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Modal Logics in the Vicinity of S1

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Abstract We define *prenormal* modal logics and show that **S1**, **S1**$^\circ$, **S0.9**, and **S0.9**$^\circ$ are *Lewis versions* of certain prenormal logics, determination and decidability for which are immediate. At the end we characterize *Cresswell logics* and ponder C. I. Lewis's idea of strict implication in **S1**.

1 Introduction The modal logics **S1**, **S2**, **S3**, **S4**, and **S5**—the "Lewis systems"—made their first collective appearance in [14].[1] The logics were presented syntactically, by means of axioms and rules of inference, and it was established that they are distinct and form a chain under inclusion, with **S1** being the weakest, or "strictest" in Lewis's accounting. The appearance of possible worlds models in the late 1950s expedited a systematic attempt to analyze modal logics semantically. As is well known, the normal logics **S4** and **S5** were the first of the series to receive satisfactory treatment in possible worlds terms. The others, being nonnormal, had to wait longer. The last to succumb was **S1**, for which Cresswell [4] provided a possible worlds model theory.[2]

Most modal logics studied in recent years have been at least classical, in the sense that they permit replacement of logically equivalent formulas. The logic **S1**, however, is nonclassical, as are certain others in its vicinity, notably **S1**$^\circ$, **S0.9**, and **S0.9**$^\circ$. In this paper we present a uniform approach to **S1** and these close relatives by way of a class of classical "prenormal" logics, interesting in their own right.

Following some preliminary definitions we introduce prenormal logics, develop their semantics, and establish determination and decidability theorems. Then we define certain nonclassical extensions of prenormal logics, of a kind we call Lewis logics, determination and decidability results for which are thus ready to hand. Once we describe **S1**, **S1**$^\circ$, **S0.9**, and **S0.9**$^\circ$ axiomatically and note some of their salient principles, the upshot is an identification of **S1** and its relatives as Lewis logics. Finally, we

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generalize a key schema and rule and in terms of them characterize a class of what we call Cresswell logics; this raises a question about Lewis's idea of strict implication.

As will become evident, our approach in this paper is inspired by Cresswell's work and is based on a central idea due to him, viz., models of the kind he introduced in [4], which combine features of the familiar "Kripkean" relational models with their more general neighborhood counterparts first introduced by Montague and Scott.

2 Preliminary definitions In the course of the paper we consider a host of axiom schemata and rules. Naming them all in a mnemonically satisfactory way presents problems. Certain conventions will help. We begin with these two formulas:[3]

(N) $\Box\top$
(Q) $\Diamond\bot$.

Then given a schema S we write nS for the schema N $\to$ (S), qS for the schema Q $\to$ (S), and $\Box$S for the schema $\Box$(S). For example, among the schemata to be considered are the following:

(K) $\Box(A \to B) \to (\Box A \to \Box B)$
(X) $(\Box(A \to B) \wedge \Box(B \to C)) \to \Box(A \to C)$
(T) $\Box A \to A$.

Thus by our convention we also have, inter alia:

(nK) $\Box\top \to (\Box(A \to B) \to (\Box A \to \Box B))$
(qX) $\Diamond\bot \to ((\Box(A \to B) \wedge \Box(B \to C)) \to \Box(A \to C))$
($\Box$T) $\Box(\Box A \to A)$.

(To aid memory, this is K for Kripke, X for chain-rule, T for Feys's *logique t*, N for normal, and Q for queer.) If we agree to list

(PL) the set of all tautologies,

as a kind of schema, we can stretch our convention a little by registering as well the set of necessitations of tautologies:

($\Box$PL) $\{\Box A : A \in \text{PL}\}$.

Another convention is to give rules a label beginning with the letter R. Examples are rules of necessitation and denecessitation, Scott's rule, a rule of congruence called RE for historical reasons, and a general rule of replacement:

(RN*) $A \,/\, \Box A$
(RN$_*$) $\Box A \,/\, A$
(RK) $(A_0 \wedge \cdots \wedge A_{m-1}) \to B \,/\, (\Box A_0 \wedge \cdots \wedge \Box A_{m-1}) \to \Box B (m \geq 0)$[4]
(RE) $A \leftrightarrow B \,/\, \Box A \leftrightarrow \Box B$
(RRE) $A \leftrightarrow B, C \,/\, C^{A}/_{B}$.

—where $C^{A}/_{B}$ results from replacing one occurrence of A in C by B.[5] Apart from substitution the only deviation from this policy comes with a couple of rules of modus ponens—the usual version and a "strict implication" counterpart:

(MP) $A \to B, A \,/\, B$
(SMP) $\Box(A \to B), A \,/\, B$.

We also occasionally use "n" and "q", not always with obvious meanings, in names of rules. Some examples:

(RnN*) $A \,/\, \Box\top \to \Box A$
(RnN$_*$) $\Box\top \to \Box A \,/\, A$
(RnK) $(A_0 \wedge \cdots \wedge A_{m-1}) \to B \,/\, \Box\top \to ((\Box A_0 \wedge \cdots \wedge \Box A_{m-1}) \to \Box B)(m \geq 0)$
(RqX) $A \vee B \,/\, (\Box A \wedge \Box B) \to \Box\top$.

We should emphasize that to say that a logic provides a rule of the form

$$A_0, \ldots, A_{n-1} \,/\, B$$

means that the logic contains the conclusion B whenever it contains all the hypotheses $A_0, \ldots, A_{n-1}$. In some cases we state a condition on a rule. One example is replacement for tautological equivalents:

(RRTE) $C \,/\, C^A/_B$—whenever A and B are tautologically equivalent.

Another example is the rule RNAx mentioned later on.[6]

Much of the early work in modal logic concerns the idea of *strict implication* and its concomitant, *strict equivalence*. Strict implication is nowadays usually defined as necessitation of the ("material") conditional, $\Box(A \to B)$, which is the way we shall construe it.[7] But there are at least two plausible ways to represent strict equivalence. The one used by Lewis, Feys, Lemmon, and most other traditional authors conjoins strict implications: $\Box(A \to B) \wedge \Box(B \to A)$. Another, simpler form necessitates the biconditional: $\Box(A \leftrightarrow B)$. These forms are not in general interchangeable, and certainly not in the logics to which this paper is largely devoted.[8] Thus when it comes to representing a strict analogue of the rule RRE there are two choices. In the simpler fashion, where the analogy is clearer, we write:

(RRSE) $\Box(A \leftrightarrow B), C \,/\, C^A/_B$.

In the traditional mode we write:

(RRSE$_\mathrm{T}$) $\Box(A \to B) \wedge \Box(B \to A), C \,/\, C^A/_B$.

Likewise for strict counterparts of RE:

(RSE) $\Box(A \leftrightarrow B) \,/\, \Box(\Box A \leftrightarrow \Box B)$
(RSE$_\mathrm{T}$) $\Box(A \to B) \wedge \Box(B \to A) \,/\, \Box(\Box A \to \Box B) \wedge \Box(\Box B \to \Box A)$.

For the logics we shall consider, it fortunately makes no difference which form is preferred—something we shall prove in Section 10.

3 Logics A set of formulas is a *logic* if it includes the set PL and is closed under MP and substitution. Thus PL is itself a logic—ordinary propositional logic—and, being the smallest logic, it lies within all the logics in this paper. This means that every propositionally correct inference is available in every logic. We often signal such inferences simply by PL.[9]

We distinguish between theses and theorems. A formula belonging to a logic is a *thesis*, whereas a *theorem* is a formula provable in an axiom system. The distinction is fine, but on occasion it will be useful. A logic is *consistent* just in case it excludes some formula.

A schema or rule is said to be *derivable* in a logic if all instances of the schema are theses of the logic or the logic is closed under the rule. If a schema or rule is derivable in a logic we also say that the logic *provides* or, simply, *has* the schema or the rule in question.

A *classical* logic is one that provides the rule RRE or, equivalently, RE, and a logic is *monotonic*, *regular*, or *normal* if it provides RK respectively for $m = 1$, $m \geq 1$, or $m \geq 0$. Clearly every monotonic logic—hence also every regular or normal logic—is classical. Alternatively, a classical logic is monotonic if it has $\Box(A \wedge B) \to (\Box A \wedge \Box B)$, regular if it has the schema

(F) $\quad\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$,

and normal if it has N (or RN*) and either F or K. The smallest classical, monotonic, regular, and normal logics are called **E, M, R,** and **K** respectively.[10] The following lemma records some further elementary facts.

Lemma 3.1 *(1) Classical logics provide both* RnN* *and* RRTE. *(2) A logic that provides* RRTE *and* X *also has* K. *(3) A logic that provides* RN_* *also provides* SMP, *and hence a logic that has* T *also provides* SMP. *(4) A logic that has* RRTE *provides* SMP *if and only if it provides* RN_*.

Proof: For (1): If A is a thesis of a classical logic then so is $\top \leftrightarrow A$ by PL and hence $\Box\top \to \Box A$ by way of RRE. Moreover, if A and B are tautologically equivalent then $A \leftrightarrow B$ is a tautology and hence a thesis of every logic. So by RRE if C is a thesis so is $C^A/_B$. We make frequent appeal to the rule RRTE in the pages that follow. For (2): Assume that the logic provides X and so too $(\Box(\top \to A) \wedge \Box(A \to B)) \to \Box(\top \to B)$. By two uses of RRTE we infer the thesis $(\Box A \wedge \Box(A \to B)) \to \Box B$, which is a PL-variant of K. The schemata K and X play important roles in the next few sections, either by themselves or as components of other schemata such as nK and qX, and this reasoning can be imitated in these other contexts. For (3): If both $\Box(A \to B)$ and A are theses of a logic then by RN_* so is $A \to B$, whence B by MP. For (4): It is enough to show that the logic provides RN_* if it provides SMP. Assume that $\Box A$ is a thesis. From this we obtain $\Box(\top \to A)$ via RRTE and then A from SMP and the thesis $\top$. $\Box$

The logic **S1** and certain others in its vicinity are not classical. Our aim, however, is to show that they are systematically related to a class of logics that *are* classical. In anticipation of our introduction of these logics we register the following proposition.

