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Cumulative versus Noncumulative Ramified Types

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Abstract In this paper I examine the nature of Russell's ramified type theory resolution of paradoxes. In particular, I consider the effect of construing the types in Church's *cumulative* sense, that is, the range of a variable of a given type includes the range of every variable of directly lower type. Contrary to what seems to be generally assumed, I show that the decision to make the levels cumulative and allow this to be reflected in the semantics is *not* neutral with respect to the solution of the paradoxes. I introduce a distinction between *syntactical* and *semantical* cumulativeness. It turns out that noncumulative type theories (in either sense) are equally capable of dealing with the paradoxes. Furthermore, whether cumulativeness is appropriate appears to be context dependent.

1 Introduction In this paper I examine the nature of Russell's ramified type theory resolution of paradoxes. In particular, I consider the effect of construing the types in the *cumulative* sense of Church [1], that is, the range of a variable of a given type includes the range of every variable of directly lower type. Contrary to what seems to be generally assumed, I show that the decision to make the levels cumulative and allow this to be reflected in the semantics is *not* neutral with respect to the solution of the paradoxes. I introduce a distinction between *syntactical* and *semantical* cumulativeness. It turns out that noncumulative type theories (in either sense) are equally capable of dealing with the paradoxes. Furthermore, whether cumulativeness is appropriate appears to be context dependent.

I begin by presenting Russell's ramified type theory, for the most part following Church's reconstruction [1]. Next I examine how Church's (cumulative) theory deals with a semantical paradox (the Grelling) and a nonsemantical paradox (the Bouleus), followed by an analysis of the role of the cumulativeness assumptions in each. Finally, I discuss the issue of cumulativeness with respect to Russell's more general needs in *Principia Mathematica*.

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2 Russell's ramified type theory Russell's ramified theory of types (r-*types*) is intended to preclude the possibility of paradox by prohibiting *impredicative* definitions. Precisely what one should understand by "impredicative" is problematic; Russell was not of one voice in his diagnosis of the paradoxes. What is clear, however, is what ramified types *do*: they systematically partition the domains of quantification in a way that prevents quantification from "generating" new elements into the domain of quantification.[1]

The ramified types are defined recursively. Individual variables belong to r-type i and (by stipulation) have order 0. If $m \geq 0$, $n \geq 1$, and $\beta_1, \beta_2, \ldots, \beta_m$ are r-types, then $(\beta_1, \beta_2, \ldots, \beta_m)/n$ is an r-type to which belong m-ary functional variables of level n. Propositions are taken to be the 0-ary propositional functions.[2] Given a propositional function φ of r-type $(\beta_1, \beta_2, \ldots, \beta_m)/n$, the *order* of φ is defined recursively to be $N + n$, where N is the greatest of the orders of the types $\beta_1, \beta_2, \ldots, \beta_m$ (and $N = 0$ if $m = 0$). The *level* of φ is n, which is related to Russell's (nonsyntactic) notion of a *bound* variable in the following way: if N is the greatest of the orders of $\beta_1, \beta_2, \ldots, \beta_m$ and k is the greatest of the orders of the bound variables occurring in φ, then $n = 1$ if $k \leq N$, and $n = k + 1$ if $k > N$. In Russell's terminology, φ is *predicative* if and only if $n = 1$.[3]

The r-type notation will be abbreviated as follows. The number m will stand for $(i, i, \ldots, i)$, where there are m i's between the parentheses. So for example, $(\,)/n$ is abbreviated as $0/n$, $(i, i, i)/n$ as $3/n$, and $((i)/2, (\,)/2)/1$ as $(1/2, 0/2)/1$. One may take the levels and orders to be *cumulative* in the sense that the range of a variable of r-type $(\beta_1, \beta_2, \ldots, \beta_m)/n$ is taken to include the range of every variable *directly lower*, that is, every variable with r-type of the form $(\alpha_1, \alpha_2, \ldots, \alpha_m)/k$, where $k < n$, and $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_m = \beta_m$. For now I will follow Church's "convention" of understanding r-types as being cumulative in this *syntactical* sense. Church's rationale [1], p. 289 for construing r-types in this way is that it "facilitate[s] comparison with both the simple types and . . . Tarski's resolution of the semantical antinomies." It does *more* than this, as I argue below; it actually *affects* the answer given by the type theory to the paradox-generating questions.

