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A Syntactic Approach to Maksimova's Principle of Variable Separation for Some Substructural Logics

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Abstract Maksimova's principle of variable separation says that if propositional formulas $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common and if a formula $A_1 \wedge B_1 \supset A_2 \vee B_2$ is provable, then either $A_1 \supset A_2$ or $B_1 \supset B_2$ is provable. Results on Maksimova's principle until now are obtained mostly by using semantical arguments. In the present paper, a proof-theoretic approach to this principle in some substructural logics is given, which analyzes a given cut-free proof of the formula $A_1 \wedge B_1 \supset A_2 \vee B_2$ and examines how the formula is derived. This analysis will make clear why Maksimova's principle holds for these logics.

1 Introduction In her paper [8] (see also [10]), Maksimova proved a theorem on some relevant logics, including **R** and **E**, which implies the following:

> Suppose that propositional formulas $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common. If a formula $A_1 \wedge B_1 \supset A_2 \vee B_2$ is provable, then either $A_1 \supset A_2$ or $B_1 \supset B_2$ is provable.

When the above property holds for a given logic **L**, we say that *Maksimova's principle of variable separation* (or simply *Maksimova's principle*) holds for **L**. (In this case, **L** is said to be *Maksimova-complete* in Chagrov and Zakharyaschev [4].) In [8], she gave also an example of a relevant logic for which Maksimova's principle doesn't hold. Some relationships among Maksimova's principle, the disjunction property and Halldén-completeness for intermediate logics are studied in [4]. An algebraic characterization of Maksimova's principle is given in [11].

Most of the results on Maksimova's principle obtained so far are proved by using semantical methods. In the present paper, by using a syntactic method, we will show that Maksimova's principle holds for many of the basic substructural logics,

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all of which are extensions of the logic **FL** which has no structural rules. First, we will show Maksimova's principle for logics without weakening. By using the same idea but slightly modifying the proof, we will next show Maksimova's principle for logics with weakening. We will show Maksimova's principle also for some distributive substructural logics including the relevant logics $\mathbf{R}_+$, $\mathbf{RW}_+$ and $\mathbf{TW}_+$, in which the distributive law between *additive* conjunctions and disjunctions holds. Although we will discuss here only Maksimova's principle for some propositional logics, the proof can be naturally extended to their predicate extensions. All the results on Maksimova's principle shown in the present paper except that for $\mathbf{R}_+$ are new.

The basic calculus **FL** is, roughly speaking, the system obtained from the sequent calculus **LJ** for the intuitionistic logic by deleting all of **LJ**'s structural rules. The language of **FL** consists of logical constants t, f, $\top$, and $\bot$, logical connectives $\supset$, $\wedge$, $\vee$, and $*$ (*multiplicative conjunction* or *fusion*). (We can dispense with $\top$, as it can be defined by $\bot \supset \bot$.) To make the present paper self-contained, we will give here the definition of **FL**.

Definition 1.1 For consistency of notation throughout the present paper, we assume that any sequent in **FL** is of the form $A_1; \ldots; A_m \to B$ where $m \geq 0$ and B may be empty. Also, different from the notation in Ono [14], we will use the constant symbols t and f instead of 1 and 0.

FL consists of the following initial sequents:

Initial sequents

1. $A \to A$,
2. $\Gamma; \bot; \Delta \to C$,
3. $\Gamma \to \top$,
4. $\to t$,
5. $f \to$,

and the following rules of inference:

Cut rule

$$\frac{\Gamma \to A \quad \Delta; A; \Sigma \to C}{\Delta; \Gamma; \Sigma \to C}$$

Rules for logical constants

$$\frac{\Gamma; \Delta \to C}{\Gamma; t; \Delta \to C}\ (tw) \qquad\qquad \frac{\Gamma \to}{\Gamma \to f}\ (fw)$$

Rules for logical connectives

$$\frac{\Gamma; A \to B}{\Gamma \to A \supset B}\ (\to\supset) \qquad\qquad \frac{\Gamma \to A \quad \Delta; B; \Sigma \to C}{\Delta; A \supset B; \Gamma; \Sigma \to C}\ (\supset\to)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee B}\ (\to \vee 1) \qquad\qquad \frac{\Gamma \to B}{\Gamma \to A \vee B}\ (\to \vee 2)$$

$$\frac{\Gamma;\ A;\ \Delta \to C \quad \Gamma;\ B;\ \Delta \to C}{\Gamma;\ A \vee B;\ \Delta \to C}\ (\vee \to)$$

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}\ (\to \wedge)$$

$$\frac{\Gamma;\ A;\ \Delta \to C}{\Gamma;\ A \wedge B;\ \Delta \to C}\ (\wedge 1 \to) \qquad \frac{\Gamma;\ B;\ \Delta \to C}{\Gamma;\ A \wedge B;\ \Delta \to C}\ (\wedge 2 \to)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma;\ \Delta \to A * B}\ (\to *) \qquad \frac{\Gamma;\ A;\ B;\ \Delta \to C}{\Gamma;\ A * B;\ \Delta \to C}\ (* \to)\,.$$

Sequent calculi $\mathbf{FL_w}$, $\mathbf{FL_c}$, and $\mathbf{FL_e}$ are defined to be the systems obtained from **FL** by adding the following weakening, contraction, and exchange rules, respectively:

$$\frac{\Gamma;\ \Sigma \to C}{\Gamma;\ A;\ \Sigma \to C}\ (w \to) \qquad \frac{\Gamma \to}{\Gamma \to C}\ (\to w)$$

$$\frac{\Gamma;\ A;\ A;\ \Sigma \to C}{\Gamma;\ A;\ \Sigma \to C}\ (con) \qquad \frac{\Gamma;\ B;\ A;\ \Sigma \to C}{\Gamma;\ A;\ B;\ \Sigma \to C}\ (ex)\,.$$

We will use any combination of suffixes e, c, and w to denote the calculus obtained from **FL** by adding structural rules corresponding to these suffixes. For instance, $\mathbf{FL_{ew}}$ denotes the system **FL** with both the exchange and the weakening rules. For more information on substructural logics introduced here, see, for example, Ono [13] and [14]. Since all logics discussed in this paper are formulated as sequent calculi, we will sometimes identify a sequent calculus with the logic determined by it. We can prove the following theorems. (See Ono and Komori [15] and [13].)

Theorem 1.2 *Cut elimination theorem holds for* **FL**, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, *and* $\mathbf{FL_{ecw}}$.

Theorem 1.3 *Craig's interpolation theorem holds for* **FL**, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$, *and* $\mathbf{FL_{ecw}}$.

Note here that the cut elimination theorem doesn't hold for $\mathbf{FL_c}$, as shown in Bayu Surarso and Ono [3].

2 Maksimova's principle for logics without weakening To explain the idea of our proof of Maksimova's principle, in this section we will discuss Maksimova's principle for the substructural logics without weakening. As shown in the next section, more complicated arguments will be necessary to show Maksimova's principle for logics with weakening. *Throughout this section, we assume that our language does not contain any propositional constant.* This assumption will eliminate nonessential complications in expressing our main theorem (Theorem 2.3) in this section, since the weakening rule becomes admissible for some particular constants, for example, the rule (tw) for the constant t. In the following, $S(A)$ denotes the set of subformulas of a formula A.

Lemma 2.1 *Suppose that formulas $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common. If $\Gamma \to D$ is a sequent satisfying the following three conditions*

1. *all formulas occurring in the sequent are subformulas of either $A_1 \wedge B_1$ or $A_2 \vee B_2$,*
2. *at least one of them belongs to $S(A_1) \cup S(A_2)$,*
3. *at least one of them belongs to $S(B_1) \cup S(B_2)$,*

then it is not provable in $\mathbf{FL_{ec}}$.

