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The Expressive Power of Second-Order Propositional Modal Logic

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Abstract It is shown that the expressive power of second-order propositional modal logic whose modalities are **S4.2** or weaker is the same as that of second-order predicate logic.

It has been shown by Fine in [1] that second-order arithmetic can be interpreted in second-order propositional modal logic, denoted **SOPML**, when the modality is **S4.2** or weaker. In this paper we show that actually the expressive power of **SOPML**, when the modality is **S4.2** or weaker, is the same as that of the full second-order predicate logic. This result immediately extends to the logic **Q2**, which is first-order modal logic based on the world-relative domain semantics introduced by Thomason [10]. Since **SOPML** is interpretable in **Q2** (see §2 below), second-order predicate logic can be interpreted in **Q2** as well, when the modality is **S4.2** or weaker and, of course, vice versa.

The paper is organized as follows. In the next section we recall the definition of **SOPML** and show how the second-order predicate logic can be embedded into this logic when the modality is **S4.2** or weaker. In §2 we show that **SOPML** and **Q2** are each interpretable in the other. The last section contains remarks about the expressive power of **SOPML** and **Q2** when the modality is stronger than **S4.2**.

1 Second-order propositional modal logic The language of second-order propositional modal logic, **SOPML**, is that of the propositional modal logic extended with the existential quantifier ∃. The definition of a **SOPML** formula is obtained by extending the inductive step of the definition of a propositional modal formula with the following rule.

> If φ is a **SOPML** formula and p is a propositional variable, then $\exists p\varphi$ is also a **SOPML** formula.

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Next we define the semantics of **SOPML**.

A *frame* is a pair $\langle W, R \rangle$, where W is a nonempty set of *possible worlds* and $R \subseteq W \times W$ is an *accessibility* relation on W.

Let $F = \langle W, R \rangle$ be a frame, and let $T = \{T_w\}_{w \in W}$ be a set of truth assignments for propositional variables in the worlds of F. The *satisfiability* of a **SOPML** formula φ by u under assignment T, denoted $(u, T) \models_{\mathbf{SOPML}} \varphi$, is the following extension to the definition of satisfiability of propositional modal formulas.

> If φ is a propositional variable p, then then $(u, T) \models_{\mathbf{SOPML}} \varphi$ if and only if p is true under the truth assignment T_u.
>
> $(u, T) \models_{\mathbf{SOPML}} \varphi \supset \psi$ if and only if $(u, T) \not\models_{\mathbf{SOPML}} \varphi$ or $(u, T) \models_{\mathbf{SOPML}} \psi$.
>
> $(u, T) \models_{\mathbf{SOPML}} \neg\varphi$ if and only if $(u, T) \not\models_{\mathbf{SOPML}} \varphi$.
>
> $(u, T) \models_{\mathbf{SOPML}} \exists p \varphi(p)$ if and only if there exists a set of truth assignments $T' = \{T'_w\}_{w \in W}$, such that for each $w \in W$, T'_w differs from T_w at most at p, and $(u, T') \models_{\mathbf{SOPML}} \varphi(p)$. That is, we adopt what Fine [1] calls the *platonistic* interpretation of propositional quantifiers, on which propositional variables range over the full power set of worlds.
>
> $(u, T) \models_{\mathbf{SOPML}} \Box\varphi$ if and only if for each w satisfying uRw, $(w, T) \models_{\mathbf{SOPML}} \varphi$.

We say that a formula φ is *valid* in a frame $\langle W, R \rangle$ if and only if for any set of truth assignments T and for any $u \in W$, $(u, T) \models_{\mathbf{SOPML}} \varphi$. For a class of frames $\mathcal{F}$, the *logic defined by* $\mathcal{F}$ consists of all formulas which are valid in all frames of $\mathcal{F}$.

Below we prove that the second-order predicate logic is interpretable in **SOPML** when the modality is **S4.2** or weaker.[1] We shall describe a **SOPML** formula PAIRING that defines pairing of worlds of a frame, thus allowing us to express second-order dyadic predicates on the worlds instead of monadic ones (which correspond to worlds satisfying propositional variables).