In Church's syntactically cumulative system, a propositional variable (that is, one of r-type $0/n$) is a well-formed formula when standing alone. A formula $\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ is well-formed if and only if $\mathbf{f}$ is a variable or constant of r-type $(\beta_1, \beta_2, \ldots, \beta_m)/n$, where $m > 0$ and $\mathbf{x}_j$ is a variable or constant whose r-type is β_j or *directly lower* than β_j, for $j = 1, 2, \ldots, m$. In addition to an infinite alphabet of variables in each r-type, the primitive symbols include a (unspecified) list of primitive constants, each of definite r-type, the usual notation for negation, disjunction, and the universal quantifier. The following standard formation rules are assumed: given that $\mathbf{P}$ and $\mathbf{Q}$ are well-formed and $\mathbf{a}$ is a variable, then $\neg\mathbf{P}$, $[\mathbf{P} \vee \mathbf{Q}]$, and $(\mathbf{a})\mathbf{P}$ are well-formed. A comprehension axiom (for both propositional variables and functional variables) that reflects the orders of the variable is also assumed. In particular,

$$(\exists \mathbf{p})[\mathbf{p} \equiv \mathbf{P}], \ \ \mathbf{p} \text{ not free in } \mathbf{P},$$

where $\mathbf{p}$ is a propositional variable of r-type $0/n$, the bound variables of $\mathbf{P}$ are all of order less than n, and the free variables and constants of $\mathbf{P}$ are all of order not greater than n, and

$$(\exists \mathbf{f})(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)[\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) \equiv \mathbf{P}],\ \mathbf{f} \text{ not free in } \mathbf{P},$$

where **f** is a functional variable of r-type $(\beta_1, \beta_2, \ldots, \beta_m)/n$, and the $\mathbf{x}_j$ are distinct variables of r-type β_j ($j = 1, 2, \ldots, m$), and the bound variables of **P** are all of order less than the order of **f**, and the free variables of **P** (including perhaps some or all of the $\mathbf{x}_j$) and the constants occurring in **P** are all of order not greater than the order of **f**.

Finally, a standard set of rules and axioms for propositional calculus and quantifier logic will be assumed including the following rules of inference: *modus ponens*, the laws of propositional calculus, the deduction theorem, and the following rules.

Universal instantiation: from (**a**)**P** to infer the result of substituting **b** for all free occurrences of **a** throughout **P**, where **a** is a variable, **b** is a variable or constant that is either of the same r-type as **a** or of r-type *directly lower* than **a**.

Existential generalization: from **Q** to infer ($\exists$**a**)**P**, where **Q** is the result of substituting **b** for all free occurrences of **a** throughout **P**, and where **a**, **b**, and **P** satisfy the same conditions laid out for UI.

Existential instantiation: from **P**, **Q** $\vdash$ **S** to infer **P**, ($\exists$**a**)**Q** $\vdash$ **S**, where the variable **a** is free only in **Q**.

I note for future reference that the syntactical cumulativeness of this system is reflected in the preceding formation, comprehension, and inference rules in that the relevant restrictions on r-types do not require *identical* r-types, but rather require identical or *directly lower* r-types.

3 Grelling's paradox Grelling's paradox has to do with *adjectives*. An adjective is called autological if it has the property it expresses; if not, it is called heterological. For example, 'polysyllabic' is autological, and 'long' is heterological. The paradox-generating question is whether the adjective 'heterological' is autological or heterological.

In the formal language, the individuals that are closest to adjectives are propositional forms with one free variable. Thus the formal language analog of an *adjective* expressing a *property* will be that the *propositional form* have a *value* for each value of its free variable. An infinite list of primitive constants val^2, val^3, val^4, ... will be introduced to formally capture this idea.

The term $val^{n+1}(a^i, v^i, F^{1/n})$ should be understood as stating that the *value* of v^i is $F^{1/n}(x^i)$ for every value x^i of the variable a^i, where a^i is an individual variable and v^i is a propositional form (well-formed formula) having a^i as its only free variable. More intuitively, the idea is that $val^{n+1}(a^i, v^i, F^{1/n})$ means that the well-formed formula v^i with a^i as its only free (individual) variable *designates* the propositional function $\lambda x F^{1/n}(x)$.

All of the symbols and expressions of (whatever version of) the ramified type theory are assumed to be "individuals," that is, objects of r-type i. The constants val^{n+1} will be taken to be of level 1. Thus the r-type of val^{n+1} is $(i, i, 1/n)/1$, and $val^{(i,i,1/n)/1}$ will be abbreviated by val^{n+1}. The order of val^{n+1} is $n + 1$.

Based on this, the following postulates seem reasonable. First there is the principle of univocacy in its weak (extensional) form:

Postulate 3.1 $val^{m+1}(a, v, F^{1/m}) \longrightarrow [val^{n+1}(b, v, G^{1/n}) \longrightarrow (x)(F(x) \longleftrightarrow G(x))]$.