Proof: To the contrary, suppose that $\Gamma \to D$ is provable. Then there must be a cut-free proof Π in $\mathbf{FL_{ec}}$ whose endsequent is $\Gamma \to D$. It is easily seen that in any application of a rule of inference in Π, if the lower sequent satisfies the above three conditions then at least one of its upper sequents must also satisfy these conditions. Notice here that this holds for any application of either $(\to *)$ or $(\supset\to)$ since its principal formula must be a member of the set $S(A_1) \cup S(A_2) \cup S(B_1) \cup S(B_2)$. So, at least one of initial sequents of Π must satisfy these three conditions. But, clearly no initial sequent can satisfy all of these conditions. Clearly, the above argument doesn't hold when we have the weakening rule. □

Corollary 2.2 *Suppose that $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common and that* $\mathbf{L}$ *is any one of* $\mathbf{FL}$, $\mathbf{FL_e}$ *and* $\mathbf{FL_{ec}}$. *If Π is a cut-free proof in* $\mathbf{L}$ *of a sequent $\Gamma \to D$ such that*

(∗) *all formulas in it are subformulas of either $A_1 \wedge B_1$ or $A_2 \vee B_2$, and at least one of them belongs to $S(A_1) \cup S(A_2)$,*

then every sequent in Π satisfies also (∗). Moreover, no applications of the following rules of inference appear in Π.

$$\frac{\Gamma; B_1; \Delta \to E}{\Gamma; A_1 \wedge B_1; \Delta \to E}\ (\wedge 2 \to) \qquad \frac{\Delta \to A_1 \quad \Delta \to B_1}{\Delta \to A_1 \wedge B_1}\ (\to \wedge)$$

$$\frac{\Gamma; A_2; \Delta \to E \quad \Gamma; B_2; \Delta \to E}{\Gamma; A_2 \vee B_2; \Delta \to E}\ (\vee \to) \qquad \frac{\Delta \to B_2}{\Delta \to A_2 \vee B_2}\ (\to \vee 2)\,.$$

Proof: We can show that

> for any application I of rules of inference in Π, if the lower sequent of I satisfies condition (∗) then the upper sequent also satisfies (or both of its upper sequents satisfy) condition (∗).

This can be proved without difficulty, except in the case where I is either $(\to *)$ or $(\supset\to)$. Suppose that I is an application of $(\to *)$ of the following form.

$$\frac{\Gamma \to D \quad \Delta \to E}{\Gamma; \Delta \to D * E}\ .$$

By the subformula property, the formula $D * E$ must be either in $S(A_1) \cup S(A_2)$ or in $S(B_1) \cup S(B_2)$. Suppose that the latter holds. Then, both D and E belong to $S(B_1) \cup S(B_2)$. By our assumption, some formulas in $\Gamma; \Delta$ belong to $S(A_1) \cup S(A_2)$. Hence, either $\Gamma \to D$ or $\Delta \to E$ satisfies all of three conditions in Lemma 2.1 and thus it is not provable. This is a contradiction. Thus, $D * E$, and hence both D and E belong to $S(A_1) \cup S(A_2)$. Therefore, the above statement holds in this case. Similarly, we can show that this holds also for $(\supset\to)$. Thus, every sequent in Π also satisfies $(*)$.

Now suppose that any one of the applications stated in Corollary 2.2 appears in Π. Then, by what we have shown in the above, its upper sequent(s) must satisfy $(*)$. On the other hand, (at least one of) the upper sequent(s) contains either B_1 or B_2. Then the sequent, which is of course provable, satisfies all three of the conditions in Lemma 2.1. This is a contradiction. □

Theorem 2.3 *Maksimova's principle holds for* **FL**, $\mathbf{FL_e}$, *and* $\mathbf{FL_{ec}}$. *More precisely, suppose that formulas $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common. Then the following hold for each logic in the above.*

1. *If a sequent $A_1 \wedge B_1 \to A_2 \vee B_2$ is provable, then either $A_1 \to A_2$ or $B_1 \to B_2$ is provable.*
2. *If a sequent $A_1 \wedge B_1 \to A_2$ is provable, then the sequent $A_1 \to A_2$ is provable.*
3. *If a sequent $A_1 \to A_2 \vee B_2$ is provable, then the sequent $A_1 \to A_2$ is provable.*

Proof: Suppose that the sequent $A_1 \wedge B_1 \to A_2 \vee B_2$ is provable in the logic **L**, where **L** is any one of **FL**, $\mathbf{FL_e}$, and $\mathbf{FL_{ec}}$. Clearly, it is not an initial sequent. So we can assume that its cut-free proof Π in **L** (and consequently in $\mathbf{FL_{ec}}$) is of the following form, where I is a rule of inference other than exchange and contraction.

$$\cfrac{\cfrac{\vdots}{A_1 \wedge B_1; \ldots; A_1 \wedge B_1 \to A_2 \vee B_2}\,(I)}{\cfrac{(\textit{some exchanges and contractions})}{A_1 \wedge B_1 \to A_2 \vee B_2}}$$

Then, I must be one of the following rules of inference; $(\wedge 1 \to)$, $(\wedge 2 \to)$, $(\to \vee 1)$, and $(\to \vee 2)$. Suppose that I is $(\wedge 1 \to)$. That is,

$$\frac{A_1 \wedge B_1; \ldots; A_1; \ldots; A_1 \wedge B_1 \to A_2 \vee B_2}{A_1 \wedge B_1; \ldots; A_1 \wedge B_1; \ldots; A_1 \wedge B1 \to A_2 \vee B_2}\,(\wedge 1 \to).$$

Here, the left side of the upper sequent of I contains only one A_1 and others are $A_1 \wedge B_1$. Then by Corollary 2.2, the proof of the upper sequent and hence the whole proof doesn't contain any application of the following rules of inference:

$$\frac{\Gamma; B_1; \Delta \to E}{\Gamma; A_1 \wedge B_1; \Delta \to E}\,(\wedge 2 \to) \qquad \frac{\Delta \to A_1 \quad \Delta \to B_1}{\Delta \to A_1 \wedge B_1}\,(\to \wedge)$$

$$\frac{\Gamma; A_2; \Delta \to E \quad \Gamma; B_2; \Delta \to E}{\Gamma; A_2 \vee B_2; \Delta \to E}\,(\vee \to) \qquad \frac{\Delta \to B_2}{\Delta \to A_2 \vee B_2}\,(\to \vee 2).$$

It means that when an occurrence of the formulas $A_1 \wedge B_1$ and $A_2 \vee B_2$ is introduced in the proof Π, it must be introduced only by rules of the following form:

$$\frac{\Gamma;\, A_1;\, \Delta \to E}{\Gamma;\, A_1 \wedge B_1;\, \Delta \to E}\ (\wedge 1 \to) \qquad\qquad \frac{\Delta \to A_2}{\Delta \to A_2 \vee B_2}\ (\to \vee 1)\,.$$

(Note that these $A_1 \wedge B_1$ and $A_2 \vee B_2$ may be introduced in several places in Π.) Now, we replace first all occurrences of $A_1 \wedge B_1$ by A_1 and of $A_2 \vee B_2$ by A_2 in Π, and then remove every redundant application that occurs by this replacement. The figure thus obtained is in fact a proof in **L** whose endsequent is $A_1 \to A_2$. When I is any one of the other rules, by using the similar argument we can get the proof of either $A_1 \to A_2$ or $B_1 \to B_2$. □

As we mentioned at the beginning of the present section, it is necessary to modify Theorem 2.3 slightly when our language contains propositional constants. For instance, it is easy to see that the sequent $p \wedge (r \wedge t) \to q \supset q$ is provable in **FL**, where t is the propositional constant introduced in Section 1 and $p, q,$ and r are mutually distinct constants. On the other hand, $p \to q \supset q$ is not provable in it. Thus, Case 2 of Theorem 2.3 doesn't hold in the present form.