Lemma 3.2 *A classical logic provides any one of (i)* nK, *(ii)* RnK, *(iii)* nF, *and (iv)* nX *if and only if it provides them all.*

Proof: The reasoning is like that for the "n-less" forms of the schemata and rule in the realm of normal logics. From (i) to (ii) we argue inductively: When $m = 0$, RnK is just RnN*, which every classical logic provides (Lemma 3.1(1)). So assume that the rule is provided up to $m > 0$ and that $(A_0 \wedge \cdots \wedge A_m) \to B$ is a thesis. Then so is $(A_0 \wedge \cdots \wedge A_{m-1}) \to (A_m \to B)$, from which by the inductive hypothesis we reach $\Box\top \to ((\Box A_0 \wedge \cdots \wedge \Box A_{m-1}) \to \Box(A_m \to B))$. Using nK in the form $\Box\top \to (\Box(A_m \to B) \to (\Box A_m \to \Box B))$, we infer $\Box\top \to ((\Box A_0 \wedge \cdots \wedge \Box A_{m-1}) \to (\Box A_m \to \Box B))$, which is equivalent to the desired conclusion. From

(ii) to (iii), note that by way of RnK and various tautologies nF follows (as, indeed, do nK and nX). From (iii) to (iv), suppose now that the logic has nF. To see that it has nX, note first that it provides the rule RnK for $m = 1$: the hypothesis $A \to B$ is PL-equivalent to $A \leftrightarrow (A \wedge B)$, which yields $\Box A \leftrightarrow \Box(A \wedge B)$ by RE, from which the conclusion $\Box\top \to (\Box A \to \Box B)$ follows via nF. So by this rule on a tautology, $\Box\top \to (\Box((A \to B) \wedge (B \to C)) \to \Box(A \to C))$ is a thesis, and via an instance of nF this becomes nX. Finally, from (iv) to (i), an argument like that for Lemma 3.1(2) shows that nX yields nK. $\Box$

4 Strict classical logics As we remarked, many of the logics to be considered are not classical, i.e., they do not provide for replacement of equivalents. These logics do, however, provide for replacement of strict equivalents. Let us delimit as *strict classical* those logics that include $\Box$PL and are closed under RRSE. In anticipation of our needs in Section 10, and so as not to beg any questions, we also define a logic to be *strict*$_{\mathrm{T}}$ *classical* ("traditionally strict classical") if and only if it includes $\Box$PL and is closed under RRSE$_{\mathrm{T}}$.

Lemma 4.1 *A strict or strict*$_{\mathrm{T}}$ *classical logic provides* N *and* RRTE.

Proof: N is of course in such a logic. On the other hand, suppose that A and B are tautologically equivalent—so that $A \leftrightarrow B$, $A \to B$, and $B \to A$ are tautologies—and that C is a thesis of the logic. By $\Box PL$, $\Box(A \leftrightarrow B)$ is also a thesis, as are $\Box(A \to B)$ and $\Box(B \to A)$ and hence too $\Box(A \to B) \wedge \Box(B \to A)$. Then $C^{A}/_{B}$ follows by RRSE or by RRSE$_{\mathrm{T}}$. $\Box$

When is it a matter of indifference which of RRSE and RRSE$_{\mathrm{T}}$ we use to characterize strict (or strict$_{\mathrm{T}}$) classical logics? One weak, sufficient condition is the presence of RRTE and the following rule (more precisely, pair of rules):

(RF) $\Box(A \wedge B) \,/\, \Box A \wedge \Box B \qquad \Box A \wedge \Box B \,/\, \Box(A \wedge B)$.

Thus it should be evident that a logic that has both RRTE and RF provides RRSE if and only if it provides RRSE$_{\mathrm{T}}$. So by Lemma 4.1 strict and strict$_{\mathrm{T}}$ classicality coincide whenever a logic provides RF. More pertinent to our interests below are the next two lemmas.

Lemma 4.2 *A logic that has* N *and* RRTE *(and hence* $\Box$PL*) and* K *also provides* RF.

Proof: For left-to-right, suppose that $\Box(A \wedge B)$ is one of the logic's theses, for some A and B. By $\Box$PL so are $\Box((A \wedge B) \to A)$ and $\Box((A \wedge B) \to B)$. Using PL, these plus some instances of K—

$$\Box((A \wedge B) \to A) \to (\Box(A \wedge B) \to \Box A)$$

$$\Box((A \wedge B) \to B) \to (\Box(A \wedge B) \to \Box B)$$

—deliver $\Box A \wedge \Box B$. Conversely, if $\Box A \wedge \Box B$ is a thesis for some A and B, then by $\Box$PL so is $\Box(A \to (B \to (A \wedge B))$. All this and two instances of K—

$$\Box(A \to (B \to (A \wedge B))) \to (\Box A \to \Box(B \to (A \wedge B)))$$

$$\Box(B \to (A \land B)) \to (\Box B \to \Box(A \land B))$$

—deliver $\Box(A \land B)$. □

From Lemmas 3.1(2), 4.1, and 4.2, therefore, we have this corollary:

Lemma 4.3 *A logic that provides* K *or* X *is strict classical if and only if it is strict*$_{\mathrm{T}}$ *classical.*

It must be emphasized that, regardless of our use of “F” in its name, a logic that has the rule RF may yet lack the schema F or either of its conditional halves:

$$\Box(A \land B) \to (\Box A \land B) \qquad (\Box A \land \Box B) \to \Box(A \land B).$$

For example, notwithstanding their classicality the logics defined below in Section 5 lack both these schemata and so are not even monotonic. The strict and strict$_{\mathrm{T}}$ classical logics in Sections 9 and 10 likewise lack both the schemata.

One more result before we move on.

Lemma 4.4 *A strict classical logic provides* RSE*, and a strict*$_{\mathrm{T}}$ *classical logic provides* RSE$_{\mathrm{T}}$.

Proof: Suppose that $\Box(A \leftrightarrow B)$ and $\Box(A \to B) \land \Box(B \to A)$ are theses. Via PL and $\Box$PL we reach $\Box(\Box A \leftrightarrow \Box A)$ and $\Box(\Box A \to \Box A) \land \Box(\Box A \to \Box A)$, from which $\Box(\Box A \leftrightarrow \Box B)$ and $\Box(\Box A \to \Box B) \land \Box(\Box B \to \Box A)$ follow by RRSE and RRSE$_{\mathrm{T}}$ respectively. □

It should be understood that, unlike the situation in classical logics, where RE does as well as RRE, strict classical logics cannot be characterized simply as logics including $\Box$PL and closed under RSE; the same goes for strict$_{\mathrm{T}}$ classical logics and RSE$_{\mathrm{T}}$.

5 Prenormal logics We call a classical logic *prenormal* just in case it provides the schema

(nK) $\quad \Box\top \to (\Box(A \to B) \to (\Box A \to \Box B)).$

So Lemma 3.2 gives a good feeling for the strength of prenormal logics. The smallest such logic we dub **P**. As a considerable portion of this paper deals with prenormal logics definable by adding to **P** any combination of K, X, and T (including the null combination), let us call such a logic a *prenormal KXT-logic*. The following result is obvious inasmuch as nX is in every prenormal logic.

Lemma 5.1 *A prenormal logic provides* K *if and only if it provides* qK*, and it provides* X *if and only if it provides* qX.

Next note this analogue of Lemma 3.1(2):

Lemma 5.2 *A classical logic—and hence a prenormal logic—that provides* qX *also provides* qK.

Though there is thus some redundancy in characterizing KXT-logics in terms of K or X rather than qK or qX, we shall continue to use the simpler nomenclature. At the same time we note that the defining axiom nK is redundant in the axiomatizations **PK, PX, PKT,** and **PXT**. It follows that there are at most six prenormal KXT-logics:

P, PK, PX, PT, PKT, and **PXT**. As we shall see, they are all distinct; so their number is exactly six.

We call **P** and other such logics prenormal because they are separated from normal logics only by a logical hair's breadth—viz., the formula N. The smallest classical logic containing both nK and its antecedent, N, is obviously **EKN**, i.e., **K**, the smallest normal logic.

Before moving to semantics we note the equivalence of the schema qX to a rule mentioned in Section 2.

Lemma 5.3 *A classical logic provides* qX *if and only if it provides* RqX.

