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Free Algebras Corresponding to Multiplicative Classical Linear Logic and Some of Its Extensions

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Abstract In this paper, constructions of free algebras corresponding to multiplicative classical linear logic, its affine variant, and their extensions with n-contraction ($n \geq 2$) are given. As an application, the cardinality problem of some one-variable linear fragments with n-contraction is solved.

1 Introduction The topic of substructural logics, i.e., logics with restricted structural rules, already has a long tradition (witness relevance logic, BCK-logic, Lambek calculus). However, with the birth of Girard's linear logic in [3], it has regained the attention of researchers with various motivations and different traditions. An extensive survey covering the subject of substructural logics can be found in Došen and Schröder-Heister [2].

The present paper continues our investigation of intuitionistic and classical Gentzen systems with bounded contraction. In particular, n-contraction ($n \geq 2$) is a version of the contraction rule, where $(n+1)$ occurrences of a formula may be contracted to n occurrences. Our motivation for exploring these systems has its roots in the observation that substituting n-contraction for full contraction in the systems considered already results in the splitting of logical operations familiar from linear logic. However, most of the systems with n-contraction do not enjoy the cut-elimination property. Thus, standard proof-theoretic techniques of investigating metaproperties of these systems are not available. In spite of that, the desire to acquire better insights into the effects of bounded contraction led to a number of different papers on this topic. To start with, in [7] we showed that the linear models for $(n+1)$-valued Łukasiewicz logics are suitable models of n-contraction. We also presented a new complete axiomatization for these logics, essentially by means of n-contraction. Further, Hori, Ono, and Schellinx [4] extensively studied syntactic and semantical properties of extensions of the intuitionistic linear logic with knotted structural rules, re-

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sulting in cut-elimination theorems, decidability, and undecidabily results as well as finite model property theorems. Briefly, knotted structural rules permit n copies of a formula to be contracted to k copies, when $n > k$, and to be weakened to k copies, when $n < k$ (with $n, k \geq 1$). Knotted structural rules can be seen as a generalization of those discussed earlier. And finally, in [6] we introduced connectification operators for intuitionistic and classical linear algebras corresponding to linear logic and to some of its extensions with n-contraction. As a useful application of these operators we established the disjunction property for both intuitionistic and classical affine linear logics with n-contraction.

The present paper had its origin in the problem of describing the structure and determining the cardinality of one-variable fragments of some extensions of linear logic with n-contraction and weakening. First, we point out that for the intuitionistic case this task is easy. It is well known that the one-variable fragment of intuitionistic propositional logic (in our notation $\mathbf{IPL}_1^a$) is infinite, due to the Rieger-Nishimura lattice being the Lindenbaum algebra of the one-variable fragment in question (see Troelstra and van Dalen [10]). Consider now the system $\mathbf{IPL}_n^a$ ($n \geq 2$) of affine intuitionistic linear logic with n-contraction (see Appendix 1). We extend the Rieger-Nishimura lattice with $\star$ and $\multimap$, interpreted as meet and relative pseudocomplementation respectively. Clearly, under this interpretation multiplicative connectives collapse with the respective additive ones. Therefore, the structure obtained is not the Lindenbaum algebra of the one-variable fragment of $\mathbf{IPL}_n^a$ but just an infinite $\mathbf{IPL}_n^a$-algebra (see [6]) corresponding to the system under consideration. Nonetheless, since the canonical valuation of one-variable $\mathbf{IPL}_n^a$-formulas with values in this infinite algebra is surjective, we may conclude that the one-variable fragment of $\mathbf{IPL}_n^a$ is infinite.

Far more interesting and involved is the problem of the structure and cardinality of one-variable fragments of some classical linear logic extensions with n-contraction ($n \geq 2$). The Lindenbaum algebra of the one-variable fragment of ordinary classical propositional logic is a lattice of exactly four elements. However, in this paper we show that the Lindenbaum algebras of the one-variable fragments of purely multiplicative (i.e., tensor, par) classical linear logic extended with n-contraction and its affine version are infinite.

Here we make a brief digression to mention some of the recent papers investigating similar questions. It is interesting to note that each of them is supplemented by a computer program designed for a specific generation of models. De Jongh, Hendriks, and Renardel de Lavalette [5] examined the structure of finite diagrams of intuitionistic propositional logic fragments. The diagram of a fragment is nothing but the set of equivalence classes of its formulas partially ordered by the derivability relation. And finally, Slaney [8] showed that natural systems close to relevance logic, but weaker, have infinitely many nonequivalent Ackermann constants. The nondistributive version of relevance logic considered by Slaney is in fact equivalent to propositional classical linear logic extended with full contraction.

In the study of the one-variable fragments of classical logics with n-contraction, we encounter the following obstacles. First, the systems with n-contraction considered here do not enjoy cut-elimination (for a counterexample, see Section 2). Thus, the usual proof-theoretic methods to examine provable equivalence of formulas in these systems are not available. The other approach to this problem is a model-theoretic one. As far as we know, there are only two classes of models suitable to mimic the effects of n-contraction in the logical systems. These are the models for $(n+1)$-valued Łukasiewicz logics (see Appendix 4) and the algebraic models given in Section 2. For our purposes, Łukasiewicz models are useless, since they do not distinguish sufficiently many formulas. On the other hand, there is a difficulty with the algebraic models as well, namely, how to determine nonequivalent expressions in a partially ordered algebraic structure. In what follows, we shall give a partial answer to this, sufficient to solve the cardinality problem discussed earlier. Briefly, we construct a free algebra on one generator corresponding to the one-variable fragment of affine multiplicative classical linear logic with 2-contraction. Further, we elaborate some lemmas and propositions in order to show the existence of two infinite chains in the free algebra introduced. The free algebra on one generator is isomorphic to the Lindenbaum algebra of the one-variable fragment considered, yielding that this fragment is infinite. As an immediate consequence, we see also that one-variable fragments of multiplicative classical linear logic with n-contraction (for any $n \geq 2$) and its affine version are infinite.

Moreover, we wish to emphasize that our construction of the free algebra is of interest in its own right. It will be adapted to respective free algebras on one generator corresponding to multiplicative classical linear logic (a new result in this field), as well as to its affine variant, and their extensions with n-contraction. Also, it will become evident later that any of these constructions can be generalized to the corresponding free algebra on an arbitrary set of generators.

At the end of Section 4, we shall indicate why our present strategy for solving the cardinality problem of one-variable fragments is not directly applicable to the systems extended with constants, negation, and additive connectives. We shall also give some suggestions for further research.

We use Troelstra's [9] notation for the operators of linear logic.

2 The system $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$ ***and monoidal*** $\mathbf{CPL}_2^a$***-algebras*** A multiplicative system of affine classical linear logic with 2-contraction, $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$, is given by the following axioms and rules. Throughout the sequel, Γ, Γ_1, Γ_2, Δ, Δ_1, Δ_2 denote finite multisets of formulas.

