not have an arbitrary large grade (recall that the grade is the number of arguments of the head variable). Now we consider the leftmost branch B of the Böhm tree of [s]. Suppose that B is an infinite chain of occurrences of the variable x_1 as head variable at every node. Then we have in [s] $\mathbf{d}_n$ an infinite leftmost sequence of the form:

$\langle \ldots \rangle (\langle \ldots \rangle (\langle \ldots \rangle \ldots$

so that [s] $\mathbf{d}_n$ cannot reduce to nf because it has an infinite leftmost reduction sequence.

It follows that for some m the mth node of B is a variable $\xi \neq x_1$. Now we introduce some definitions to describe the situation after substitution of $\mathbf{d}_n$ for x_1 in [s], and after reductions of the form:

($\star$) $\mathbf{d}_n$X = X($\mathbf{K}^{2^n}\mathbf{I}$)... ($\mathbf{K}^{2^i}\mathbf{I}$)... (**KI**).

Let $\mathrm{M}_{1,j}$ denote the jth term, for $1 \leq j \leq m$, of the following sequence: let

$\mathrm{M}_{1,1} = \mathrm{M}_1$;

for $j < m$ assume inductively that $\mathrm{M}_{1,j}$ has the form:

$\lambda x_{j_1} \ldots x_{j_{r_j}}.\xi_j \mathrm{M}_{j_1} \ldots \mathrm{M}_{j_{t_j}}$,

then we put

$\mathrm{M}_{1,j+1} = \mathrm{M}_{j_1}$.

Notice now that ξ_j is x_1 if $j < m$ and it is ξ otherwise (i.e., for $j = m$). We abbreviate with $\mathcal{X}_j$ the sequence $x_{j_1} \ldots x_{j_{r_j}}$; and with $\mathcal{M}_j$ the sequence $\mathrm{M}_{j_2} \ldots \mathrm{M}_{j_{r_j}}$ after substitution of $\mathbf{d}_n$ for x_1. Moreover we abbreviate with $\mathcal{X}_0$ the sequence $x_2 \ldots x_r$, with $\mathcal{M}_0$ the sequence $\mathrm{M}_2 \ldots \mathrm{M}_t$ after substitution of $\mathbf{d}_n$ for x_1, and with $\mathcal{K}_n$ the sequence ($\mathbf{K}^{2^n}\mathbf{I}$)...($\mathbf{K}^{2^i}\mathbf{I}$)...(**KI**). Then after reductions of the form ($\star$) we obtain:

$$\begin{aligned}[\mathrm{s}]\,\mathbf{d}_n \;=\;& \lambda \mathcal{X}_0.(\lambda \mathcal{X}_1.(\lambda \mathcal{X}_2 \ldots (\lambda \mathcal{X}_{m-2}. \\ & (\lambda \mathcal{X}_{m-1}.(\lambda \mathcal{X}_m.\xi \mathcal{M}_m)\mathcal{K}_n \mathcal{M}_{m-1})\mathcal{K}_n \mathcal{M}_{m-2}) \ldots \mathcal{K}_n \mathcal{M}_2)\mathcal{K}_n \mathcal{M}_1)\mathcal{M}_0.\end{aligned} \tag{1}$$

Now we have different cases:

Case 1: ξ is a free variable. This is impossible because [s] $\mathbf{d}_n$ must reduce to a closed term;

Case 2: ξ is one of the x_{j_k} for $1 \leq j \leq m$, $1 \leq k \leq r_j$, then a term of the form $\mathbf{K}^{2^u}\mathbf{I}$ is substituted for ξ so that for n sufflciently large [s] $\mathbf{d}_n$ reduces to $\mathbf{K}^{t(n)}\mathbf{I}$ where:

$$t(n) \geq 2^{(n-m\star)} - \Sigma_j t_j - t - n \cdot (m-1) \tag{2}$$

and $m\star = max_j r_j$.

Now everything is fixed, with the exception of n, so that for $n \Rightarrow \infty$ the right side of (2) is a well defined positive number, however

$$\mathbf{K}^{t(n)}\mathbf{I} \neq \mathbf{d}_{n+1},$$

and we obtain a contradiction.

Case 3: If ξ, is one of the variables $x_2, \ldots, x_r$, then after substitution of $\mathbf{d}_n$ for x_1, M_{m_1} must reduce to $\mathbf{K}^{2^{n+1}}\mathbf{I}$; we can now repeat on M_{m_1} the same reasoning we followed for [s] in Cases 1 and 2. For brevity we give only a sketch, noticing that:

- The leftmost branch B' of the Böhm tree of M_{m_1} cannot be an infinite chain of occurrences of the variable x_1;
- After reductions of the form ($\star$) we obtain for $\mathrm{M}_{m_1}[\mathbf{d}_n/x_1]$ an expanded form strictly analogous to equation (1), to be put inside (1) (expanding $\mathrm{M}_{m_1}[\mathbf{d}_n/x_1]$);
- Now consider the new head variable $\xi\star$:

  1. $\xi\star$ cannot be a free variable;
  2. $\xi\star$ cannot be an abstracted variable different from $x_2, \ldots, x_r$ (only constants change in the right side of formula (2) and we have to add a constant number of abstractions, but this cannot give us $\mathbf{K}^{2^{n+1}}\mathbf{I}$);
  3. Assume now that $\xi\star$ is one of the variables $x_2, \ldots, x_r$, then for every n it is possible to obtain in $\mathrm{M}_{m_1}[\mathbf{d}_n/x_1]$ only a fixed number of head abstracted variables, as is easily seen from formula (1). However this is impossible since $\mathrm{M}_{m_1}[\mathbf{d}_n/x_1]$ must reduce to $\mathbf{K}^{2^{n+1}}\mathbf{I}$.