Proof: For left-to-right, suppose that $A \vee B$ is a thesis. Then so are $A \leftrightarrow (B \to A)$ and $B \leftrightarrow (A \to B)$, as is the tautology $\top \leftrightarrow (B \to B)$. Hence by RRE on an instance of qX so is $\Diamond\bot \to ((\Box A \wedge \Box B) \to \Box\top)$. Via RRE and PL this is equivalent to $(\Box A \wedge \Box B) \to \Box\top$. For right-to-left, apply the rule to the tautology $(A \to B) \vee (B \to C)$ to infer $(\Box(A \to B) \wedge \Box(B \to C)) \to \Box\top$—and use RRE and PL to reach $\Diamond\bot \to \neg(\Box(A \to B) \wedge \Box(B \to C))$, which at once yields qX by PL.[11] $\Box$

6 Semantics for prenormal logics We begin with the idea of a *frame*, by which we understand a structure $\mathcal{F} = (U, N, Q, R, S)$ in which U is a set, N and Q are disjoint subsets of U that exhaust it (i.e., $N \cup Q = U$ and $N \cap Q = \varnothing$), R is a binary relation in $N \times U$, and S is a family $\{S(x) : x \in Q\}$, where each $S(x)$ is a collection of subsets of U subject to the condition that $U \notin S(x)$. The set U is the *universe* of the frame, and we think of the elements in N as *normal* and of those in Q as nonnormal, or *queer*. R is an *alternativeness* relation, and $S(x)$ collects the *neighborhoods* of x. For brevity's sake we reduce the frames to (U, Q, R, S), but throughout the paper we continue to use N for $U - Q$.

A *valuation* in a set U is a mapping V from the set of atomic formulas to $\mathcal{P}U$, the power set of U. *Models* are structures $\mathcal{M} = (U, Q, R, S, V)$ consisting of a frame (U, Q, R, S) together with a valuation in its universe.

A frame or model is said to be *finite* when its universe is.

The definition of *truth* at a point in a model is as usual except for the modal clauses, where the truth conditions depend on whether the point is normal or queer:

For $x \in N$: $\mathcal{M} \models_x \Box A$ if and only if $\forall y(\langle x, y\rangle \in R \Rightarrow \mathcal{M} \models_y A)$.
For $x \in Q$: $\mathcal{M} \models_x \Box A$ if and only if $\|A\|^{\mathcal{M}} \in S(x)$.

Here $\|A\|^{\mathcal{M}}$ is $\{u \in U : \mathcal{M} \models_u A\}$, the "truth set" of A in $\mathcal{M}$.

We shall usually drop references to $\mathcal{F}$ or $\mathcal{M}$ when it is clear what frame or model is intended.

A formula is *true in a model* if true at all points in the model; *valid in a frame* if true in all models on the frame. A set of formulas is *satisfiable in a frame* if all its members are true together at some point in some model on the frame. A logic is said to be *strongly determined by* a class of frames just in case every set of formulas consistent in the logic is satisfiable in a frame in the class (completeness) and vice versa (soundness). A logic is *weakly determined by* a class of frames if and only if the formulas valid throughout the class are theses of the logic (weak completeness) and vice versa. By $\mathbf{L}(\mathcal{F})$ we mean the *logic* of the frame $\mathcal{F}$, i.e., the set of formulas valid in $\mathcal{F}$.

To those familiar with the contemporary semantic analyses of modal logics it will be apparent that the apparatus presented above combines the relational and neighborhood approaches. Within the domain of normal points the alternativeness relation holds sway, and the frames, models, and truth conditions are relational; for queer points the structures and truth conditions are of a neighborhood type. In brief, if $N = U$ and $Q = \varnothing$ the frames become completely normal, as does their logic; when the situation is reversed we are in the realm of neighborhood semantics.[12] The fruitfulness of this way of doing things shows itself in our determination theorems.

7 Determination for prenormal logics To obtain determination theorems for the prenormal KXT-logics the first task is to define the *corresponding property* of each of the schemata K, X, and T—more precisely, of qK, qX, nT, and qT. Let $\mathcal{F} = (U, Q, R, S)$ be any frame.

Corresponding to K is what we call the *modus ponens property*:

(qk) $\quad x \in Q \Rightarrow ((U - X) \cup Y, X \in S(x) \Rightarrow Y \in S(x)).$

For X we have the *syllogism property*:

(qx) $\quad x \in Q \Rightarrow ((U - X) \cup Y, (U - Y) \cup Z \in S(x) \Rightarrow (U - X) \cup Z \in S(x)).$[13]

We say further that R is *normal-reflexive* if it is reflexive in the ordinary sense with in N; i.e., if $\langle x, x\rangle \in R$ whenever $x \in N$. And we say that S is *queer-reflexive* if whenever $x \in Q$, $X \in S(x)$ only if $x \in X$. Normal-reflexivity corresponds to nT, and queer-reflexivity corresponds to qT.

When R or S has any of these properties we also say that the frame $\mathcal{F}$ fdoes, and that any model on $\mathcal{F}$ does.[14] Finally, we say that a frame or model is *omnireflexive*—or has the property (t)—if R is normal-reflexive and S is queer-reflexive. Omnireflexivity corresponds to T.

It will be evident that each prenormal KXT-logic is sound with respect to any class of frames having the appropriate corresponding properties, and hence **P, PK, PX, PT, PKT**, and **PXT** are all distinct.

To prove determination we employ canonical models.

Let $\mathbf{L}$ be any prenormal logic. Where $\mathrm{Max}_{\mathbf{L}}$ is the set of all maximal consistent sets of formulas in $\mathbf{L}$, we write $|A|_{\mathbf{L}}$ for the set $\{u \in Max_{\mathbf{L}} : A \in u\}$ (the "proof set" of A in $\mathbf{L}$).[15] We say that $\mathcal{M}_{\mathbf{L}} = (U_{\mathbf{L}}, Q_{\mathbf{L}}, R_{\mathbf{L}}, S_{\mathbf{L}}, V_{\mathbf{L}})$ is a canonical model for $\mathbf{L}$ if it satisfies these conditions:

(i) $U_{\mathbf{L}} = \mathrm{Max}_{\mathbf{L}}$.
(ii) $Q_{\mathbf{L}} = \{x \in U_{\mathbf{L}} : \Box\top \notin x\}$.
(iii) $R_{\mathbf{L}} = \{\langle x, y\rangle \in N_{\mathbf{L}} \times U_{\mathbf{L}} : \forall A(\Box A \in x \Rightarrow A \in y)\}$.
(iv) $|A|_{\mathbf{L}} \in S_{\mathbf{L}}(x)$ if and only if $\Box A \in x$, for every $x \in Q_{\mathbf{L}}$.
(v) $V_{\mathbf{L}}(P) = |P|_{\mathbf{L}}$, for every atomic formula P.

We usually drop the reference to $\mathbf{L}$ when it is clear which logic $\mathbf{L}$ is.

Our definition allows for a variety of canonical models for a prenormal logic, ranging from the smallest (where each $S(x)$ contains just the proof sets $|A|$ such that

$\Box A \in x$) to the largest (where each $S(x)$ contains all such proof sets together with every nonproof set).[16]

Lemma 7.1 *Let $\mathcal{M}$ be a canonical model for a prenormal logic* **L**. *Then, for every formula* A *and every $x \in U$, $\mathcal{M} \models_x A$ if and only if $A \in x$.*

Proof: The proof is an induction of the usual kind, and the nonmodal cases are unproblematic. We will only highlight a detail in the proof for $x \in N$—i.e., for normal x—that $\mathcal{M} \models_x \Box A$ if and only if $\Box A \in x$. This involves proving that $\Box A \in x$ if and only if A is **L**-deducible from $\{B : \Box B \in x\}$. The only-if part is immediate. So suppose A is thus deducible. Then there are formulas $\Box B_0, \ldots, \Box B_{m-1} \in x$ such that $(B_0 \wedge \cdots \wedge B_{n-1}) \to A$ is a thesis of **L**. Consequently, by the rule RnK, $\Box\top \to ((\Box B_0 \wedge \cdots \wedge \Box B_{n-1}) \to \Box A)$ is also a thesis of **L**. Since x is normal, $\Box\top \in x$. It follows that $\Box A \in x$, since x is a maximal consistent set. □

It is immediate that **P** is strongly determined by the class of all frames. To show determination for the rest of the prenormal KXT-logics it is sufficient to show that they have canonical models that satisfy the appropriate conditions.

For **PX** and **PT** it is enough to check that their smallest canonical models respectively have the properties of syllogism (qx) and omnireflexivity (t), and that that for **PXT** has both. This is almost trivial for (t), and it is easy enough for (qx).[17] For **PK** and **PKT** something more is needed, what Benton [1] calls an "overlay" canonical model—i.e., one in which the neighborhoods of queer points are defined for nonproof sets X by:

$$X \in S(x) \text{ if and only if } \exists A \exists B(|A| \subseteq X \subseteq |B| \& \forall C(|A| \subseteq |C| \subseteq |B| \Rightarrow \Box C \in x)).$$

In terms of these constructions and Lemma 7.1 we reach:

Theorem 7.2 *Each prenormal* KXT-*logic is strongly determined by the class of frames with the appropriate corresponding properties.*

We can further obtain finite determination results for the prenormal KXT-logics.

Theorem 7.3 *Each prenormal* KXT-*logic is (weakly) determined by the class of finite frames with the appropriate corresponding properties.*

Theorem 7.3 can be shown by means of filtrations and Theorem 7.2. But because the KXT-logics are all noniterative—i.e., there is no nesting of modalities in their axioms—the results are in fact a consequence of Lewis's principal theorem in [15].[18]

From Theorem 7.3 we conclude that each of the prenormal KXT-logics has the *finite model property*: each nonthesis is false in some finite model for the logic. Of course an axiomatizable logic that has the finite model property is decidable.[19] Therefore:

Theorem 7.4 *The prenormal* KXT-*logics are all decidable.*

8 *The rule RnN$_*$*

We devote this section to showing that all of the prenormal logics definable in terms of K, X, and T provide the rule RnN$_*$, which will play an important role later on.