Next there is an *extensional* postulate schema which can be informally verified (for each **P**) by taking v to be the propositional form **P** and a to be the individual variable x:

Postulate 3.2 (schema) $(\exists a)(\exists v)(\exists F^{1/n})[val^{n+1}(a, v, F) \wedge (x)(F(x) \longleftrightarrow \mathbf{P})]$, *where* **P** *is a well-formed formula with no free variable other than 'x', with all of the bound variables of order less than n, and constants not greater than order n.*

The following postulate allows the constants val^{n+1} to reflect the cumulative character of our r-typed variables.[4]

Postulate 3.3 $val^{n+1}(a, v, F^{1/n}) \longrightarrow val^{m}(a, v, F^{1/n})$, *where* $m > n + 1$.

The analog of the word 'heterological' in the informal version of Grelling's paradox may then defined as

Definition 3.4 $het^{n+1}(v) =_{\text{df}} (\exists a)(\exists F^{1/n})[val^{n+1}(a, v, F) \wedge \neg F(v)]$.

The following three theorems may then be proven.

Theorem 3.5 $het^{n+1}(v) \longrightarrow het^{m+1}(v)$, *if* $m \geq n$.

Theorem 3.6 $[val^{m+2}(a, v, G^{1/m+1}) \wedge (x)(G(x) \longleftrightarrow het^{m+1}(x))] \longrightarrow \neg het^{n+1}(v)$, *if* $m \geq n$.

Theorem 3.7 $[val^{m+2}(a, v, G^{1/m+1}) \wedge (x)(G(x) \longleftrightarrow het^{m+1}(x))] \longrightarrow het^{n+1}(v)$, *if* $m < n$.

The proofs of these theorems are straightforward; I will not present them here. (See [1], pp. 296–97, for analogous proofs.) It is important to note, however, that (3.3) is needed to prove each of these three theorems. Finally, using (3.2) with $het^{m+1}(x)$ as the propositional form **P** we get the following.

Theorem 3.8 $(\exists a)(\exists v)(\exists G^{1/m+1})\ [val^{m+2}(a, v, G) \wedge (x)(G(x) \longleftrightarrow het^{m+1}(x))]$.

To see how the contradiction is produced in the simple type theory, simply disregard the r-type indicators: the constants val^{n+1} collapse into a single constant, Postulates 3.1 and 3.2 remain evident, Postulate 3.3 becomes a tautology, and the proofs of Theorems 3.5 – 3.8 still go through. Theorems 3.6 – 3.8 then constitute a contradiction. With the r-types in place, however, (3.6) – (3.8) do not constitute a contradiction. Moreover the ramified theory of types also provides an *answer* to the question of whether the propositional form $het^{m+1}(x)$ is autological or heterological. The answer is that by (3.6) it is autological for all orders $n \leq m+1$, and by (3.7) it is heterological for all orders $n > m+1$.[5]

4 The Grelling resolution: cumulativeness As I have claimed above, supplementing Russell's r-type theory with cumulative levels/orders affects the solution the theory offers. In order to see this I will consider how a *noncumulative* ramified theory can (if in fact it can) deal with Grelling's paradox. The Grelling discussion suggests two senses in which a type theory may be cumulative. The first is a *syntactical* sense: the range of a variable of a given type includes the range of every variable of directly lower type. Cumulativeness in this sense was introduced (by Church) with the *syntax* of the ramified type theory. The second sense is a *semantical* sense: the constants val^{n+1} are such that if they are true of a form/function pair of a given order, then they are true for all higher orders. Cumulativeness in this second sense was introduced in (3.3), which characterized the cumulative nature of the *semantical* constants val^{n+1}.

The syntactical sense is more fundamental in the following way. While syntactical cumulativeness may be seen as a motivation for introducing semantical cumulativeness, it would still be consistent to retain syntactical cumulativeness and reject semantical cumulativeness, that is, simply dropping (3.3) would accomplish this. On the other hand, semantical cumulativeness without syntactical cumulativeness makes no sense at all—(3.3) is not well-formed without syntactical cumulativeness.

Consider first the effect of dropping only the *semantical* cumulativeness of the type theory, which was introduced by (3.3). The effect of omitting (3.3) on (3.5)–(3.7) is as follows. (3.5) which held for $m \geq n$ now can be seen directly to hold (trivially) only for $m = n$. Call this version (3.5′). (3.6) which also held for $m \geq n$ and whose proof made use of (3.5), still goes through for $m = n$. Call this version (3.6′). (3.7) which held for $m < n$ and whose proof used (3.5) has no restricted version directly provable from (3.5′). Notice first that the (semantically) noncumulative type theory still prevents the direct attempt to restore the paradox, since as before it cannot be (directly) shown that for $m = n$

$$(4.1) \qquad [val^{m+2}(a, v, G^{1/m+1}) \wedge (x)(G(x) \longleftrightarrow het^{m+1}(x))] \longrightarrow het^{n+1}(v).$$

Now consider what this version yields as an answer to the question of whether the propositional form $het^{m+1}(x)$ is autological or heterological. By (3.6′) $het^{m+1}(x)$ is autological for order $m+1$, while for orders $\neq m+1$ the theory does not provide an answer. This result agrees with the cumulative version for order $m+1$. We see that while accepting (3.3) is equivalent to accepting the answer given by the theory for orders $\neq m+1$, the answer the theory gives for order $m+1$ is independent of accepting (3.3).