The frames for PAIRING consist of six "groups" of worlds. The first group contains only one world—the root. The second group contains the worlds which constitute the domain of pairing, the third and fourth groups contain identical copies of the domain elements (which are the first and second pair components, respectively), and the fifth one contains the pairs themselves. The sixth group contains only one world that is accessible from all the worlds.[2] Each world u in the second group is connected (by means of the accessibility relation) to a unique world u' in the third group and to a unique world u'' in the fourth group (the copies of u), and a world w in the fifth group is a pair (u_1, u_2), if both u'_1 (the copy of u_1 in the third group) and u'_2 (the copy of u_2 in the fourth group) are connected to w. We use six propositional constants $\{L_i\}_{1 \le i \le 6}$ to distinguish among the groups (see Axiom 1 below).[3] The relative position of the groups is shown in Figure 1 on the next page.

We shall need the "uniqueness modality," $\Diamond!$, stating that there is a unique world reachable from a given state where a given formula holds. A formula $\Diamond!\varphi$ is defined by $(\Diamond\varphi) \wedge \forall q(\Box(\varphi \supset q) \vee \Box(\varphi \supset \neg q))$ (see Garson [4], p. 296, where $\Diamond!$ is denoted by I).

Next we introduce the axioms defining the interpretation.

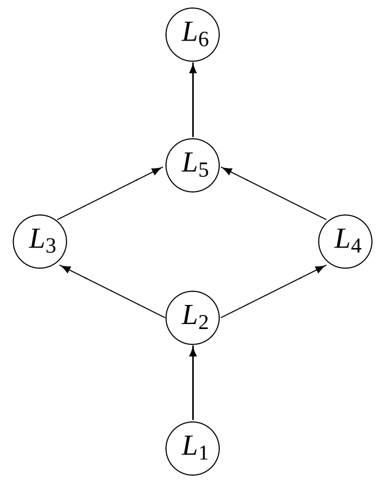

Figure 1: The relative position of the group elements in the frame.

1. $\Diamond!L_6 \wedge \Box \bigvee_{i=1}^{6} L_i$. This axiom states that there are at most six groups of worlds, and that the sixth group consists of exactly one world.
2. $\bigwedge_{i=1}^{6} \Box(L_i \supset \Diamond!L_i)$. This axiom states that distinct worlds in each group are each inaccessible from the other.
3. $\bigwedge_{i=2}^{6} \Box(L_i \supset \bigwedge_{j<i} \Box\neg L_j)$. This axiom states that the groups are mutually disjoint. Moreover, for $j < i$, no element of the jth group is accessible from an element of the ith group.
4. $\Box(L_3 \supset \Box\neg L_4)$. This axiom states that no element of the fourth group is accessible from an element of the third group.
5. $L_1 \wedge \bigwedge_{i\in\{1,2,4,5\}} \Box(L_i \supset \Diamond L_{i+1}) \wedge \Box(L_2 \supset \Diamond L_4) \wedge \Box(L_3 \supset \Diamond L_5)$. This axiom together with Axioms 3 and 4 implies that the groups can be divided into five nonempty levels in the following manner. The first group lies on the first (ground) level, the second group lies on the second level, the third and fourth groups lie on the third level, the fifth group lies on the fourth level, and the sixth group lies on the fifth level (see Figure 1).
6. $\Box(L_2 \supset (\Diamond!L_3 \wedge \Diamond!L_4))$. This axiom states that each element of the second group is connected to unique elements (its copies) of the third and the fourth groups.
7. $\forall p(\Diamond!(L_3 \wedge p) \supset \Diamond!(L_2 \wedge \Diamond(L_3 \wedge p))) \wedge \forall p(\Diamond!(L_4 \wedge p) \supset \Diamond!(L_2 \wedge \Diamond(L_4 \wedge p)))$. This axiom states that each element of the third or the fourth group is accessible from a unique element of the second group. Thus Axioms 6 and 7 imply that the accessibility relation imposes a bijection between the second and the third (fourth) groups.

