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Recursive Models and the Divisibility Poset

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Abstract In contrast to Tennenbaum's theorem asserting that there are no nonstandard models of Peano Arithmetic in which either addition or multiplication is recursive, the result here is that there is a nonstandard model of Peano Arithmetic having a recursive divisibility poset and a recursive divisibility lattice.

According to the well-known theorem of Tennenbaum [17], there are no recursive nonstandard models of Peano Arithmetic. Usual proofs of this theorem yield the following refinement. If $\mathcal{M} = (M, +, \times, 0, 1, \leq)$ is a nonstandard model of PA, then neither its additive semigroup $(M, +)$ nor its multiplicative semigroup $(M, \times)$ is a recursive structure. This suggests considering other reducts of models of PA such as the divisibility poset $(M, |)$, where we understand $x|y$ to mean that x is a divisor of y. It will be proved here (see Corollary 14) that, in contrast to Tennenbaum's theorem, there do exist nonstandard models $\mathcal{M}$ of PA whose divisibility poset $(M, |)$ is recursive. There even exist nonstandard models $\mathcal{M}$ of PA whose divisibility lattice $(M, \wedge, \vee)$ is recursive, where we understand $x \wedge y$ and $x \vee y$ to be, respectively, the greatest common divisor and the least common multiple of x and y. These examples give a positive answer to the second part of Problem 9 in Jensen-Ehrenfeucht [7].

All theories considered here are for finite languages with no function symbols. If a theory does not appear to be of this sort, make the appropriate modification so that it is. Several first-order theories will be considered here, but we will mainly be concerned with theory $\mathsf{DP} = \mathrm{Th}(\omega, |)$, where $(\omega, |)$ is the divisibility poset of the standard model $\mathbb{N} = (\omega, +, \times, 0, 1, \leq)$, and the theory $\mathsf{DL} = \mathrm{Th}(\omega, \wedge, \vee)$, where $(\omega, \wedge, \vee)$ is the divisibility lattice of the standard model. Cegielski [1], [2] has shown that DP is finitely axiomatizable, and, therefore, DL also is.

Let T be an arbitrary theory. Then T is *rich* if, for some $n < \omega$, there is a recursive sequence $\langle \varphi_i(\bar{x}) : i < \omega \rangle$ of n-ary formulas such that for any disjoint, finite

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subsets $I, J \subseteq \omega$, the sentence

$$\exists \bar{x}\, [\bigwedge_{i \in I} \varphi_i(\bar{x}) \wedge \bigwedge_{j \in J} \neg\varphi_j(\bar{x})]$$

is a consequence of T. If, in addition, all the formulas $\varphi_i(\bar{x})$ are existential formulas, then T is *existentially rich*.

The properties of being rich and existentially rich are important in the study of reducts of PA. (See Kaye [8].) The two prototypical examples of existentially rich theories are Presburger Arithmetic and Skolem Arithmetic. Recall that Presburger Arithmetic is the theory $\mathsf{Pr} = \mathrm{Th}(\omega, +)$ and that Skolem Arithmetic is the theory $\mathsf{Sk} = \mathrm{Th}(\omega, \times)$. To see that Pr is existentially rich, consider the sequence of 1-ary formulas, the ith one in the sequence being $\exists y(p_i y = x)$, where p_i is the ith prime. Similarly for Sk, let the ith formula be $\exists y(y^{p_i} = x)$. The existential richness of these theories provides an approach to proving Tennenbaum's Theorem. Had Tennenbaum's proof been published, the following theorem would probably have been implicit in it. As it is, this theorem seems to be a result from the folklore.