Where $\mathcal{M} = (U, Q, R, S, V)$ is a model, we say that $\mathcal{M}^\# = (U^\#, Q^\#, R^\#, S^\#, V^\#)$ is a *safe extension* of $\mathcal{M}$ if the following conditions hold:

(i) $U^\# \supseteq U$.
(ii) $Q^\# = Q$.
(iii) $R^\# \cap (U \times U) = R$.
(iv) $\langle x, y\rangle \in R^\# \Rightarrow (x \in U \Rightarrow y \in U)$, for every $x, y \in U^\#$.
(v) $X \in S^\#(x) \Leftrightarrow X \cap U \in S(x)$, for every $x \in Q^\#$ and every $X \subseteq U^\#$.
(vi) $V^\#(P) \cap U = V(P)$, for every atomic formula P.[20]

Theorem 8.1 *Let $\mathcal{M}^\#$ be a safe extension of $\mathcal{M}$. Then for every formula A and every $x \in U$, $\mathcal{M}^\# \models_x A$ if and only if $\mathcal{M} \models_x A$.*

Proof: Let us write $\|A\|$ for $\|A\|^{\mathcal{M}}$, and $\|A\|^\#$ for $\|A\|^{\mathcal{M}\#}$. Then the theorem asserts that $\|A\| = \|A\|^\# \cap U$. We prove it by induction on A. If A is atomic or boolean there is no difficulty. Suppose that $A = \Box B$ and—as the induction hypothesis—that the result holds for B. Let x be any element of U.

First suppose that x is normal. Note that it follows from the definition of a safe extension that, for every $y \in U^\#$, $\langle x, y\rangle \in R^\#$ if and only if $\langle x, y\rangle \in R$. With the help of this observation and the induction hypothesis we see that $\forall y \in U^\#(\langle x, y\rangle \in R^\# \Rightarrow \mathcal{M}^\# \models_y B)$ if and only if $\forall y \in U(\langle x, y\rangle \in R \Rightarrow \mathcal{M} \models_y B)$. Therefore $\mathcal{M}^\# \models_x \Box B$ if and only if $\mathcal{M} \models_x \Box B$. Next suppose that x is queer. Using the definition of $S^\#$ and the induction hypothesis we see that $\|B\|^\# \in S^\#(x)$ if and only if $\|B\|^\# \cap U \in S(x)$ if and only if $\|B\| \in S(x)$. Therefore $\mathcal{M}^\# \models_x \Box B$ if and only if $\mathcal{M} \models_x \Box B$. □

Note that new points in a safe extension $\mathcal{M}^\#$, i.e., those in $U^\# - U$, are always normal. It is easy to see that when $\mathcal{M}$ is omnireflexive and $\langle u, u\rangle \in R^\#$ for every new u, $\mathcal{M}^\#$ is normal-reflexive. It is also omnireflexive: for any $x \in Q^\#$, *if* $X \in S^\#(x)$, then $x \in Q$ and $X \cap U \in S(x)$. So if $S(x)$ is queer-reflexive, $x \in X \cap U \subseteq X$.

Lemma 8.2 *If $\mathcal{M}^\#$ is a safe extension of $\mathcal{M}$, then $\mathcal{M}^\#$ has the property (qk) or (qx), respectively, if $\mathcal{M}$ does.*

Proof: As (qk) is actually a special case of the syllogism property, (qx), let us argue just for the latter. Suppose that $\mathcal{M}$ has (qx). Where $x \in Q^\# = Q$, take any subsets X, Y, and Z of $U^\#$ such that both $(U^\# - X) \cup Y$ and $(U^\# - Y) \cup Z$ are elements of $S^\#(x)$. Then by the definition of safe extension:

$$((U^\# - X) \cup Y) \cap U \in S(x). \tag{1}$$

$$((U^\# - Y) \cup Z) \cap U \in S(x). \tag{2}$$

Given that $U \subseteq U^\#$, we can rewrite (1) and (2) thus:

$$((U - (X \cap U)) \cup (Y \cap U) \in S(x). \tag{3}$$

$$((U - (Y \cap U)) \cup (Z \cap U) \in S(x). \tag{4}$$

Hence by the syllogism condition for S, $((U - (X \cap U)) \cup (Z \cap U) \in S(x)$, which works out to mean that $((U^\# - X) \cup Z) \cap U \in S(x)$. So by the definition of $S^\#$, $(U^\# - X) \cup Z \in S^\#$. □

Theorem 8.3 *All the prenormal* KXT-*logics provide the rule* RnN_*.

Proof: Let **L** be one of the logics, and suppose that A is not a thesis of **L**. By Theorem 7.3, A is false at some point x in some model $\mathcal{M}$ having the appropriate properties from among (qk), (qx), and (t). Where u is new to $\mathcal{M}$, let $\mathcal{M}^{\#}$ be the safe extension of $\mathcal{M}$ obtained by adding $\langle u, x\rangle$ to R to form $R^{\#}$; add $\langle u, u\rangle$ as well in case **L** provides T. Then by Lemma 8.2 $\mathcal{M}^{\#}$ has the relevant properties among (qk), (qx), and (t), and so is a model for **L**. But $\Box A$ is false at u, and so, as u is normal,$\Box\top \to \Box A$ is also false at u. Hence $\Box\top \to \Box A$ is not a thesis of **L**. □

9 *Lewis logics and their semantics* By the *Lewis version* **Lew(L)** of a logic **L** we understand the smallest logic that extends **L** and contains $\Box\top$. Thus **Lew(L)** is the closure of $\mathbf{L} \cup \{\Box\top\}$ under modus ponens. Notice that **Lew(L)** is closed under substitution and so is a logic, but that **Lew(L)** may not be closed under other rules under which **L** is closed. For example, if **L** has the rule RRTE so does **Lew(L)**, whereas the rule RRE is not generally inherited by Lewis versions of classical logics. We define a *Lewis logic* as a logic that is the Lewis version of some classical logic.

Lemma 9.1 *Let* **L** *be a classical logic and A be any formula. (1) A is a thesis of* **Lew(L)** *if and only if $\Box\top \to A$ is a thesis of* **L**. *(2) If* **L** *provides* RnN_* *then A is a thesis of* **L** *if and only if $\Box A$ is a thesis of* **Lew(L)**. *(3) If* **L** *provides* RnN_* *then* **Lew(L)** *is strict classical and provides RN_*.*

Proof: (1) is simply a matter of definition and the deduction theorem. For the left-to-right of (2), suppose A to be a thesis of **L**. Then so too is $\Box\top\to\Box A$(Lemma 3.1(1)). Hence by part (1) **Lew(L)** has $\Box A$. For right-to-left, suppose that $\Box A$ is a thesis of **Lew(L)**. By part (1) $\Box\top \to \Box A$ is a thesis of **L**, and hence by RnN_* so is A. In the case of (3), for $\Box$PL, suppose that A is a tautology. Then A is a thesis of **L**, and by part (2) $\Box A$ is a thesis of **Lew(L)**. For RRSE, suppose that both $\Box(A \leftrightarrow B)$ and C are in **Lew(L)**. Then $A \leftrightarrow B$ is a thesis of **L** by part (2), and so is $\Box\top \to C$ by part (1). Since **L** is classical, it has $\Box\top \to C^A/_B$ and so $C^A/_B$ is a thesis of **Lew(L)** by part (1) again. Therefore **Lew(L)** is strict classical. For RN_*, suppose that $\Box A$ is a thesis of **Lew(L)**. Then **L** has A by part (2) and so does **Lew(L)** since $\mathbf{L} \subseteq \mathbf{Lew}(\mathbf{L})$. □

It follows from Theorem 8.3 and Lemma 9.1 that each prenormal KXT-logic is strict classical. Moreover, by Lemmas 3.1(3), 4.1, and 4.4 we find this corollary to Lemma 9.1:

Lemma 9.2 *Where* **L** *is a classical logic that provides* RnN_**—e.g., by Theorem 8.3, where* **L** *is a prenormal* KXT*-logic—***Lew(L)** *provides* RRTE, RSE, *and* SMP.

Notice that if **L** is normal then **L** = **Lew(L)**. The converse is not generally true. For example, the smallest classical logic that contains $\Box\top$ is a Lewis version of itself, but it is not normal. However, if **L** is prenormal, then **L** = **Lew(L)** only if **L** is normal.

The semantics for prenormal logics is readily adapted to suit their Lewis versions. Frames and models are as before, but the ideas of validity and satisfiability are relativized to normal points. We say that a set of formulas is *Lewis-satisfiable* in a frame if at some normal point in some model on the frame all the formulas are true, and that a formula is *Lewis-valid* in a frame if it is true at every normal element in every model on the frame. The concepts of strong and weak Lewis-determination are obvious. The fundamental result is this.

Theorem 9.3 *Where* **L** *is a classical logic strongly determined by a class of frames,* **Lew(L)** *is strongly Lewis-determined by the class.*

Proof: Let **L** be a classical logic determined by a class of frames. Lewis-soundness is evident. For Lewis-completeness, suppose Γ to be a **Lew(L)**-consistent set of formulas. Then $\Gamma \cup \{\Box\top\}$ is **Lew(L)**-consistent, and also **L**-consistent (since **L** is a sublogic of **Lew(L)**). By Theorem 7.2 the formulas in $\Gamma \cup \{\Box\top\}$ are true together at some point in some model on a frame in the class. Because of $\Box\top$ this point is normal. □

Using models like those in the case of prenormal KXT-logics (in Section 7) one can readily prove the following proposition.