Now we define formulas $EL(p)$ and $PAIR(p)$ which state that a propositional variable p is an element of the domain and a pair respectively, and a formula $REL(p,q,r)$ stating that a propositional variable r is a pair consisting of propositional variables p

and q:

$$\begin{array}{lll} EL(p) & \text{is} & \Diamond !p \wedge \Diamond(L_2 \wedge p), \\ PAIR(p) & \text{is} & \Diamond !p \wedge \Diamond(L_5 \wedge p), \text{ and} \\ REL(p,q,r) & \text{is} & EL(p) \wedge EL(q) \wedge PAIR(r) \wedge \Diamond(L_2 \wedge p \wedge \Diamond(L_3 \wedge \Diamond r)) \wedge \\ & & \wedge \Diamond(L_2 \wedge q \wedge \Diamond(L_4 \wedge \Diamond r)). \end{array}$$

Finally, we define the usual axioms for pair enumeration, i.e.,

8. $\forall p \forall q (EL(p) \wedge EL(q)) \supset \exists ! r(PAIR(r) \wedge REL(p,q,r))$, and
9. $\forall r PAIR(r) \supset \exists ! p \exists ! q(EL(p) \wedge EL(q) \wedge REL(p,q,r))$.

Note that we have "equality" on elements and pairs defined by $\Box(p \equiv q)$. Thus the quantifier $\exists!$ is well defined.

Let PAIRING be the conjunction of Axioms 1–9.

For the definability result below we need one more bit of notation. Let $F = \langle W, R \rangle$ be a frame and let $u \in W$. Then $F^u = \langle W^u, R^u \rangle$ denotes the frame whose set of worlds consists of the worlds of W which are different from u and are reachable from u by means of of R, and R^u is the restriction of R on W^u.

For a set D not containing 2, 3, 4 or 6 let $F_D = \langle W_D, R_D \rangle$ be a frame such that

$$W_D = (D \times \{2\}) \cup (D \times \{3\}) \cup (D \times \{4\}) \cup (D \times D) \cup \{6\},$$

where R_D is the reflexive and transitive closure of

$$\{[(d,2),(d,3)]\}_{d \in D} \cup \{[(d,2),(d,4)]\}_{d \in D} \cup \{[(d_1,3),(d_1,d_2)]\}_{d_1,d_2 \in D} \cup \\ \cup \{[(d_2,4),(d_1,d_2)]\}_{d_1,d_2 \in D} \cup \{[(d_1,d_2),6]\}_{d_1,d_2 \in D}.$$

We shall call D and F_D a *pairing domain* and the *pairing frame* of D, respectively.

Theorem 1.1 *Let F be a frame and let u be a world of F. Then u satisfies* PAIRING *if and only if the following holds. There exists a pairing domain D and an isomorphism ι between F^u and F_D such that for every $w \in W^u$, $w \models L_i$ if and only if $\iota(w) \in D \times \{i\}$, $i = 2, 3, 4$, $w \models L_5$ if and only if $\iota(w) \in D \times D$, and $w \models L_6$ if and only if $\iota(w) = 6$.*

Proof: The "if" part of the theorem is immediate. For the "only if" part, assume that u satisfies PAIRING. Let $D = \{w \in W^u : w \models L_2\}$. Then we can define ι as follows. If $w \in D$, then $\iota(w) = (w, 2)$. If $w \models L_3$ ($w \models L_4$), then, by Axioms 6 and 7, there exists a unique $w' \in D$ such that $w' R w$, and we put $\iota(w) = (w', 3)(\iota(w) = (w', 4))$.