Theorem 1 *If T is existentially rich, then T does not have any recursive, recursively saturated models.*

Proof: Let M be a recursive, recursively saturated model of T, and let $\varphi_i(\bar{x})$ be the ith formula witnessing the existential richness of T. Let A and B be recursively inseparable r.e. sets. By recursive saturation, there are $\bar{a}, \bar{b}$ in M such that for all $i < \omega$, $M \models \varphi_i(\bar{a})$ if $i \in A$, $M \models \varphi_i(\bar{b})$ if $i \in B$, and $M \models \varphi_i(\bar{a}) \longleftrightarrow \neg\varphi_i(\bar{b})$. By the recursiveness of M, the sets $X = \{i < \omega : M \models \varphi_i(\bar{a})\}$ and $Y = \{i < \omega : M \models \varphi_i(\bar{b})\}$ are recursively enumerable. But then X and Y, being complements of each other, are recursive. This contradicts the recursive inseparability of A and B. $\square$

The properties of Pr and Sk needed to deduce Tennenbaum's Theorem from Theorem 1 are contained in the following lemma. This lemma is a consequence, not just of the quantifier-elimination for Pr and Sk, but also of the facts that, in each case, the quantifier-elimination is provable in PA. This property of Pr follows easily from the original proof of Presburger [14]. For Sk it was proved by Nadel [12] (also see Cegielski [3] and Chatzidakis [4]) who recognized that Skolem's [16] original proof of quantifier elimination for Sk did not yield that every multiplicative reduct is a model of Sk.

Lemma 2 *Let $\mathcal{M}$ be a nonstandard model of PA. Then its additive semigroup $(M, +)$ is a recursively saturated model of Pr and its multiplicative semigroup $(M, \times)$ is a recursively saturated model of Sk.*

Corollary 3 (Tennenbaum's Theorem) *If $\mathcal{M}$ is a nonstandard model of PA, then neither $(M, +)$ nor $(M, \times)$ is a recursive structure.*

We can get both the DP and DL analogs of Lemma 2 as a corollary of just the Sk part of that lemma.

Corollary 4 *If $\mathcal{M}$ is a nonstandard model of PA, then its divisibility poset $(M, |)$ is a recursively saturated model of DP and its divisibility lattice $(M, \wedge, \vee)$ is a recursively saturated model of DL.*

Proof: It suffices to observe that $|$, $\wedge$, and $\vee$ are all definable in $(M, \times)$. $\square$

The following was noted in Lemma 4.1 of [7].

Proposition 5 *Both of the theories* DP *and* DL *are rich.*

Proof: We introduce some formal definitions (one for each positive integer n): x is the nth power of a prime if and only if the set of predecessors of x, including x itself, is linearly ordered by $|$ and has exactly $n+1$ elements. Let $\langle \varphi_i(x) : i < \omega \rangle$ be the sequence of 1-ary formulas in which $\varphi_i(x)$ asserts: there is y such that $y|x$ and y is the $(i+1)$th power of a prime, but there is no z which is the $(i+2)$th power of a prime and for which $y|z|x$. This shows that DP is rich. The theories DP and DL are interdefinable, so DL is also rich. $\square$

The richness of reducts of PA has consequences about countable, recursively saturated models of PA. For example, the richness of DP and DL implies the following: if $\mathcal{M} = (M, +, \times, 0, 1, \leq)$ and $\mathcal{N} = (N, +, \times, 0, 1, \leq)$ are elementarily equivalent, countable, recursively saturated models of PA, and $(M, |) \cong (N, |)$ or $(M, \wedge, \vee) \cong (N, \wedge, \vee)$, then $\mathcal{M} \cong \mathcal{N}$.

The formulas appearing in the proof of Proposition 5 are not existential; in fact, they seem to be $\exists_3$. Thus, if $\mathcal{M}$ is a nonstandard model of PA such that $(M, |)$ is recursive, then each $X \in \mathrm{SSy}(\mathcal{M})$ is Δ_3. In particular, there is no nonstandard model $\mathcal{M}$ of True Arithmetic for which $(M, |)$ is recursive.

Do the theories DP and DL have the stronger property of being existentially rich? This question is answered in Proposition 9 and, from another point of view, in Corollary 14.