3 Maksimova's principle for logics with weakening When we have weakening, the situation becomes different from what we mentioned in the previous section. For instance, $p \to \neg p \vee (q \supset q)$ is provable in $\mathbf{FL_w}$, as shown by the following.

$$\cfrac{\cfrac{\cfrac{q \to q}{\to q \supset q}\ (\to\supset)}{\to \neg p \vee (q \supset q)}\ (\to \vee)}{p \to \neg p \vee (q \supset q)}\ (w \to)\,.$$

The sequent $p \to \neg p$ is not provable in it (cf. the third case of Theorem 2.3). So it will be necessary to modify the statement of the principle of variable separation. Basically, our proof of Maksimova's principle for logics without weakening still works. So, when a cut-free proof of a sequent $A_1 \wedge B_1 \to A_2 \vee B_2$ is given, from the endsequent upward in the proof we will search for such an application of rules by which either $A_1 \wedge B_1$ or $A_2 \vee B_2$ is introduced, that is, the last application of either $(\wedge \to)$ whose principal formula is $A_1 \wedge B_1$, or $(\to \vee)$ whose principal formula is $A_2 \vee B_2$. When **L** is one of logics without weakening discussed in Section 2, if $A_1 \wedge B_1$ is obtained from A_1 by an application of $(\wedge \to)$ then $A_2 \vee B_2$ must be obtained from A_2 but not from B_2, by an application of $(\to \vee)$, as shown in Corollary 2.2. Hence, we can transform the original proof into a proof with the endsequent $A_1 \to A_2$, by replacing all occurrences of $A_1 \wedge B_1$ by A_1 and of $A_2 \vee B_2$ by A_2, respectively.

But this argument doesn't work well for logics *with weakening*, since the weakening rule may cause various possibilities. To avoid this, we will consider a cut-free proof of the sequent $A_1;\, B_1 \to A_2 \vee B_2$, instead of that of $A_1 \wedge B_1 \to A_2 \vee B_2$, since the provability of the former sequent follows from the provability of the latter by using weakening rule (though the converse doesn't always hold). Then we can focus our attention only on when and how $A_2 \vee B_2$ will be introduced. The previous argument seems to work well except the case where $(\vee \to)$ is applied before $A_2 \vee B_2$ is introduced, or more precisely, there exists an application of $(\vee\rightarrow)$ below the application of $(\rightarrow\vee)$ whose principal formula is $A_2 \vee B_2$. In such a case $A_2 \vee B_2$ will appear in different *branches* in the proof, and therefore it might be obtained from A_2 in one place but from B_2 in other places. If this happens, then it is impossible to replace the formula $A_2 \vee B_2$ by only one of A_2 and B_2 throughout the proof, preserving the correctness of the proof. In the following, however, we will show that such an application of $(\vee\rightarrow)$ is avoidable.

In this section, we suppose that our language may contain some propositional constants. We note that when we have the weakening rule, constants 0 and 1 are logically equivalent to $\perp$ and $\top$, respectively. The following lemma is proved as a special case of Craig's interpolation theorem given in [15] (Theorem 2.4).

Lemma 3.1 *Let Γ, Σ, and Π be finite sequences of formulas and E be a formula. Suppose that the sequence Σ and the sequence Γ, Π, E have no propositional variables in common. Then for each calculus $\mathbf{FL_w}, \mathbf{FL_{ew}}$, and $\mathbf{FL_{ecw}}$, if the sequent $\Gamma; \Sigma; \Pi \rightarrow E$ is provable then either $\Sigma \rightarrow$ or $\Gamma; \Pi \rightarrow E$ is provable.*

Lemma 3.2 *Let $\mathbf{L}$ be any one of $\mathbf{FL_w}, \mathbf{FL_{ew}}$, and $\mathbf{FL_{ecw}}$. Suppose that $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common and that Λ (and Θ) is an arbitrary finite (possibly empty) sequence of subformulas of A_1 (and B_1, respectively). If the sequent $\Lambda; \Theta \rightarrow A_2 \vee B_2$ is provable in $\mathbf{L}$, then*

1. *there exists a cut-free proof of this sequent in $\mathbf{L}$ which has no applications of $(\vee\rightarrow)$ to sequents with $A_2 \vee B_2$ on the right-hand side,*
2. *either $\Lambda; \Theta \rightarrow A_2$ or $\Lambda; \Theta \rightarrow B_2$ is provable in $\mathbf{L}$.*

Proof: We will prove our lemma for $\mathbf{FL_w}$. By a slight modification of the proof, we can prove it for other cases. Let Π be a cut-free proof of the sequent $\Lambda; \Theta \rightarrow A_2 \vee B_2$. We will prove our lemma using the induction on the height n of Π. When $n = 1$, $\Lambda; \Theta \rightarrow A_2 \vee B_2$ must be an initial sequent. But this happens only when $\perp$ occurs in the sequence Λ, Θ. Then replacing $A_2 \vee B_2$ by A_2, we will get a sequent $\Lambda; \Theta \rightarrow A_2$, which is still an intial sequent. Thus, we have (2). Clearly, this proof satisfies (1).

Next suppose that $n > 1$. Since Π is a cut-free proof, if a sequent in it has $A_2 \vee B_2$ on the right-hand side then its left-hand side must always be of the form $\Lambda^*; \Theta^*$, where Λ^* and Θ^* are finite (possibly empty) sequences of subformulas of A_1 and B_1, respectively. Now, let I be the last inference in Π. We suppose first that the principal formula of I is $A_2 \vee B_2$. Then I must be one of $(\rightarrow w)$, $(\rightarrow \vee 1)$ and $(\rightarrow \vee 2)$. Obviously, (1) is satisfied in this case, since the upper sequent is one of the following; $\Lambda; \Theta \rightarrow$, $\Lambda; \Theta \rightarrow A_2$ and $\Lambda; \Theta \rightarrow B_2$. Neither of them has $A_2 \vee B_2$ on the right-hand side. Thus (1) holds. Also, (2) follows immediately.

Suppose next that the principal formula of I is not $A_2 \vee B_2$ and moreover that I is not $(\vee\rightarrow)$. Then, the upper sequent (or only one of the upper sequents when I is $(\supset\rightarrow)$) is of the form $\Lambda^\dagger; \Theta^\dagger \rightarrow A_2 \vee B_2$, where $\Lambda^\dagger$ and $\Theta^\dagger$ are finite (possibly empty) sequences of subformulas of A_1 and B_1. Then, by the hypothesis of induction, this sequent has a cut-free proof which has no applications of $(\vee\rightarrow)$ to sequents with $A_2 \vee B_2$ on the right-hand side. Since I is not $(\vee\rightarrow)$, (1) holds. Also, by the hypothesis of induction, either $\Lambda^\dagger; \Theta^\dagger \rightarrow A_2$ or $\Lambda^\dagger; \Theta^\dagger \rightarrow B_2$ is provable. By applying I to either of them, we can get the proof of either $\Lambda; \Theta \rightarrow A_2$ or $\Lambda; \Theta \rightarrow B_2$.

Finally suppose that I is $(\vee\rightarrow)$. Without loss of generality, we can assume that Λ is of the form $\Lambda_1;\ A'\vee A'';\ \Lambda_2$ (and hence $A'\vee A''$ is a subformula of A_1) such that $A'\vee A''$ is the principal formula of I. Then the inference I will be of the following form:

$$\frac{\Lambda_1;\ A';\ \Lambda_2;\ \Theta\rightarrow A_2\vee B_2 \quad \Lambda_1;\ A'';\ \Lambda_2;\ \Theta\rightarrow A_2\vee B_2}{\Lambda_1;\ A'\vee A'';\ \Lambda_2;\ \Theta\rightarrow A_2\vee B_2}\ (\vee\rightarrow).$$