Axioms

$$A \Rightarrow A$$

Logical rules

$$\text{L}\star \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \star B \Rightarrow \Delta}$$

$$\text{R}\star \quad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \star B, \Delta_1, \Delta_2}$$

$$\text{L}+ \quad \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A + B \Rightarrow \Delta_1, \Delta_2}$$

$$\text{R}+ \quad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A + B, \Delta}$$

Structural rules

$$\text{LW}\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}$$

$$\text{RW}\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\text{LC}_2\quad \frac{\Gamma, A, A, A \Rightarrow \Delta}{\Gamma, A, A \Rightarrow \Delta}$$

$$\text{RC}_2\quad \frac{\Gamma \Rightarrow A, A, A, \Delta}{\Gamma \Rightarrow A, A, \Delta}$$

$$\mathit{Cut}\quad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Note that for any formulas A, B, and C, the sequent $(B + C) \star A \Rightarrow (B \star A) + C$ is derivable in $\langle\star, +\rangle$-$\mathbf{CPL}_2^a$, however, $\nvdash (B \star A) + C \Rightarrow (B + C) \star A$. The latter is due to the fact that the sequent $(B \wedge A) \vee C \Rightarrow (B \vee C) \wedge A$ is not derivable in $\{\wedge, \vee\}$-classical propositional logic of which $\langle\star, +\rangle$-$\mathbf{CPL}_2^a$ is just a fragment. However, the fragment considered differs from classical propositional logic since $\langle\star, +\rangle$-$\mathbf{CPL}_2^a \nvdash A \Rightarrow A \star A$. To see this, choose for A a propositional variable p. As a counter-model of the given sequent we may take the Łukasiewicz 3-valued model $\mathbf{M}_3(\llbracket\ .\ \rrbracket)$ (see Appendix 4), with $\llbracket p \rrbracket = \frac{1}{2}$.

Next, we shall give a counterexample for cut-elimination in the system $\langle\star, +\rangle$-$\mathbf{CPL}_2^a$. Let p be a propositional variable in the language of $\langle\star, +\rangle$-$\mathbf{CPL}_2^a$. Then,

$$\langle\star, +\rangle\text{-}\mathbf{CPL}_2^a \vdash p, p + p \Rightarrow (p + p)^3,$$

where $(p + p)^3$ stands for $(p + p) \star (p + p) \star (p + p)$. The derivation of the given sequent can be obtained as follows.

Applying R$\star$ to two axiom instances $p + p \Rightarrow p + p$ yields $(p + p)^{(2)} \Rightarrow (p + p)^2$. Another application of R$\star$ to the latter sequent and $p + p \Rightarrow p + p$ yields $(p + p)^{(3)} \Rightarrow (p + p)^3$, and so, with LC_2, applied next, gives $(p + p)^{(2)} \Rightarrow (p + p)^3$. On the other hand, RW applied to $p \Rightarrow p$ yields $p \Rightarrow p^{(2)}$ and, with R+ next, $p \Rightarrow p + p$. Cutting the latter sequent with $(p + p)^{(2)} \Rightarrow (p + p)^3$ results in $p, p + p \Rightarrow (p + p)^3$, with $A^{(k)} \equiv A, A, \ldots, A$, i.e., k copies of formula A in the derivation above.

However, it is easy to check that the sequent $p, p + p \Rightarrow (p + p)^3$ has no cut-free derivation in the system considered. Moreover, note that this is also a counterexample for cut-elimination in the one-variable fragment of $\langle\star, +\rangle$-$\mathbf{CPL}_2^a$.

We now introduce monoidal $\mathbf{CPL}_2^a$-algebras corresponding to this system.

Definition 2.1 $\mathbf{X} = \langle X, \star, +, \leq\rangle$ is a monoidal $\mathbf{CPL}_2^a$-algebra, if:

1. $\langle X, \star\rangle$ and $\langle X, +\rangle$ are commutative monoids (also referred to as semigroups);
2. $\leq$ is a partial order on X, satisfying the following clauses for all $x, y, z \in X$:
   (a) $x \star y \leq x$ and $x \leq x + y$, corresponding to weakening;
   (b) $x \star x \leq x \star x \star x$ and $x + x + x \leq x + x$, corresponding to 2-contraction;
   (c) $(y + z) \star x \leq (y \star x) + z$, i.e., sub-commutativity of $\star$ and $+$;
   (d) if $x \leq y$, then $x \star z \leq y \star z$ and $x + z \leq y + z$, i.e., monotonicity of $\star$ and $+$ with respect to $\leq$.

This formulation of a monoidal $\mathbf{CPL}_2^a$-algebra will turn out to be particularly suitable for our purpose, which is to construct the corresponding free algebra on one generator. Note that (2c) of Definition 2.1 covers precisely the derivability of the sequent $(B+C)\star A \Rightarrow (B\star A)+C$ in $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$. Next, consider an extension of a monoidal $\mathbf{CPL}_2^a$-algebra with 0 and 1 as the respective units for $+$ and $\star$, with involution $\sim$ and the clauses $x\star\sim x=0$, $x+\sim x=1$. We shall show later that in this algebraic structure sub-commutativity of monoidal operations is just equivalent to adjointness (witness Appendix 3).

Further, a monoidal $\mathbf{CPL}_2^a$-model, $\mathbf{M}$, is a pair $\langle \mathbf{X}, [\![\ .\ ]\!]\rangle$, where $\mathbf{X}$ is a monoidal $\mathbf{CPL}_2^a$-algebra and $[\![\ .\ ]\!]$ is a valuation defined in a usual way (see Appendix 3). For validity of a $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$-sequent $\Gamma\Rightarrow\Delta$ (see Appendix 3) in a given monoidal $\mathbf{CPL}_2^a$-model $\mathbf{M}$, we use the standard notation: $\models_{\mathbf{M}} \Gamma\Rightarrow\Delta$.

Soundness and completeness hold for the system and models under the following consideration.

Proposition 2.2 *Given a* $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$- *sequent* $\Gamma\Rightarrow\Delta$,

$$\textit{if}\ \ \langle\star,+\rangle\text{-}\mathbf{CPL}_2^a\vdash\Gamma\Rightarrow\Delta,\ \ \textit{then}\ \ \models_{\mathbf{M}}\Gamma\Rightarrow\Delta,$$

for every monoidal $\mathbf{CPL}_2^a$*-model* $\mathbf{M}$.

Proof: By induction on the length of a derivation of $\Gamma\Rightarrow\Delta$.

To show soundness of *Cut* the use of sub-commutativity of $\star$ and $+$, i.e., (2c) of the definition of a monoidal $\mathbf{CPL}_2^a$-algebra, is essential. The rest of the proof is straightforward. □

Proposition 2.3 *There exists a monoidal* $\mathbf{CPL}_2^a$*-model* $\mathbf{M}_L$*, such that*

$$\textit{if}\ \ \models_{\mathbf{M}_L}\Gamma\Rightarrow\Delta,\ \ \textit{then}\ \ \langle\star,+\rangle\text{-}\mathbf{CPL}_2^a\vdash\Gamma\Rightarrow\Delta,$$

for any $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$*-sequent* $\Gamma\Rightarrow\Delta$.