Let $\mathbf{K}$ be a class of finite $\mathcal{L}$-structures. We say that $\mathbf{K}$ is *hereditary* if whenever $M \in \mathbf{K}$ and N is embeddable in M, then $N \in \mathbf{K}$. If $\mathbf{K}$ is hereditary, then its *barrier bar* $(\mathbf{K})$ is the class (of isomorphism types) of $\mathcal{L}$-structures M such that $M \notin \mathbf{K}$ but every proper substructure of M is in $\mathbf{K}$. Then we say that $\mathbf{K}$ is $\forall$-*finite* if whenever $\mathbf{L}$ is a hereditary subclass of $\mathbf{K}$, then *bar* $(\mathbf{L})$ is finite. There is an alternate approach to defining $\forall$-finiteness using well-quasi-ordered theory. The hereditary class $\mathbf{K}$ is $\forall$-finite if and only if, whenever $M_0, M_1, M_2, \ldots$ is a sequence of structures in $\mathbf{K}$, then there are $j < i < \omega$ such that M_j is embeddable in M_i. If, for each $n < \omega$, the class of structures $(M, \bar{a})$, where $M \in \mathbf{K}$ and $\bar{a}$ is an n-tuple from M, is $\forall$-finite, then $\mathbf{K}$ is *strongly* $\forall$-*finite*. For any $\mathcal{L}$-theory T, let $Sub(T)$ be the collection of all finite substructures of models of T. Then, Ershov [5] defines T to be $\forall$-finite or strongly $\forall$-finite just in case $Sub(T)$ is $\forall$-finite or strongly $\forall$-finite.

Ershov proved that every r.e., strongly $\forall$-finite theory has a recursive model. The principal application of this is to the theory of trees, which is strongly $\forall$-finite as a consequence of Kruskal's theorem [10]. Thus, every r.e. theory of trees has a recursive model. Ershov's theorem improved an earlier result of Peretyat'kin [13] that every r.e. extension of the theory of linear order has a recursive model. For related results and improvements, see Lerman and Schmerl [11], Schmerl [15], and Knight [9]. The following proposition shows that Ershov's theorem cannot be applied to either of the theories DP or DL.

Proposition 6 *Neither of the theories* DP *and* DL *is* $\forall$-*finite.*

Proof: Notice that every finite poset is embeddable in the standard divisibility poset $(\omega, |)$. Let $(P_n, <)$ be the poset where $P_n = \{a_0, a_1, \ldots, a_{n+2}, b_0, b_1, \ldots, b_{n+2}\}$ and $x < y$ if and only if for some $i < n+3$, $x = a_i$ and $y \in \{b_i, b_{i+1}\}$ (where b_{n+3} is understood to be b_0). The poset P_n is known as an $(n+3)$-*crown*. Let $\mathbf{K}$ be the hereditary class of all finite posets not embedding any one of the $(n+3)$-crowns. Clearly, if $j < i < \omega$, then P_j is not embeddable in P_i. Thus, *bar* $(\mathbf{K})$ is not finite, so $\mathbf{K}$ and also DP are not $\forall$-finite. Since $|$ is definable in a divisibility lattice by a quantifier-free formula, we also get that DL is not $\forall$-finite. □

Proposition 7 implies that every strongly $\forall$-finite theory T fails to be existentially rich; actually, the conclusion says that T fails to have a property which is even weaker than its being existentially rich. Applying Proposition 7 to the theory of trees yields that this theory has no completions which are existentially rich. However, this theory does have decidable completions which are rich. Of course, Proposition 7 is of no use in showing that DP or DL fails to be existentially rich.

We say that the theory T is *existentially order-rich* if, for some $n < \omega$, there is a recursive sequence $\langle \varphi_i(\bar{x}) : i < \omega \rangle$ of existential n-ary formulas such that each of the sentences

$$\exists \bar{x}\, [\varphi_i(\bar{x}) \wedge \bigwedge_{j<i} \neg \varphi_j(\bar{x})]$$

is a consequence of T. Clearly, every existentially rich theory is existentially order-rich. There are theories which are existentially order-rich but not existentially rich: the theory of the countable graph whose components are precisely the n-cycles (one for each $n \geq 3$) is a complete and decidable example.