If cumulativeness is dropped altogether, it is easily seen that type theory still saves us from directly producing the contradiction, since again (4.1) cannot be (directly) shown for $m = n$. Even so, there is an intuitive difference lurking here between this (noncumulative) theory and the (merely) syntactically cumulative theory. The syntactically cumulative theory naturally urges a question to which it cannot (directly) provide an answer. In asking whether $het^m(x)$ is n-heterological ($m \neq n$), one naturally follows the reasoning backward from (3.6)–(3.8) to (3.5), and finally to (3.3). Thus on the syntactically cumulative theory, one is naturally lead to ask (3.3) which the theory is unable to answer. On the noncumulative theory, however, the same line of questioning is not left open; one cannot ask (3.3), since it on this theory it is "ill-posed." In this sense, the noncumulative theory "disallows" the question of whether $het^m(x)$ is n-heterological ($m \neq n$), in contrast to the syntactically cumulative theory, which "allows" the question but fails to answer it.

So which of the three possibilities, (fully) noncumulative, partially (syntactically only) cumulative, or (fully) cumulative, deals most satisfactorily with the paradox? All three avoid the direct production of the paradox and provide an answer to whether $het^m(x)$ is heterological for order m. Based on other considerations *internal* to the paradox, however, the noncumulative version seems to be the most reasonable. The partially cumulative version is clearly inferior to the noncumulative for the reason discussed immediately above. The problem with (Church's original) cumulative version is that it provides what amounts to ad hoc answers to a myriad of questions that it unnecessarily proliferated in the first place by construing the r-types cumulatively. The answers provided for orders $n \neq m$ appear ad hoc since they follow directly for the otherwise unmotivated (3.3); whereas the answer provided for order m is independent of (3.3). The myriad of mixed-order questions are unhelpful because they constitute a further departure from the Grelling question as it naturally arises.

By its very nature, a formal (even more so a *typed* formal) solution to a natural language problem introduces elements that were not present in the original problem. So while it is not reasonable to reject such solutions merely because they go beyond the natural language, still it is reasonable to insist that they remain as close to the original problem as possible. Considerations of this kind favor the noncumulative solution. Introducing the typed notion of "heterological," $het^{m+1}(x)$, is already a step away from the natural language inspiration of the paradox, but to allow different orders at which $het^{m+1}(x)$ may or may not be heterological is to take yet another step away. The fully noncumulative solution "recognizes" the question of whether $het^{m+1}(x)$ is heterological only for its own order; it does not further proliferate "type questions" as do the cumulative versions. On the noncumulative version, we get one question; construe it at whatever order you will, it is still the same question with the same answer for the same reasons.

5 The Bouleus paradox The Bouleus paradox concerns *beliefs*.[6] Bouleus believes that some of his beliefs are mistaken, but all his other beliefs are in fact true, (except possibly for some that are logically implied by this belief together with his true beliefs). The paradox generating question is whether it is true that Bouleus is sometimes mistaken.

To formalize the paradox we will need the primitive constants, $bel^2, bel^3, \ldots$ where $bel^{n+1}(\mathcal{S}^i, P^{(0/n)})$ should be interpreted as meaning person $\mathcal{S}^i$ believes proposition $P^{(0/n)}$. It has r-type $(i, 0/n)/1$ which will be abbreviated as above. For the present, the *syntax* should be understood as cumulative. The formal rendition of the critical proposition P_0 is then

$$P_0^{n+1}(\mathcal{S}^i) =_{\text{df}} (\exists P^n)[bel^{n+1}(\mathcal{S}^i, P^n) \wedge \neg P^n], \tag{5.1}$$

for $n \geq 1$. To formalize the fact that the scope of the critical proposition P_0 concerns