By the hypothesis of induction, both of the upper sequents have proofs which have no applications of $(\vee\rightarrow)$ to sequents having $A_2\vee B_2$ on the right-hand side. Also, either $\Lambda_1;\ A';\ \Lambda_2;\ \Theta\rightarrow A_2$ or $\Lambda_1;\ A';\ \Lambda_2;\ \Theta\rightarrow B_2$ is provable, and either $\Lambda_1;\ A'';\ \Lambda_2;\ \Theta\rightarrow A_2$ or $\Lambda_1;\ A'';\ \Lambda_2;\ \Theta\rightarrow B_2$ is provable. When $\Theta\rightarrow B_2$ is provable, then by applying $(w\rightarrow)$ repeatedly, we have $\Lambda_1;\ A'\vee A'';\ \Lambda_2;\ \Theta\rightarrow B_2$. So, by $(\rightarrow\vee 2)$ we get $\Lambda_1;\ A'\vee A'';\ \Lambda_2;\ \Theta\rightarrow A_2\vee B_2$. Thus, both (1) and (2) hold. Next suppose that $\Theta\rightarrow B_2$ is not provable. When $\Lambda_1;\ A';\ \Lambda_2;\ \Theta\rightarrow B_2$ is provable, then $\Lambda_1;\ A';\ \Lambda_2\rightarrow$ must be provable by Lemma 3.1, and hence $\Lambda_1;\ A';\ \Lambda_2;\ \Theta\rightarrow A_2$ is provable, by using weakening. But, since either $\Lambda_1;\ A';\ \Lambda_2;\ \Theta\rightarrow A_2$ or $\Lambda_1;\ A';\ \Lambda_2;\ \Theta\rightarrow B_2$ is provable by our assumption, $\Lambda_1;\ A';\ \Lambda_2;\ \Theta\rightarrow A_2$ must be provable in either case. Similarly, we can show that $\Lambda_1;\ A'';\ \Lambda_2;\ \Theta\rightarrow A_2$ is provable. So, applying $(\vee\rightarrow)$, we get $\Lambda_1;\ A'\vee A'';\ \Lambda_2\rightarrow A_2$. Note that in this application of $(\vee\rightarrow)$, the right-hand side of the sequent is A_2, not $A_2\vee B_2$. So, $\Lambda_1;\ A'\vee A'';\ \Lambda_2;\ \Theta\rightarrow A_2\vee B_2$ follows from this. Hence, both (1) and (2) also hold in this case. □

Using these lemmas, we have the following theorem on Maksimova's principle for logics with weakening in a stronger form.

Theorem 3.3 *Suppose that $A_1\supset A_2$ and $B_1\supset B_2$ have no propositional variables in common. Then the following hold for* $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, *and* $\mathbf{FL_{ecw}}$.

1. *If a sequent $A_1 * B_1\rightarrow A_2\vee B_2$ is provable, then either $A_1\rightarrow A_2$ or $B_1\rightarrow B_2$ is provable.*
2. *If a sequent $A_1 * B_1\rightarrow A_2$ is provable, then either $A_1\rightarrow A_2$ or $B_1\rightarrow$ is provable.*
3. *If a sequent $A_1\rightarrow A_2\vee B_2$ is provable, then either $A_1\rightarrow A_2$ or $\rightarrow B_2$ is provable.*

Proof: We will give a proof of (1). It is clear that $A_1 * B_1\rightarrow A_2\vee B_2$ is provable if and only if $A_1;\ B_1\rightarrow A_2\vee B_2$ is provable. Taking A_1 and B_1 for Λ and Θ, respectively, in Lemma 3.2, we have that either $A_1;\ B_1\rightarrow A_2$ or $A_1;\ B_1\rightarrow B_2$ is provable. Then, by Lemma 3.1 (and then by applying the weakening rule, if necessary), either $A_1\rightarrow A_2$ or $B_1\rightarrow B_2$ is provable. Similarly, we can prove our theorem for (2) and (3). □

Corollary 3.4 *Maksimova's principle holds for* $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, *and* $\mathbf{FL_{ecw}}$. *More precisely, suppose that $A_1\supset A_2$ and $B_1\supset B_2$ have no propositional variables in common. Then the following holds for each logic in the above.*

1. *If a sequent $A_1\wedge B_1\rightarrow A_2\vee B_2$ is provable, then either $A_1\rightarrow A_2$ or $B_1\rightarrow B_2$ is provable.*
2. *If a sequent $A_1\wedge B_1\rightarrow A_2$ is provable, then either $A_1\rightarrow A_2$ or $B_1\rightarrow$ is provable.*

3. *If a sequent $A_1 \to A_2 \vee B_2$ is provable, then either $A_1 \to A_2$ or $\to B_2$ is provable.*

Proof: When we have the weakening, $A_1 * B_1 \to A_1 \wedge B_1$ is provable. So our corollary follows immediately from Theorem 3.3. □

We have discussed Maksimova's principle for various intuitionistic substructural logics, that is, substructural logics obtained from the intuitionistic logic by deleting some of structural rules. But our method works well also for the classical substructural logics. Let **GL** be Girard's linear logic (without exponentials). Also let $\mathbf{GL_c}$ and $\mathbf{GL_w}$ be logics obtained from **GL** by adding the contraction and the weakening, respectively. Then we have the following.

Theorem 3.5 *Maksimova's principle holds for* **GL**, $\mathbf{GL_c}$, *and* $\mathbf{GL_w}$.

4 Adding the distributive law In this and the next sections, we will discuss Maksimova's principle for the positive relevant logics $\mathbf{R}_+$ and $\mathbf{RW}_+$ and also for the logics **DBCC** and **DBCK** introduced in [15]. A common feature among them is that the following distributive law

$$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$$

holds in all of them. In fact, the logics **DBCC** and **DBCK** can be obtained from $\mathbf{FL_w}$ and $\mathbf{FL_{ew}}$, respectively, by adding the distributive law as initial sequents.

In Dunn [5] and Giambrone [7], sequent systems *without the cut rule* for $\mathbf{R}_+$ and $\mathbf{RW}_+$ are introduced and discussed. Also, Slaney introduced in [16] sequent systems *without the cut rule* $\mathbf{LL_{DBCC}}$ and $\mathbf{LL_{DBCK}}$ for **DBCC** and **DBCK**, respectively.

In the following, we will give a definition of these systems, but in a slightly modified form. We will take the same set of logical connectives as that introduced in §1. On the other hand, we will take no logical constants for both $\mathbf{R}_+$ and $\mathbf{RW}_+$, and take only the logical constant $\perp$ for the other distributive logics. First, we will introduce *structures*, which are called *bunches* in [16], recursively as follows:

1. any formula is a structure;
2. for $n \geq 2$, if X_i is a structure for $i = 1, \dots, n$, then both sequences $(X_1, \dots, X_n)$ and $(X_1; \dots; X_n)$ are structures.

Structures of the form $(X_1, \dots, X_n)$—and of the form $(X_1; \dots; X_n)$—are said to be *extensional* (and *intensional*, respectively). Each structure X_i is called an *immediate constituent* of $(X_1, \dots, X_n)$ and $(X_1; \dots; X_n)$. For simplicity's sake, we assume that no immediate constituents of an extensional (and an intensional) structure are extensional (and intensional, respectively). (Thus, a structure of the form $(X; (Y; Z); W)$ will be identified with the structure $(X; Y; Z; W)$.) In other words, extensional structures and intensional structures will appear alternately in a given structure. We will omit parentheses when no confusions will occur.

Intuitively, a structure $A_1, \dots, A_n$ (and $A_1; \dots; A_n$) expresses the formula $A_1 \wedge \cdots \wedge A_n$ (and $A_1 * \cdots * A_n$, respectively). In the following, capital letters X, Y, and Z, and so on, with or without subscripts will denote structures. *Substructures* of a given structure Z can be defined in the usual way. Sometimes, we will pay special attention to a particular occurrence of a substructure X of Z. In such a case, the occurrence X is called a *structure-occurrence* of X (in Z) which is *indicated*. An expression such as $\Gamma(X)$ will be used for denoting a structure with an indicated structure-occurrence of X in it. Sequents in the following calculi are expressions of the form $X \to A$, where X is a structure (possibly empty) and A is a formula.

Now, we will introduce a sequent system **LDFL** for the basic distributive logic **DFL** as follows.