Proposition 7 *If T is strongly $\forall$-finite, then T is not existentially order-rich.*

Proof: We will prove the contrapositive, so assume that T is existentially order-rich, and let $\varphi_i(x)$ be the ith existential formula in the sequence witnessing that T is existentially order-rich. For each i, let (B_i, a_i) be such that B_i is a model of T, $B_i \models \varphi_i(a_i)$ and for all $j < i$, $B_i \models \neg\varphi_j(a_i)$. Let M_i be a finite substructure of B_i such that a_i is in M_i and $M_i \models \varphi_i(a_i)$. Notice that for $j < i$, $M_i \models \neg\varphi_j(a_i)$ since $\varphi_j(x)$ is existential. Let $\mathbf{K}$ be the smallest hereditary class containing each (M_i, a_i). It easily follows from the finiteness of *bar* $(\mathbf{K})$ that there are $j < i < \omega$ such that (M_j, a_j) is embeddable in (M_i, a_i). But then $(M_i, a_i) \models \neg\varphi_j(a_i)$ since (M_j, a_j) is a model of that sentence. This is a contradiction. Thus, T is not strongly $\forall$-finite. □

We next define a certain type of poset. For $1 \leq n < \omega$, let $[n]$ be $n = \{0, 1, 2, \ldots, n-1\}$, considered as a linearly ordered set with the usual ordering $\leq$ on it. We will consider product posets $(B, \leq) = ([n_0] \times [n_1] \times \cdots \times [n_d], \leq)$. We refer to any poset which is isomorphic to such a $(B, \leq)$ as a *box*. A subset $E \subseteq B$ is a *line* if for some $j \leq d$ there is $a \in B$ such that $E = \{x \in B : \text{ if } j \neq i \leq d, \text{ then } x_i = a_i\}$. An abstract characterization of lines can be given: if $(B, \leq)$ is a box and $E \subseteq B$, then E is a line if and only if E is a maximal linearly ordered subset of B of the form $\{x \in B : a \leq x \leq b\}$. An embedding of one box into another which preserves lines is a *box-embedding*. An important point to observe is that if we view boxes as distributive lattices, then box-embeddings are also lattice embeddings.

Every model of DP has lots of subboxes. If $M \models \mathsf{DP}$ and $A \subseteq M$ is finite, then there is a box B such that $A \subseteq B \subseteq M$. There is more that can be said. If $M \models \mathsf{DL}$ and $A \subseteq M$ is finite, then there is a sublattice $B \subseteq M$ such that $A \subseteq B$ and B is a box.

Let $\mathcal{B}_n$ be the set of structures $(B, \leq, a)$, where $(B, \leq)$ is a box and a is an n-tuple from B. The following is a result from well-quasi-ordered theory.

Lemma 8 *Let $n < \omega$. If $B_0, B_1, B_2, \ldots$ is a sequence from $\mathcal{B}_n$, then there are $j < i < \omega$ and a box-embedding of B_j into B_i.*

Proof: The lemma can be rephrased in terms of *wqo*. Recall that a structure $(A, \leq)$ is a *quasi order* if $\leq$ is a transitive, reflexive (but not necessarily anti-symmetric) binary relation on A. A quasi order $(A, \leq)$ is a *well-quasi-order (wqo)* if whenever $a_0, a_1, a_2, \ldots$ is a sequence from A, there are $j < i < \omega$ such that $a_j \leq a_i$. The lemma asserts that for each n, the set $\mathcal{B}_n$ is a wqo under box-embeddability.

If A and B are wqos, then their product $A \times B$ is a wqo. More subtle is the fundamental theorem of Higman [6]. If A is a quasi order, then the set $A^{<\omega}$ of finite sequences from A can be considered as a quasi order with the following definition. Let $a, b \in A^{<\omega}$ where the length of a is m and the length of b is n. Then $a \leq b$ if and only if there is an injection $f : m \to n$ such that for each $i < m$, $a_i \leq b_{f(i)}$. Then Higman's theorem asserts: if A is a wqo, then so is $A^{<\omega}$.