Bouleus's *other* beliefs (excludes the critical belief itself), we have

$$P_0^{n+1}(\mathcal{S}^i) \longrightarrow (\exists P^n)[(P^n \neq P_0^m(\mathcal{S}^i)) \wedge bel^{n+1}(\mathcal{S}^i, P^n) \wedge \neg P^n], \tag{5.2}$$

where $n \geq 1$ and $m \geq 2$. For the present, I will assume that the maximum order of the *propositions* believed by Bouleus is finite; let N be this maximum.[7] Bouleus believes P_0^{n+1} for some n, hence the order of this proposition should be the maximum order N, since its scope is all of his other beliefs. To assert then that Bouleus believes this is

$$bel^{n+1}(\mathcal{B}, P_0^N), \tag{5.3}$$

where $\mathcal{B}$ stands for Bouleus. In the interest of readability, I will generally suppress (as I have here) the argument "$(\mathcal{B})$" in writing the propositional function $P_0^{n+1}(\mathcal{S}^i)$ evaluated at $\mathcal{B}$. Also, I will assume that Bouleus knows that he is Bouleus so that Bouleus's belief that $P_0^N(\mathcal{B})$ is a belief about *himself*. Finally the fact that his other beliefs are true yields

$$(P^n)[bel^{n+1}(\mathcal{B}, P^n) \wedge (P^n \neq P_0^N) \longrightarrow P^n], \tag{5.4}$$

The above three conditions set up the paradox; the paradox-generating question is whether P_0^{N+1} or $\neg P_0^{N+1}$.

To show how the contradiction arises in simple type theory I will disregard the r-type superscripts. Then if we assume

$$P_0^{N+1}, \tag{5.5}$$

from (5.1), (5.2), (5.5), modus ponens, and existential instantiation ($n = N$, $m = N$, and P_1^N for P^n) we get

$$(P_1^N \neq P_0^N) \wedge bel^{N+1}(\mathcal{B}, P_1^N) \wedge \neg P_1^N, \tag{5.6}$$

which by rules for propositional calculus yields

$$\neg P_1^N. \tag{5.7}$$

From (5.4) with universal instantiation (P_1^N for P^n) we get

$$bel^{N+1}(\mathcal{B}, P_1^N) \wedge (P_1^N \neq P_0^N) \longrightarrow P_1^N, \tag{5.8}$$

and (5.6) with rules for propositional calculus gives

$$(P_1^N \neq P_0^N) \wedge bel^{N+1}(\mathcal{B}, P_1^N). \tag{5.9}$$

Finally from (5.8), (5.9), and modus ponens we get

$$P_1^N, \tag{5.10}$$

and this contradicts (5.7). Conversely if we assume

$$\neg P_0^{N+1}, \tag{5.11}$$

then negating the definition of P_0 in (5.1) we get

$$(5.12) \qquad (P^N)\neg[bel^{N+1}(\mathcal{B}, P^N) \wedge \neg P^N],$$

universal instantiation (with P_0^{N+1} for P^N) yields

$$(5.13) \qquad \neg[bel^{N+1}(\mathcal{B}, P_0^{N+1}) \wedge \neg P_0^{N+1}],$$

but (5.3) and (5.11) with rules for propositional calculus give us

$$(5.14) \qquad bel^{N+1}(\mathcal{B}, P_0^{N}) \wedge \neg P_0^{N+1},$$

which contradicts (5.13).

Now consider how the r-types prevent the contradiction. Going back through the steps of the first deduction from P_0^{N+1}, (5.5)–(5.10), the r-types allow the deduction to go through. The second deduction, however, from $\neg P_0^{N+1}$, (5.11)–(5.14), does not go through. In moving from (5.12) to (5.13), the constant P_0^{N+1} was substituted for the variable P^N, but with the r-types in place this substitution is not allowed since the order of P_0^{N+1} is *greater* than the order of P^N. Thus the r-type theory blocks the contradiction and yields the result that $\neg P_0^{N+1}$. This may be interpreted as saying that it is false that Bouleus is N-mistaken—all his beliefs of order $n \leq N$ are true, but this means that he is $(N+1)$-mistaken because he believes wrongly the $N+1$ order proposition that he is N-mistaken, that is, P_0^{N+2} is true.

Notice also that since, by assumption, Bouleus has no beliefs of order greater than $N+1$, the first conjunct in (5.1), $bel^{n+1}(\mathcal{B}, P^n)$ will be (trivially) false for all P^n, $n > N+1$, and hence P_0^n will be false for $n > N+2$, that is, Bouleus is not n-mistaken for $n \geq N+2$.

6 The Bouleus resolution: cumulativeness Again it turns out that the solution one obtains depends on whether (and how) the levels/orders of the type theory are taken to be cumulative. Recall that in the Bouleus application discussed above, the r-type theory was taken to be only syntactically cumulative, that is, the levels/orders of the type theory itself were taken to be cumulative. To facilitate comparison with the *semantically* cumulative version it will be convenient to have more "control" of the relevant r-types in the definition of the critical belief than (5.1) allows. Consider the following generalization of (5.1):

$$(6.1) \qquad P_0^{n+1,m}(\mathcal{S}^i) =_{\text{df}} (\exists P^n)[bel^m(\mathcal{S}^i, P^n) \wedge \neg P^n], \qquad \text{where } m > n.$$

This definition allows the order of the belief to be different from (higher than the next above) the order of the proposition believed. The following postulate allows the *semantics* to reflect the cumulative nature of the syntax.