Theorem 9.4 *The Lewis versions of* **P, PK, PX, PT, PKT**, *and* **PXT** *are all distinct.*

The Lewis versions of the prenormal KXT-logics are of course axiomatizable, and by means of filtrations they can all be shown to have the finite model property. So they are all decidable.[21] But more simply, the question of whether or not a formula A is a thesis of the Lewis version boils down to that of whether $\Box\top \to A$ is a thesis of the prenormal KXT-logic itself; since the latter is decidable so is the former. The result is worth recording:

Theorem 9.5 *The Lewis versions of* **P, PK, PX, PT, PKT**, *and* **PXT** *are all decidable.*

10 S1 and S0.9, S1° and S0.9° Our aim now is to show that the four logics mentioned in the title of this section—we refer to them as "the squadron"—are simple examples of Lewis logics. In fact, we shall be able to identify them as follows:

$$\mathbf{S1}^\circ = \mathbf{Lew(PX)}. \qquad \mathbf{S1} = \mathbf{Lew(PXT)}.$$
$$\mathbf{S0.9}^\circ = \mathbf{Lew(PK)}. \qquad \mathbf{S0.9} = \mathbf{Lew(PKT)}.$$

To prepare for these results we first revise various formulations of **S0.9**$^\circ$, **S1**$^\circ$, **S0.9**, and **S1** found in the literature so as to obtain a uniform set of axiomatizations that reveal at a glance certain important properties.

In [11] Lemmon provided a simple axiomatization of **S1** that, unlike earlier formulations, clearly separated its modal elements from its basis in propositional logic.[22] Specifically, Lemmon took as axioms PL and all instances of the schemata X and T, and by way of rules he used MP, a limited form of necessitation,

(RNAx) $A \,/\, \Box A$ —whenever A is one of the axioms,

and the traditional replacement rule for strict equivalents, $\mathrm{RRSE_T}$.

In [11] Lemmon also introduced the system **S0.9**, defining it by taking as axioms PL and all instances of K and T and as rules MP, RNAx, and the rule $\mathrm{RSE_T}$ of congruence for strict equivalence.

The other two systems, **S1**$^\circ$ and **S0.9**$^\circ$, are often loosely described as **S1** and **S0.9** minus the schema T. However, care is needed here, for it cannot be assumed a priori that deleting an axiom schema shared by two different axiomatizations of the same old logic will always result in the same new logic. Slightly modifying the definitive axiomatization of **S1**$^\circ$ given in [6] we note that **S1**$^\circ$ can be axiomatized by taking as axioms PL and all instances of X and by adopting as rules MP, RNAx, $\mathrm{RRSE_T}$, and the strict version of modus ponens, SMP.

Similarly, we may consider that **S0.9**$^\circ$ is a logic that can be axiomatized by replacing $\Box$X in Feys's axiomatization of **S1**$^\circ$ by $\Box$K. Then in the same way we may describe **S0.9**$^\circ$ as axiomatized by the system that has PL and all instances of K as axioms, and MP, RNAx, $\mathrm{RRSE_T}$, and SMP as rules.

Thus all the logics in the squadron have PL, MP, and RNAx; the "**1**-logics" have X where the "**0.9**"s have K; the "$^\circ$-logics" have SMP where the "$^\circ$-less" ones have T. The situation is pictured in Figure 1. The figure reveals the anomaly that **S0.9** uses

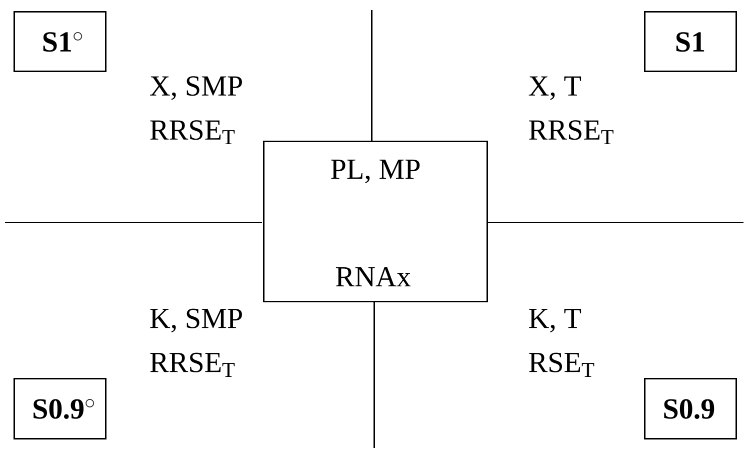

Figure 1

$\mathrm{RSE_T}$ where the others have $\mathrm{RRSE_T}$. In what follows we propose to regularize this and otherwise recast the axiomatizations of the logics in the squadron.

First the rule RNAx. Attractive for its summarizing properties, it is too radically context-dependent for a project such as ours, where we are dealing with a number of logics. We therefore replace RNAx by the combination of RRTE, N, and $\Box$S for each non-PL axiom schema S in the squadron's logics. The first two of these yield $\Box$PL, the necessitations of all tautologies.

Next notice that since **S1** and **S0.9** provide T, by Lemma 3.1(3) all the logics in the squadron have the rule SMP (we shall shortly improve on this result).

Lemmon himself remarked that the rule $\mathrm{RRSE_T}$ is derivable in **S0.9**.[23] Indeed, a routine induction proves this lemma.

Lemma 10.1 *A logic that provides* K, $\Box$PL, *and* $\mathrm{RSE_T}$ *also provides this rule of replacement:* $\Box(A \to B) \wedge \Box(B \to A) \,/\, \Box(C \to C^A/_B) \wedge \Box(C^A/_B \to C)$.

From this we obtain the following proposition, from which the desired result for **S0.9** follows.

Lemma 10.2 *A logic that provides* K, $\Box$PL, RSE$_{\mathrm{T}}$, *and* SMP *also provides* RRSE$_{\mathrm{T}}$.

Proof: Suppose that $\Box(A \to B) \wedge \Box(B \to A)$ and C are theses. Then by the rule in Lemma 10.1 so is $\Box(C \to C^A/_B) \wedge \Box(C^A/_B \to C)$ and hence, by PL, $\Box(C \to C^A/_B)$. From this $C^A/_B$ follows via SMP. $\Box$

To see that in Lemmon's axiomatization of **S0.9** RSE$_{\mathrm{T}}$ is indeed equivalent to RRSE$_{\mathrm{T}}$, note that if RSE$_{\mathrm{T}}$ is replaced by RRSE$_{\mathrm{T}}$, then given $\Box$PL the logic is strict$_{\mathrm{T}}$ classical and by Lemma 4.4 RSE$_{\mathrm{T}}$ is derivable.

Thus we see that the logics in the squadron are all strict$_{\mathrm{T}}$ classical. Note, moreover, that they all provide K or X; indeed, by Lemmas 3.1(2) and 4.1, they all provide K. Therefore, by Lemma 4.3 the squadron's logics are all strict classical. This means that we may adopt—as we hereby do—RRSE in place of RRSE$_{\mathrm{T}}$ or RSE$_{\mathrm{T}}$.

Furthermore, Lemmas 3.1(4) and 4.1 together tell us that each logic in the squadron provides the rule RN$_*$. So we are also free to use this simpler rule instead of SMP in axiomatizing **S1**$^\circ$ and **S0.9**$^\circ$. Indeed, if we stipulate RN$_*$ in each of the four axiomatizations, we can eliminate the schemata K, X, and T throughout the formulations and make do with just their necessitations, $\Box$K, $\Box$X, and $\Box$T.

This leads to the following axiomatizations: each of the squadron's logics has PL, MP, N, RRTE, RRSE, and RN$_*$; the "**1**-logics" have $\Box$X where the "**0.9**-logics" have $\Box$K; and the "$^\circ$-less" logics in addition have $\Box$T.[24] To help to keep track of the squadron's logics as thus formulated we offer the chart in Figure 2, in which a logic's more inclusive relatives are reached by traveling directly upward or rightward.

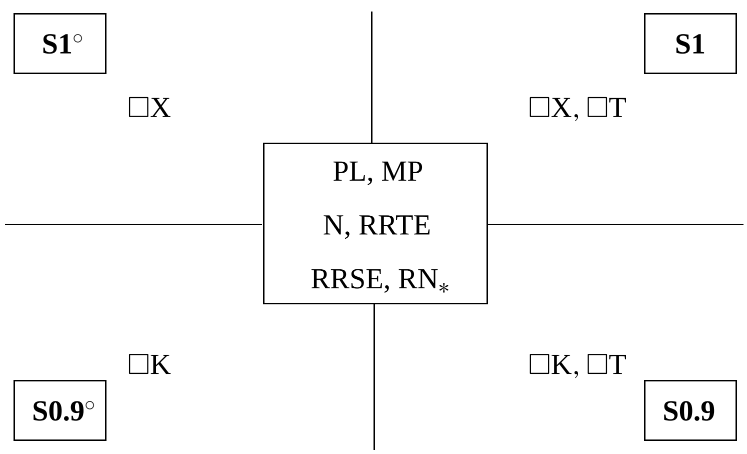

Figure 2

11 Lewis logics and the logics of the squadron We are nearly ready to demonstrate the identities advertised at the beginning of Section 10. The result appears as Theorem 11.2, the way to which is paved by the following:

Lemma 11.1 *If A is a theorem of* **PK, PKT, PX**, *or* **PXT**, *then $\Box A$ is a theorem respectively of* **S0.9**$^\circ$,**S0.9**, **S1**$^\circ$, *or* **S1**.

Proof: For the purposes of the proofs of this and the theorem to follow we write **SL** for a logic in the squadron (**S0.9**$^\circ$, **S0.9**, **S1**$^\circ$, **S1**) and **L** for the corresponding logic in the "presquadron" (**PK, PKT, PX, PXT**). Let **L** be one of the latter. As the defining axiom nK for **L** is redundant, we take **L** to be axiomatized by PL, all instances of the relevant schemata (K, X, T), and the rules MP and RRE.