This completes Case 3 and we have proved that a successor cannot exist. To prove that a test for zero cannot exist we invoke Corollary 4.7 of the next section.

4 A trip into the zoology of numeral systems NSs, although they are the simplest infinite sets of terms, can still be very complicated. In [7] the problem of a general classification of NSs was posed and partially solved. However there seems to be room for the "zoological" stadium of research (i.e., the description of "strange specimens"). In this section, after giving a few well known instrumental results, we give some examples of the complexity of this subject.

4.1 CH-sets We recall that QNS always means “normal, closed QNS.” As before, we use the notation [. . .] to denote operations on QNSs.

Definition 4.1 A QNS **d**:

- Has an *iterator* if there exists a term $[it]$ such that $[it]\mathbf{d}_n = \underline{n}$;
- has a *test for equality* if there exists a term $[eq]$ such that:

$$\begin{aligned} [eq]\,\mathbf{d}_n\mathbf{d}_m &= \mathbf{T} \quad \text{if } \mathbf{d}_n = \mathbf{d}_m; \\ &= \mathbf{F} \quad \text{otherwise.} \end{aligned}$$

Now we prove the characterization of NSs stated in the introduction.

Lemma 4.2 *Let Q be a r.e. set of terms in normal form, then there is a term F such that:*

$$Q = \{F\underline{n} \mid n \in \mathcal{N}\}.$$

Remark on notation: equality modulo convertibility is extended for brevity to sets.

Proof: This follows easily from the existence of an universal generator **E** (see paragraph 8.1.6 of [1]). We can also require that F is injective on Church numerals.

Theorem 4.3 *A QNS* **d** *is a NS iff it is r.e. and has a test for equality.*

Proof: One direction follows easily. For the other assume **d** to be a r.e. QNS and let $[eq]$ be a test for equality of **d**. Moreover let F be such that $F\underline{n} = \mathbf{d}_n$. Obviously **d** has a test for zero. We show that **d** has a successor as follows. Let G be defined by the following fixed point equation:

$$Gxy = [eq]x(Fy)(F(\underline{s}y))(Gx(\underline{s}y)).$$

It is easily seen that $G\,\mathbf{d}_n\underline{0} = \mathbf{d}_{n+1}$. The existence of a predecessor is proved in a similar way.

Definition 4.4 A QNS **d** is a *CH-set* if there is a term H such that:

(i) For every $\mathbf{d}_n$ $H\mathbf{d}_n$ is a Church numeral;

(ii) For every $\mathbf{d}_n$, $\mathbf{d}_m$ if $\mathbf{d}_n \neq \mathbf{d}_m$ then $H\mathbf{d}_n \neq H\mathbf{d}_m$.

Remark 4.5 It is sometimes easier to test whether a QNS is a CH-set than to find an iterator. However, as far as r.e. QNSs are considered, these conditions are equivalent. In a sense the CH-sets are those enumerable “from inside the λ-calculus.” We have the following “Post-like” result:

Theorem 4.6 *The r.e. CH-sets are exactly the NSs.*

Proof: Immediate by Theorem 4.3.

Corollary 4.7 *A r.e. QNS* **q** *with test for zero and predecessor is a NS.*

Proof: By a standard fixed point construction **q** has an iterator. So it is a CH-set and the result follows.

Remark 4.8 Notice that it is not too difficult to find a QNS not r.e. with an iterator.

4.2 Böhm-Wadsworth unusual NSs In [10] the problem of finding NSs such that numerals have unbounded head abstractions is posed and solved. NSs with this property were found independently by Böhm. With the aid of previous results it easy to find a plethora of such NSs, like the following one:

Example 4.9 $\mathbf{d}_n = \lambda y x_1 \ldots x_n . y \underline{n} (\mathbf{K} x_1) \ldots (\mathbf{K} x_n)$.

Proof: We have to show that $\mathbf{d} = \mathbf{d}_0, \ldots, \mathbf{d}_n, \ldots$ is a NS. Obviously $\mathbf{d}$ is r.e., and moreover it is not difficult to find an iterator.

Remark 4.10 Examples like the previous one—but not those examples of Böhm and Wadsworth—are clearly "redundant." However the question arises as to how one can formally discriminate between the two kinds of NS.

4.3 Paradoxical NSs In [4] the following problem is posed (recall that we are assuming that numerals are in nf):

Problem Let $\mathbf{d}$ be a NS. Is it always possible to find a successor [s], a predecessor [p] and a test for zero [z] such that:

1. [s] $\mathbf{d}_n \gg \mathbf{d}_{n+1}$;
2. [p] $\mathbf{d}_{n+1} \gg \mathbf{d}_n$;
3. [z] $\mathbf{d}_0 \gg \mathbf{T}$, [z] $\mathbf{d}_{n+1} \gg \mathbf{F}$?

Recall that $\gg$ means strong normalization, i.e., every sequence of reductions is finite.

We exhibit a counterexample below. Independently Statman [9] found a different counterexample which, in a sense, furnishes a stronger negative result. In fact Statman's NS cannot have a test for zero in nf.