Initial sequents

1. $A \to A$
2. $\perp \to$

Structural rules for extensional structures

$$\frac{\Gamma(Y, X) \to C}{\Gamma(X, Y) \to C}\ (E-ex) \qquad \frac{\Gamma(X) \to C}{\Gamma(X, Y) \to C}\ (E-weak)$$

$$\frac{\Gamma(X, X) \to C}{\Gamma(X) \to C}\ (E-con)\,.$$

Rules for logical connectives

$$\frac{X; A \to B}{X \to A \supset B}\ (\to\supset) \qquad \frac{X \to A \quad \Gamma(B) \to C}{\Gamma(A \supset B; X) \to C}\ (\supset\to)$$

$$\frac{X \to A}{X \to A \vee B}\ (\to \vee 1) \qquad \frac{X \to B}{X \to A \vee B}\ (\to \vee 2)$$

$$\frac{\Gamma(A) \to C \quad \Gamma(B) \to C}{\Gamma(A \vee B) \to C}\ (\vee \to)$$

$$\frac{X \to A \quad Y \to B}{X, Y \to A \wedge B}\ (\to \wedge) \qquad \frac{\Gamma(A, B) \to C}{\Gamma(A \wedge B) \to C}\ (\wedge \to)$$

$$\frac{X \to A \quad Y \to B}{X; Y \to A * B}\ (\to *) \qquad \frac{\Gamma(A; B) \to C}{\Gamma(A * B) \to C}\ (* \to)\,.$$

Some comments would be necessary here for understanding some expressions that appear in the above rules. Let us take the rule $(\supset\to)$ in the above, for instance. Here, $\Gamma(A \supset B; X)$ means the structure obtained from $\Gamma(B)$ by replacing the indicated occurrence of B by the expression $A \supset B; X$. As mentioned in the above, we assume that extensional structures and intensional structures must appear alternately. So, when the indicated occurrence of B appears in an intensional substructure such as $Y; B; Z$ in $\Gamma(B)$, the substructure resulting from this replacement is not $Y; (A \supset B; X); Z$, but $Y; A \supset B; X; Z$. By our definition of sequents, it may happen that X is empty. In such a case, the lower sequent $\Gamma(A \supset B; X) \to C$ must be understood as $\Gamma(A \supset B) \to C$. Similar considerations will be necessary also for $(\to \wedge)$ and $(\to *)$ when at least one of X and Y is empty.

In $(E - weak)$, we must assume that $\Gamma(X)$ is nonempty. Otherwise, it will work just like the weakening rule for sequents with empty antecedents. We allow $(E - con)$ to apply to a sequent of the form $Y, X, X, Z \to C$ and to get the sequent $Y, X, Z \to C$. Thus, X, X in $\Gamma(X, X)$ will be understood not as a substructure but as a *subexpression*. We will use these sloppy definitions simply to avoid unnecessary complications. (See footnotes 28 and 29 in Dunn [6].)

Next, we will define some extensions of **LDFL**. To do so, let us consider the following structural rules for intensional structures:

$$\frac{\Gamma(Y; X) \to C}{\Gamma(X; Y) \to C}\ (I - ex) \qquad \frac{\Gamma(X; X) \to C}{\Gamma(X) \to C}\ (I - con)$$

$$\frac{\Gamma(X) \to C}{\Gamma(X; Y) \to C}\ (I - weak) \qquad \frac{X \to}{X \to C}\ (\to w)\,.$$

Different from $(E - weak)$, it is possible to apply $(I - weak)$ when $\Gamma(X)$ is empty. In this case, $\Gamma(X; Y)$ must be understood as Y.

The sequent system $\mathbf{LRW}_+$ (and $\mathbf{LR}_+$) for the relevant logics $\mathbf{RW}_+$ (and $\mathbf{R}_+$) is obtained from **LDFL** by adding $(I - ex)$ (and both $(I - ex)$ and $(I - con)$, respectively). In the sequent systems **LDBCC** and **LDBCK** for the distributive logic **DBCC** and **DBCK**, we will allow any sequent of the form $X \to$ where X is a structure. They are obtained from **LDFL** by first adding the initial sequent of the form:

2. $\quad \perp \to$

and then adding $(I - weak)$ and $(\to w)$ for **LDBCC**, and $(I - weak)$, $(\to w)$, and $(I - ex)$ for **LDBCK**, respectively. The cut rule in these calculi is a rule of the following form:

$$\frac{X \to A \quad \Gamma(A) \to C}{\Gamma(X) \to C}\ .$$

Although the cut rule is admissible in both **LDBCC** and **LDBCK**, it is not necessarily admissible in $\mathbf{LRW}_+$ and $\mathbf{LR}_+$ (when we don't have the constant t). But still, they are adequate systems for $\mathbf{RW}_+$ and $\mathbf{R}_+$, respectively. (See [7] for $\mathbf{RW}_+$, and [5] and Mints [12] for $\mathbf{R}_+$.)

Now we will show Maksimova's principle for $\mathbf{RW}_+$ and $\mathbf{R}_+$. In the following, we will prove several lemmas which hold for both $\mathbf{LRW}_+$ and $\mathbf{LR}_+$. In the rest of this section, we assume that formulas $A_1 \supset A_2$ and $B_1 \supset B_2$ are given and that they have no propositional variables in common. What we want to show now is that if the sequent $A_1 \wedge B_1 \to A_2 \vee B_2$ is provable, then either $A_1 \to A_2$ or $B_1 \to B_2$ is provable. A formula D is an *A-formula* (a *B-formula*) if D belongs to the set $S(A_1) \cup S(A_2)$ (the set $S(B_1) \cup S(B_2)$, respectively). Similarly, a structure X is an *A-structure* (a *B-structure*) if only A-formulas (B-formulas, respectively) appear in it.

The basic idea of our proof comes from the proof given in §2. But there seem to be difficulties which are peculiar to these sequent calculi introduced in this section.

To see this, consider a proof of the sequent $A_1 \wedge B_1 \to A_2 \vee B_2$. The sequent may be obtained from the sequent $(A_1, B_1) \to A_2 \vee B_2$ by applying $(\wedge \to)$. If we look at the proof of this sequent from the bottom upward, one would expect the existence of an application of $(E - weak)$ by which (A_1, B_1) is *decomposed* into either A_1 or B_1. But this may not always happen in these calculi, since a formula A_1 or B_1 (in (A_1, B_1)) may be decomposed prior to an application of $(E - weak)$.

The following result is known for **R** as the relevance principle or the variable-sharing property. (See Anderson and Belnap [1], p. 417.) Obviously, this also holds for both $\mathbf{RW}_+$ and $\mathbf{R}_+$ since they are subsystems of **R**.

Lemma 4.1 *Suppose Z is a structure and D is a formula such that they have no propositional variables in common. Then $Z \to D$ is not provable.*

Lemma 4.2 *Let $Z; C$ be a structure and D a formula such that they have no propositional variables in common. Then if $Z \to C \vee D$ is provable, $Z \to C$ is provable.*

Proof: Let Π be a (cut-free) proof of $Z \to C \vee D$. Clearly there is no sequent in Π which contains $C \vee D$ in its antecedent. So $C \vee D$ must be introduced in Π by applying $(\to \vee 1)$ or $(\to \vee 2)$ of the following form:

$$\frac{Z' \to C}{Z' \to C \vee D}\ (\to \vee 1) \qquad\qquad \frac{Z' \to D}{Z' \to C \vee D}\ (\to \vee 2)\,.$$

By the subformula property of Π, we can show that $Z'; C$ and D have no propositional variables in common. But $Z' \to D$ cannot be provable by Lemma 4.1. So, Π doesn't contain any application of $(\to \vee 2)$ of the above form. Thus, in each branch of the proof Π, $C \vee D$ must be introduced by an application of $(\to \vee 1)$ whose upper sequent is of the form $Z' \to C$. Now, by replacing every occurrence of formula $C \vee D$ in Π by the formula C and removing redundant sequents, we can get the proof of $Z \to C$.