If $w \models L_5$, then, by Axioms 6,7,8, and 9, there exist a unique pair $(w_1, w_2) \in D \times D$ such that w is reachable from w_1 through a world satisfying L_3 and is reachable from w_2 through a world satisfying L_4. We put $\iota(w) = (w_1, w_2)$. Finally, if $w \models L_6$, we put $\iota(w) = 6$. Now it follows from Axioms 1–5, that ι satisfies the conditions of the theorem. □

Corollary 1.2 *We can embed second-order predicate logic into* **SOPML** *when the modality is* **S4.2** *or weaker.*

Proof: We can use propositional variables which are sets of pairs as dyadic second-order predicates, and propositional variables which are elements as their arguments. For propositional variables R, p, and q, we define $R(p, q)$ as $\exists r(REL(p, q, r) \wedge \Diamond(L_5 \wedge R \wedge r))$, which means that p and q are related by R. Moreover, using dyadic predicates to define a tuple enumeration, the full second-order predicate logic can be interpreted in this logic. Now the corollary follows from Theorem 1.1. □

In particular, Corollary 1.2 implies that **SOPML** is not recursively axiomatizable when the modality is **S4.2** or weaker.[4]

Corollary 1.3 *Second-order predicate logic and* **SOPML** *(when the modality is* **S4.2** *or weaker) are interpretable one in the other.*

Proof: The proof follows from Corollary 1.2 and the fact that validity in a frame can be defined in second-order predicate logic, where quantifiers on truth assignments are expressible. □

Remark 1.4 Note that it follows from the definition of PAIRING that second-order monadic theory of a reflexive, transitive, and convergent binary relation with first point is equivalent to second-order predicate logic.

2 The world-relative domain semantics This section is organized as follows. First we recall the definition of the logic **Q2** based on the world-relative domain semantics (cf. [10]). Then we reproduce the proof from Kamp [8] of the fact that **SOPML** is interpretable in **Q2**. An immediate corollary to this fact is interpretability of second-order predicate logic in **Q2** and vice versa when the modality is **S4.2** or weaker. We start with the definition of the world-relative domain semantics.

An *interpretation* σ consists of a nonempty set D_σ, called the *domain* of σ, and an assignment to each n-place predicate symbol P an n-place relation P^σ in D_σ.[5] A (**Q2**) *model* is a triple $\mathcal{M} = \langle W, R, S\rangle$, where $\langle W, R\rangle$ is a frame and S is a mapping from W into the class of interpretations.

Let $V = \{V_w\}_{w\in W}$ be a set of assignments for variables in the interpretations $S(w)$, $w \in W$. The *satisfiability* of a formula φ at u under assignments V, denoted $(u, V) \models_{\mathbf{Q2}} \varphi$, is defined inductively as follows.

> If φ is an atomic formula $P(x_1, \ldots, x_n)$, then $(u, V) \models_{\mathbf{Q2}} \varphi$ if and only if $(V_u(x_1), \ldots, V_u(x_n)) \in P^{S(u)}$, where $V_u(x)$ is the the element of $D_{S(u)}$ assigned to variable x by assignment V_u.
>
> $(u, V) \models_{\mathbf{Q2}} \varphi \supset \psi$ if and only if$(u, V) \not\models_{\mathbf{Q2}} \varphi$ or $(u, V) \models_{\mathbf{Q2}} \psi$.
>
> $(u, V) \models_{\mathbf{Q2}} \neg\varphi$ if and only if$(u, V) \not\models \varphi$.
>
> $(u, V) \models_{\mathbf{Q2}} \exists x\varphi(x)$ if and only if there exists a set of assignments $V' = \{V'_w\}_{w\in W}$, such that for each $w \in W$, V'_w differs from V_w at most at x, and $(u, V') \models_{\mathbf{Q2}} \varphi(x)$. That is, in **Q2** we quantify over *individual concepts*.
>
> $(u, V) \models_{\mathbf{Q2}} \Box\varphi$ if and only if for every w such that uRw, $(w, V) \models_{\mathbf{Q2}} \varphi$.