Theorem 4.11 *There exists a NS* $\mathbf{d}$ *such that:*

1. $\mathbf{d}$ *has a successor [s], a predecessor [p] and a test for zero [z] in normal form;*
2. *For every choice of [s], [p] and [z] none of the strong normalizability relations (1) – (3) above hold.*

Informal sketch of the proof: We start informally explaining the ideas involved in the proof. We want to disprove Böhm's conjecture, so we look for a NS $\mathbf{d}$ such that, though numerals are in nf, for every choice of a successor [s], [s] $\mathbf{d}_n$ reduces to $\mathbf{d}_{n+1}$ but not strongly. This means that the reduction process of [s] $\mathbf{d}_n$ creates subterms without nf, which are subsequently erased. To make sure that this will be the case for every choice of [s], observe that we can obviously assume that [s] is already in nf. So to get the counterexample we encode in the numerals of $\mathbf{d}$ larger and larger sequences $\mathcal{Z}_n$ of terms suitable to generate Ω (i.e., $(\lambda x.xx)(\lambda x.xx)$) when substituted in terms in nf of a given complexity. So for every nf N there will be a numeral large enough to create an Ω subterm when substituted inside N.

Proof: For every n, we abbreviate with $\mathcal{Z}_n$ the following sequence:

$$\mathcal{Z}_n = \underbrace{(\mathbf{K}^{u_n}\mathbf{I}) \ldots (\mathbf{K}^{u_n}\mathbf{I})}_{v_n\text{-times}} \underbrace{\mathbf{I}\,\mathbf{I} \ldots \mathbf{I}\,\mathbf{I}}_{w_n\text{-times}} \,\omega\,\omega$$

where u_n, v_n and w_n are to be defined, and $\omega \equiv \lambda x.xx$.

Now we set:

$$\mathbf{d}_n = \lambda x.x\underline{n}\underbrace{(x\mathcal{Z}_n)\ldots x\mathcal{Z}_n)}_{z_n\text{-times}} \tag{3}$$

and we put:

$z_n = 2^{2^n}$, $u_n = 2^{z_n}$, $v_n = 2^{u_n}$ and $w_n = 2^{v_n}$.

We claim that the QNS **d** determined by (3) is the required NS. We subdivide the proof into two different parts:

1. **d** is a NS with successor, predecessor and test for zero in nf.
2. For every successor [s] for **d** there exist numerals $\mathbf{d}_n$ such that [s] $\mathbf{d}_n \gg \mathbf{d}_{n+1}$ does not hold.

Proof: Part (1): Obviously **d** is r.e. and has an iterator [it] in normal form. In fact we can put:

$[\text{it}]= \lambda y.y(\lambda y_1.y_1\underline{2}\,\underline{2}\mathbf{K}y_1)$

using the well known fact that n^m is represented, inside Church numerals, by application of $\underline{m}$ to $\underline{n}$. It follows from Theorem 4.6 that **d** is a numeral system. Observe that the $\mathbf{d}_n$ terms are uniformly solvable by the sequence

$\lambda y_1.y_1\underline{2}\,\underline{2}\mathbf{K}y_1,\mathbf{I},\mathbf{I}$.

With the aid of this sequence it is possible, by a standard construction (or by the antlion paradigm of Böhm and Intrigila [5]) to "make a normal" successor, predecessor and test for zero.

Proof: Part (2): Suppose that [s] is a successor for **d**. We can assume:

$[\text{s}]= \lambda x_1 \ldots x_r.x_i\,\text{M}_1 \ldots \text{M}_t$.

We note that i must be 1 for otherwise [s] $\mathbf{d}_n = \mathbf{d}_{n+1}$ could not have an arbitrary large grade. As in the proof of Theorem 3.13 we consider the leftmost branch B of the Böhm tree of [s]. We assume that [s] $\gg$ nf (otherwise we would be done); so that we can directly assume that [s] is in normal form. Hence B is finite and we let k be the length of B. Two cases can occur:

Case 1: for some $m \le k$, the mth element of B is a variable $\xi \ne x_1$.

To prove Case 1 we shall introduce some new definitions: let $\text{M}_{1,j}$ denote the jth term, for $1 \le j \le m$ of the following sequence: let $\text{M}_{1,1} = \text{M}_1$. For $j < m$ assume inductively that M has the form $\lambda x_{j_1} \ldots x_{j_{r_j}}.\xi_j \text{M}_{j_1} \ldots \text{M}_{j_{t_j}}$. Then we put $\text{M}_{1,j+1} = \text{M}_{j_1}$. Observe that ξ_j is the $(j+1)$th element of B, and it is the variable x_1 if $j+1 < m$ and $\xi_j \ne x_1$ if $j+1 = m$. We abbreviate with $\mathcal{X}_j$ the sequence $x_{j_1} \ldots x_{j_{t_j}}$ and with $\mathcal{M}_j$ the sequence $\text{M}_{j_2} \ldots \text{M}_{j_{t_j}}$ after substitution of $\mathbf{d}_n$ for x_1. Moreover we abbreviate with $\mathcal{X}_0$ the sequence $x_2 \ldots x_r$, and with $\mathcal{M}_0$ the sequence $\text{M}_2 \ldots \text{M}_t$ after substitution of $\mathbf{d}_n$ for x_1. We consider the situation in [s] $\mathbf{d}_n$ after reductions of following form:

$(\star)$ $\mathbf{d}_n\text{Y} = \text{Y}\underline{n}(\text{Y}\mathcal{Z}_n)\ldots(\text{Y}\mathcal{Z}_n)$.