Postulate 6.2 $bel^{n+1}(\mathcal{S}^i, P^n) \longrightarrow bel^m(\mathcal{S}^i, P^n)$, *where $m > n$.*

This postulate introduces semantical cumulativeness into the formal language in a way analogous to (3.3) of the Grelling. The following theorem is an immediate consequence of (6.2) and Bouleus's critical belief $P_0^{n+1,m}$ as defined in (6.1).

$$(6.3) \qquad P_0^{n+1,n+1} \longrightarrow P_0^{n+1,m}, \qquad \text{where } m > n.$$

This theorem has the effect that if Bouleus mistakenly believes at order $n+1$ a proposition of order n, then he mistakenly believes the proposition at order m, where $m > n$, (i.e., he believes it at all higher orders). I will use the expression 'm-mistaken' to mean that $P_0^{n+1,m}$ is true for *some* $n < m$. Thus (6.3) can be interpreted as follows: if Bouleus is n-mistaken, then he is m-mistaken for all $m > n$. It should be clear that the cumulativeness principles (6.1) – (6.3) do not affect the r-type system's ability to deal with the paradox; the contradiction is blocked regardless of whether the type theory is taken to be semantically cumulative.

Even prior to considering the details of the effect of the semantical cumulativeness principles on the *resolution* of the Bouleus, there is reason to question the plausibility of the cumulativeness principles. Postulate 6.2 goes beyond what is called for with respect to beliefs. It is most natural to have the order of the *bel* constant tied immediately (one order higher) to the order of the proposition it takes as its (second) argument. In other words, the order of a belief should reflect the order of the proposition believed. Postulate 6.2 severs this connection and introduces the possibility of having two beliefs that differ, not in the object/sense of the belief, but only in the order of the belief—to whatever that amounts. If (6.2) is allowed, then Bouleus might have two distinct beliefs that, say, the earth is round; these beliefs would be distinct even though the proposition believed and the sense under which it is believed would be identical. This is too fine-grained a system to be used in formalizing beliefs.

Recall that in the presentation of the Bouleus it was assumed that there was a maximum order of Bouleus's beliefs. The cumulativeness principle (6.2) requires us to rethink this assumption since it guarantees beliefs of all orders greater than n, given at least one belief of order n. Were it worthwhile to pursue the fully cumulative system in great detail, it would be necessary to replace the assumption of a maximum order of belief with the notion of the *reduced form* of a belief $bel^m(\mathcal{S}^i, P^n)$, which could be defined as

$$(6.4) \qquad bel^r(\mathcal{S}^i, P^n), \qquad \text{where} \quad r =_{\mathrm{df}} \min\{j \in \mathbb{N} | bel^j(\mathcal{S}^i, P^n)\}.$$

In words, the reduced form of a given belief, $bel^m(\mathcal{S}^i, P^n)$, would be the subject's belief of least order that had the proposition in question as its object. Thus if one were to assume a maximum order N of the *propositions* believed as per the previous section and hence that the order of the critical proposition P_0 is N, then the order of the reduced form of this belief would serve as the critical order below which Bouleus is not mistaken and at which he is. Bouleus would also be mistaken on all orders *above* this critical order as an immediate consequence of (6.3) as discussed above. This is an interesting difference from the syntactically cumulative version of the previous section, which had Bouleus *not* mistaken on orders above the critical order at which he was (first) mistaken. Were the semantically cumulative system a viable candidate, there would, of course, be further complications to be dealt with in fully developing it.[8] I take it, however, that the considerations of the previous paragraph are sufficient to dispense with the syntactically cumulative version without further ado.

In considering the effect of moving to a (fully) noncumulative system, the first thing to note is that such a system, which is more restrictive than either of the cumulative versions, is equally effective as a means of circumventing the paradox. The telling question is whether the lack of cumulativeness affects the language's ability to formulate any of the crucial beliefs. While it does turn out that no expressibility is lost with respect to the relevant beliefs and conditions, the noncumulative theory may be seen as representing them in a less "natural" way. In this case the proposition P_0^{n+1} would not be interpreted as "I have a mistaken belief of order less than or equal to n"; but rather as "I have a mistaken belief of (exactly) order n." One would then make the following definition

$$Q_0^n =_{\text{df}} (P_0^n \vee P_0^{n-1} \vee \cdots \vee P_0^1), \qquad \text{where } n \geq 1. \tag{6.5}$$

Bouleus's critical belief would then be that Q_0^n. As a belief Q_0^n is arguably less "natural" in that it involves the belief of an explicit disjunction of typed propositions.