An easy application of Higman's theorem yields that $\omega^{<\omega}$ is wqo. Each sequence in $\omega^{<\omega}$ can be identified with a box, and this shows that $\mathcal{B}_0$ is wqo. Next we want to show that $\mathcal{B}_n$ is wqo. Consider an n-augmented box $(B, a_0, a_1, \ldots, a_{n-1})$ in $\mathcal{B}_n$. With each pair $i, j < n$ associate the subbox $S_{ij} = \{x \in B : a_j \leq x \leq a_i\}$, and with each $i < n$ associate $T_i = \{x \in B : x \leq a_i\}$ and $U_i = \{x \in B : x \geq a_i\}$. Then associate with $(B, a_0, a_1, \ldots, a_{n-1})$ the $(n+1)^2$-tuple $\langle B, \langle S_{ij} : i, j < n\rangle, \langle\langle T_i, U_i\rangle : i < n\rangle\rangle \in \mathcal{B}_0^{(n+1)^2}$. Since $\mathcal{B}_0^{(n+1)^2}$ is wqo, it easily follows that $\mathcal{B}_n$ also is. This proves the lemma. □

The following proposition answers our earlier question by showing that DP and DL are not existentially rich. An apparently stronger result will be given in Theorem 10.

Proposition 9 *The theories DP and DL are not existentially order-rich.*

Proof: It suffices to prove that DL is not existentially order-rich, since $|$ is definable from (either one of) $\wedge$ and $\vee$ by a quantifier-free formula. Suppose that DL is existentially order-rich, and let $\varphi_i(x)$ be the ith existential formula in the sequence witnessing that DL is existentially order-rich. For each i, let (M_i, a_i) be such that M_i is a model of DL, $M_i \models \varphi_i(a_i)$, and for all $j < i$, $M_i \models \neg\varphi_j(a_i)$. Each M_i has a finite sublattice B_i such that B_i is a box, a_i is a tuple from B_i, and $B_i \models \varphi_i(a_i)$. Notice that if $j < i < \omega$, then $B_i \models \neg\varphi_j(a_i)$. By Lemma 8, there are $j < i < \omega$ such that (B_j, a_j) is box-embeddable in (B_i, a_i), and thus, as lattices, (B_j, a_j) is embeddable in (B_i, a_i). Since $\varphi_j(x)$ is existential and $B_j \models \varphi_j(a_j)$, it follows that $B_i \models \varphi_j(a_i)$, which is a contradiction. □

As noted previously, Ershov's theorem does not help in showing that DP and DL have recursive models, although their standard models are recursive. The proof of the following theorem owes much to Ershov's proof.

Theorem 10 *Each of the theories* DP *and* DL *has a recursive, recursively saturated model.*

Proof: It suffices to show that DL has a recursive, recursively saturated model, so we will just consider that case. Let $\mathcal{L}$ be the language of DL. Let $c_0, c_1, c_2, \ldots$ be new, distinct constant symbols, and then let $\mathcal{L}_n = \mathcal{L} \cup \{c_0, c_1, \ldots c_{n-1}\}$ for each $n < \omega$ and $\mathcal{L}_\omega = \mathcal{L} \cup \{c_0, c_1, c_2, \ldots\}$. We will construct a completion D of DL in the language $\mathcal{L}_\omega$ which will be the elementary diagram of a recursively saturated model of DL. Even though D may turn out not to be recursive, the set of atomic sentences in D will be recursive.

Fix a recursive sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$ of the sentences in DL. If $n < \omega$ and Σ is a set of formulas, we say that Σ is *n-consistent* if no contradiction can be derived from Σ in n or fewer steps. We can easily construct a doubly-indexed recursive sequence $\langle \gamma_{ij}(x) : i, j < \omega \rangle$ of 1-ary $\mathcal{L}_\omega$-formulas such that

1. each $\gamma_{ij}(x)$ is an $\mathcal{L}_i$-formula;
2. if $n < \omega$ and $\Gamma(x)$ is a nonempty recursive set of $\mathcal{L}_n$-formulas, then there is some $i < \omega$ such that $\Gamma(x) = \{\gamma_{ij}(x) : j < \omega\}$.