Starting from M and considering reduction of type ($\star$) we obtain the following sequence:

$$\mathcal{N}_0 = \lambda\mathcal{X}_m.\xi\mathcal{M}_m$$
$$\mathcal{N}_{i+1} = \lambda\mathcal{X}_{m-(i+1)}.\mathcal{N}_i\underline{n}(\mathcal{N}_i\mathcal{Z}_n)\dots(\mathcal{N}_i\mathcal{Z}_n)\mathcal{M}_{m-(i+1)}$$
$$\mathcal{N}_m = \lambda\mathcal{X}_0.\mathcal{N}_{m-1}\underline{n}(\mathcal{N}_{m-1}\mathcal{Z}_n)\dots(\mathcal{N}_{m-1}\mathcal{Z}_n)\mathcal{M}_0.$$

Observe that [s] $\mathbf{d}_n = \mathcal{N}_m$. Now ξ, cannot be free, so that we have:

Subcase 1: $\xi = x_i$ for x_i in $\mathcal{N}_0$. By induction on construction of the sequence $\mathcal{N}_i$ we have the following upper bound for the grade $g(\xi)$ of ξ in [s] $\mathbf{d}_n$:

$$g(\xi) \leq (2^{2^n} + 1)\cdot m + \Sigma_j t_j + r. \tag{4}$$

But this implies $g(\xi) < 2^{2^{n+1}}$, which is impossible.

Subcase 2: $\xi = x_{j_h}$ for $1 \leq j \leq m$ and $1 \leq h \leq r_j$. Then we consider the subterm L = $\mathcal{N}_{m-j}\mathcal{Z}_n$ in the term $\mathcal{N}_{(m-j)+1}$. Clearly L can be β-reduced, so that a term of the form $\mathbf{K}^{u_n}\mathbf{I}$ will be eventually substituted for ξ. But $u_n = 2^{z_n}$ and the upper bound (4) holds *a fortiori* for $g'(\xi)$, the grade of ξ in $\mathcal{N}_{m-j}$. It follows that:

$$\mathrm{L} > \underbrace{(\mathbf{K}^{u_n}\mathbf{I})\dots(\mathbf{K}^{u_n}\mathbf{I})}_{a\text{-times}}\underbrace{\mathbf{I}\,\mathbf{I}\dots\mathbf{I}\,\mathbf{I}}_{w_n\text{-times}}\,\omega\,\omega$$

where $a > 2^{u_n} - 2\cdot u_n$. By the choice of u_n and w_n, L reduces to Ω, where $\Omega \equiv \omega\omega$. This completes the proof of Case 1.

Case 2: For every $m \leq k$, the mth element of B is x_1;

We can use the same setting of Case 1, and define a sequence $\mathrm{M}_{1,j}$ of terms, for $1 \leq j \leq k-1$, as follows: let $\mathrm{M}_{1,1} = \mathrm{M}_1$. For $j < k-1$ assume inductively that $\mathrm{M}_{1,j}$ has the form $\lambda x_{j_1}\dots x_{j_{r_j}}\cdot\mathrm{M}_{j_1}\dots\mathrm{M}_{j_{t_j}}$. Then we put $\mathrm{M}_{1,j+1} = \mathrm{M}_{j_1}$. Observe that ξ_j is the $(j+1)$th element of B, and is x_1 for every j. Moreover $\mathrm{M}_{1,k-1}$ has the form: $\lambda x_{(k-1)_1}\dots x_{(k-1)_{r_{(k-1)}}}\cdot x_1$.

As in Case 1, we abbreviate with $\mathcal{X}_j$ the sequence $x_{j_1}\dots x_{j_{r_j}}$ and with $\mathcal{M}_j$ the sequence $\mathrm{M}_{j_2}\dots\mathrm{M}_{j_{t_j}}$ after substitution of $\mathbf{d}_n$ for x_1. Now we are in a situation similar to Case 1, but much more simple. Substitution of $\mathbf{d}_n$ for x_1 in $\mathrm{M}_{1,k-2}$ gives us: $\lambda\mathcal{X}_{(k-2)}\cdot\lambda\mathcal{X}_{(k-1)}\cdot\mathbf{d}_n)\underline{n}(\lambda\mathcal{X}_{(k-1)}\cdot\mathbf{d}_n\mathcal{Z}_n)\dots(\lambda\mathcal{X}_{(k-1)}\cdot\mathbf{d}_n\mathcal{Z}_n)\mathcal{M}_{k-2}$. Now we consider one subterm L of the form: $(\lambda\mathcal{X}_{(k-1)}\cdot\mathbf{d}_n\mathcal{Z}_n$. We have:

$$\mathrm{L} > \underbrace{\mathbf{d}_n(\mathbf{K}^{u_n}\mathbf{I})\dots(\mathbf{K}^{u_n}\mathbf{I})}_{a\text{-times}}\underbrace{\mathbf{I}\,\mathbf{I}\dots\mathbf{I}\,\mathbf{I}}_{w_n\text{-times}}\,\omega\,\omega$$

where $a = v_n - r_{k-1} > 0$. Now consider the following subterm L$'$ of L: L$'$ = $\mathbf{d}_n(\mathbf{K}^{u_n}\mathbf{I})$ as is immediately seen in the reduction of L$'$ the following subterm L$''$ occurs: L$''$ = $\mathbf{K}^{u_n}\mathbf{I}\mathcal{Z}_n$. But then by the form of $\mathcal{Z}_n$ we obtain L$'' > \Omega$. This completes the proof of Case 2, and Part 2 follows.

The cases of predecessor and test for zero are similar to the one just developed, and will be omitted. This completes the proof of Theorem 4.11.

Now we show that we can find a NS similar to the one considered in the proof of Theorem 4.11 which is such that its numerals are λ-**I**-terms.