Proof: Clearly, $\mathbf{M}_L=\langle L,[\![\ .\ ]\!]\rangle$, where $L=\langle F/\equiv,\star',+',\leq'\rangle$ is the Lindenbaum algebra of the system $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$. The Lindenbaum algebra considered is obtained by the standard definition for linear logic systems. The partial order $\leq'$ in $F/\equiv$ is given by: *for any formulas* A *and* B, $[A]_\equiv\leq'[B]_\equiv$ *iff* $\langle\star,+\rangle$-$\mathbf{CPL}_2^a\vdash A\Rightarrow B$. *Clearly, then* $A\equiv B$ *iff* $[A]_\equiv\leq'[B]_\equiv$ *and* $[B]_\equiv\leq'[A]_\equiv$. It is now easy to check that L is indeed a monoidal $\mathbf{CPL}_2^a$-algebra. The rest of the proof is standard (see [9]). □

3 A construction of the free monoidal $\mathbf{CPL}_2^a$***-algebra on one generator*** A double commutative monoid is a triple $\langle S,\star,+\rangle$, where S is a nonempty set, $\star$ and $+$ are binary associative and commutative operations in S.

We shall now give an explicit construction of the free double commutative monoid $\mathcal{P}(\gamma)=\langle P,\star,+\rangle$ generated by $\{\gamma\}$. First, we shall define for any $d=0,1,2,\ldots$ the sets $P_d^\star$, (i.e., $\star$-$+$-polynomials of depth d, with a principal operation $\star$, when $d\geq 1$), and their duals P_d^+, inductively on the depth of a $\star$-$+$-polynomial.

1. The only polynomial of depth 0 is the generator γ, so we put $P_0^\star=P_0^+=\{\gamma\}$.

2. For $d \geq 1$:

   (a) Let $P_d^\star$ be the set of all formal products
$$(p_1) \star \cdots \star (p_k), \quad (k \geq 2),$$
   such that
$$p_j \in P_{d_j}^+ \ \ (j = 1, \ldots, k) \quad \text{and} \quad \max_{j=1}^{k} d_j = d - 1.$$

   (b) Let P_d^+ be the set of all formal sums
$$(p_1) + \cdots + (p_k), \quad (k \geq 2),$$
   such that
$$p_j \in P_{d_j}^\star \ \ (j = 1, \ldots, k) \quad \text{and} \quad \max_{j=1}^{k} d_j = d - 1.$$

   The operations $\star$ and $+$ between $\star$-$+$-polynomials are defined in the obvious way.

We now define an equivalence relation, $\simeq$, in $P_d^\star$ and in P_d^+ inductively by:

1. $\gamma \simeq \gamma$;
2. for $d \geq 1$:
   $(p_1) \star \cdots \star (p_k) \simeq (p'_1) \star \cdots \star (p'_s)$ iff $k = s$ and for some permutation θ of $\{1, \ldots, k\}$, $p_j \simeq p'_{\theta(j)}$ $(j = 1, \ldots, k)$. And analogously, $(p_1) + \cdots + (p_k) \simeq (p'_1) + \cdots + (p'_s)$ iff $k = s$ and for some permutation θ of $\{1, \ldots, k\}$, $p_j \simeq p'_{\theta(j)}$ $(j = 1, \ldots, k)$.

Moreover, since
$$p \simeq p' \quad \text{implies} \quad p \star q \simeq p' \star q \quad \text{and} \quad p + q \simeq p' + q,$$
for all $\star$-$+$-polynomials q, clearly $\simeq$ is a congruence relation on the algebra of $\star$-$+$-polynomials.

Put now $P_0 = P_0^\star/\simeq\, = P_0^+/\simeq$ and $P_d = (P_d^\star/\simeq) \bigcup (P_d^+/\simeq)$, for any $d \geq 1$, and take
$$P = \bigcup_{d=0}^{\infty} P_d.$$
Finally, we define $\mathcal{P}(\gamma) = \langle P, \star, + \rangle$, with $\star$ and $+$ being induced from $\star$ and $+$ on $\star$-$+$-polynomials.

Since $\simeq$ is a congruence relation on the algebra of $\star$-$+$-polynomials, the operations $\star$ and $+$ are, indeed, well-defined on $\simeq$-equivalence classes. Moreover, note that $\star$ and $+$ are associative and commutative operations in P. It is now easy to see, using standard arguments (Birkhoff [1]), that $\mathcal{P}(\gamma)$ is indeed a free double commutative monoid on one generator γ. Next, observe the following important property of $\mathcal{P}(\gamma)$.

Fact 3.1 *By the construction of $\mathcal{P}(\gamma)$, it follows that every $x \in P_d$ ($d \geq 1$) has a unique decomposition up to a permutation of factors or summands, in the following sense:*

(i) *if* $x \in P_d^\star/\simeq$ *(for some* $d = 1, 2, \ldots$*), then*

$$x = x_1 \star \cdots \star x_k,\ (k \geq 2),\ \textit{with}\ \ x_j \in \bigcup_{i=0}^{d-1} (P_i^+/\simeq)(j = 1, \ldots, k).$$

(ii) *if* $x \in P_d^+/\simeq$ *(for some* $d = 1, 2, \ldots$*), then*

$$x = x_1 + \cdots + x_k,\quad (k \geq 2),\ \textit{with}\ \ x_j \in \bigcup_{i=0}^{d-1} (P_i^\star/\simeq)\quad (j = 1, \ldots, k).$$

Our intention in what follows is to construct the free monoidal $\mathbf{CPL}_2^a$-algebra on one generator from $\mathcal{P}(\gamma)$. The construction is carried out in four steps.

I. A binary relation $\leq_0$ is introduced in P, by clauses (a)–(d) being satisfied for all $x, y, z \in P$:

   (a) $x \leq_0 x$;
   (b) $x \star y \leq_0 x$ and $x \leq_0 x + y$;
   (c) $x \star x \leq_0 x \star x \star x$ and $x + x + x \leq_0 x + x$;
   (d) $(y + z) \star x \leq_0 (y \star x) + z$.

II. We construct the monotonic closure of $\leq_0$, as follows.
A sequence of binary relations $\{\leq_n\}_{n=0}^{\infty}$ in P is defined inductively by:
for every $n \geq 1$, let $\leq_n$ be the extension of $\leq_{n-1}$, determined by:
for all $x, y, z \in P$, if $x \leq_{n-1} y$, then $x \star z \leq_n y \star z$ and $x + z \leq_n y + z$.
We define

$$\leq_\infty = \bigcup_{n=0}^{\infty} (\leq_n),$$

i.e., given $x, y \in P$, $x \leq_\infty y$ iff $x \leq_n y$, for some $n \geq 0$.

III. Let $\leq$ be the transitive closure of $\leq_\infty$ in P, i.e., given $x, y \in P$, $x \leq y$ iff there exists a finite chain $x \leq_\infty x_1 \leq_\infty \cdots \leq_\infty x_k \leq_\infty y$ in P.