Suppose that A is a theorem of **L** and so appears on a line in a proof in the axiom system for the logic. We argue inductively.

Case 1: A is a tautology, an instance of X, or an instance of T. Recall that **SL** has $\Box$PL and, as axioms, $\Box$X and $\Box$T.

Case 2: A is inferred by modus ponens from previous formulas B and $B \to A$. By the inductive hypothesis the necessitations of both of these are theorems of **SL**. Since **SL** provides K, it follows by two uses of MP in **SL** that $\Box A$ is a theorem.

Case 3: A has the form $B^C/_D$ and is inferred by RRE from earlier formulas $C \leftrightarrow D$ and B. By the inductive hypothesis, $\Box(C \leftrightarrow D)$ and $\Box B$ are **SL** theorems. Hence the rule RRSE applies, so $(\Box B)^C/_D$, i.e., $\Box(B^C/_D)$, i.e., $\Box A$ is also a theorem of **SL**. $\Box$

Theorem 11.2 **Lew(PK)** = **S0.9**$^\circ$, **Lew(PKT)** = **S0.9**, **Lew(PX)** = **S1**$^\circ$, *and* **Lew(PXT)** = **S1**.

Proof: Again let **L** be one of **PK, PKT, PX**, and **PXT**. Suppose that A is a thesis of **Lew(L)**. Then **L** has $\Box\top \to A$ (Lemma 9.1). By Lemma 11.1, **SL** has $\Box(\Box\top \to A)$ as a thesis. So by RN$_*$, $\Box\top \to A$ is also a thesis of **SL**, and hence so is A, since N is. This shows that **Lew**(**L**) $\subseteq$ **SL**. To prove the converse it is enough to note that **Lew(L)** includes PL and is closed under MP (by definition); that **Lew(L)** contains all instances of the relevant axiom schemata (by Lemma 9.1(2), since their "denecessitations" are all in **L**); and that **Lew(L)** provides $\Box$PL, RRSE, and RN$_*$ (by Lemma 9.1(3)). $\Box$

Determination and decidability for **S0.9**$^\circ$, **S0.9**, **S1**$^\circ$, and **S1** follow from Theorems 9.3, 9.5, and 11.2. Credit for completeness, finite model property, and decidability of **S1**$^\circ$ and **S1** goes to Shukla [19] and independently Cresswell [4]. The proofs in those papers are algebraic; the arguments in Cresswell's recent [5] use canonical models.

The distinctness of the logics of the squadron is part of Theorems 9.4 and 11.2. The fact that **S0.9** $\neq$ **S1** is worth a comment. The question was raised and left open by Lemmon in [11] and remained open until Girle gave the answer in [7]. However, Scott had an earlier, unpublished proof that he presented in lectures at Stanford University in 1967. The details of this proof seem to have been lost, but a simple proof is easily constructed, given our results.

Consider the frame (U, Q, R, S) in which $U = \{0, 1, 2\}$, $Q = \{2\}$, $R = \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$ and $S(2) = \mathcal{P}U - \{U\}$. We wish to argue that (i) this frame validates all theses of **S0.9**, yet (ii) the formula $\Box((\Box(A \to B) \land \Box(B \to A)) \to \Box(A \to A))$, where A and B are distinct atomic formulas, fails at 2 under any valuation V such that $V(A) = \{0\}$ and $V(B) = \{1\}$.

That (i) holds follows from the fact that this frame has the property (qk) and is omnireflexive. To see that the former claim is true, suppose that X and Y are subsets of U such that both $(U - X) \cup Y$ and X are elements of $S(2)$. If $Y \notin S(2)$, then $Y = U$, contradicting the assumption that $(U - X) \cup Y \in S(2)$; hence $Y \in S(2)$.

To establish (ii), let V be any valuation in U such that $V(A) = \{0\}$ and $V(B) = \{1\}$. Then $\|A \to B\| = (U - \|A\|) \cup \|B\| = \{1, 2\} \cup \{1\} = \{1, 2\}$ and $\|B \to A\| = (U - \|B\|) \cup \|A\| = \{0, 2\} \cup \{0\} = \{0, 2\}$, while $\|A \to A\| = \{0, 1, 2\}$. Consequently, $\Box(A \to B)$ and $\Box(B \to A)$ are true at 2 in the model defined by V, whereas $\Box(A \to A)$ is not; hence (ii) holds.

An eight-valued matrix of the kind Lemmon sought in [11] can be extracted from this example. As truth values take 1, 2, 3, 4, 5, 6, 7, and 8, with 1 as sole designated value. The truth tables of, for example, conjunction, negation and necessity are as in Figure 3.

$\wedge$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	2	5	6	5	6	8	8
3	3	5	3	7	5	8	7	8
4	4	6	7	4	8	6	7	8
5	5	5	5	8	5	8	8	8
6	6	6	8	6	8	6	8	8
7	7	8	7	7	8	8	7	8
8	8	8	8	8	8	8	8	8

	$\neg$
1	8
2	7
3	6
4	5
5	4
6	3
7	2
8	1

	$\Box$
1	4
2	2
3	3
4	1
5	5
6	6
7	2
8	8

Figure 3

We conclude this section with the diagram in Figure 4, which gives us a certain perspective on the logics discussed so far. The convention again is that a logic's more inclusive relatives are upward or rightward. The structure shown is in fact a lattice if the meet of two logics is taken as their set-theoretical intersection and their join is taken as the smallest logic including both. By **C2**, first defined by Lemmon in [11], we understand the smallest regular logic, i.e., **R**, which can be viewed as the smallest prenormal logic to contain all instances of the schema $\Box A \to \Box\top$. (This says in effect that all necessitations are false at queer points in any model, something that secures the truth of all instances of the schema F throughout any model; so the logic is at least regular. Normality is still lacking, however, since $\Box\top$ remains false at queer points.)

By **E2**, also introduced in [11], we understand the smallest prenormal logic to extend **C2** and provide T. It is easy to show, by the safe extension technique Lemmon demonstrated in [12], that **C2** and **E2** provide the rule RnN_*. Therefore, as Lemmon noted, **S2**$^\circ$ = **Lew(C2)** and **S2** = **Lew(E2)**; also see [10].

12 Cresswell logics No doubt many logicians, frustrated in their efforts to understand the system **S1**, would be tempted to agree with Cresswell's opinion in [3], repeated in [5], that it is "on almost any account a very silly system."[25] This view of the man to whom we owe what insight into **S1** there is cannot be taken lightly. But whether or not **S1** can be defended on philosophical grounds, it is not without interest from a formal point of view, as we hope this section will demonstrate.

The fruitful new semantic condition on a frame (U, Q, R, S) that Cresswell introduced in his analysis of **S1** was

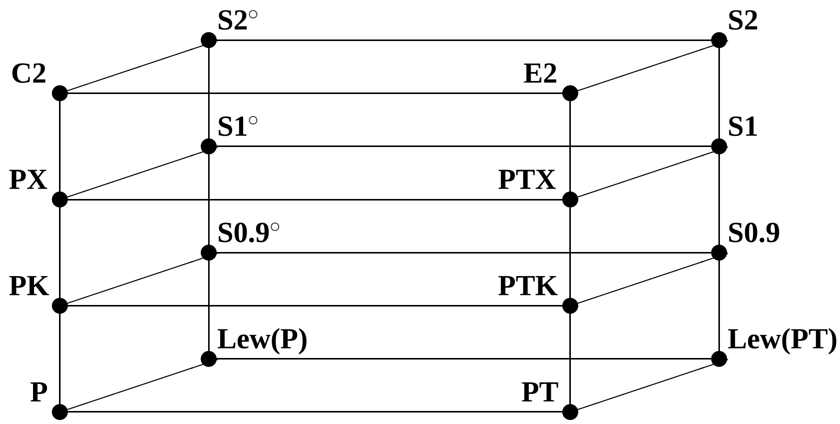

Figure 4

(c$_2$) $\qquad x \in Q \Rightarrow (X, Y \in S(x) \Rightarrow X \cup Y \neq U).$[26]

Cresswell proved, in effect, that **S1** is Lewis-determined by the class of omnireflexive frames that satisfy this condition. Since conditions (qx) and (c$_2$) are equivalent, it follows by Theorem 7.2 that any logic determined by a class of frames satisfying (c$_2$) is prenormal.

The condition (c$_2$) is a special case of the following more general condition:

(c$_n$) $\qquad x \in Q \Rightarrow (X_0, \ldots, X_{n-1} \in S(x) \Rightarrow X_0 \cup \cdots \cup X_{n-1} \neq U) \quad (n \geq 2).$

We shall say that $X_0, \ldots, X_{n-1} \subseteq U$ *cover* U if $X_0 \cup \cdots \cup X_{n-1} = U$; in that case $X_0 \cup \cdots \cup X_{n-1}$ is said to be an *n-element covering*. Thus (c$_n$) expresses the condition that $S(x)$ possesses no n-element covering. In particular, Cresswell's condition is that no neighborhood system for a queer point has a 2-element covering. We note the following fact.

Lemma 12.1 *(c$_m$) implies (c$_n$) if and only if $m \geq n$.*

Thus for $n \geq 2$ any logic determined by a class of frames satisfying (c$_n$) is prenormal.