This aspect of the noncumulative system must be weighed against the following potential "drawback" of the syntactically cumulative version. The noncumulative system does not allow expressions such as

$$bel^m(\mathcal{S}^i, P^n), \qquad \text{where } m > n, \tag{6.6}$$

since P^n is of lower order than is required by the r-type of bel^m. The syntactically cumulative system, however, does allow such expressions as well-formed, but it lacks any sort of semantical principles to relate such expressions to ones that have obvious interpretations, for example, $bel^{n+1}(\mathcal{S}^i, P^n)$. Of course, it may be that the sensible way to deal with a question such as whether (6.6) is true is to disregard it as ill-formed for *nonsyntactic reasons*, (e.g., the ones offered in the discussion of the semantically cumulative system). Nonetheless this aspect of the syntactically cumulative system may be seen as something of a drawback in that it gives rise to unnecessary questions that it cannot answer.

7 Reducibility, classes, and cumulativeness Having examined the affects of cumulativeness on the details of paradox resolution in ramified type theory, I now step back and consider briefly how the issue of the cumulativeness of the r-types fits with the general needs of *PM*.

Consider first the relationship between (syntactic) cumulativeness and Russell's (in)famous Axiom of Reducibility. Two questions that come naturally to mind are (1) whether *cumulative* levels/orders might alleviate the need for reducibility and (2) whether *noncumulative* levels/orders render the expression of a reducibility schema problematic. With respect to the first concern, both Church [1] and Myhill [6] in their respective reconstructions of Russell's ramified type theory have incorporated cumulativeness and explored therein the role of the Axiom of Reducibility. It is clear in these systems that reducibility is still needed in order to reconstruct classical mathematics (see [1], p. 303 and [6], p. 82). Thus the answer to the first question is "no."

The answer to the second question is also "no." Consider the following statement of reducibility following [1]:

$$(F^{(\beta_1,\ldots,\beta_m)/n})(\exists G^{(\beta_1,\ldots,\beta_m)/1})(x_1^{\beta_1},\ldots,x_m^{\beta_m})[F(x_1,\ldots,x_m) \equiv G(x_1,\ldots,x_m)],$$

where $m = 1, 2, 3, \ldots$ and $\beta_1, \ldots, \beta_m$ are arbitrary r-types. This axiom schema does not violate syntactic noncumulativeness. Certainly it is true that the r-type of G is directly lower than the r-type of F, and consequently if syntactic noncumulativeness is in place, then the range of a variable of the r-type of F would include F and not G (i.e., the same variable could not quantify over both F and G). But clearly this is not required in order to state reducibility—the *arguments* to F and G have precisely the same r-types, $\beta_1, \ldots, \beta_m$, which is all that is required.

Being able to *state* a reducibility axiom is a first step, but the need for the axiom arises in the development of classical mathematics via Russell's contextually defined classes. Thus a natural next question is how cumulativeness affects Russell's theory of classes; more specifically, whether a fully cumulative grammar would hinder contextual class definition since it does restrict what may be expressed in the system. Indeed, a noncumulative grammar does render problematic Russell's scheme for contextually defining at least *some* class symbols. In particular, the class expression $\{x^i : \mathbf{A}\} \in \{z^{(i)/2} : \mathbf{B}\}$ which on *PM*'s scheme would be rendered as

$$(\exists G)[\ (x)[G(x) \equiv \mathbf{B}(x)]\ \&\ (\exists F)((y)[F(y) \equiv \mathbf{A}(y)]\ \&\ G(F))\ ],$$

would be ill-formed, since the r-type of $G(x)$ could not be assigned consistently so as to agree with the r-type required by the class expression, $((i)/2)/1$, and the r-type required by the expression $G(F)$, which is $((i)/1)/1$ since the r-type of F is $(i)/1$. This difficulty makes obvious one motivation for reconstructing *PM* (as Church does) with a *cumulative* grammar, for with a syntactically cumulative grammar G may be of r-type $((i)/2)/1$ and still allow $G(F)$, where F is of r-type $(i)/1$. For a noncumulative grammar, however, this presents a genuine, though not intractable, problem.