With the following two lemmas we shall justify monotonicity of $\star$ and $+$ with respect to $\leq$ in P and show that $\leq$ is a preorder on P.

Lemma 3.2 *The operations $\star$ and $+$ are monotone with respect to $\leq_\infty$ in P.*

Proof: Assume $x \leq_\infty y$ for some $x, y \in P$. By definition, there is $n \geq 0$, such that $x \leq_n y$. Take any $z \in P$. Then, $x \star z \leq_{n+1} y \star z$ and $x + z \leq_{n+1} y + z$. Hence, $x \star z \leq_\infty y \star z$ and $x + z \leq_\infty y + z$. □

Lemma 3.3 *The relation $\leq$ is a preorder on P. Moreover, the operations $\star$ and $+$ are monotone with respect to $\leq$ in P.*

Proof: Indeed, $\leq$ is reflexive and transitive, due to I(a) and (III) above.

To show monotonicity, assume $x \leq y$ for some $x, y \in P$. Hence there is a finite chain in P, $x \leq_\infty x_1 \leq_\infty \cdots \leq_\infty x_k \leq_\infty y$. Then for any $z \in P$, $x \star z \leq_\infty x_1 \star z \leq_\infty \cdots \leq_\infty x_k \star z \leq_\infty y \star z$ and $x + z \leq_\infty x_1 + z \leq_\infty \cdots \leq_\infty x_k + z \leq_\infty y + z$ are again finite chains in P due to Lemma 3.2. Therefore, $x \star z \leq y \star z$ and $x+z \leq y+z$. $\square$

In the sequel, we shall generate two infinite chains in the preordered algebraic structure $\langle P, \star, +, \leq\rangle$ constructed so far. In order to show that these two infinite chains exist also in the free monoidal $\mathbf{CPL}_2^a$-algebra, we continue with two crucial lemmas.

Lemma 3.4 *$\forall n \geq 0$, $\forall z \in P$: $\gamma \leq_n z$ if and only if $z = \gamma$ or $z = \gamma + z'$ for some $z' \in P$.*

Proof: Note that $\forall n \geq 0$ and $\forall z \in P$, $\gamma \leq_n z$ iff $\gamma \leq_0 z$. Moreover, following the definition of $\leq_0$ we get $\gamma \leq_0 z$ iff $z = \gamma$ or $z = \gamma + z'$ for some $z' \in P$. $\square$

Lemma 3.5 *$\forall n \geq 0$, $\forall y, z \in P$: if $\gamma + y \leq_n z$, then $z = \gamma + z'$ for some $z' \in P$ and $y \leq_n z'$ or $\gamma \leq_n z'$.*

Proof: By induction on n.

1. ($n = 0$):
Assume $\gamma + y \leq_0 z$, for some $y, z \in P$. The following three cases are to be considered.
If I(a) occurs, then $z = \gamma + y$, and hence $z' = y$, yielding $y \leq_0 z'$.
If I(b) occurs, then $z = \gamma + y + u$, for some $u \in P$. Hence, $z' = y + u$ and thus, indeed, $y \leq_0 z'$.
If I(c) occurs, then two subcases occur.
(1) $y = \gamma + \gamma$ and $z = \gamma + \gamma$, resulting in $z' = \gamma$. By Lemma 3.4, $\gamma \leq_0 z'$.
(2) $y = \gamma + \gamma + u + u + u$ and $z = \gamma + \gamma + u + u$, for some $u \in P$. Therefore, $z' = \gamma + u + u$, and by Lemma 3.4 we are done.
2. ($n \geq 1$):
Assume Lemma 3.5 to hold for $k = 0, 1, \ldots, (n-1)$. Assume further that for some $y, z \in P$, $\gamma + y \leq_n z$ and $\gamma + y \not\leq_{n-1} z$. This means that there are $a, b, c \in P$, satisfying
$$(M) \quad a \leq_{n-1} b,$$
and hence, $a + c \leq_n b + c$, such that $\gamma + y = a + c$, and $z = b + c$. Now, due to Fact 3.1, the following possibilities are to be distinguished.
   (a) $a = \gamma$, hence $z = b + y$. Then, by (M) above, $\gamma \leq_{n-1} b$. Using Lemma 3.4 we get $b = \gamma$ or $b = \gamma + b'$, for some $b' \in P$. Thus, $z = \gamma + y$ or $z = \gamma + b' + y$, yielding $z' = y$ or $z' = b' + y$. And hence, $y \leq_n z'$.
   (b) $c = \gamma$. Then $y = a$ and $z' = b$. Using (M) we get $y \leq_n z'$.
   (c) $a = \gamma + u$ (for some $u \in P$), resulting in $y = u + c$. Now by (M) $\gamma + u \leq_{n-1} b$, therefore we can use induction hypothesis and get $b = \gamma + b'$ with the options
   $$(O) \quad u \leq_{n-1} b' \text{ or } \gamma \leq_{n-1} b'.$$
   Since $z = b + c = \gamma + b' + c$, we get $z' = b' + c$. We finally have to consider each option in (O).
   If $u \leq_{n-1} b'$, then $y = u + c \leq_n b' + c = z'$, and we are done.
   If $\gamma \leq_{n-1} b'$, then $b' = \gamma$ or $b' = \gamma + b''$ (for some $b'' \in P$) due to

Lemma 3.4. Hence, $z' = \gamma + c$ or $z' = \gamma + b'' + c$, resulting in $\gamma \leq_n z'$ by Lemma 3.4.

(d) $c = \gamma + u$ (for some $u \in P$), yielding $y = a + u$. Hence, $z = \gamma + u + b$ and so $z' = u + b$. Due to (M), we get $y = a + u \leq_n b + u = z'$, and we are done. □

Proposition 3.6 $\forall y, z \in P$:

1. *if $\gamma + y \leq z$, then $z = \gamma + z'$ for some $z' \in P$ and $y \leq z'$ or $\gamma \leq z'$.*
2. *$\gamma \leq z$ if and only if $z = \gamma$ or $z = \gamma + z'$ for some $z' \in P$.*

Proof: Following the definitions of $\leq$ and $\leq_\infty$ the proof goes by induction on the length of the chain using essentially Lemmas 3.4 and 3.5. □

By duality the following proposition can be established.

Proposition 3.7 $\forall y, z \in P$:

1. *if $z \leq \gamma \star y$, then $z = \gamma \star z'$ for some $z' \in P$ and $z' \leq y$ or $z' \leq \gamma$.*
2. *$z \leq \gamma$ if and only if $z = \gamma$ or $z = \gamma \star z'$ for some $z' \in P$.*

Consider now the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ of elements of P, given inductively by:

$$x_0 = \gamma + \gamma \ , \ y_0 = \gamma \star (\gamma + \gamma) \ , \ x_{n+1} = y_n + \gamma \ , \ y_{n+1} = x_{n+1} \star \gamma.$$

Due to I(b) of the definition of $\leq_0$ and monotonicity of $\star$ and $+$ with respect to $\leq$ (see Lemma 3.3), it is easy to see that:

$$x_{n+1} \leq x_n \text{ and } y_{n+1} \leq y_n \text{ for all } n \in \mathbf{N}.$$

In order to show later that none of these two chains collapses in the free monoidal $\mathbf{CPL}_2^a$-algebra, we need one more lemma.