In order to study the covering conditions we introduce a family of rules each of which we refer to as a *Cresswell rule*:[27]

(RC$_n$) $\qquad A_0 \vee \cdots \vee A_{n-1} \,/\, (\Box A_0 \wedge \cdots \wedge \Box A_{n-1}) \rightarrow \Box\top \quad (n \geq 2).$

In terms of Cresswell rules we have a syntactic counterpart to Lemma 12.1, telling us that classical logics having these rules are always prenormal:

Lemma 12.2 *A logic that provides* RC$_n$ *also provides* RC$_m$*, for every $m \leq n$.*

Proof: Assume that we have a logic providing RC$_n$, and that $A_0 \vee \cdots \vee A_{m-1}$ is a thesis. To ignore trivial cases, say that $m < n$. By PL, $A_0 \vee \cdots \vee A_{m-1} \vee B$ is also a thesis, where B is a disjunction of $n - m$ occurrences of A_{m-1}. By RC$_n$ we have the thesis $(\Box A_0 \wedge \cdots \wedge \Box A_{m-1} \wedge C) \rightarrow \Box\top$, where C is the $(n-m)$-termed conjunction of the formula $\Box A_{m-1}$. But $C \leftrightarrow \Box A_{m-1}$ is a tautology. So by PL, $(\Box A_0 \wedge \cdots \wedge \Box A_{m-1}) \rightarrow \Box\top$ is a thesis, as we wished to show. $\Box$

The Cresswell rules correspond closely to the covering conditions as the following results show.

Lemma 12.3 *If a frame $\mathcal{F}$ satisfies* (c_n)*, then the logic* $\mathbf{L}(\mathcal{F})$ *provides* RC_n.

Proof: Assume that $\mathcal{F} = (U, Q, R, S)$ satisfies (c_n). Suppose that $A_0 \vee \cdots \vee A_{n-1}$ is a thesis of $\mathbf{L}(\mathcal{F})$ and therefore is valid in $\mathcal{F}$. Let $\mathcal{M}$ be any model on $\mathcal{F}$. Suppose that x is an element of U such that $\mathcal{M} \models_x \Box A_i$, for every $i < n$. We wish to prove that $\mathcal{M} \models_x \Box\top$. If x is normal this is trivial. If x is queer, then $\|A_i\| \in S(x)$, for every $i < n$. By (c_n), $\|A_0\| \cup \cdots \cup \|A_{n-1}\| \neq U$. But the validity of $A_0 \vee \cdots \vee A_{n-1}$ implies that $\|A_0 \vee \cdots \vee A_{n-1}\| = U$. This is impossible, so the case that x is queer cannot arise. □

Lemma 12.4 (c_n) *is satisfied by the smallest canonical frame for a classical logic that provides* RC_n.

Proof: Let $\mathcal{M} = (U, Q, R, S, V)$ be a canonical model for a classical logic that provides RC_n, and suppose that (c_n) is not satisfied by the frame of $\mathcal{M}$. Then there is a point $x \in Q$ and sets $X_0, \ldots, X_{n-1} \subseteq U$ that are elements of $S(x)$. This means that there are formulas $A_0, \ldots, A_{n-1}$ such that:

$$|A_0|, \ldots, |A_{n-1}| \in S(x). \tag{1}$$

$$|A_0| \cup \cdots \cup |A_{n-1}| = U. \tag{2}$$

From (2) it follows that $|A_0 \vee \cdots \vee A_{n-1}| = U$, so $A_0 \vee \cdots \vee A_{n-1}$ is a thesis of the logic, whence by RC_n

$$(\Box A_0 \wedge \cdots \wedge \Box A_{n-1}) \to \Box\top \text{ is a thesis.} \tag{3}$$

But (1) implies that $\Box A_0, \ldots, \Box A_{n-1} \in x$. By (3), then, $\Box\top \in x$. This contradicts the assumption that x is queer. Thus condition (c_n) is satisfied. □

Prenormal logics closed under RC_n we shall call *Cresswell logics*. Define $\mathbf{PC}_n$ as the smallest Cresswell logic to provide RC_n, and $\mathbf{PC}_n\mathbf{T}$ as the smallest Cresswell logic to have both RC_n and T. The following is an immediate consequence of Lemmas 12.3 and 12.4.

Theorem 12.5 $\mathbf{PC}_n$ *is determined by the class of frames satisfying* (c_n)*, and* $\mathbf{PC}_n\mathbf{T}$ *is determined by the class of omnireflexive frames that satisfy* (c_n).

Of course one corollary to Theorem 12.5 we have had for some time, given the equivalence of qX and RqX. But it is worth noting formally.

Theorem 12.6 $\mathbf{PC}_2 = \mathbf{PX}$ *and* $\mathbf{PC}_2\mathbf{T} = \mathbf{PXT}$*, and so* $\mathbf{Lew}(\mathbf{PC}_2) = \mathbf{S1}^\circ$ *and* $\mathbf{Lew}(\mathbf{PC}_2\mathbf{T}) = \mathbf{S1}$.

Thus **PX** and **PXT** are Cresswell logics.[28]

Rules are usually more difficult to work with than schemata, and therefore Theorem 12.6 is welcome. Indeed it suggests there are many schemata with the same effect as the Cresswell rules; we shall now proceed to make good this claim.[29] The schemata we have in mind are the members of the following family:

$$(\mathrm{X}_n) \qquad (\Box(A \to (B_1 \wedge \cdots \wedge B_n)) \wedge \Box(B_1 \to C_1) \wedge \cdots \wedge \Box(B_n \to C_n)) \to \Box\top.$$

(Here we begin the numbering with 1.)

Theorem 12.7 *For each $n \geq 1$, a classical logic provides* RC_{n+1} *if and only if it provides* X_n.

Proof: Assume that the logic is classical and suppose, first, that it provides RC_{n+1}. Then it also provides X_n, inasmuch as

$$(A \to (B_1 \wedge \cdots \wedge B_n)) \vee (B_1 \to C_1) \vee \cdots \vee (B_n \to C_n)$$

is a tautology. For the reverse, suppose that the logic provides X_n. Suppose that

$$A_0 \vee \cdots \vee A_n \tag{1}$$

is a thesis. For $i \leq n$ let $\bigwedge_{j \neq i} \neg A_j$ be the conjunction in some order of all formulas $\neg A_j (j \leq n)$ except $\neg A_i$. Then it is easy to see the following tautological consequences of (1):

$$A_i \leftrightarrow (\neg A_i \to \bigwedge_{j \neq i} \neg A_j). \tag{2}$$

From (2) by RRE the logic has these theses:

$$\Box A_i \leftrightarrow \Box(\neg A_i \to \bigwedge_{j \neq i} \neg A_j). \tag{3}$$

Next we have an instance of X_n:

$$(\Box(\neg A_0 \to (\neg A_1 \wedge \cdots \wedge \neg A_n)) \wedge \Box(\neg A_1 \to \bigwedge_{j \neq 1} \neg A_j) \wedge \cdots \wedge \Box(\neg A_n \to \bigwedge_{j \neq n} \neg A_j)) \to \Box\top. \tag{4}$$

Hence from (3) and (4) by PL we arrive at the desired thesis:

$$(\Box A_0 \wedge \cdots \wedge \Box A_n) \to \Box\top. \tag{5}$$

□

As a corollary to Theorem 12.7 we have the following.

Theorem 12.8 $\mathbf{PC}_{n+1} = \mathbf{PX}_n$ *and* $\mathbf{PC}_{n+1}\mathbf{T} = \mathbf{PX}_n\mathbf{T}$.

The first two members of the X_n family are:

$$(\mathrm{X}_1) \qquad (\Box(A \to B) \wedge \Box(B \to C)) \to \Box\top$$
$$(\mathrm{X}_2) \qquad (\Box(A \to (B_0 \wedge B_1)) \wedge \Box(B_0 \to C_0) \wedge \Box(B_1 \to C_1)) \to \Box\top.$$

Schema X_1 throws a certain light on **S1**, for it follows from Theorems 12.6 and 12.8 that **S1** can be axiomatized by replacing schema $\Box\mathrm{X}$ in our axiomatization of that logic by the schema $\Box\mathrm{X}_1$. It is noteworthy that in his 1967 lectures Scott showed that replacing $\Box\mathrm{X}$ by $\Box\mathrm{X}_1'$ will also yield **S1**, where X_1' is

$$(\Box(A \to B) \wedge \Box A) \to \Box\top$$

—a schema that is deductively equivalent in classical modal logics to an instance of X_1. Another example of a schema that does the job is:

$$(\Box(A \to B) \land \Box(B \to A)) \to \Box\top.$$

It is interesting that **S1** can be axiomatized in such different ways.

In [4] Cresswell remarks that "Lewis's reasons for arriving at **S1** have only an accidental connection with its formal structure."[30] Perhaps in hindsight one may say that we now finally understand why **S1** proved so difficult to understand. Lewis happened to hit upon a schema whose structure leads one to suppose that **S1** has something to do with strict implication. In fact it does not, or at least if it does, it does so indirectly and opaquely. Schemata such as X do little to suggest their essential connection with the covering condition (c_2); it took the ingenuity of Cresswell to uncover that.

