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# Peeking at the Impossible

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**Abstract** The question of the interpretation of impossible pictures is taken up. Penrose's account is reviewed. It is argued that whereas this account makes substantial inroads into the problem, there needs to be a further ingredient. An inconsistent account using heap models is proposed.

***1 Introduction*** Given that the mathematics of the inconsistent has developed to the point of self-subsistence, it becomes essential to look to applications. Anomalies in physics and pure mathematics are an intriguing prospect. But one very obvious example remains unaddressed: the impossible pictures such as are found in Escher's works, for example, the inconsistent triangle, ascending and descending, and the like. The embarrassment for the paraconsistency program is that it took a thoroughly classical mathematician, Roger Penrose, to make the first significant inroads on the problem.

Penrose applies the theory of cohomology groups to the problem. He shows necessary and sufficient conditions for a two-dimensional picture to represent a consistent three-dimensional object. This paper begins by setting out Penrose's account in Section 2. In Section 3 it is seen that there remains one ingredient to be added to Penrose's solution. A theory is described which extends Penrose's account by means of the theory of heaps. The inconsistent theory of heaps has been studied by Priest, Restall, van Bendegem, and others. It proves necessary to modify that theory, though an inconsistent version remains the most intuitive. The result provides a sense in which looking at inconsistent drawings is peeking at the impossible. However, the existence of a stable theory also tends to show that the inconsistent may not be so impossible after all.

***2 Penrose's account*** Consider the inconsistent triangle.[1] Penrose notes correctly that it can be a picture of a three-dimensional structure, in fact many different structures. This could happen if the structure were in fact three disassembled parts lined up behind one another so that the distances between them could not be seen.[2] Indeed, there is an infinite collection of such 3-D structures which "project down" onto

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a front screen, so that the projection forms the 2-D picture. These are readily obtained by thinking of moving the three disassembled parts closer or farther from the picture while allowing that their sizes can increase or decrease depending on whether they are farther or closer from the 2-D image on the front screen, so as to make them have that image as their projection. This requires that objects expand uniformly the farther away they are, in such a way as to keep the same size and shape of the image.

The difference between this case and the case where the 2-D picture is an image of a consistent 3-D object is thus: in the former case the parts cannot be assembled

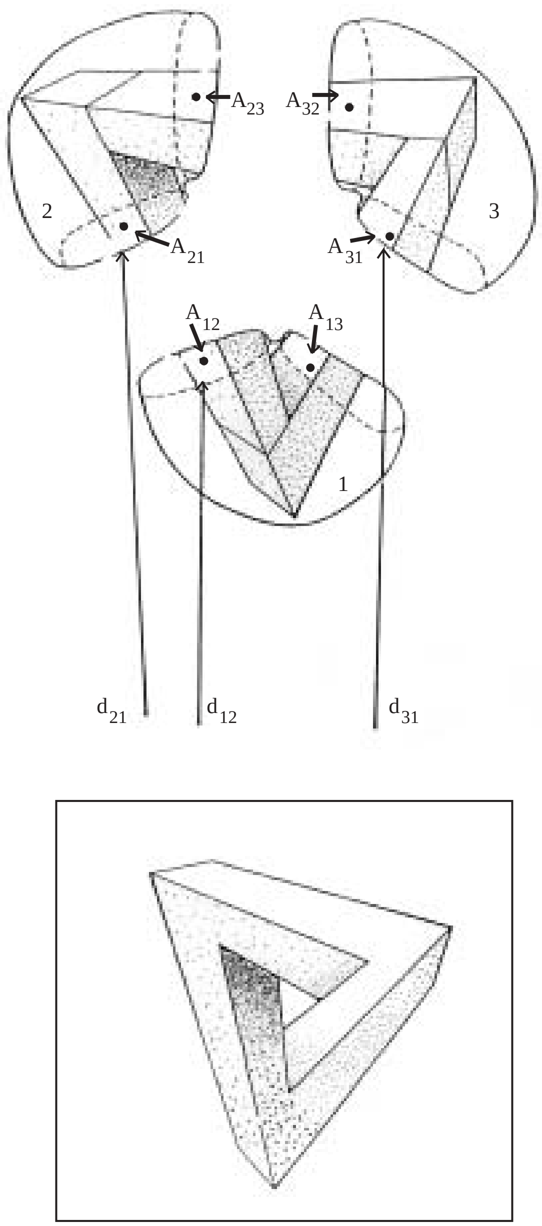


Figure 1: The inconsistent triangle and a 3-D disassembly (adapted from Penrose)