Lemma 3.8 *Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be as above. Then, $x_n \not\leq x_{n+1}$ and $y_n \not\leq y_{n+1}$, for any $n \in \mathbf{N}$.*

Proof: By induction on n.

1. ($n = 0$:) we want to prove
   (i) $\gamma + \gamma \not\leq (\gamma \star (\gamma + \gamma)) + \gamma$
   (ii) $\gamma \star (\gamma + \gamma) \not\leq ((\gamma \star (\gamma + \gamma)) + \gamma) \star \gamma$.
   Assume first that $\gamma + \gamma \leq (\gamma \star (\gamma + \gamma)) + \gamma$. By Proposition 3.6.1, we get $\gamma \leq \gamma \star (\gamma + \gamma)$. This contradicts Proposition 3.6.2, and we are done.
   Assume next that $\gamma \star (\gamma + \gamma) \leq ((\gamma \star (\gamma + \gamma)) + \gamma) \star \gamma$. Proposition 3.7.1 yields the following options:

$$\gamma + \gamma \leq (\gamma \star (\gamma + \gamma)) + \gamma,$$

   contradicting (i) above, or

$$\gamma + \gamma \leq \gamma,$$

   violating Proposition 3.7.2.

2. ($n \geq 1$:)
   (i) Assume $x_n \leq x_{n+1}$, i.e., $y_{n-1} + \gamma \leq y_n + \gamma$. By Proposition 3.6.1, we get $y_{n-1} \leq y_n$, contradicting the induction hypothesis, or $\gamma \leq y_n = x_n \star \gamma$, violating proposition 3.6.2.
   (ii) Finally, assume $y_n \leq y_{n+1}$, i.e., $x_n \star \gamma \leq x_{n+1} \star \gamma$. By Proposition 3.7.1 we get $x_n \leq x_{n+1}$, contradicting the first part of the lemma already proved, or $y_{n-1} + \gamma = x_n \leq \gamma$, which contradicts Proposition 3.7.2. □

IV. Next, we define a congruence relation $\equiv$ on $\langle P, \star, +, \leq \rangle$ by:

$$\text{for any } x, y \in P,\ x \equiv y \text{ iff } x \leq y \text{ and } y \leq x.$$

Clearly, due to Lemma 3.3, $\equiv$ is a congruence relation on the structure considered. We now define

$$\mathcal{F}(\gamma) = \langle P/\equiv, \star', +', \leq' \rangle,$$

with $\star'$,$+'$, and $\leq'$ being induced from P.

Since $\equiv$ is a congruence relation, $\star'$ and $+'$ are well-defined in $\mathcal{F}(\gamma)$. Moreover, $\leq'$ is a partial order on $\mathcal{F}(\gamma)$. Taking into account also the given constructions of P and $\leq$ on P, it is now easy to see that $\mathcal{F}(\gamma)$ is a monoidal $\mathbf{CPL}_2^a$-algebra. Moreover, $\mathcal{F}(\gamma)$ is the free $\mathbf{CPL}_2^a$-algebra on one generator γ (see [1]). To verify the latter, we have to show that for every monoidal $\mathbf{CPL}_2^a$-algebra $\mathbf{X}$ and for every $x \in X$, there exists a unique morphism $f : \mathcal{F}(\gamma) \to \mathbf{X}$, such that $f([\gamma]) = x$. Define, for any $q \in P$, $f([q]) = q(x)$, where $q(x)$ is the evaluation of the polynomial q at x in X. To see, first, that f is well-defined (i.e., independent of representatives) we have to show that for any $q', q \in P$, if $q' \equiv q$, then $q'(x) = q(x)$ in X. In fact, it is enough to prove that for all $n \in \mathbf{N}$ and $\forall q, q' \in P$, if $q' \leq_n q$, then $q'(x) \leq q(x)$ in X. A diligent reader is now invited to work out the proof following the definition of $\leq_n$ by induction on n. From the above, it follows that f preserves the partial order. Hence, indeed, f is a morphism between the two algebras considered, since it also satisfies the following: for any $q, q' \in P$, $f([q \circ q']) = (q \circ q')(x) = q(x) \circ q'(x) = f([q]) \circ f([q'])$, with $\circ \in \{\star, +\}$. Also, $f([\gamma]) = x$, by definition. And finally, the uniqueness of f follows directly from f being a morphism and $f([\gamma]) = x$. Thus, $\mathcal{F}(\gamma)$ is the free monoidal $\mathbf{CPL}_2^a$-algebra on one generator γ and isomorphic to the Lindenbaum algebra of the one-variable fragment of $\langle \star, + \rangle$-$\mathbf{CPL}_2^a$.

We emphasize, however, that our construction of $\mathcal{F}(\gamma)$ has a much wider scope, establishing also the free algebras (on one generator) corresponding to the multiplicative fragment of classical linear logic (i.e., tensor, par fragment without constants and modalities), in our notation $\langle \star, + \rangle$-$\mathbf{CPL}$, its affine version, $\langle \star, + \rangle$-$\mathbf{CPL}^a$, and their extensions with n-contraction ($n \geq 2$): $\langle \star, + \rangle$-$\mathbf{CPL}_n$ and $\langle \star, + \rangle$-$\mathbf{CPL}_n^a$ respectively. To be specific, we state the following fact.

Fact 3.9 *A construction of the free algebra, on one generator γ, corresponding to $\langle \star, + \rangle$- $\mathbf{CPL}$, $\langle \star, + \rangle$-$\mathbf{CPL}^a$, $\langle \star, + \rangle$-$\mathbf{CPL}_n$, and $\langle \star, + \rangle$-$\mathbf{CPL}_n^a$, ($n \geq 2$), can be obtained from the given construction of $\mathcal{F}(\gamma)$ respectively by:*

1. *omitting I(b),(c);*

2. *omiting I(c);*
3. *omitting I(b) and replacing I(c) by I(c′):*

$$x^n \leq_0 x^{n+1} \text{ and } (n+1)x \leq_0 nx,$$

where x^n and nx denote n copies of x in the product and in the sum respectively;

4. *replacing I(c) with I(c′), as above.*

Besides, note that the construction of $\mathcal{F}(\gamma)$ and its variants, in the fact above, can easily be generalized to the corresponding free algebras generated by an arbitrary set of generators.

4 One-variable fragments of multiplicative classical linear logic with n-contraction are infinite We are now ready to prove the existence of the two infinite chains in $\mathcal{F}(\gamma)$, with its application to the one-variable fragment of $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$.