If Lewis was attracted to **S1** as a logic of strict implication because it provided the schema

$$((A \multimap B) \land (B \multimap C)) \multimap (A \multimap C),$$

where $\multimap$ stands for strict implication, one wonders whether he would also have been attracted to the schema

$$((A \multimap (B_0 \land B_1)) \land (B_0 \multimap C_0) \land (B_1 \multimap C_1)) \multimap (A \multimap (C_0 \land C_1)),$$

or, more generally, to

$$((A \multimap (B_0 \land \cdots \land B_{n-1})) \land (B_0 \multimap C_0) \land \cdots \land (B_{n-1} \multimap C_{n-1})) \\ \multimap (A \multimap (C_0 \land \cdots \land C_{n-1})).$$

In our view these schemata are not much less plausible, from an intuitive point of view, than the defining schema of **S1**. But they are not provided by **S1**, as simple semantic arguments show. If Lewis had pursued this direction in his inquiries he would have been led to the definition of a whole family of new logics, viz., **Lew(PC**$_{n+1}$**T)** = **Lew(PX**$_n$**T)**. In Figure 5 we present a diagram where these logics appear with their relatives **PX**$_n$**, PX**$_n$**T**, and **Lew(PX**$_n$**)**. For each a completeness result exists along the lines drawn earlier. Thus **PX**$_n$ is determined by the class of frames in which no neighborhood system possesses an n-element covering, and **PX**$_n$**T** is determined by the class of those frames that share this property but in addition are omnireflexive. The Lewis versions are of course Lewis-determined by the same classes.

In Figure 5 we have also included the logic **PX**$_\omega$ and its relatives. This logic, defined as the join of the logics **PX**$_n$ where $n < \omega$, is determined by the class of all frames in which no neighborhood system possesses a finite covering; thus it represents a limiting case—the ultimate Cresswell logic. It is included in Lemmon's **C2** but not conversely: as noted in Section 11, **C2** (i.e., the smallest regular logic, **R**) provides the schema $\Box A \to \Box\top$; but this not in **PX**$_\omega$ or its relatives.

In closing let us remark that a safe extension of a model satisfies the condition (c_n) whenever the model does; the reasoning generalizes that in the proof of Lemma 8.2. It follows from this that the Cresswell logics **PX**$_n$ and **PX**$_n$**T** all provide the rule RnN_*.

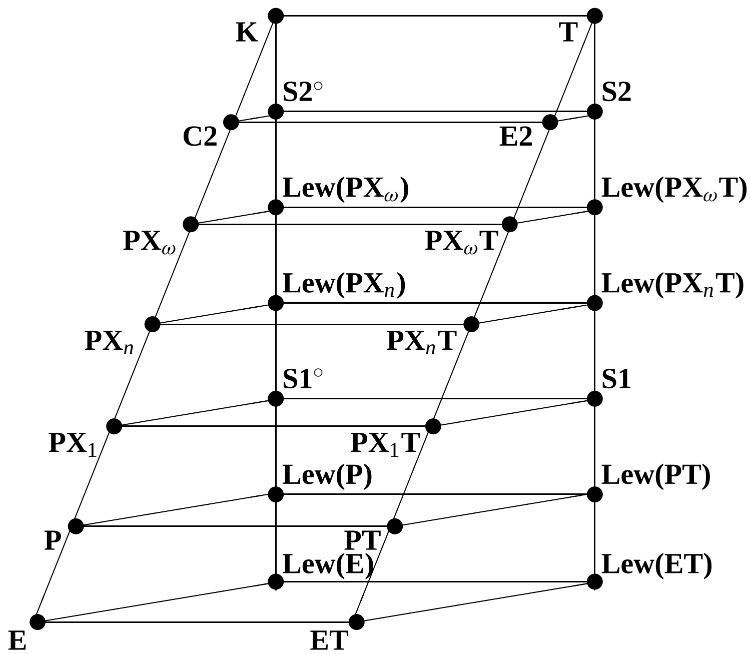

Figure 5

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We dedicate our paper to Dana Scott, under whose guidance we wrote our dissertations more than twenty-five years ago, and who more than anyone else has influenced our thinking about modal logic.

NOTES

1. *Collective* appearance. Lewis formulated **S3** more than a decade earlier, in [13].
2. Some subsequent misgivings about Cresswell's results (see [17] and [21]) have been allayed by his [5].
3. For convenience' sake, throughout the paper the possibility operator $\Diamond$ is regarded as defined by $\neg\Box\neg$.
4. When the antecedent is devoid of conjuncts, a conditional $(A_1 \wedge \cdots \wedge A_m) \to B$ is identified with its consequent. In other cases empty conjunctions are identified with $\top$, and empty disjunctions with $\bot$.
5. So long as we have PL this rule can equally well be formulated: $A \leftrightarrow B \,/\, C \leftrightarrow C^A/_B$. Similarly for the other replacement rules in the paper, RRTE, RRSE, and RSE$_{\mathrm{T}}$.

6. Note that we might adopt a rule of necessitation for tautologies in this fashion: $/\ \Box A$—whenever $A \in$ PL. We prefer not to do so, since this would in effect be the "schema" $\Box$PL.

7. Lewis, Feys, and others define strict implication in terms of (im)possibility: $\neg\Diamond(A \wedge \neg B)$. But for all traditional authors strict implication is interchangeable with $\Box(A \to B)$ however it is defined.

8. A point noted, e.g., in [6], p. 45.

9. As we shall see, PL need not appear in the list of axioms for a logic. The rule of substitution is presumed for every logic we discuss in this paper, and so it will rarely be mentioned.

10. See, e.g., [2] for these and other identifications.

11. In this connection we may note a rule-alternative RqK to the schema qK in classical logics: $A \vee B, A \to (B \leftrightarrow C)\ /\ (\Box A \wedge \Box B) \to (\Box C \vee \Box \top)$.

12. A relational frame is a structure (U, R) in which U is a set and R is a binary relation in U. The truth condition for the operator $\Box$ with respect to a point x in a model $\mathcal{M}$ on such a frame is: $\mathcal{M} \models_x \Box A$ if and only if $\forall y \in U(\langle x, y\rangle \in R \Rightarrow \mathcal{M} \models_y A)$. The class of all relational frames determines **K**, the smallest normal logic. Neighborhood frames are structures (U, S) with U as before, S a set $\{S(x) : x \in U\}$ such that $S(x) \subseteq \mathcal{P}U$, and truth condition: $\mathcal{M} \models_x \Box A$ if and only if $\|A\|^{\mathcal{M}} \in S(x)$. The smallest classical logic **E** is determined by the class of all neighborhood frames. For more on normal logics, see, e.g., [8], [9], [12], [2], or [18]; for classical logics, the last two of these.

13. It is noteworthy, and easily proved, that the syllogism property is equally well expressed: $x \in Q \Rightarrow (X, Y \in S(x) \Rightarrow X \cup Y \neq U)$.

14. The modus ponens and syllogism properties are so called because they say in effect that the class of propositions necessary at a point in effect obeys the rules of modus ponens and hypothetical syllogism, respectively. Actually, the properties say this only with respect to queer points, but we can be content with the unrefined names.

15. Note that $|A|_{\mathbf{L}} = |B|_{\mathbf{L}}$ whenever $A \leftrightarrow B$ is a thesis of **L**, since in that case A and B belong to the same maximal consistent sets in **L**. This fact secures the correctness of clause (iv) in the definition, to follow, of a canonical model.

16. Compare the notions of canonical model in [2], [12], and [18].

17. Particularly in the equivalent guise mentioned in Note 13. The proof for Lemma 12.4 contains a general argument.

18. Indeed, Theorem 7.2 itself follows from a generalization of Lewis's result due to Surendonk [20].

19. At least so long as the logic is finitely axiomatizable or the finite frames for it form a recursively enumerable class—properties satisfied by the prenormal KXT-logics.

20. The idea of safe extensions originated with Lemmon, in [12]. The results that follow should be compared to Theorem 3.5 in that volume.

21. See Note 19.

22. Lemmon omitted an axiom that had been shown to be redundant in [16]; cf. [6], p. 44.

23. [11], p. 180. Lemmon used $\mathrm{RSE_T}$ rather than $\mathrm{RRSE_T}$, it seems, because he was moving from the logic **P2** to **S0.9** by weakening "Becker's rule"—$\Box(A \to B)\ /\ \Box(\Box A \to \Box B)$—to $\mathrm{RSE_T}$.

24. For the sake of uniformity we retain the formulas in PL, though they are derivable via N, RRTE, and RN_*. It is perhaps worth mentioning that in the axiomatizations of the squadron's logics presented in Figure 1 RRSE cannot be replaced by RSE. Such replacement works, however, wherever the schema K is already derivable without the use of RRSE—i.e., in the "**0.9**-logics". This is because (as arguments like those for Lemmas 10.1 and 10.2 will show) a logic that provides K, N, RRTE, and RSE also provides RRSE. Thus if K were to be stipulated as an additional axiom in the "**1**-logics" RSE could replace RRSE in all the axiomatizations of the squadron's logics.

25. The reference in [3] is n. 3 on p. 199.

26. This is the alternative to (qx) mentioned in Note 13.

27. The cases $n = 0$ and $n = 1$ are worth noting, if only to dispose of them. RC_0 is trivial, holding vacuously in every logic, and RC_1 says that $\Box A \to \Box\top$ is a thesis whenever A is, which holds in every classical logic. Thus only for $n \geq 2$ do we get something interesting (RC_2 is of course the rule RqX, which figures in Lemma 5.3). Note in this connection that the condition (c_n) is similarly uninteresting when $n < 2$.

28. This is perhaps a good place to note a condition equivalent to (qk): if $U \notin S(x)$, $X \cup Y = U$, and $(U - X) \cup Y = (U - X) \cup Z$, then $X, Y \in S(x)$ only if $Z \in S(x)$. Compare the rule RqK mentioned in Note 11.

29. The general issue we touch on here is of considerable interest: when is it possible to express a rule by an axiom schema? To the best of our knowledge there are only relatively few particular answers to this question and no general one.

30. [4], p. 495, n. 4.

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