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Predicative Logic and Formal Arithmetic

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Abstract After a summary of earlier work it is shown that elementary or Kalmar arithmetic can be interpreted within the system of Russell's *Principia Mathematica* with the axiom of infinity but without the axiom of reducibility.

1 Historical introduction After discovering the inconsistency in Frege's *Grundgesetze der Arithmetik*, Russell proposed two changes: first, dropping the assumption that to every higher-order entity there corresponds a first-order entity; and second, restricting the assumptions on the existence of higher-order entities, so that instead of a simple hierarchy of first-order, second-order, third-order, and so on, one has a ramified hierarchy in which each order is subdivided into various types in such a way that a condition involving quantification over all entities of one type is never assumed to determine another entity of the same type, but only of a higher type. But Russell found that with these two changes he could not derive classical mathematics, so in *Principia Mathematica* he partially compensated for the first change by assuming the axiom of infinity and, for all mathematical purposes, wholly undid the second change by assuming his axiom of reducibility.

The *predicativist* tradition from Weyl [21] to Feferman [2] and beyond accepts infinity but rejects reducibility and is willing to give up parts of classical mathematics. However, predicativists have been unable to derive classical arithmetic and unwilling to give it up and so have simply assumed it as axiomatic. This assumption has its defenders, as with Feferman and Hellman [3], and also its detractors, as with C. Parsons [15]. It is, therefore, of some philosophical as well as historical interest to ask how large a fragment of classical arithmetic can be developed within the Russellian system of *Principia Mathematica* with infinity but without reducibility.

Now many subsystems of classical or Peano arithmetic have been recognized since the work of Skolem [18], Kalmar [9], Grzegorczyk [4], and other pioneers. Among these the most studied have been the *subprimitive* or *Grzegorczyk* arithmetics $\mathbf{\Gamma}_n$. These agree in allowing definitions by primitive recursion, but only when the function F being defined recursively is bounded by some function already given; or

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what is essentially equivalent, they agree in allowing proofs by mathematical induction but only when the statement φ being proved inductively contains only bounded quantifications, that is, quantifications of the forms $\forall x \leq u$ and $\exists x \leq u$, meaning $\forall u(u \leq v \rightarrow \ldots)$ and $\exists u(u \leq v \wedge \ldots)$. They disagree only as to how many of the fundamental operations of addition $+$, multiplication $*$, exponentiation $\uparrow$, superexponentiation $\Uparrow$, and so on, they admit. Thus, $\mathbf{\Gamma}_2$ or *subelementary* arithmetic admits only $+$ and $*$, $\mathbf{\Gamma}_3$ or *elementary* or *Kalmar* arithmetic admits also $\uparrow$, and $\mathbf{\Gamma}_4$ or *super-elementary* arithmetic admits also $\Uparrow$. The union of all of $\mathbf{\Gamma}_n$ amounts to *primitive recursive* or *Skolem* arithmetic.

Which if any of these systems can be developed or interpreted within the Russellian system? Russell's own later attempt to develop arithmetic within the system (Whitehead and Russell [22], Appendix B) was a failure, but several other logicians over the years have been more successful. We will describe briefly such previous work on the question as is known to us. (Given the frequency of independent rediscovery and of unpublished work in this area, it would not be surprising if there were also other work unknown to us.) All positive results require only the second-order part of the system and only a few types of predicative second-order entities.[1]

To begin with, Skolem ([19], ch. 14), using an ordinal approach to number reminiscent of Dedekind, and later Kripke (unpublished), using a cardinal approach reminiscent of Russell himself, both succeeded in developing the basic laws of addition and multiplication. Skolem seems to have hoped this positive result could be extended to all of primitive recursive arithmetic—perhaps everyone who has ever thought about the problem and got anywhere at all with it has hoped this initially—but Kripke before long realized that it could not, since the consistency of the Russellian system can be proved within primitive recursive arithmetic (to which Gödelian considerations apply).