into a consistent 3-D object (no bending permitted), whereas in the latter case the parts can be so assembled. Note that disassembly and reassembly are required to be done in such a way that the same 2-D projection is preserved in all motions of the 3-D parts. This, in a nutshell, is Penrose's suggestion: it is necessary and sufficient for a 2-D picture to be a picture of a consistent 3-D object, that the collection of 3-D configurations which project onto it has amongst it a consistent connected object which is a reassembly of any of the 3-D structures. What is interesting, of course, is the possible failure of the necessary and sufficient condition. Then the 2-D image fails to be a projection of any consistent connected 3-D object, though it remains a projection of an infinite number of disassembled 3-D structures.

Penrose describes this in the language of cohomology groups. We begin with the multiplicative group of positive real numbers $\mathbb{R}^+$. Penrose calls this the *ambiguity group* of the structure. It represents the fact that a 2-D picture ambiguously represents a class of 3-D structures because the (single) eye cannot distinguish points which are in a direct line with one another from the eye. For any such pair of points, the distance to one from the eye can be represented as a multiple of the distance to the other, where the multiplier is from $\mathbb{R}^+$.

If the 3-D structure is in three parts, joining it up will require identifying (at least) two points on each part with a point on each of the two other parts. (We ignore the point that strictly surfaces, not points, will need to be identified, by taking the single points as representative of the surfaces.) Let the parts be numbered 1, 2, 3 and let the point on part $i$ which is to be identified with a point on part $j$ be called $A_{ij}$. Thus, $A_{ij}$ joins with $A_{ji}$. Let the distance from the front screen to $A_{ij}$ be called $d_{ij}$ (see diagram).[3] The requirement of assembly can thus be expressed by the condition $d_{ij} = d_{ji}$, all $i,\ j$, (where the $d_{ij}$ belong to $\mathbb{R}^+$). There are an infinite number of structures which satisfy this condition for any given 2-D picture. It is convenient to reduce this condition to one involving equivalence classes. Introduce the ratios $r_{ij}$ defined by $d_{ij}/d_{ji}$ . (Note that $r_{ij} = 1/r_{ji}$). Then each triple $(r_{12}, r_{23}, r_{13})$ determines an infinite equivalence class of structures having the same ratio of distances between corresponding points on the three bodies. These triples are called *cocycles*. The condition of assembly, that $d_{ij} = d_{ji}$, is obviously equivalent to the condition that the cocyle = $(1, 1, 1)$. If this happens, the cocycle is called a *coboundary*. A coboundary can obviously function as the unit of a group, where group multiplication is defined as pointwise multiplication of the three components of the cocycles, that is, $(a, b, c) * (d, e,\ f) = (ad, be, cf)$. The group operation evidently has physical significance in that any pair of equivalence classes of configurations under the operation produces a unique equivalence class of configurations. When the set of cocycles contains a coboundary, the cocycles form a *cohomology* group. Thus, the cocycles fail to be a group just when they lack a coboundary, that is, just when they cannot be assembled into a consistent connected 3-D structure.

***3 Inconsistent heaps*** One question remains with this approach. Why is it that it seems to the eye that the picture represents an impossible object, and not any one of the possible disassembled objects which look the same? The nearest Penrose comes to an answer seems to be the observation that the eye cannot distinguish between points directly behind one another. Still, this applies as much to the consistent case as the inconsistent case. What is it for the eye to *preferentially interpret* what it sees as inconsistent, and not any one of the possible structures? These questions are answered in the next section, where it is proposed that the inconsistent interpretation is represented by an inconsistent theory. But first we must do some preliminary work.