□

Lemma 4.3 *Suppose that D is an A-formula and Z is a structure which consists only of the formula $A_1 \wedge B_1$, A-formulas, and B-formulas. Let Z' be an arbitrary structure obtained from Z by replacing each occurrence of $A_1 \wedge B_1$ by A_1 and each occurrence of a B-formula by any A-structure. Then, if $Z \to D$ is provable, $Z' \to D$ is also provable.*

Proof: Let Π be a (cut-free) proof of $Z \to D$. We will prove our lemma by induction on the length of Π. Here we will give a proof only when the last inference I of Π is one of $(E - weak)$, $(\supset\to)$, and $(\wedge \to)$.

Case 1: The last inference is $(E - weak)$. We can assume that $Z \to D$ is of the form $\Gamma(X, Y) \to D$ and I is of the following form.

$$\frac{\Gamma(X) \to D}{\Gamma(X, Y) \to D}\ (E - weak)\,.$$

We can assume that by any replacement mentioned in the above, the lower sequent of I will change into the sequent of the form $\Gamma'(X', Y') \to D$. By the hypothesis of induction, $\Gamma(X') \to D$ is provable. Therefore, $\Gamma'(X', Y') \to D$ is also provable by applying $(E - weak)$.

Case 2: The last inference is $(\supset\rightarrow)$. In this case, we can assume that $Z \rightarrow D$ is of the form $\Gamma(C_1 \supset C_2; X) \rightarrow D$ and the last inference is of the following form.

$$\frac{X \rightarrow C_1 \quad \Gamma(C_2) \rightarrow D}{\Gamma(C_1 \supset C_2; X) \rightarrow D} \ (\supset\rightarrow).$$

By the subformula property, the formula $C_1 \supset C_2$ is either an A-formula or a B-formula. Suppose first that it is an A-formula. Then the result of the lower sequent by a given replacement will be of the form $\Gamma'(C_1 \supset C_2; X') \rightarrow D$. By the hypothesis of induction, both $X' \rightarrow C_1$ and $\Gamma'(C_2) \rightarrow D$ are provable. Thus, by $(\supset\rightarrow)$ we can get a proof of $\Gamma'(C_1 \supset C_2; X') \rightarrow D$.

On the other hand, when $C_1 \supset C_2$ is a B-formula, we will get a sequent of the form $\Gamma'(U; X') \rightarrow D$ by a replacement, where U is an A-structure. By using the hypothesis of induction for the right upper sequent of I, we can show that $\Gamma'(V) \rightarrow D$ is provable for *any* A-structure V. Thus, $\Gamma'(U; X') \rightarrow D$ is provable by taking $U; X'$ for V since $U; X'$ is an A-structure.

Case 3: The last inference is $(\wedge \rightarrow)$. When the principal formula of the inference is different from $A_1 \wedge B_1$, the proof goes essentially in the same way as Case 2. When the principal formula is $A_1 \wedge B_1$, the last inference is of the following form.

$$\frac{\Gamma(A_1, B_1) \rightarrow D}{\Gamma(A_1 \wedge B_1) \rightarrow D} \ (\wedge \rightarrow).$$

Then the result of the lower sequent by a replacement will be of the form $\Gamma'(A_1) \rightarrow D$. By the hypothesis of induction, $\Gamma(A_1, U) \rightarrow D$ is provable for any A-structure U. In particular, by taking A_1 for U, we have that $\Gamma(A_1, A_1) \rightarrow D$ is provable. Using $(E - con)$, we can derive that $\Gamma'(A_1) \rightarrow D$ is provable. □

Lemma 4.4 *Suppose Z is a structure that consists only of the formula $A_1 \wedge B_1$, A-formulas, and B-formulas. Let Z_A (and Z_B) be an arbitrary structure obtained from Z by first replacing each occurrence of $A_1 \wedge B_1$ by A_1 (and B_1) and then replacing each occurrence of a B-formula (and an A-formula) in Z by an A-structure (and a B-structure, respectively). Then if $Z \rightarrow A_2 \vee B_2$ is provable, either $Z_A \rightarrow A_2 \vee B_2$ is provable for any such Z_A or $Z_B \rightarrow A_2 \vee B_2$ is provable for any such Z_B.*

Proof: Let Π be a (cut-free) proof of $Z \rightarrow A_2 \vee B_2$. We will prove our lemma by induction on the length of the proof Π. In the following, we will give a proof here when the last inference I is one of $(\supset\rightarrow)$, $(\vee \rightarrow)$, $(\rightarrow \vee 1)$, and $(\wedge \rightarrow)$.

Case 1: The last inference is $(\supset\rightarrow)$. Here $Z \rightarrow A_2 \vee B_2$ is of the form $\Gamma(C_1 \supset C_2; X) \rightarrow A_2 \vee B_2$ and the last inference is of the following form.

$$\frac{X \rightarrow C_1 \quad \Gamma(C_2) \rightarrow A_2 \vee B_2}{\Gamma(C_1 \supset C_2; X) \rightarrow A_2 \vee B_2} \ (\supset\rightarrow).$$

Without a loss of generality, we can assume that $C_1 \supset C_2$ is an A-formula. Also, we can suppose that Z_A and Z_B are of the form $\Gamma_A(C_1 \supset C_2; X_A)$ and $\Gamma_B(U_B; X_B)$, respectively, where U_B is an arbitrary B-structure. Let us consider the right upper sequent of I. By the hypothesis of induction, either $\Gamma_A(C_2) \to A_2 \vee B_2$ is provable, or $\Gamma_B(V_B) \to A_2 \vee B_2$ is provable for any B-structure V_B. Suppose first that $\Gamma_A(C_2) \to A_2 \vee B_2$ is provable. By Lemma 4.3, $X_A \to C_1$ is provable. Hence, by using $(\supset\to)$ we can get a proof of $\Gamma_A(C_1 \supset C_2; X_A) \to A_2 \vee B_2$. Suppose otherwise. Then, by taking V_B for $U_B; X_B$ we can get a proof of $\Gamma_B(U_B; X_B) \to A_2 \vee B_2$ for any U_B.

Case 2: The last inference is $(\vee \to)$. In this case, $Z \to A_2 \vee B_2$ is of the form $\Gamma(C_1 \vee C_2) \to A_2 \vee B_2$ and the last inference is of the following form.

$$\frac{\Gamma(C_1) \to A_2 \vee B_2 \quad \Gamma(C_2) \to A_2 \vee B_2}{\Gamma(C_1 \vee C_2) \to A_2 \vee B_2}\ (\vee \to)\,.$$

Without a loss of generality, we can assume that $C_1 \vee C_2$ is an A-formula. Also, we suppose that Z_A and Z_B are of the form $\Gamma_A(C_1 \vee C_2)$ and $\Gamma_B(U_B)$ for a B-structure U_B, respectively. Taking both of the upper sequents and using the hypothesis of induction, we have that either

1. $\Gamma_A(C_1) \to A_2 \vee B_2$ is provable

or

2. $\Gamma_B(V_B) \to A_2 \vee B_2$ is provable for any B-structure V_B,

and also that either

3. $\Gamma_A(C_2) \to A_2 \vee B_2$ is provable

or

4. $\Gamma_B(W_B) \to A_2 \vee B_2$ is provable for any B-structure W_B.

Now suppose that either (2) or (4) is the case. Then by taking U_B for V_B or W_B, we can get a proof of $\Gamma_B(U_B) \to A_2 \vee B_2$ for an arbitrary U_B. Suppose otherwise. Then both (1) and (3) hold. By applying $(\vee \to)$ to these sequents, we can get a proof of $\Gamma_A(C_1 \vee C_2) \to A_2 \vee B_2$.

Case 3: The last inference is $(\to \vee 1)$. Here the last inference is of the following form.

$$\frac{Z \to A_2}{Z \to A_2 \vee B_2}\ (\to \vee 1)\,.$$

Let us consider the upper sequent. By Lemma 4.3, $Z_A \to A_2$ is provable (for any Z_A). Now by using $(\to \vee 1)$, we have a proof of $Z_A \to A_2 \vee B_2$.