For $d < \omega$, a set C is an $\mathcal{L}_d$*-diagram* if the following hold.

1. The set C consists of $\mathcal{L}_d$-sentences each one of which is either an atomic sentence or the negation of an atomic sentence.
2. If α is an atomic $\mathcal{L}_d$-sentence, then $\alpha \in C$ iff $\neg\alpha \notin C$.
3. For $i, j < d$, the sentence $c_i = c_j$ is in C iff $i = j$.

For each $\mathcal{L}_d$-diagram C, there is a unique $\mathcal{L}$-structure on the set $\{0, 1, 2, \ldots, d-1\}$ whose diagram is C, and we denote this structure by $M(C)$.

We will say that the 5-tuple $P = \langle n, d, I, f, C \rangle$ is a *promise* if the following hold.

1. $I \subseteq n \leq d < \omega$ and $f : I \to d$.
2. C is an $\mathcal{L}_d$-diagram and $M(C)$ is a box.
3. $C \cup T$ is consistent.
4. $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\} \cup C \cup \{\gamma_{ij}(c_{f(i)}) : i \in I$ and $j < n\}$ is n-consistent.
5. If $r < n$ and $r \notin I$, then $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\} \cup C \cup \{\gamma_{ij}(c_{f(i)}) : i \in I,\ i < r$ and $j < n\} \cup \{\gamma_{rj}(x) : j < n\}$ is not n-consistent.

Notice that there is an effective procedure for determining, being given n, d, I, f, C, whether or not $\langle n, d, I, f, C \rangle$ is a promise. A promise is really just an encoded description of a set of $\mathcal{L}_\omega$-sentences. If $P = \langle n, d, I, f, C \rangle$ is a promise, then let

$$\Sigma(P) = \{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\} \cup C \cup \{\gamma_{ij}(c_{f(i)}) : i \in I \text{ and } j < n\}$$

be the set of sentences it encodes. The definition of a promise requires that $\Sigma(P)$ be n-consistent.

We will say that the promise $P = \langle n, d, I, f, C \rangle$ is an n-promise of range d. For any n, d, there are only finitely many n-promises of range d. It is obvious that $\langle 0, 0, \varnothing, \varnothing, \varnothing \rangle$ is a 0-promise .

Now suppose that $n < m < \omega$. If $P_n = \langle n, e, I_n, f_n, C_n \rangle$ is an n-promise and $P_m = \langle m, d, I_m, f_m, C_m \rangle$ is an m-promise, then P_m is an *extension* of P_n if $C_n \subseteq C_m$.

Consequently, if P_n has range e and its extension P_m has range d, then $e \leq d$. Every n-promise has an extension which is an m-promise. This is a consequence of the following claim.

Claim 11 *For each n-promise $P_n = \langle n, e, I_n, f_n, C_n \rangle$, there is an $(n+1)$-promise $P_{n+1} = \langle n+1, d, I_{n+1}, f_{n+1}, C_{n+1} \rangle$ which is an extension of P_n.*

To see this claim, consider some n-promise $P_n = \langle n, e, I_n, f_n, C_n \rangle$. We will construct an $(n+1)$-promise $P_{n+1} = \langle n+1, d, I_{n+1}, f_{n+1}, C_{n+1} \rangle$ which is an extension P_n.