In point of fact, there *are* assembled 3-D structures which project onto the inconsistent triangle. First, it is plain that one can *almost* assemble three parts. One can identify *two* of the pairs of points, say $A_{12}$ with $A_{21}$ and $A_{23}$ with $A_{32}$, leaving the third pair separate. If the third pair of points were only "virtually separate," but identical because *they* were painted on a *back* screen, that would be enough to produce the image on the front screen. This suggests a general strategy for producing 3-D objects which project to the desired image on the front screen. Suppose that space is finite, with a backdrop or back boundary. Assemble as much as one can, put the pieces against the backdrop, and *draw in* the remainder of the lines on the backdrop. All the world's a stage (or a Hollywood set).

One can imagine an objection that the backdrop universe is not an acceptable 3-D structure for projection onto the front screen because it fails to force separate points in 3-D space to remain separate. Algebraically, this is the stipulation that the ambiguity group corresponds to, or maps one-one to, the set of possible distances from the front screen. In the language of model theory, it is the condition that the theory describing the geometrical solution preserves all statements about the ambiguity group which deny identity. This is a reasonable requirement, and it leads us thus to an inconsistent theory, as we will see.

The theory of *heap* models projects the positive integers with addition onto a primitive concept of counting, for example, "1, 2, 3, Heap." (The presence of zero and negatives is optional, see below.) Thus we have that, in addition to all true equations of positive integer arithmetic, all of $4 = 5 = \cdots =$ Heap hold. If we think of the positive integers as like the ambiguity group, and additionally impose the condition of the previous paragraph that the map from the ambiguity group to the heap be 1-1, we also have that $4 \neq 5 \neq 6 \cdots$ all hold, which is inconsistent. That is, there is an inconsistent theory which satisfies this condition. Heap $H$ functions as an indeterminate upper limit to counting, a kind of infinity in that $a + H = H + a = H$ holds (except where the model also contains the additive inverse of $H$, where $H + (-H) = 0$). However, $H$ can be reached by finite means: $1 + 3 = H$. Obviously, there are an infinite number of heap models, one for every maximal nonheap element.[4]

Adapting heap theory to the present case of the multiplicative group of positive reals requires some modifications. The natural view of the "backdrop" universe described earlier is to allow only the distances up to a certain distance $d_{ij}^{max}$, the distance to the backdrop. This might be represented initially as a mapping of the ambiguity group $\mathbb{R}^+$ to $\{x \in \mathbb{R}^+ : x \leq d_{ij}^{max}\}$, where the restriction of the domain to $x \leq d_{ij}^{max}$ is the identity mapping. However, if we impose the condition that the mapping from the ambiguity group to the set of distances be 1-1, we have an inconsistent theory.

To focus, let $d_{ij}^{max}$ have a specific real number value, say 4.1. Then, for example $2 = 3 = 3.1$ do not hold, but $4.1 = 4.2 = \cdots$ all hold. Since these identicals are intersubstitutable in all contexts, we can introduce the name '$H$' to refer ambiguously to all of them, that is, $4.1 = 4.2 = \cdots = H$. Since the mapping from the ambiguity group is one-one, also $4.1 \neq 4.2 \neq \cdots \neq H$. To multiply two numbers $a, b$ in the heap: first determine whether either $= H$. If not, then multiply $a * b$ normally, determine whether the result $= H$ and if so identify $a * b$ in addition with all numbers which $= H$. Otherwise, if one or both of $a, b = H$, then the result $= H$ and all its identicals as well. For example, since $2 * 4 = 8$, then by substitution of identicals $2 * 4 = H = 4.1$ also: as with the integer models it is possible to reach the backdrop by operation on items in front of the backdrop. But also $a * H = H * b = H$, for example, $0.5 * H = H * 2 = H$. Evidently, $*$ is commutative. Also, the structure has a unit: $a * 1 = 1 * a = a$.

But the heap is not a group for two reasons. First, it lacks natural inverses for some elements, those which are greater than 4.1. Inverses can be produced by a further inconsistent extension of the theory of the ambiguity group. Identify all members of the class $\{x \in \mathbb{R}^+ : x \leq (4.1)^{-1}\}$ with one another and call them $H^{-1}$. If two numbers strictly between $H^{-1}$ and $H$ are multiplied together, take their product in $\mathbb{R}^+$ and determine whether it is identical with $H$, $H^{-1}$ or something in between, identifying with all identicals in the former two cases. Otherwise, $H^{-1}$ behaves like a zero, with $a * H^{-1} = H^{-1} * a = H^{-1}$; except for the case where $a = H$, where $H^{-1}$ behaves as the inverse of $H$, with $H * H^{-1} = H^{-1} * H = 1$. Thus $H^{-1}$ functions as a lower limit on distances.