Theorem 4.1

1. *$\mathcal{F}(\gamma)$ is infinite.*
2. *There are infinitely many provably nonequivalent formulas built from one propositional variable in the system $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$.*

Proof:

1. Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be the sequences of elements of P, as given above. Then, clearly,

$$[x_{n+1}]_{\equiv} \leq' [x_n]_{\equiv} \text{ and } [y_{n+1}]_{\equiv} \leq' [y_n]_{\equiv}, \text{ for all } n \in \mathbf{N}.$$

Now by Lemma 3.8 we may conclude that $x_{n+1} \not\equiv x_n$ and $y_{n+1} \not\equiv y_n$, thus proving the existence of two strictly decreasing infinite chains in $\mathcal{F}(\gamma)$.

2. Let p be a single propositional variable in the language of $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$. Take now the monoidal $\mathbf{CPL}_2^a$-model $\mathbf{M} = \langle \mathcal{F}(\gamma), [\![\ .\ ]\!]\rangle$ and put $[\![p]\!] = \gamma$. Since $\mathbf{M}$ is the Lindenbaum model of the fragment considered, we are done by (1) of the present theorem. □

Figure 1 below illustrates a part of $\mathcal{F}(\gamma)$ with the first two elements of each of the two strictly decreasing infinite chains: $x_0 = 2\gamma$, $x_1 = \gamma + (\gamma \star 2\gamma)$, $y_0 = \gamma \star 2\gamma$ and $y_1 = \gamma \star (\gamma + (\gamma \star 2\gamma))$. Moreover, for any $\star$-$+$-polynomial p of γ, $2p$ and p^2 stand for $p+p$ and $p \star p$ respectively. From now on, we will use these abbreviations whenever convenient.

Explicit constructions of the two infinite sequences $\{F_n\}_{n=0}^{\infty}$ and $\{G_n\}_{n=0}^{\infty}$ of $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$ provably nonequivalent formulas of one-variable can be obtained by analogy with the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ given above. To be specific:

$$F_0 = p + p \ \ , \ \ G_0 = p \star (p+p) \ \ , \ \ F_{n+1} = G_n + p, \ , \ \ G_{n+1} = F_{n+1} \star p,$$

where p is the single propositional variable in the language of $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$.

Finally, observe that the respective sets of derivable sequents (i.e., theorems) in $\mathbf{CPL}_2^a$, $\mathbf{CPL}_n^a$, $\mathbf{CPL}_n$, $(n \geq 2)$, $\mathbf{CPL}^a$, and $\mathbf{CPL}$ are in the following relation:

$$Th(\mathbf{CPL}_2^a) \supseteq Th(\mathbf{CPL}_n^a) \supseteq Th(\mathbf{CPL}_n) \supseteq Th(\mathbf{CPL}),$$

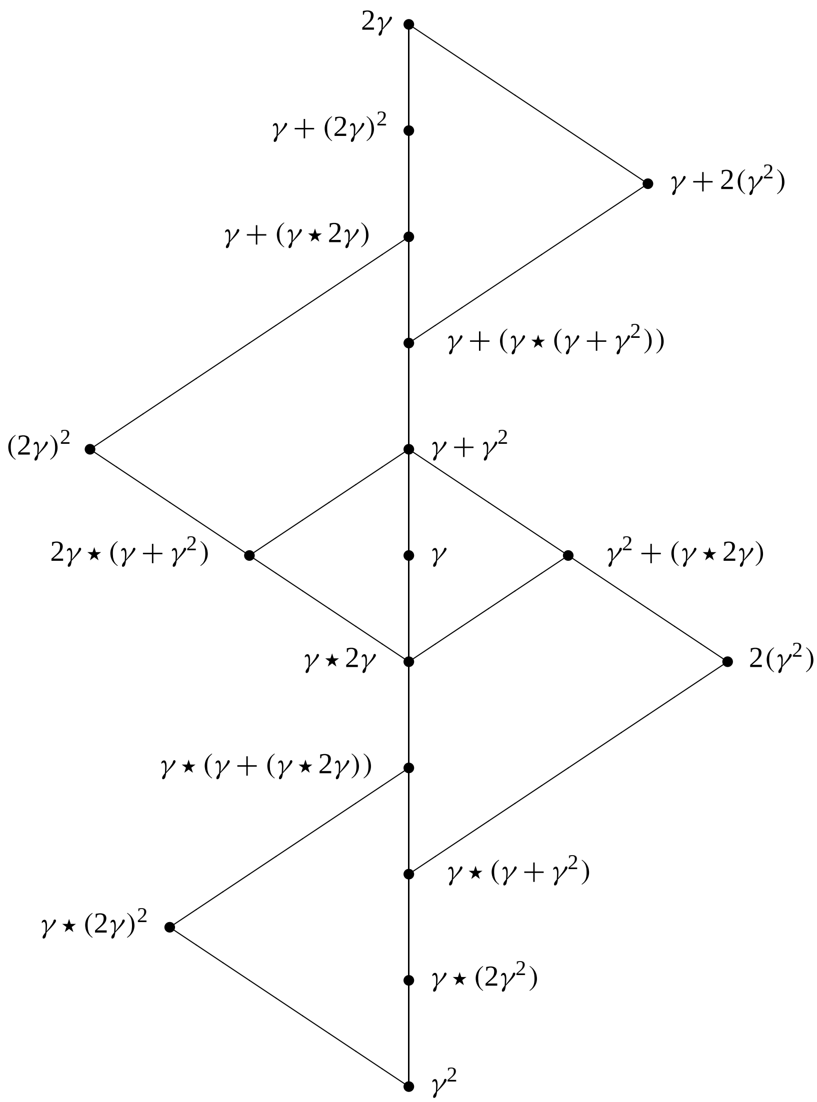

Figure 1: A part of $\mathcal{F}(\gamma)$.

$$\text{and also, } Th(\mathbf{CPL}_2^a) \supseteq Th(\mathbf{CPL}^a).$$

Clearly this fact remains valid for purely multiplicative fragments of the systems considered. Thus, the two infinite sequences of formulas $\{F_n\}_{n=0}^{\infty}$ and $\{G_n\}_{n=0}^{\infty}$ are also provably nonequivalent in the multiplicative fragment of any of the systems $\mathbf{CPL}_n^a$, $\mathbf{CPL}_n$, $(n \geq 2)$, $\mathbf{CPL}^a$, and **CPL**.

We can now sum up the results obtained in this section in the following.

Corollary 4.2 *There are infinitely many provably nonequivalent formulas built from one propositional variable in any of the systems:* $\langle\star, +\rangle$- $\mathbf{CPL}_n^a$, $\langle\star, +\rangle$-$\mathbf{CPL}_n$ *(*$n \geq 2$*),* $\langle\star, +\rangle$-$\mathbf{CPL}^a$*, and* $\langle\star, +\rangle$- **CPL**.

We emphasize, however, that the corollary above presents a nontrivial result only with respect to the systems with *n*-contraction. The systems $\mathbf{CPL}^a$ and **CPL**, as well as their respective multiplicative fragments, all enjoy cut-elimination (as opposed to the systems with *n*-contraction). Therefore, for these systems the statement of Corollary 4.2 can be obtained very easily by purely syntactic reasoning.

A natural question arising at this point is whether the results obtained so far can be extended to the systems which include also constants, negation, and additive connectives. We shall now briefly comment on this problem.