Theorem 4.12 *There exists a NS* $\mathbf{d}'$ *such that* $\mathbf{d}'$ *satisfies conditions (1) and (2) of Theorem 4.12 and moreover every* $\mathbf{d}'_n$ *is a* λ-**I**-*term.*

Proof: For every n we abbreviate with $\mathcal{S}_n$ the following sequence:

$$\mathcal{S}_n = \underbrace{\mathrm{V}_n \ldots \mathrm{V}_n}_{v_n\text{-times}} \underbrace{\mathbf{I}\,\mathbf{I} \ldots \mathbf{I}\,\mathbf{I}}_{w_n\text{-times}}$$

where V_n is the following term:

$$\mathrm{V}_n = \lambda x_1 \ldots x_{u_n} \cdot x_1 \omega\, \omega (x_2 \omega\, \omega) \ldots (x_{u_n} \omega\, \omega)$$

and u_n, v_n and w_n are natural numbers to be defined. We define:

$$\mathbf{d}'_n = \lambda x.x\underline{n}\, \underbrace{(x\mathcal{S}_n) \ldots (x\mathcal{S}_n)}_{z_n\text{-times}}$$

and we put $z_n = 2^{2^n}$, $u_n = 2^{z_n}$, $v_n = 2^{u_n}$ and $w_n = 2^{v_n}$. Now the proof follows strictly the one of Theorem 4.11, observing that if in V_n some x_i is substituted with **I** or V_n, then (sub)terms of the form Ω arise.

Corollary 4.13 *There exist QNSs of the* λ-**I**-*calculus which are not NS in the* λ-**I**-*calculus but are NS in the* λ-**K**-*calculus.*

Proof: By Theorem 9.1.5 of [1], the NS $\mathbf{d}'$ of Theorem 4.12 cannot have a successor or a predecessor or a test for zero in the λ-**I**-calculus.

Remark 4.14

1. If NS with numerals without nf are admitted then it is easy to find examples of NS with the properties of Corollary 4.13. However we stress that the example we give does have numerals in nf.
2. As noted above, Statman has found NSs without test for zero in nf. Let us call **d** a *Statman NS* if **d** is a NS (with numerals in normal form as we always assume) which does not have a successor, a predecessor and a test for zero all in nf. Questions now arise similar to those considered in Section 3, except that where before we were concerned with *existence* we are now concerned with *having nf*. We conjecture that these conditions are also highly independent of each other. Finally we observe that it is possible, with the trick used in the proof of Theorem 4.12, to obtain a Statman NS such that its numerals are λ-**I**-terms.

4.4 Church-sequences We now generalize some results of [7].

Definition 4.15 For a given term N, the *Church sequence uniform in* N (in short SU(N)) is a sequence s_{N} of terms such that:

$$\begin{array}{lcl} s_{\mathrm{N},0} & = & \mathrm{N} \\ s_{\mathrm{N},n+1} & = & \underbrace{\langle \mathrm{N},\ldots,\mathrm{N}\rangle}_{n+1\text{-times}} = \lambda x.x\mathrm{N}^{\sim n+1}. \end{array}$$

Theorem 4.16 *For every closed normal term N, SU(N) is a NS.*

To prove this Theorem we need the following lemma:

Lemma 4.17 *For every closed nf N there is a nf U such that UN = U.*

Proof: We use the “ant-lion paradigm” from [5]. Let $\mathcal{S} = \mathrm{M}_1 \ldots \mathrm{M}_k$ be a finite sequence of terms in nf such that $\mathrm{N}\mathcal{S} = \mathbf{I}$. Let V be such that $\mathrm{V}x = \langle \mathcal{S}, x, x\rangle$. Then we set U = VV.

Proof of Theorem 4.16: Clearly SN(N) is a QNS with successor and predecessor. To find a test for zero let t be a natural number such that for every nf X we have $\mathrm{N}(\mathbf{K}^t\mathrm{X}) >$ nf. That such a t exists is easily proved by induction on the complexity of N. Moreover, by Lemma 4.17, let U be a closed nf such that UN = U. Notice now that:

$$s_{\mathrm{N},n}(\mathbf{K}^k\mathrm{U}) = \begin{cases} \mathrm{N}[\mathbf{K}^t\mathrm{U}/x_1] & \text{if } n = 0; \\ \mathbf{K}^{t-n}\mathrm{U} & \text{if } t > n; \\ \mathrm{U} & \text{otherwise.} \end{cases}$$

Notice moreover that by a suitable choice of U and t, we can always make $\mathrm{N}[\mathbf{K}^t\mathrm{U}/x_1]$ different from U. Now we can discriminate $s_{\mathrm{N},0}$ from $s_{\mathrm{N},n}$ for $n > t$ and therefore, by two applications of the (generalized) Böhm theorem, we are done.

Definition 4.18 For a given term N, *the descending sequence from* N (in short SD(N)) is a sequence s'_{N} of terms such that:

$$\begin{array}{lcl} s'_{\mathrm{N},0} & = & \mathrm{N} \\ s'_{\mathrm{N},n+1} & = & \underbrace{\langle\langle\ldots\langle\langle}_{n+1\text{-times}} \ \mathrm{N}\ \rangle\rangle\ldots\rangle\rangle. \end{array}$$

Theorem 4.19 *For every closed normal term N, SD(N) is a NS.*

Proof: Clearly SD(N) is a QNS with successor and predecessor. To find a test for zero, let $\mathrm{N} = \lambda x_1 \ldots x_q.x_i\mathrm{M}_1 \ldots \mathrm{M}_m$. We have different cases:

Case 1: $m = 0$ so that $\mathrm{N} = \lambda x_1 \ldots x_r.x_i$. This case is known, see [7].

Case 2: $m > 0$ and $q \neq m$. Let $t > 1$ such that NV > nf; where $\mathrm{V} = \mathbf{K}^t\mathbf{I}$. Now we have $s'_{\mathrm{N},n+1}\mathrm{V} = \mathbf{K}^{t-1}\mathbf{I}$ and $\mathrm{NV} \neq \mathbf{K}^{t-1}\mathbf{I}$ for a suitable choice of $t (t > q - 1)$; therefore we can apply the Böhm Theorem.