We will construct I_{n+1} and $f_{n+1} : I_{n+1} \to \omega$ recursively; that is, given $i \leq n$ and knowing $I_{n+1} \cap i$ and $f_{n+1}(j)$ for $j \in I_{n+1} \cap i$, we will determine whether or not $i \in I_{n+1}$ and, if $i \in I_{n+1}$, the value of $f_{n+1}(i)$. At the same time, we will construct a sequence $e \leq d_0 \leq d_1 \leq d_2 \leq \cdots \leq d_n < \omega$ and a sequence $D_0, D_1, D_2, \ldots, D_n$ of diagrams, where each D_i is an $\mathcal{L}_{d_i}$-diagram of a box and $C_n \subseteq D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n$. First we consider $i = 0$. There are two alternatives: either there exists $d \geq e$ and an $\mathcal{L}_d$-diagram D such that the following hold:

1. $D \supseteq C_n$ and $M(D)$ is a box;
2. $D \cup T$ is consistent;
3. $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n\} \cup D \cup \{\gamma_{0j}(x) : j \leq n\}$ is not $(n+1)$-consistent;

or there are no such d and D. If there are such a d and D, then they can be found effectively, and we let $d_0 = d$ and $D_0 = D$, and then decree that $0 \notin I_{n+1}$. And if there are no such d and D, then we decree that $0 \in I_{n+1}$, and we effectively get $d_0 \geq e$, $f_{n+1}(0) < d_0$ and an $\mathcal{L}_{d_0}$-diagram D_0 such that $D_0 \supseteq C_n$, $D_0 \cup T$ is consistent, $M(D_0)$ is a box, and $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n\} \cup D_0 \cup \{\gamma_{0j}(c_{f_{n+1}(0)}) : j \leq n\}$ is $(n+1)$-consistent.

For $0 < i \leq n$, proceed in a very similar way. Again, there are two alternatives: either there are $d \geq d_{i-1}$ and an $\mathcal{L}_d$-diagram D such that the following hold:

1. $D \supseteq D_{i-1}$ and $M(D)$ is a box;
2. $D \cup T$ is consistent;
3. $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n\} \cup D \cup \{\gamma_{rj}(c_{f_{n+1}(r)}) : r \in I_{n+1}, r < i \text{ and } j < n\} \cup \{\gamma_{ij}(x) : j \leq n\}$ is not $(n+1)$-consistent;

or there are no such d and D. If there are such a d and D, then they can be found effectively, and we let $d_i = d$ and $D_i = D$, and decree that $i \notin I_{n+1}$. And if there are no such d and D, then we decree that $i \in I_{n+1}$, and we effectively get $d_i > d_{i-1}$, $f_{n+1}(i) < d_i$ and an $\mathcal{L}_{d_i}$-diagram D_i such that $D_i \supseteq D_{i-1}$, $D_i \cup T$ is consistent, $M(D_i)$ is a box, and $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n\} \cup D_i \cup \{\gamma_{rj}(c_{f_{n+1}(r)}) : r \in I_{n+1} \cap (i+1) \text{ and } j < n\}$ is $(n+1)$-consistent.

We have constructed I_{n+1} and f_{n+1}. Let $C_{n+1} = D_n$ and $d = d_n$. The construction guarantees that $P_{n+1} = \langle n+1, d, I_{n+1}, f_{n+1}, C_{n+1} \rangle$ is an $(n+1)$-promise which extends C_n. This proves Claim 11.

If $P_n = \langle n, e, I_n, f_n, C_n \rangle$ is an n-promise and $P_m = \langle m, d, I_m, f_m, C_m \rangle$ is an m-promise extending P_n, then we say that P_m is a *k-extension* of P_n if $k = \max\{r \leq n : I_n \cap r = I_m \cap r \text{ and } f_n|(I_n \cap r) = f_m|(I_m \cap r)\}$. The following claim is easily proved.

Claim 12 *Given n and e, we can effectively find d such that whenever P is an n-promise of range e which has a k-extension which is an $(n+1)$-promise, then it has one with range at most d.*

Putting Claims 11 and 12 together, we can obtain a recursive sequence $P_0, P_1, P_2, \ldots$ such that the following hold for each $n < \omega$.

1. P_n is an n-promise.
2. There is $k \leq n$ such that P_{n+1} is a k-extension of P_n and for no $k' > k$ is there an $(n+1)$-promise which is a k'-extension of P_n.

Having obtained this sequence of promises, let $k_0, k_1, k_2, \ldots$ be the sequence defined as follows: k_n is the least $k < \omega$ such that for some $m > n$, P_m is a k-extension of P_n. Clearly, $k_0 \leq k_1 \leq k_2 \leq \cdots$. The crux of the proof is the following claim.