We say that a formula φ is *valid* in a model $\mathcal{M}$, denoted $\mathcal{M} \models_{\mathbf{Q2}} \varphi$, if and only if for any set of assignments V and for any $u \in W$, $(u, V) \models_{\mathbf{Q2}} \varphi$, and we say that a set of formulas Γ is *valid* in $\mathcal{M}$, denoted $\mathcal{M} \models_{\mathbf{Q2}} \Gamma$, if and only if $\mathcal{M} \models_{\mathbf{Q2}} \varphi$, for all

$\varphi \in \Gamma$. We say that Γ *semantically entails* φ, denoted $\Gamma \models_{\mathbf{Q2}} \varphi$, if $\mathcal{M} \models_{\mathbf{Q2}} \Gamma$ implies $\mathcal{M} \models_{\mathbf{Q2}} \varphi$ for each model $\mathcal{M}$.

Next it is shown that the expressive power of **SOPML** is equivalent to that of **Q2** with only one monadic predicate P and the axiom

$$TWO : \exists x P(x) \wedge \exists x \neg P(x),$$

stating that there are two different elements in the corresponding world.

The validity-preserving transformation of **SOPML** formulas into **Q2** formulas is obtained by replacing each propositional variable p with $P(x_p)$ and every formula of the form $\exists p \psi(p)$ with $\exists x_p \psi(P(x_p))$.[6] Formally, for a **SOPML** formula φ we define a **Q2** formula $\varphi^{\mathbf{Q2}}$ by induction as follows. Let $p \leftrightarrow x_p$ be a bijection between propositional variables of **SOPML** and individual variables of **Q2**.

If φ is a propositional variable p, then $\varphi^{\mathbf{Q2}}$ is $P(x_p)$.

$(\varphi \supset \psi)^{\mathbf{Q2}}$ is $\varphi^{\mathbf{Q2}} \supset \psi^{\mathbf{Q2}}$, $(\neg\varphi)^{\mathbf{Q2}}$ is $\neg\varphi^{\mathbf{Q2}}$, $(\Box\varphi)^{\mathbf{Q2}}$ is $\Box\varphi^{\mathbf{Q2}}$, and $(\exists p\varphi)^{\mathbf{Q2}}$ is $\exists x_p \varphi^{\mathbf{Q2}}$.

Conversely, for a **Q2** formula φ we define a **SOPML** formula $\varphi^{\mathbf{SOPML}}$ by induction as follows. Let p_x be the propositional variable corresponding to the individual variable x under the bijection $p \leftrightarrow x_p$. That is, $(p_x)_p$ is p, and $(x_p)_x$ is x.

If φ is an atomic formula $P(x)$, then $\varphi^{\mathbf{SOPML}}$ is p_x.

$(\varphi \supset \psi)^{\mathbf{SOPML}}$ is $\varphi^{\mathbf{SOPML}} \supset \psi^{\mathbf{SOPML}}$, $(\neg\varphi)^{\mathbf{SOPML}}$ is $\neg\varphi^{\mathbf{SOPML}}$, $(\Box\varphi)^{\mathbf{SOPML}}$ is $\Box\varphi^{\mathbf{SOPML}}$, and $(\exists x\varphi)^{\mathbf{SOPML}}$ is $\exists p_x \varphi^{\mathbf{SOPML}}$.

Note that for a **SOPML** formula φ, $(\varphi^{\mathbf{Q2}})^{\mathbf{SOPML}}$ is φ, and for a **Q2** formula φ, $(\varphi^{\mathbf{SOPML}})^{\mathbf{Q2}}$ is φ. Let TWO^n denote $\bigwedge_{i=0}^{n} \Box^i TWO$.[7]