Case 3: $m > 0$ and $q = m$. We can assume that $i = 1$, otherwise we can simply use the argument of Case 2. Now, let $V' = \lambda z y_1 \ldots y_t.z$ where t is great enough to have N[V'/x1] > nf. Then we have:

$$s'_{N,n} V' \underbrace{\mathbf{I}\,\mathbf{I} \ldots \mathbf{I}}_{t\text{-times}} = \begin{cases} s'_{N,n-1} & \text{if } n > 0; \\ M_1[V'/x_1] & \text{otherwise;} \end{cases}$$

and we obtain a new sequence: $M_1[V'/x_1]$, N, $\langle$ N $\rangle$, $\langle\langle$ N $\rangle\rangle \ldots$ If $M_1[V'/x_1]$ has the form: $M_1[V'/x_1] = \lambda x_1 \ldots x_p.x_1 Q_1 \ldots Q_p$ then we can iterate the process and after a finite number of steps we arrive at a term M′ of the form: $M' = \lambda x_1 \ldots x_p \,.\, x_j Q_1 \ldots Q_{p'}$ such that either $p \neq p'$, or $j \neq 1$, or $M' = \lambda x_1 \ldots x_p.x_j$. Now we consider the final sequence:

$$P_k, P_{k-1}, \ldots, P_1, N, \langle N \rangle, \langle \langle N \rangle \rangle \ldots \tag{5}$$

where P_k =M′. For a suitable choice of t, every term T in (5) is such that $T(\mathbf{K}^t\mathbf{I}) = \mathbf{K}^{t-1}\mathbf{I}$ with the exception of $M'(\mathbf{K}^t\mathbf{I}) \neq \mathbf{K}^{t-1}\mathbf{I}$, but again in nf. Then we can apply the Böhm Theorem. This completes the proof of Theorem 4.19.

5 *The Curry-Hindley-Seldin abstract approach*

5.1 Generalities For completeness of exposition, we recall some definitions and results from Chapter 13A of [6].

Definition 5.1 We say that **c** is *a set of combinatory numerals* (SCN) if there exist closed terms F, G such that:

$$\mathbf{c} = \{F^n G | n \in \mathcal{N}\}.$$

Notation Following [6] we introduce the metavariables [s] and [0] for terms F and G respectively. We let [n] stand for $F^n G$; that is, the nth numeral of **c**. But we also make use of the notation $\mathbf{c}_n$. Observe that this is coherent with the terminology of Sections 2 and 3. We use **c**, **d**, **q** as metavariables for SCN. As before we append subscripts to operations or numerals (as $[s]_{\mathbf{d}}$, $[0]_{\mathbf{d}}$, etc.) only when it is needed to avoid confusion. To link this new approach to the previous one *we assume that all numerals* [n] *reduce to nf*. Thus all the SCNs that we shall consider in this section are assumed to have numerals in nf. *Par abus de langage*, they will be indicated simply with SCN.

Theorem 5.2 *A SCN* **c** *is a finite set or a QNS with successor.*

Proof: Assume that for some m, n, with $m > n$, [n] = [m]. Then for every $q \geq m$ we have $F^q G = F^s G$ where $s = n + re(q - n, m - n)$, and $re(n_1, n_2)$ = the remainder of the division of n_1 by n_2. Therefore **c** is a finite set.

Now we define operations on SCNs. Definitions of successor [s], predecessor [p], test for zero [z] and iterator [it] are done in the obvious way. If we consider QNSs as a particular case of SCNs the old definitions agree with the new ones.

Definition 5.3 A SCN **c**:

1. has a *recursor* if there exists a term [R] such that for all terms T_1, T_2:

$$\begin{aligned}[R]T_1T_2[0] &= T_1 \text{ and}\\ [R]T_1T_2([s][n]) &= T_2[n]([R]T_1T_2[n]);\end{aligned}$$

2. has a μ*-operator* if there exists a term [pe] such that for all terms T_1, T_2:

$$\begin{aligned}[pe]T_1T_2 = T_2 &\quad \text{if } T_1T_2 = [0]; \text{ and}\\ [pe]T_1T_2 = [pe]T_1([s]T_2) &\quad \text{if } T_1T_2 = [n+1] \text{ for some } n.\end{aligned}$$

The following theorem is taken from [6], see Chapter 13A Paragraph 3, Theorems 1 and 2.

Theorem 5.4 *For SCNs:*

(i) each of the terms [it] and [R] is interdefinable with the pair [p], [z];

(ii) none of the terms [p], [z], [it], and [R] are definable only by [0] and [s].

Remark 5.5 The content of Theorem 5.4 (i) is included in that of Theorem 2.6.1, which is slightly stronger. In fact, by Lemma 3.7 SCN are r.e. sets of terms.

Remark 5.6 Theorem 5.4 (ii) follows from the results of Section 3. We observe that to prove Theorem 5.4 (ii) in [6] SCNs are given that turn out to be finite. Now the general question arises as to whether one can obtain the results of [6] with the additional requirement of the mutual nonconvertibility of numerals. We shall be concerned with an example of this general question below.

5.2 Indefinability of the μ operator We observe that [pe] is a weak form of test for zero. In fact the existence of [z] implies the existence of [pe] by the following fixed point construction: $[pe]xy = [z](xy)y([pe]x([s]y))$.

On the other side we know from Section 3 that [z] cannot be defined by [0], [s] and [p], even in the QNS case. In [6], Chapter 13A, Paragraph 3, Remark 1, it is asked whether [pe] can be defined by [0], [s] and [p]. We answer this question negatively below, proving that for the Barendregt-Wadsworth QNS, $\mathbf{d}_n = \mathbf{K}^n\mathbf{I}$, there is not a term satisfying the properties of [pe].