Claim 13 *The sequence $k_0, k_1, k_2, \ldots$ is unbounded.*

To prove the claim, assume it is false so that there are $m, s < \omega$ such that $k_i = m$ if and only if $i \geq s$. Then there is a subsequence $k_{i_0}, k_{i_1}, k_{i_2}, \ldots$ such that $s = i_0 < i_1 < i_2 < \cdots$ and that whenever $j < r < \omega$, then $P_{k_{i_r}}$ is an m-extension of $P_{k_{i_j}}$. Let $\{j_0, j_1, \ldots, j_{n-1}\} = I_s \cap m$, where $j_0 < j_1 < \cdots < j_{n-1} < m$, and let $\bar{a}$ be the n-tuple $\langle c_{f_s(j_0)}, c_{f_s(j_1)}, \ldots, c_{f_s(j_{n-1})} \rangle$. For each $j < \omega$ let $M_j = M(C_{k_{i_j}})$ and let $a_j = f_{k_{i_j}}(m)$. By Lemma 8, there are $j < r < \omega$ such that $(M_j, \bar{a}, a_j)$ is box-embeddable (and thus, also embeddable as lattices) in $(M_r, \bar{a}, a_r)$. Identifying $(M_j, \bar{a}, a_j)$ with its image under this embedding, we see that there is a $k > m$ such that $(M_r, \bar{a}, a_r)$ is a k-extension of $(M_j, \bar{a}, a_j)$. This contradiction proves Claim 13.

Now let D be the set of $\mathcal{L}_\omega$-sentences σ such that for all sufficiently large n, $\sigma \in \Sigma(P_n)$. Clearly, $D \supseteq T$ and D is consistent.

We show that D is complete. Let σ be an $\mathcal{L}_\omega$-sentence. Then there is $i < \omega$ such that $\sigma = \gamma_{ij}(x)$ for all $j < \omega$. (Of course, the free variable x does not actually occur in $\gamma_{ij}(x)$.) Then, if $\sigma \notin D$, it must be that $D \cup \{\sigma\}$ is inconsistent. Similarly, if $\neg\sigma \notin D$, it must be that $D \cup \{\neg\sigma\}$ is inconsistent. Therefore, as D is consistent, either $\sigma \in D$ or $\neg\sigma \in D$.

In a similar way, we show that D is a Henkin theory; that is, if $\exists x \varphi(x) \in D$, then for some k, $\varphi(c_k) \in D$. Assume $\exists x \varphi(x) \in D$ and that $i < \omega$ is such that $\varphi(x) = \gamma_{ij}(x)$ for all $j < \omega$. It is clear that $i \in I_n$ for all sufficiently large n. Then let $k = f_m(i)$, where m is sufficiently large. This shows that D is the complete diagram for some model M of T. The diagram of M is $\bigcup\{C_n : n < \omega\}$, so M is recursive. The argument showing that M is recursively saturated is just like the one showing that D is a Henkin theory. Consider some recursive set $\Gamma(x)$ of $\mathcal{L}_n$-formulas. There is $i < \omega$ such that $\Gamma(x) = \{\gamma_{ij}(x) : j < \omega\}$. If $\Gamma(x)$ is consistent with D, then $i \in I_n$ for all sufficiently large n. Then, for sufficiently large m, $\Gamma(c_{f_m(i)}) \subseteq D$, so $\Gamma(x)$ is realized in M. This completes the proof of Theorem 10. □

Corollary 14 *There is a nonstandard (even recursively saturated) model $\mathcal{M}$ of* PA *whose divisibility poset $(M, |)$ and divisibility lattice $(M, \wedge, \vee)$ are recursive.*

Proof: Let $(M, \wedge, \vee)$ be a recursive, recursively saturated model of DL. Since it is countable, $(M, \wedge, \vee)$ is resplendent, so it can be expanded to a recursively saturated model $\mathcal{M}$ of PA. □

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