Case 4: The last inference is $(\wedge \to)$. When the principal formula of the inference is different from $A_1 \wedge B_1$, the proof goes essentially in the same way as the above cases. When the principal formula is $A_1 \wedge B_1$, the last inference is of the following form.

$$\frac{\Gamma(A_1, B_1) \to A_2 \vee B_2}{\Gamma(A_1 \wedge B_1) \to A_2 \vee B_2}\ (\wedge \to)\,.$$

Here, Z is $\Gamma(A_1 \wedge B_1)$ and hence we can suppose that Z_A and Z_B are of the form $\Gamma_A(A_1)$ and $\Gamma_B(B_1)$, respectively. By the hypothesis of induction, either

$\Gamma_A(A_1, U) \to A_2 \vee B_2$ is provable for any A-structure U or $\Gamma_B(V, B_1) \to A_2 \vee B_2$ is provable for any B-structure V. If the former holds, then we can show that $Z_A \to A_2 \vee B_2$ is provable, by taking A_1 for U and applying $(E - con)$. Similarly, $Z_B \to A_2 \vee B_2$ is provable when the latter holds. $\square$

Theorem 4.5 *Maksimova's principle holds for* $\mathbf{RW}_+$ *and* $\mathbf{R}_+$. *More precisely, suppose that formulas* $A_1 \supset A_2$ *and* $B_1 \supset B_2$ *have no propositional variables in common. Then the following holds.*

1. *If a sequent* $A_1 \wedge B_1 \to A_2 \vee B_2$ *is provable, then either* $A_1 \to A_2$ *or* $B_1 \to B_2$ *is provable.*
2. *If a sequent* $A_1 \wedge B_1 \to A_2$ *is provable, then* $A_1 \to A_2$ *is provable.*
3. *If a sequent* $A_1 \to A_2 \vee B_2$ *is provable, then* $A_1 \to A_2$ *is provable.*

Proof: (1) Applying Lemma 4.4 to the sequent $A_1 \wedge B_1 \to A_2 \vee B_2$, we have that either $A_1 \to A_2 \vee B_2$ is provable or $B_1 \to A_2 \vee B_2$ is provable. Then by Lemma 4.2, $A_1 \to A_2$ is provable in the former case and $B_1 \to B_2$ is provable in the latter case. (2) and (3) follow immediately from Lemmas 4.3 and 4.2, respectively. $\square$

By a slight modification of the above proof, we can show the following result for the positive relevant logic $\mathbf{TW}_+$. As for a sequent system for $\mathbf{TW}_+$, see [7].

Theorem 4.6 *Maksimova's principle holds for* $\mathbf{TW}_+$.

It may be interesting to compare results in this section with Maksimova's negative result on the relevant logic **RM**. In [8] (and also in [10]), she showed that Maksimova's principle doesn't hold for **RM** which is obtained from **R** by adding the *mingle axiom*: $A \supset (A \supset A)$. In fact, Maksimova showed that for mutually distinct propositional variables p, q, r, and s, the formula $(\neg(p \supset p) \wedge q) \supset (s \vee (r \supset r))$ is provable in **RM**, but neither $\neg(p \supset p) \supset s$ nor $q \supset (r \supset r)$ is provable in it. Our method of proving Maksimova's principle for a logic **L** depends highly on the existence of a cut-free system for **L**. At this moment, it is not so clear where the limitation of our method lies.

5 Distributive logics with weakening Next, we will prove that Maksimova's principle holds for the distributive logics **DBCC** and **DBCK**, both of which have the weakening rule. In Bayu Surarso [2], the author proved Craig's interpolation theorem for both **DBCC** and **DBCK**. In fact, we can show a stronger form of Craig's interpolation theorem which is given below. To explain it, we will introduce some notations. Suppose that for $i = 1, \dots, n$, Y_i is a structure-occurrence in a given structure U such that Y_j and Y_k do not intersect each other when $j \neq k$. Let Z_i be a structure for each i. Then $U_{\{Z_i/Y_i\}_i}$ denotes the structure obtained from U by replacing Y_i by Z_i for each $i = 1, \dots, n$. Also, $U_{\{-/Y_i\}_i}$ denotes the structure obtained from U by simply omitting every Y_i. (In the latter case, we must also omit one of the occurrences of connections ' , ' or ' ; ' (if any) at the end of each Y_i, to make the resulting expression a structure.) The symbol $\mathrm{V}(X)$ denotes the set of propositional variables in X in the following.

Now the following theorem holds for both **LDBCC** and **LDBCK** from which the usual Craig's interpolation theorem for **DBCC** and **DBCK** follows immediately. (See [2] for details.)

Theorem 5.1 *Let U be a structure and Y_i is a structure-occurrence of U for $i = 1, \ldots, n$. Suppose that (1) Y_j and Y_k do not intersect each other when $j \neq k$ and (2) if Y_j and Y_k are substructures of structure-occurrences Z and Z' of U, respectively, then they never appear as the form $Z; Z'$ in U.*

If the sequent $U \to A$ is provable, then there exist formulas C_i for $i = 1, \ldots, n$ such that

1. *$Y_j \to C_j$ are provable for each j,*
2. *$U_{\{C_i/Y_i\}_i} \to A$ is provable,*
3. *$\mathrm{V}(C_j) \subset \mathrm{V}(Y_j) \cap [\mathrm{V}(U_{\{-/Y_i\}_i}) \cup \mathrm{V}(A)]$ for each j.*

To understand the conditions on U in the above theorem, first consider the structure $X; Y_1; (V, Y_2); W$, for instance. Take Y_1 for Z and (V, Y_2) for Z'. Then the structure has a subexpression of the form $Z; Z'$, and hence the conditions in our theorem are not satisfied. On the other hand, when U is of the form $X, Y_1, (V; Y_2), W$, it will satisfy these conditions. If $U \to A$ is provable for this U, then the theorem says that there exist formulas C_1 and C_2 such that

1. both $Y_1 \to C_1$ and $Y_2 \to C_2$ are provable,
2. $X, C_1, (V; C_2), W \to A$ is provable,
3. $\mathrm{V}(C_j) \subset \mathrm{V}(Y_j) \cap [\mathrm{V}(X, V, W) \cup \mathrm{V}(A)]$ for $j = 1, 2$.

We will show in the following that Maksimova's principle holds for **DBCC** and **DBCK** by using these sequent systems **LDBCC** and **LDBCK**. First we will show that the following lemmas hold for both **LDBCC** and **LDBCK**.

Lemma 5.2 *Let X_i and Y_i be (possibly empty) structures for $i = 1, \ldots, n$ and D be a formula. Suppose that the structure $Y_1; \ldots; Y_n$ and the structure $X_1; \ldots; X_n; D$ have no propositional variables in common. If the sequent $(X_1; Y_1), \ldots, (X_n; Y_n) \to D$ is provable, then either $X_1, \ldots, X_n \to D$ is provable or $Y_k \to$ is provable for some k such that $1 \leq k \leq n$. This also holds for the sequent $(Y_1; X_1), \ldots, (Y_n; X_n) \to D$.*

Proof: Suppose that $(X_1; Y_1), \ldots, (X_n; Y_n) \to D$ is provable. By Theorem 5.1, there are formulas C_i for $i = 1, \ldots, n$, each of which consists only of propositional constants such that $Y_i \to C_i$ is provable for each i and $(X_1; C_1), \ldots, (X_n; C_n) \to D$ is provable. On the other hand, it can be easily shown by induction on the complexity of C_i that either $\to C_i$ or $C_i \to$ is provable for each i. Suppose that $C_k \to$ is provable for some k. Then we can show that $Y_k \to$ is provable by using the admissibility of the cut rule. Otherwise, $\to C_i$ is provable for each i. Then, also by using the cut rule, we have that $X_1, \ldots, X_n \to D$ is provable. The proof goes similarly when $(Y_1; X_1), \ldots, (Y_n; X_n) \to D$ is provable. □

The following lemma is an analogue of Lemma 3.2.