The second reason heaps are not groups is that multiplication fails to be associative. For example, $(0.5 * 2) * 3 = 1 * 3 = 3$, but $0.5 * (2 * 3) = 0.5 * H = H$, while $3 = H$ does not hold. There are other multiplications which fail to be associative, for example, vector cross product. Furthermore, the failure of associativity is well motivated by the intended interpretation in a space with a backdrop and a least size: if you reach either of these limits you're stuck there, unless you're multiplied by your inverse; so the order you associate matters. However, these heaps are almost groups: commutative groupoids with a unit and inverses. Also, the subalgebra $\{H, 1, H^{-1}\}$ is plainly the limiting case of the heap where all $x > 1$ are identified with $H$ and all $x < 1$ with $H^{-1}$.

It is clear that there are both consistent and inconsistent theories here. The difference between the two is the absence or presence of $4.1 \neq 4.2 \neq \cdots H$ and their inverses. The latter is the one-one condition, that distinct elements of the ambiguity group correspond to distinct distances in any acceptable reassembly of the structure. Now corresponding to cocycles as equivalence classes of triples of distances from $\mathbb{R}^+$, there are obviously classes of triples from heaps. These structures satisfy the condition for a coboundary among cocycles, namely the existence of a unit. Clearly, given any parts of the assembly which are done consistently, in front of the backdrop, then for them certainly $d_{ij} = d_{ji}$, that is $r_{ij} = 1$. But also for any pair of distances which are identified at the backdrop, again $d_{ij} = d_{ji} = H$, and $r_{ij} = d_{ij} * (d_{ji})^{-1} = H * H^{-1} = 1$. Thus the triple of ratios $(r_{12}, r_{23}, r_{13}) = (1, 1, 1)$, which is the unit.

For the purposes of describing formally an appropriate model in which the theory of the inconsistent heap holds, closed set logic would seem to be the most natural, since it has the advantage of representing contradictions as holding on closed sets and particularly their boundaries. Consider the topological space with a basis of four closed sets: $\mathbb{R}^+$, $\{x \in \mathbb{R}^+ : x \leq (d_{ij}^{max})^{-1}\}$, $\{x \in \mathbb{R}^+ : x \geq d_{ij}^{max}\}$, $\{\}$. For convenience, we can rename the middle two of these as $H^{-1}$ and $H$, respectively. The closed sets serve as semantic values for closed set logic and theories thereof, since they are closed with respect to unions (disjunction) and intersections (conjunction). Quantifiers are interpreted as respectively generalized union and intersection as usual. For negation, one takes closed complement, that is, the least closed set containing the Boolean complement. Thus the closed complement of both $H^{-1}$ and $H$ is $\mathbb{R}^+$ itself. This means that if $x = y$ is stipulated to hold on $H$ but nowhere else, (i.e. $x = y$ takes $H$ as its semantic value) then its negation $x \neq y$ holds everywhere, that is, $\mathbb{R}^+$. In particular, since $H$ is a subset of $\mathbb{R}^+$, the contradiction $x = y \;\&\; x \neq y$ holds on $H$. The inconsistent theory of the heap can then be generated by the condition: *if* $x = y$ is true in the classical theory of $\mathbb{R}^+$ *then* assign $x = y$ to $\mathbb{R}^+$, *else if* $x$ and $y$ belong to at least one nonnull closed set, *then* assign $x = y$ to the least closed set to which both $x$ and $y$ belong, *else* assign $x = y$ to $\{\}$. To also add $H$ to the theory, if both $x$ and $y$ belong to $H$, assign $x = H$ and $y = H$ to $H$, and similarly for $H^{-1}$. Then we define a sentence to hold in the theory of the heap, if it is assigned to some nonnull closed set. Then, for example, $4.1 = 4.2 = H \neq 4.1 \neq 4.2$ all hold; but whereas $2 \neq 3 \neq H$ and $4 = 2 * 2$ hold, they hold consistently, that is, neither $2 = 3 = H$ nor $4 \neq 2 * 2$ hold. The interested reader is left to fill in further formal details. For further description of theories of closed set logics, see Mortensen [2].