Theorem 2.1 ([8]) *Let φ be a* **SOPML** *formula and let n be the maximum depth of nested modalities of φ. Let $F = \langle W, R \rangle$ be a frame, and let $\mathcal{M} = \langle W, R, S \rangle$ be a* **Q2** *model. Let $u \in W$ be such that $u \models_{\mathbf{Q2}} TWO^n$. Let $T = \{T_w\}_{w \in W}$ be a set of assignments for propositional variables and let $V = \{V_w\}_{w \in W}$ be a set of the truth assignments for (individual) variables such that for all $w \in W$, $T_w(p) = true$ if and only if $(V_w(x_p)) \in P^{S(w)}$. Then $(u, T) \models_{\mathbf{SOPML}} \varphi$ if and only if $(u, V) \models_{\mathbf{Q2}} \varphi^{\mathbf{Q2}}$.*

Proof: Since the satisfiability of a formula of modal depth n at possible world u depends only on the possible worlds in the set $\{w : uR^i w,\ 0 \le i \le n\}$, we may assume that $W = \{w : uR^i w,\ 0 \le i \le n\}$. Then $\mathcal{M} \models_{\mathbf{Q2}} TWO$.

The proof is by induction on the complexity of φ.

If φ is a propositional variable p, then the result follows immediately from the definition of T and V. The cases when φ is in one of the forms $\neg\psi$, $\psi_1 \supset \psi_2$, or $\Box\psi$ are straightforward.

Let φ be of the form $\exists p \psi(p)$. Assume $(u, T) \models_{\mathbf{SOPML}} \exists p \psi(p)$. Then there is a set of truth assignments $T' = \{T'_w\}_{w \in W}$, such that for each $w \in W$, T'_w differs from T_w at most at p, and $(u, T') \models_{\mathbf{SOPML}} \psi(p)$. Consider a set of assignments for variables $V' = \{V'_w\}_{w \in W}$ that is defined as follows.

If x is not x_p, then $V'_w(x) = V_w(x)$. If p is assigned "*true*" by T'_w, then $V'_w(x_p) \in P^{S(w)}$, and if p is assigned "*false*" by T'_w, then $V'_w(x_p) \notin P^{S(w)}$. By *TWO* such an assignment for x_p is always possible. By the induction hypothesis, $(w, V') \models_{\mathbf{Q2}}$

$\psi^{\mathbf{Q2}}(P(x_p))$. Thus, by definition, $(u, V) \models_{\mathbf{Q2}} \exists x_p \psi^{\mathbf{Q2}}(P(x_p))$. That is, $(u, V) \models_{\mathbf{Q2}} \varphi^{\mathbf{Q2}}$.

Conversely, assume $(u, V) \models_{\mathbf{Q2}} \exists x_p \psi^{\mathbf{Q2}}(P(x_p))$. Then there is a set of assignments for variables $V' = \{V'_w\}_{w \in W}$, such that for each $w \in W$, V'_w differs from V_w at most at $\{x_p\}$, and $(u, V') \models_{\mathbf{Q2}} \psi^{\mathbf{Q2}}(P(x_p))$. Consider a set of assignments for propositional variables $T' = \{T'_w\}_{w \in W}$ that is defined as follows.

If q is not p, then $T'_w(q) = T_w(q)$. If $(w, V') \models_{\mathbf{Q2}} P(x_p)$, then T'_w, assigns "*true*" to p; and if $(w, V') \not\models_{\mathbf{Q2}} P(x_p)$, then T'_w assigns "*false*" to p. By the induction hypothesis, $(u, T') \models_{\mathbf{SOPML}} \psi(p)$. Thus, by definition, $(u, T) \models_{\mathbf{SOPML}} \exists p \psi(p)$. That is, $(u, T) \models_{\mathbf{SOPML}} \varphi$. □

Theorem 2.1 has the following immediate corollaries.