Consider first the system $\mathbf{CPL}_2^a$ (see Appendix 2) in the absence of additive connectives. We claim that a monoidal $\mathbf{CPL}_2^a$-algebra, extended with 0, 1 and $\sim$, satisfying the folowing clauses for any $x \in X$,

- I(a) $x \star 1 = x$ and $x + 0 = x$;
- I(b) $\sim\sim x = x$;
- I(c) $x \star \sim x = 0$ and $x + \sim x = 1$,

is a partially ordered structure satisfying all clauses of the definition of a $\mathbf{CPL}_2^a$-algebra with exception of (2) (see Appendix 3). We refer to this algebraic structure, $\mathbf{X} = \langle X, \sim, \star, +, \leq, 0, 1\rangle$, as $\langle\sim, \star, +\rangle$-$\mathbf{CPL}_2^a$-algebra. To verify our claim, it remains to be shown only that clause (5) of the definition of a $\mathbf{CPL}_2^a$-algebra (i.e., adjointness) is derivable in a $\langle\sim, \star, +\rangle$-$\mathbf{CPL}_2^a$-algebra. We proceed as follows.

1. Assume $x \star y \leq z$, for some $x, y, z \in X$. Then, $x = x \star 1 = x \star (y + \sim y) \leq (x \star y) + \sim y \leq z + \sim y = \sim y + z$, due to I(a), I(c), commutativity of $\star$, 2(c), assumption (1), 2(d), commutativity of $+$, and transitivity of $\leq$ respectively.
2. Assume $x \leq \sim y + z$, for some $x, y, z \in X$. Then, $x \star y \leq (\sim y + z) \star y \leq (\sim y \star y) + z = 0 + z = z$, due to assumption (2), 2(d), 2(c), commutativity of $\star$, I(c), commutativity of $+$, I(a), and transitivity of $\leq$ respectively.

In fact adjointness is equivalent to sub-commutativity of $\star$ and $+$ in the algebra considered. Given $x, y \in X$, $y \star x \leq y \star x$ by reflexivity of $\leq$. Due to adjointness, we get $y \leq \sim x + (y \star x)$. Thus, for any $z \in X$, $y + z \leq \sim x + ((y \star x) + z)$, due to 2(d) and associativity of $+$. Using adjointness, once again, gives $(y + z) \star x \leq (y \star x) + z$, and we are done.

Due to the presence of involution in $\langle\sim, \star, +\rangle$- $\mathbf{CPL}_2^a$-algebras a form of de Morgan duality can be expressed in them.

Lemma 4.3 *In any* $\langle\sim, \star, +\rangle$-$\mathbf{CPL}_2^a$*-algebra* $\mathbf{X} = \langle X, \sim, \star, +, \leq, 0, 1\rangle$ *the following pairs of operators are dual to each other:* $(\sim, \sim)$, $(\star, +)$*, and* $(0, 1)$.

Proof: Clearly $\sim$ is dual to itself by I(b).

Next, we want to show, using anti-symmetry of $\leq$, that

$$\sim(x \star y) = (\sim x) + (\sim y).$$

(1) $\sim(x \star y) \leq (\sim x) + (\sim y)$ iff $1 \leq \sim\sim(x \star y) + (\sim x) + (\sim y) = ((x \star y) + (\sim x)) + \sim y$, due to commutativity of $\star$, I(a), adjointness, I(b), and associativity of $+$. Indeed, $((x \star y) + \sim x) + \sim y \geq ((x + \sim x) \star y) + \sim y = (1 \star y) + \sim y = y + \sim y = 1$, by 2(c), I(c), commutativity of $\star$, I(a), and I(c).
(2) $(\sim x) + (\sim y) \leq \sim(x \star y)$ can be proved analogously, and so we are done. The duality of 0 and 1 is now a trivial matter. □

One can extend the given construction of the free monoidal algebra $\mathcal{F}(\gamma)$ with the respective clauses for 0, 1, and involution. Eventually, one should construct a free lattice generated by the extended free structure. However, in this case, different lemmas and propositions to prove the existence of the two infinite chains in these extended free structures are needed. Therefore, our present strategy is not directly applicable to any of the extended systems discussed above. Retaining a positive attitude in spite of that, one should search for a suitable conservativity theorem. However, due to the non-eliminability of cut in the underlying syntactic systems, we still lack the proof-theoretic techniques to achieve that. Thus, it seems more promising to find a faithful embedding of the free monoidal $\mathbf{CPL}_2^a$-algebra in the free algebra corresponding to the system $\mathbf{CPL}_2^a$. However, this still remains to be achieved.

We conclude this paper with the conjecture that the one-variable fragment of the system $\mathbf{CPL}_2^a$ (and consequently, of any weaker system in the sense of Corollary 4.2) is infinite.

Appendix 1: Systems of affine intuitionistic linear logic with n-contraction For any $n \geq 2$, an intuitionistic system of affine propositional linear logic with n-contraction, $\mathbf{IPL}_n^a$, is given by the following axioms and rules. Throughout the below, Λ denotes the empty multiset, Φ denotes either an occurrence of an $\mathbf{IPL}_n^a$-formula or the empty multiset, and Γ, Γ_1, Γ_2 stand for finite multisets of $\mathbf{IPL}_n^a$-formulas.

Axioms

$A \Rightarrow A$

$0 \Rightarrow \Lambda$

$\Lambda \Rightarrow 1$

Logical rules

L$\star$ $\dfrac{\Gamma, A, B \Rightarrow \Phi}{\Gamma, A \star B \Rightarrow \Phi}$

R$\star$ $\dfrac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \star B}$

L$\multimap$ $\dfrac{\Gamma_1 \Rightarrow A \qquad \Gamma_2, B \Rightarrow \Phi}{\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \Phi}$

R$\multimap$ $\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B}$

L$\sqcap$ $$\frac{\Gamma, A_i \Rightarrow \Phi}{\Gamma, A_1 \sqcap A_2 \Rightarrow \Phi} \quad (i = 1, 2)$$

R$\sqcap$ $$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcap B}$$

L$\sqcup$ $$\frac{\Gamma, A \Rightarrow \Phi \qquad \Gamma, B \Rightarrow \Phi}{\Gamma, A \sqcup B \Rightarrow \Phi}$$

R$\sqcup$ $$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \sqcup A_2} \quad (i = 1, 2)$$

Structural rules

LW $$\frac{\Gamma \Rightarrow \Phi}{\Gamma, A \Rightarrow \Phi}$$

RW $$\frac{\Gamma \Rightarrow \Lambda}{\Gamma \Rightarrow A}$$

LC$_n$ $$\frac{\Gamma, A^{(n+1)} \Rightarrow \Phi}{\Gamma, A^{(n)} \Rightarrow \Phi}$$

where $A^{(k)} \equiv A, A, \ldots, A$, i.e., k copies of formula A.

Cut $$\frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2, A \Rightarrow \Phi}{\Gamma_1, \Gamma_2 \Rightarrow \Phi}$$

Remark: A noninvolutive negation can be defined by $\sim A = A \multimap 0$.