Lemma 5.3 *Let* **L** *be* **LDBCC** *or* **LDBCK**. *Suppose that $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common. Let Z be a structure of the form $(U_1; V_1), \ldots, (U_m; V_m)$, where each U_i is either empty or a structure consisting only of formulas in $S(A_1)$ and each V_i is either empty or a structure consisting only of formulas of $S(B_1)$. If the sequent $Z \to A_2 \vee B_2$ is provable, then*

1. *there exists a proof of this sequent which has no applications of* $(\vee \rightarrow)$ *to sequents with* $A_2 \vee B_2$ *on the right-hand side,*
2. *either* $Z \rightarrow A_2$ *or* $Z \rightarrow B_2$ *is provable.*

Proof: We will prove our lemma for **LDBCC**. By a slight modification of the proof, we can prove it also for **LDBCK**. The proof is essentially the same as that of 3.2. Let Π be a proof (in **LDBCC**) of the sequent $Z \rightarrow A_2 \vee B_2$. We will prove our lemma using the induction on the height n of Π.

If $n = 1$, then $Z \rightarrow A_2 \vee B_2$ must be an initial sequent. But this cannot happen because of the form of initial sequents of **LDBCC**. Thus, n must be greater than 1. Since Π contains no applications of the cut rule, if any sequent in it has $A_2 \vee B_2$ on the right-hand side then its left-hand side must be of the form $(U_1^*; V_1^*), \ldots, (U_k^*; V_k^*)$, where each U_i^* is either empty or a structure consisting only of subformulas of A_1 and also each V_i^* is empty or a structure consisting only of subformulas of B_1. Now let I be the last inference in Π. We suppose first that the principal formula of I is $A_2 \vee B_2$. Then I must be one of $(\rightarrow \vee 1)$, $(\rightarrow \vee 2)$, and right weakening. Then the upper sequent is one of the following: $Z \rightarrow$, $Z \rightarrow A_2$, and $Z \rightarrow B_2$. Since none of them has $A_2 \vee B_2$ on the right-hand side, (1) holds obviously. Also (2) follows from this.

Suppose next that the principal formula of I is not $A_2 \vee B_2$ and, moreover, that I is not $(\vee \rightarrow)$. Then the upper sequent (or only one of the upper sequents, when I is $(\supset\rightarrow)$) is of the form $(U_1^\dagger; V_1^\dagger), \ldots, (U_s^\dagger; V_s^\dagger) \rightarrow A_2 \vee B_2$, where each $U_i^\dagger$ is either empty or a structure consisting only of subformulas of A_1 and each $V_i^\dagger$ is either empty or a structure consisting only of subformulas of B_1. Then, by the hypothesis of induction, this sequent has a proof which has no applications of $(\vee \rightarrow)$ to sequents with $A_2 \vee B_2$ on the right-hand side. Since I is not $(\vee \rightarrow)$, (1) holds. Also, by the hypothesis of induction, either $(U_1^\dagger; V_1^\dagger), \ldots, (U_s^\dagger; V_s^\dagger) \rightarrow A_2$ or $(U_1^\dagger; V_1^\dagger), \ldots, (U_s^\dagger; V_s^\dagger) \rightarrow B_2$ is provable. By applying I to either of them, we can get the proof of either $Z \rightarrow A_2$ or $Z \rightarrow B_2$.

Finally, suppose that I is $(\vee \rightarrow)$. Without loss of generality, we can assume that U_1 is of the form. $\Gamma(A' \vee A'')$ such that $A' \vee A''$ is the principal formula of I. Then the inference I will be of the following form.

$$\frac{(\Gamma(A'); V_1), \ldots, (U_m; V_m) \rightarrow A_2 \vee B_2 \quad (\Gamma(A''); V_1), \ldots, (U_m; V_m) \rightarrow A_2 \vee B_2}{(\Gamma(A' \vee A''); V_1), \ldots, (U_m; V_m) \rightarrow A_2 \vee B_2}\ (\vee \rightarrow)$$

By the hypothesis of induction, both of the upper sequents have proofs which have no applications of $(\vee \rightarrow)$ to sequents having $A_2 \vee B_2$ on the right-hand side. Also, (i) either $(\Gamma(A'); V_1), \ldots, (U_m; V_m) \rightarrow A_2$ or $(\Gamma(A'); V_1), \ldots, (U_m; V_m) \rightarrow B_2$ is provable, and also (ii) either $(\Gamma(A''); V_1), \ldots, (U_m; V_m) \rightarrow A_2$ or $(\Gamma(A''); V_1), \ldots, (U_m; V_m) \rightarrow B_2$ is provable. Now suppose that $V_1, \ldots, V_m \rightarrow B_2$ is provable. Then, applying $(I - weak)$ repeatedly, we can get $(\Gamma(A' \vee A''); V_1), \ldots, (U_m; V_m) \rightarrow B_2$. So, by $(\rightarrow \vee 2)$ we have a proof of $(\Gamma(A' \vee A''); V_1), \ldots, (U_m; V_m) \rightarrow A_2 \vee B_2$, which satisfies the condition in (1). Clearly, (2) also holds. Next suppose that $V_1, \ldots, V_m \rightarrow B_2$ is not provable. If $(\Gamma(A'); V_1), \ldots, (U_m; V_m) \rightarrow B_2$ is provable, then by Lemma 5.2 either $\Gamma(A') \rightarrow$ is provable or $U_k \rightarrow$ is provable for some k. In either case, by applying $(E - weak)$, $(I - weak)$, and right weakening, we will have $(\Gamma(A'); V_1), \ldots, (U_m; V_m) \to A_2$. So, by the above assumption (i), $(\Gamma(A'); V_1), \ldots, (U_m; V_m) \to A_2$ is always provable. Similarly, $(\Gamma(A''); V_1), \ldots, (U_m; V_m) \to A_2$ is provable. So, applying $(\vee \to)$ to them, we have $(\Gamma(A' \vee A''); V_1), \ldots, (U_m; V_m) \to A_2$. Now by applying $(\to \vee 2)$ we get a proof of $(\Gamma(A' \vee A''); V_1), \ldots, (U_m; V_m) \to A_2 \vee B_2$ which satisfies the condition in (1). Clearly, (2) holds. □

Using these lemmas, we have the following theorem.

Theorem 5.4 *Suppose that $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common. Then the following holds for* **LDBCC** *and* **LDBCK**.

1. *If a sequent $A_1 * B_1 \to A_2 \vee B_2$ is provable, then either $A_1 \to A_2$ or $B_1 \to B_2$ is provable.*
2. *If a sequent $A_1 * B_1 \to A_2$ is provable, then either $A_1 \to A_2$ or $B_1 \to$ is provable.*
3. *If a sequent $A_1 \to A_2 \vee B_2$ is provable, then either $A_1 \to A_2$ or $\to B_2$ is provable.*

Proof: We will give a proof of (1). By using the admissibility of cut rule, it is clear that $A_1 * B_1 \to A_2 \vee B_2$ is provable if and only if $A_1; B_1 \to A_2 \vee B_2$ is provable. Then by taking $m = 1$ and taking A_1 for U_1 and B_1 for V_1 in Lemma 5.3, we have that either $A_1; B_1 \to A_2$ or $A_1; B_1 \to B_2$ is provable. Hence, by Lemma 5.2, either $A_1 \to A_2$ or $B_1 \to B_2$ is provable. Similarly, we can also prove our theorem for Cases 2 and 3. □

Corollary 5.5 *Maksimova's principle holds for* **LDBCC** *and* **LDBCK**. *More precisely, suppose that $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common. Then the following holds for* **LDBCC** *and* **LDBCK**.

1. *If a sequent $A_1 \wedge B_1 \to A_2 \vee B_2$ is provable, then either $A_1 \to A_2$ or $B_1 \to B_2$ is provable.*
2. *If a sequent $A_1 \wedge B_1 \to A_2$ is provable, then either $A_1 \to A_2$ or $B_1 \to$ is provable.*
3. *If a sequent $A_1 \to A_2 \vee B_2$ is provable, then either $A_1 \to A_2$ or $\to B_2$ is provable.*

Proof: It is easy to show that $A_1 * B_1 \to A_1 \wedge B_1$ is provable. So our corollary follows immediately from Theorem 5.4. □

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