Theorem 5.7 *The QNS* $\mathbf{d}_n = \mathbf{K}^n\mathbf{I}$ *does not have a [pe] term.*

In order to prove the theorem we need the following lemma:

Lemma 5.8 *Let M be such that M*$\mathbf{I}$ *reduces to nf. Then there exists a* n_0 *such that, for* $n \geq n_0$*, M(*$\mathbf{K}^n\mathbf{I}$*) reduces to nf.*

Proof: We consider the Böhm tree of M. First of all M has a head nf: $M = \lambda x_1 \ldots x_q.\xi M_1 \ldots M_m$. If $\xi = x_1$ the lemma follows. If $\xi \neq x_1$ then we can assume that x_1 occurs in some M_j for $1 \leq j \leq m$.

For each M_j, we observe that M_j must have a head nf and if for all M_j, x_1 is the head variable then the lemma follows. Repeating this argument at each level of the Böhm tree of M, we have the following alternatives:

(a) for some level every subterm containing x_1 has x_1 as head variable;

(b) for every level there exists a term containing x_1, such that x_1 is not the head variable.

In (a) the lemma follows. In (b) we observe that by König's Lemma there exists an infinite branch of the Böhm tree of M in which x_1 is not the head variable of any node. But in this case M[**I**/x_1] cannot have nf, so that (b) is impossible. Now we turn to the proof of Theorem 5.7.

Proof of Theorem 5.7: Assume that a [pe] term exists. Then for each X, Y: [pe]XY = Y if XY = **I** and [pe]XY = [pe]X(**K**Y) if XY = $\mathbf{K}^n\mathbf{I}$ for some $n > 0$. Put X = **C**⋆**I**, (where **C**⋆$xy \equiv yx$) and let M = [pe]X. Then M**I** = **I**, (in fact **C**⋆**I I** = **I**); and M(**KI**) = **KI**, (in fact **C**⋆**I**(**KI**) = **I**).

Now for $m, n \geq 2$ we have M($\mathbf{K}^m\mathbf{I}$) = M($\mathbf{K}^n\mathbf{I}$). By the Lemma, M($\mathbf{K}^n\mathbf{I}$) must reduce to nf for $n \geq n_0$, for some n_0. Therefore for $n \geq 2$, M($\mathbf{K}^n\mathbf{I}$) = U for some normal form U. If U = **I** then we can find a test for zero for **d** putting [z]$\mathbf{d}_n$ = M$\mathbf{d}_{n+1}$. If U = **KI** then M would directly be a test for zero. Finally, if U ≠ **I** and U ≠ **KI**, then by Böhm's Theorem, we could again find a test for zero for **d**. So all these cases lead to a contradiction and Theorem 5.7 follows.

Now it is natural to ask if [pe] is strictly weaker than test for zero. If finite counterexamples are admitted, it is easy to see that if we put [s] = **I** and [0] = **I**, the resulting SCN **c** = {**I**} has predecessor and a [pe] term (that in both cases is **I**) but obviously not test for zero. We do not know if this is the case also for QNSs.

Open Problem 5.9 *Does there exist a QNS* **d** *with successor, predecessor and [pe] operator, but without a test for zero?*

6 Adequate, nonnormal, numeral systems without test for zero As we stated in Section 1, there exist adequate numeral systems without the usual test for zero. Since the r.e. function f such that: $f(0) = 0$ and $f(n+1) = 1$ must be representable, it follows from Böhm's Theorem that the numerals have no nf. We give an example of this kind of numeral system below. First however, we have to agree about the meaning of "undefined" in the representation of recursive functions. In fact the Barendregt-Wadsworth thesis: "undefined means an unsolvable term" must now be refined, since numerals may also not have head normal form.

Definition 6.1 Let **d** be a QNS (normal or not). A partial recursive function $f : \mathcal{N}^m \to \mathcal{N}$ *is represented on* **d** by a term F if:

(i) F$\mathbf{d}_{n_1} \ldots \mathbf{d}_{n_m}$ = $\mathbf{d}_{f(n_1 \ldots n_m)}$ if f is defined for $n_1 \ldots n_m$;
= an unsolvable term different from each $\mathbf{d}_n$ otherwise;

(ii) we can effectively recognize if a computation has terminated.

Definition 6.2 Let **d** be a QNS (normal or not), **d** is *an adequate numeral system* if every recursive function is representable on **d**, in accordance with Definition 6.1.

The following lemma has interest in its own right and states that it is sufficient that binary functions are representable.

Lemma 6.3 *Let* **d** *be a QNS (normal or not). Moreover let* **d** *be an adequate numeral system for every unary and binary recursive function. Then* **d** *is adequate for every recursive function.*

Proof: Let $f(m, n, p)$ be a tertiary recursive function (the general case is similar). As it is well known (see paragraph 5.3 of Rogers [8]) there exist a recursive pairing function j and recursive projection functions l, r. By hypothesis they are represented by terms J, L, R. Let the recursive function $g(m, n)$ be defined by $g(m, n) = f(l(m), r(m), n)$. Let G be a term representing g. It is easily seen that the term F defined by $Fxyz = G(Jxy)z$ represents f.

Theorem 6.4 *There exist adequate numeral systems without test for zero.*

Proof: Let $\mathrm{U}x \equiv (xx) \circ (xx)$ and set $\mathrm{P} \equiv \mathrm{UU}$ so that:

($\star\star$) $\mathrm{P} \circ \mathrm{P} = \mathrm{P}$.