The immediate source of the difficulty is the impredicative class variable $z^{(i)/2}$: the problem does not arise if the class variable is predicative (i.e., $z^{(i)/1}$). Thus a straightforward way around this difficulty is to restrict the system so that there are no nonpredicative predicate variables—all and only predicate terms are predicative. In this case circumflexion would not be a predicate term forming operator and reducibility would be the sole comprehension principle. This move would, of course, alleviate the difficulty with impredicative class expressions, but it is not especially amenable to Church's formulation of the Grelling paradox.[9]

There is, however, a middle ground response that falls somewhere between Church's cumulative system with unrestricted impredicative variables and the more "Russellian" version that eschews impredicative variables altogether. This response is suggested by Hatcher's reconstruction ([3], p. 114ff) in which he retains nonpredicative variables and circumflex terms as grammatical, but effectively excludes them from the system by a predicativity restriction in his abstraction principle. A variation of this move is available for a noncumulative version of Church's presentation; in this system the predicative function terms appearing in contextual definitions of class symbols are restricted to predicative function terms that take predicative terms *as arguments*. This restriction effectively excludes problematic impredicative class terms like $z^{(i)/2}$ above, and such a *predicative* theory of classes is rich enough to develop classical mathematics roughly along the lines Russell suggested (see, for example, Hatcher's [3], p. 116ff discussion). It should be noted, however, that in such a system not all propositional functions will have extensions and while this consequence does not affect the treatment of the paradoxes, to the extent that it is counterintuitive it must be counted as a possible drawback of the fully noncumulative r-type system.

8 Concluding remark We have seen that in the setting of the Grelling paradox, one obtains distinct solutions to the paradox with each of the three possibilities with respect to the cumulativeness of the type theory: (1) fully cumulative, (2) partially (syntactically) cumulative, or (3) fully noncumulative. I argued that (3) offers the most satisfactory solution. In the setting of the Bouleus paradox, I suggested that (1) is clearly the wrong choice and that choosing between (2) and (3) is more difficult. The slim edge that (3) may have with respect to the particulars of the Bouleus paradox itself vanishes in the broader context of class theory and mathematical foundations because of the unnatural consequences discussed in the previous section.

Finally, I hope to have shown that in general, the issue of whether an r-type theory is cumulative (syntactically or semantically) is not merely a matter of notational convenience. That, in fact, the particular answer given to the paradox-generating question and the plausibility of the resolution in general is dependent on issues of cumulativeness.

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NOTES

1. Russell's "self-reproductive" diagnosis in [7] fits most neatly with this characterization.

2. In this I follow Church in taking the range of a variable of r-type $0/n$ to be 0-ary propositional functions—despite Whitehead and Russell's explicit rejection that a proposition is "a single entity at all" ([8], pp. 43–44). Church's rationale for this is that "this is what is clearly demanded by the background and purpose of Russell's logic" ([1], p. 291). It should be noted that Cocchiarella argues convincingly that (at least part of) Church's case for ignoring Russell's rejection of propositions as single entities in *PM* is mistaken ([2], p. 44ff). Church seemed to have realized this as well; he replaces three sentences in the original 1976 version of [1], p. 748, with two different ones in the 1984 reprint in Martin [5], p. 291.

3. In [8], Russell's characterization of a predicative function is that "a function . . . [is] *predicative* when it is of the next order above that of its argument, i.e., of the lowest order compatible with its having that argument" (p. 53). Thus it follows that φ is predicative if and only if $n = 1$.

4. Again, I begin by following Church's cumulative treatment ([1], p. 296), and consider later the effect of dropping these assumptions. In the interest of readability, however, I do not follow Church's use of "$\equiv_x$"; I use "$(x)[\ldots \longleftrightarrow \ldots]$" instead.

5. Church took it to be a virtue of the type theory that it not only prevented the contradiction, but that it provided an *answer* to the paradox-generating question ([1], p. 298).

6. This paradox was mentioned in [1], p. 303n as an example of an antinomy that is "about intensional matters but not semantical in nature."

7. One way to motivate this finite order assumption would be to assume that Bouleus has a finite number of beliefs, but this may be too restrictive: Bouleus might be seen as having a (potentially?) infinite number of beliefs concerning the natural numbers (e.g., that $1 > 0$, $2 > 1$, $3 > 2$, etc.) since it is implausible that Bouleus's understanding of the natural numbers stops at some (arbitrary) stage of the hierarchy. While it seems plausible to maintain that all of Bouleus's beliefs concerning the natural numbers are confined to

some finite order, this assumption too is complicated since, as is well known, type theory yields a systematically ambiguous treatment of the natural numbers, representing them on all higher-orders of the type theory. I note this, however, as an issue for type theory in general.

8. The further complications would concern issues such as whether the reduced order of a belief would necessarily be the next one higher than the order of the proposition believed, and whether (5.3) should be formulated to allow more "distance" between the order of the belief and the order of the proposition believed.

9. While I am not here trying to work out the most Russellian account, I note that this reading fits the actual *PM* (1910) development quite well—at least as well as Church's cumulative reconstruction. See, for example, Hochberg [4].

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