***4 Inconsistent representations*** The question is, in looking at the inconsistent triangle, *what* is one peeking at behind it? One answer could be that one is peeking at a disassembled object, where the eyes fail to disidentify points directly behind one another. Where this seems to fall short, is that it does not tell us the positive consequences of implementing this failure to disidentify in such a way as to model the inconsistency. The eye seems to be in a positive default mode: identify those things which one fails to disidentify. But what are the mathematical consequences of literally identifying those points? Which connected structure is it that it seems to be? And what is it for it to nonetheless seem inconsistent?

It would be even more unsatisfactory to say merely that one is not peeking at anything at all. How does that differ from shutting the eyes for example? And how does it destroy the overwhelming illusion that one is peeking at something? The obvious third possibility is that one is peeking at an object in a backdrop universe, which is a heap.

As we have already seen, there are consistent and inconsistent versions of these models. The inconsistent theories satisfy the condition that the ambiguity group maps 1-1 to the set of distances of the heap: if $x \neq y$ holds in $\mathbb{R}^+$, then $x \neq y$ holds for distances $d_{ij}$, and for ratios/cocycle components $r_{ij}$. There are two ways to interpret the inconsistent models: epistemologically and ontologically (weak and strong paraconsistency, respectively). Epistemologically, what we are seeking to describe is a cognitive phenomenon: how is it that it *seems*, when it seems inconsistent? This concerns the representation of inconsistent data. It is well known that this can arise wherever there are at least two sources of data, or updates from a single source. In the present case, one might suppose that $4.1 \neq 5$ represents the *norm*, what one *knows* or *believes* about the space one is in; whereas $4.1 = 5$ represents the *default consequences* of being *unable to tell* between these distances.

Ontologically on the other hand, one is required to regard the theories representing such cognitive states as at least possible. On an intuitive level, the standard strong paraconsistentist argument is that one is required to take seriously the thought that one's inconsistent cognitive state might really be the way things are. *It really might be like that*. This is a general consequence of good theoretical practice, in any case: one should believe in the results of one's careful scientific investigations, which might contain persistent anomalies. Thus, if the world really were inconsistent in this way, then *this* is how it would look if you took a peek at it. Of course, there are other things which would look the same, but that is the way of it with eyes. *Credo ut intelligam*.

***5 Conclusion*** There is one less than complete feature of this discussion. We have discussed heaps, and heaps can provide a unit for the cohomology group. But I doubt that heaps are the only inconsistent way to understand the triangle, or even the best way. The inconsistency we perceive is more cyclical and less brute force than that. This indicates a more subtle inconsistentizing operation is in the offing, which hopefully will be the topic of a further paper. But the general method of constructing inconsistent models remains the same.

Penrose also briefly indicates how to extend his discussion to cohomology groups associated with other 2-D figures representing ambiguous or inconsistent 3-D structures. His approach is undoubtedly rich with different applications. It is to be presumed that the inconsistent approach will lend itself to a similar range of applications and it is also proposed to study these in more detail in later papers.[5]

## NOTES

1. Penrose calls this the tribar. Whereas there is an excellent case for some such short name (fewer keystrokes), the present author cannot bring himself to divorce this word from its common use among logicians, to refer to the sign for material equivalence $\equiv$.
2. Penrose and Penrose in [5] built a (consistent) 3-D structure which photographs as stairs ascending in a closed loop, but needless to say it is not as it seems. For the photograph, see Penrose [4].
3. This represents a slight departure from Penrose's symbolism, which reserves '$d_{ij}$' for what we call '$r_{ij}$'.
4. The term "heap model" seems to have been first used by Meyer. On heap theory, see, e.g., Priest [6], p. 227 or van Bendegem [7].
5. For further inconsistent structures in the general area of projective geometry, wherein an inconsistent theory of homogeneous coordinates is proposed, see Mortensen [1], Chapter 9. On closed set logic, see Chapter 11 or [2].

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