Now we put $\mathbf{d}_n = \lambda x.\mathrm{P}(x\underline{n})$. We have to show that:

(i) $\mathbf{d} = \mathbf{d}_0, \ldots, \mathbf{d}_n, \ldots$ is an adequate numeral system;

(ii) $\mathbf{d}$ cannot have a test for zero in the usual sense.

Proof of (i): By the Lemma 6.3, let f be a recursive function of two arguments. Then there exists a term F such that F represents f on Church numerals. First suppose that n, m are such that $f(n, m)$ exists. Let $\mathrm{H}_1 \equiv \lambda ywz.y(wz)$, i.e., let H_1 be the combinator $\mathbf{B}$. Then $\mathrm{H}_1\mathbf{d}_n = \lambda wz.\mathrm{P}(wz\underline{n})$. Let $\mathrm{H}_2 \equiv \lambda x_1x_2x_3.x_1(\mathrm{B}x_2(Fx_3))$, then $\mathrm{H}_2\mathbf{d}_m = \lambda x_2x_3.\mathrm{P}(x_2(Fx_3\underline{m}))$. Now we have: $\mathrm{H}_1\mathbf{d}_n(\mathrm{H}_2\mathbf{d}_m) = \lambda z.\mathrm{P}(\mathrm{P}(z(F\underline{n}\,\underline{m})))$ and by ($\star\star$):

$$= \lambda z.\mathrm{P}(z(F\underline{n}\,\underline{m}) = \mathbf{d}_{f(n,m)}.$$

We have proved that f, when defined, is represented on $\mathbf{d}$ by:

$$\mathrm{F}\star = \lambda xy.\mathrm{H}_1x(\mathrm{H}_2y).$$

Now suppose that n, m are such that $f(n, m)$ is not defined. Then $F\underline{n}\,\underline{m}$ is an unsolvable term, so that$\mathrm{F}\star\underline{n}\,\underline{m}$ does not reduce to any numeral $\mathbf{d}_k$. Furthermore it is clear that we can effectively recognize if a computation has terminated.

Proof of (ii): Now we show that $\mathbf{d}$ cannot have a test for zero [z]. By [10], [z] must have the form [z] = $\lambda x_1 \ldots x_m.x_1\mathrm{M}_1 \ldots \mathrm{M}_q$. Now, if $q = 0$, i.e., [z] = $\lambda x_1 \ldots x_m.x_1$ then $[\mathrm{z}]\mathbf{d}_n = \lambda x_2 \ldots x_m z.\mathrm{P}(z\underline{n})$ which cannot reduce to nf. If $q > 0$ then:

$$[\mathrm{z}]\ \mathbf{d}_n = \lambda x_2 \ldots x_m.\mathrm{P}(\mathrm{M}_1[\mathbf{d}_n/x_1]\underline{n})\mathrm{M}_2[\mathbf{d}_n/x_1] \ldots \mathrm{M}_q[\mathbf{d}_n/x_1]$$

which also cannot reduce to nf. This completes the proof of Theorem 6.4.

Acknowledgments I would like to thank Professor Corrado Böhm, who introduced me to this subject and gave constant help and useful suggestions. Also, I would like to thank the anonymous referee, for useful suggestions about the presentation of the results of this paper. My work was partially supported by ESPRIT WG 3220 "Common Foundations of Functional and Logic Programming" and by Research Project 60% MURST, Italy.

REFERENCES

[1] Barendregt, H., *The Lambda Calculus*, Studies in Logic, vol. 103, North-Holland, Amsterdam, 1984. Zbl 0551.03007 MR 86a:03012 1, 1, 2, 2, 2, 3.1, 3.4, 3.2, 4.1, 4.3

[2] Barendsen, E., "Theoretical pearls: an unsolvable numeral system in lambda-calculus," *Journal of Functional Programming*, vol. 1 (1991), pp. 367–72. Zbl 0941.03513 MR 93g:03012 1

[3] Böhm, C., "Cuch as a formal and description language," pp. 179–197 in *Formal Language Description Languages for Computer Programming*, edited by T. Steel, North-Holland, Amsterdam, 1966. 1

[4] Böhm, C., "Subduing self application," pp. 108–122 in *ICALP 89*, Springer Verlag, Berlin,1989. Zbl 0686.68010 MR 91a:03025 1, 4.3

[5] Böhm, C., and B. Intrigila, "The ant-lion paradigm for strong normalization," *Information and Computation*, vol. 114, (1994), pp. 30–49. Zbl 0817.03006 MR 95i:03027 4.3, 4.4

[6] Curry H., J. Hindley, and J. Seldin, *Combinatory Logic, vol. II*, North-Holland Amsterdam, 1972. Zbl 0242.02029 1, 1, 1, 1, 1, 1, 3.1, 5.1, 5.1, 5.1, 5.6, 5.6, 5.2

[7] Rezus, A., *Lambda conversion and Logic*, Ph.D. Dissertation, University of Utrecht, Utrecht, 1981. 1, 1, 1, 1, 3.1, 4, 4.4, 4.4

[8] Rogers, H., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York, 1967. Zbl 0183.01401 MR 37:61 6

[9] Statman, R." Taming the wild ant-lion, a counterexample to a conjecture of C. Böhm," Unpublished. 1, 4.3

[10] Wadsworth, C., "Some unusual λ-calculus numeral systems," pp. 215–230 in *To H. B. Curry: Essays on Combinatory Logic, Lambda-Calculus and Formalism*, edited by J. Hindley and J. Seldin, Academic Press, New York, 1980. MR 82b:03043 1, 1, 3.2, 3.3, 4.2, 6

Dipartimento di Matematica
Via Vetoio, Coppito
67100 L'Aquila, Italy
email: *INTRIGILA@AQUILA.INFN.IT*