Corollary 2.2 *Let φ be a* **SOPML** *formula. Then $\models_{\mathbf{SOPML}} \varphi$ if and only if $TWO \models_{\mathbf{Q2}} \varphi^{\mathbf{Q2}}$.*

Corollary 2.3 *Let φ be a* **SOPML** *formula and let n be the maximum depth of nested modalities of φ. Then $\models_{\mathbf{SOPML}} \varphi$ if and only if $\models_{\mathbf{Q2}} TWO^n \supset \varphi^{\mathbf{Q2}}$.*

Corollary 2.4 *We can embed second-order predicate logic into* **Q2** *when the modality is* **S4.2** *or weaker.*

Proof: The proof follows from Corollaries 1.2 and 2.3. □

In particular, Corollary 2.4 implies that **Q2** when the modality is **S4.2** or weaker, is not recursively axiomatizable.

Corollary 2.5 *If the modality is* **S4.2** *or weaker, then second-order predicate logic,* **Q2***, and* **SOPML** *are each interpretable in the others.*

Proof: The proof follows from Corollary 1.3, Corollary 2.4, and the fact that validity in a **Q2** model can be defined in second-order predicate logic. □

***3 Logics stronger than* S4.2** We conclude the paper with several notes concerning the power of **SOPML** and **Q2** when the modality is **S4.3**[8] or **S5**.

First, it follows from a very nontrivial result of Gurevich and Shelah [5] that second-order arithmetic is interpretable in **SOPML** with the **S4.3** modality.[9] By [5], Corollary 0.2, second-order arithmetic is interpretable in the monadic second-order theory of order of real numbers. Therefore, by Shelah [9], Lemma 7.12, second-order arithmetic is interpretable in the monadic second-order theory of linear order, which is definable in the **S4.3** frames.

Moreover, it is shown in Gurevich and Shelah [6] that, under a weak set-theoretic assumption, second-order predicate logic is interpretable in the monadic second-order theory of linear order, and therefore in **SOPML** with the **S4.3** modality.

Finally, **SOPML** with the **S5** modality is decidable, see [1], because it is equivalent to monadic second-order theory, and Kamp [8] presents Kripke's recursive axiomatization of **Q2** with the **S5** modality. An interesting byproduct of Kripke's proof is that *de re* modalities are eliminable in **Q2** based on **S5**. That is, in this logic, each formula is equivalent to a formula not containing modalities in the scope of quantifiers, cf. Fine [2] and [3].

NOTES

1. That is, for logics defined by a class of frames that contains all reflexive, transitive, and convergent frames. These properties of a frame are implied by the axioms $\forall p(\Box p \supset p)$, $\forall p(\Box p \supset \Box\Box p)$, and $\forall p(\Diamond\Box p \supset \Box\Diamond p)$, respectively, and vice versa, see Hughes and Cresswell [7], p. 31.
2. This world ensures that the frame is convergent.
3. These constants can be eliminated by replacing PAIRING with
$$\exists L_1 \exists L_2 \exists L_3 \exists L_4 \exists L_5 \exists L_6 \text{PAIRING}.$$
4. This result is proved in [1] by embedding second-order arithmetic into **SOPML**.
5. For simplicity we assume that the underlying language is one without equality and contains no constants or function symbols. As we shall see in a moment, all we need is one monadic predicate symbol.
6. This translation is defined in Kamp [8]. The translation in Garson [4] uses equality instead of P.
7. As usual, $\Box^0\varphi$ is φ, and $\Box^{i+1}\varphi$ is $\Box\Box^i\varphi$.
8. That is, **SOPML** defined by the class of all reflexive, transitive, and connected frames. These properties of a frame are implied by the axioms $\forall p(\Box p \supset p)$, $\forall p(\Box p \supset \Box\Box p)$, and $\forall p \forall q(\Box(\Box p \supset q) \vee \Box(\Box q \supset p))$, respectively, and vice versa, see [7], p. 30.
9. In [1] Fine uncarefully claims that **SOPML** with the **S4.3** modality is decidable. Furthermore, it is claimed in [4], Section 3.4 that "second-order modal arithmetic" is interpretable in **Q2** with the **S4.3** modality. However the proof of this result is based on a mistake statement ([4], Section 3.4, Lemma 8) that is an extension of Theorem 2.1 in which modal arithmetic operators are interpreted by function symbols.

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