Appendix 2: Systems of affine classical linear logic with n***-contraction*** For any $n \geq 2$, a classical system of affine propositional linear logic with n-contraction, $\mathbf{CPL}_n^a$, is given by the following axioms and rules. Throughout the sequel, Λ denotes the empty multiset and $\Gamma, \Gamma_1, \Gamma_2, \Delta, \Delta_1, \Delta_2$ stand for finite multisets of $\mathbf{CPL}_n^a$-formulas.

Axioms

$A \Rightarrow A$

$0 \Rightarrow \Lambda$

$\Lambda \Rightarrow 1$

Logical rules

L$\sim$ $$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \sim A \Rightarrow \Delta}$$

R$\sim$ $$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \sim A, \Delta}$$

L$\star$ $$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \star B \Rightarrow \Delta}$$

R$\star$ $$\frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \star B, \Delta_1, \Delta_2}$$

L+ $$\frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A + B \Rightarrow \Delta_1, \Delta_2}$$

R+ $$\frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A + B, \Delta}$$

$$\text{L}\sqcap \quad \frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_1 \sqcap A_2 \Rightarrow \Delta} \quad (i = 1, 2)$$

$$\text{R}\sqcap \quad \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \sqcap B, \Delta}$$

$$\text{L}\sqcup \quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \sqcup B \Rightarrow \Delta}$$

$$\text{R}\sqcup \quad \frac{\Gamma \Rightarrow A_i, \Delta}{\Gamma \Rightarrow A_1 \sqcup A_2, \Delta} \quad (i = 1, 2)$$

Structural rules

$$\text{LW} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}$$

$$\text{RW} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\text{LC}_n \quad \frac{\Gamma, A^{(n+1)} \Rightarrow \Delta}{\Gamma, A^{(n)} \Rightarrow \Delta}$$

$$\text{RC}_n \quad \frac{\Gamma \Rightarrow A^{(n+1)}, \Delta}{\Gamma \Rightarrow A^{(n)}, \Delta}$$

where $A^{(k)} \equiv A, A, \ldots, A$, i.e., k copies of formula A.

$$\textit{Cut} \quad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Remark: A linear implication can be defined by $A \multimap B = \sim A + B$.

Appendix 3: Algebraic models for $\mathbf{CPL}_n^a$

Definition $\mathbf{X} = \langle X, \sim, \star, +, \sqcap, \sqcup, 0, 1\rangle$ is a $\mathbf{CPL}_n^a$-algebra, if:

1. $\langle X, \star, 1\rangle$ and $\langle X, +, 0\rangle$ are commutative monoids with units 1 and 0 respectively;
2. $\langle X, \sqcap, \sqcup, 0, 1\rangle$ is a lattice with bottom 0 and top 1;
3. $\sim$ is involution, i.e., $\sim\sim x = x$ for all $x \in X$;
4. $\star$ and $+$ are monotone with respect to the lattice order $\leq$, i.e., for all $x, y, z \in X$, if $x \leq y$, then $x \star z \leq y \star z$ and $x + z \leq y + z$;
5. for all $x, y, z \in X$, $x \star y \leq z$ if and only if $x \leq \sim y + z$, i.e., adjointness;
6. for all $x \in X$, $x^n \leq x^{n+1}$ and $(n+1)x \leq nx$, where $x^k = x \star \cdots \star x$ and $kx = x + \cdots + x$ with k copies of x respectively.

Remark: Note that, a $\mathbf{CPL}_n^a$-algebra is just a classical linear algebra (provided $\multimap$ is taken as primitive while $\top$, $\sim$ and $+$ are defined in a usual way, see [9]), satisfying in addition:

- $\bot = 0$ and $\top = 1$ (corresponding to weakening);
- clause (6) (corresponding to n-contraction).

Definition $\mathbf{M} = \langle \mathbf{X}, [\![\ .\ ]\!]\rangle$ is a $\mathbf{CPL}_n^a$-model, if:

1. $\mathbf{X}$ is a $\mathbf{CPL}_n^a$-algebra;
2. $[\![\ .\ ]\!]$ is a valuation satisfying:

(a) $[\![P]\!] \in X$, for every propositional variable P;

(b) $[\![0]\!] = 0$, $[\![1]\!] = 1$;

$[\![\ .\ ]\!]$ is extended to $\mathbf{CPL}^a_n$-formulas inductively by:
$[\![\sim A]\!] = \sim [\![A]\!]$ and $[\![A \bullet B]\!] = [\![A]\!] \bullet [\![B]\!]$, with $\bullet \in \{\star, +, \sqcap, \sqcup\}$;
Moreover, a $\mathbf{CPL}^a_n$-sequent $A_1, \dots, A_k \Rightarrow B_1, \dots, B_m$ is valid in $\mathbf{M}$ if and only if $[\![A_1]\!] \star \cdots \star [\![A_k]\!] \leq [\![B_1]\!] + \cdots + [\![B_m]\!]$.

Appendix 4: Models for n-valued Łukasiewicz logics

Definition For any $n \geq 2$, a model for n-valued Łukasiewicz logic, $\mathbf{M}_n([\![\ .\ ]\!])$, consists of:

1. a valuation $[\![\ .\ ]\!]$ assigns to each propositional variable p an element of the set $S_n = \{\frac{k}{n-1} \mid k = 0, 1, \dots, n-1\}$;
2. $[\![\ .\ ]\!]$ is extended to arbitrary formulas (in the language $\{\sim, \sqcap, \sqcup, \star, +, \multimap\}$) inductively by:

   (a) $[\![\sim A]\!] = 1 - [\![A]\!]$;

   (b) $[\![A \sqcap B]\!] = \min\{[\![A]\!], [\![B]\!]\}$;

   (c) $[\![A \sqcup B]\!] = \max\{[\![A]\!], [\![B]\!]\}$;

   (d) $[\![A \star B]\!] = \max\{[\![A]\!] + [\![B]\!] - 1, 0\}$;

   (e) $[\![A + B]\!] = \min\{[\![A]\!] + [\![B]\!], 1\}$;

   (f) $[\![A \multimap B]\!] = \min\{1 - [\![A]\!] + [\![B]\!], 1\}$.

3. $[\![\ .\ ]\!]$ is extended to arbitrary sequent $A_1, \dots, A_m \Rightarrow B_1, \dots, B_j$ by:

$$[\![A_1, \dots, A_m \Rightarrow B_1, \dots, B_j]\!] = [\![\sim A_1 + \cdots + \sim A_m + B_1 + \cdots + B_j]\!].$$

A given sequent $\Gamma \Rightarrow \Delta$ is valid in $\mathbf{M}_n([\![\ .\ ]\!])$ if and only if $[\![\Gamma \Rightarrow \Delta]\!] = 1$. Moreover, a sequent $\Gamma \Rightarrow \Delta$ is n-valid if and only if $[\![\Gamma \Rightarrow \Delta]\!] = 1$ for every valuation $[\![\ .\ ]\!]$.

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