To reconstruct and elaborate this argument two points are to be noted. First, the consistency of the Russellian system can be proved by iterated application of the proof by Shoenfield [17] of Novak Gal's theorem [14] to the effect that the extension $\mathbf{T}+$ formed by addition of one type of predicative higher-order entity to a consistent theory $\mathbf{T}$ is still consistent. Second, Shoenfield's proof can be formalized in primitive recursive arithmetic, being an application of basic results of finitist, though nonelementary, proof theory. Since the proof can be carried out in primitive recursive arithmetic, it can be carried out in $\mathbf{\Gamma}_n$ for all sufficiently large n, and it will be impossible to develop or interpret such $\mathbf{\Gamma}_n$ in the Russellian system, though a more detailed analysis of the proof would be needed to determine the minimum value of n required.[2]

Nelson [13] would at first glance seem to be relevant to our concerns here: the very name of his enterprise, "predicative arithmetic," suggests as much. At second glance, the cited book seems less relevant, since it is concerned with a system of feasible arithmetic, interpretable in *rudimentary* or *Robinson* arithmetic $\mathbf{Q}$, as in Tarski, Mostowski, and Robinson [20], in which there are no higher-order entities at all and in which the basic laws of addition and multiplication are not deduced but assumed as axiomatic. (Moreover, even Montagna and Mancini [12], which undertakes a reduction of Nelson's starting point to a weak system of *set theory*, is not directly relevant to our present question of reduction to a predicative system of *higher-order logic*.) At third glance, the cited book is seen to be relevant after all, since once one has shown

by whatever means that the most basic laws, the axioms of **Q**, can be obtained in the Russellian system, anything shown in the cited book to be interpretable in the system **Q** will be interpretable in the Russellian system. And in the cited book it is shown, often using in the proofs ideas attributed to Solovay (unpublished), that much indeed is interpretable in the system **Q**. From such results it follows that subelementary arithmetic is interpretable in the Russellian system, in the weak sense that, for any finitely many theorems of subelementary arithmetic, there is a class of individuals (specifiable by a formula of the language of **Q**) that can be proved in the Russellian system to satisfy those theorems.

Wilkie (unpublished) has shown that in fact a system at least as strong as subelementary arithmetic is interpretable in **Q** in the usual sense that there is a *single* class of individuals (specifiable by a formula of the language of **Q**) that can be proved in **Q** to satisfy all the theorems of the system. For a proof see Hájek and Pudlák ([5], pp. 366–71).

One might well expect (or at least hope) that a shorter, simpler proof of the interpretability of subelementary arithmetic in the Russellian system ought to be possible, if one proceeded directly rather than via **Q** and exploited some of the extra strength of the second-order apparatus. Hazen [7] in effect indicates that such a comparatively short and simple proof is indeed available: the cited paper gives an exposition of the derivation of the basic laws of addition and multiplication in the Russellian system, using only a few types of predicative second-order entities, and at the end announces that it can be shown that subelementary arithmetic as a whole can be interpreted in the Russellian system in the ordinary sense (by a direct method used already in Hazen [6]). This direct method is such that if we could only get exponentiation (or super-exponentiation), it would at once give elementary (or super-elementary) arithmetic.

Leivant [10] appears to address the second-order part of the Russellian system and states without proof that in the system being addressed, super-exponentiation is provably total, whereas super-super-exponentiation is not. But to our knowledge the only subsequent published paper relevant to the cited abstract is Leivant [11], which really addresses a different system, a typed version of Church's system rather than a version of the system of types of Russell. Since the Church-style system is stronger than the Russellian system, the positive results about the former do not automatically apply to the latter, but the negative result on the unobtainability of super-super-exponentiation *would* apply to the Russell-style system or at least its second-order part.[3]

The most conspicuous question left open by the foregoing discussion is whether exponentiation (or even super-exponentiation) is obtainable. Our main result is that elementary arithmetic is interpretable in a small fragment of the Russellian system. The bulk of this paper is devoted to the proof.[4]

Section 2 describes more precisely the small fragment in question, which is close to the system originally used in [18]. Section 3 treats order. Sections 4 and 5 treat addition and multiplication in a manner adapted from [13]. Section 6 treats bounded induction in a manner adapted from [6]. Section 7 treats exponentiation. Section 8 contains a subsidiary result about super-exponentiation.[5]

2 The systems Fortunately, it will not be necessary to describe the *whole* of predicative higher-order logic but only its second-order part. The *simple* predicative second-order extension $\mathbf{U}_1(\mathbf{T})$ of a first-order theory $\mathbf{T}$ has, besides the first-order variables $x, y, z, \ldots$ for individuals as in $\mathbf{T}$, monadic second-order variables $X^1, Y^1, Z^1, \ldots$ for certain classes of individuals to be called 1-*classes*, with the membership symbol $\in$ and the following axiom scheme of *comprehension* for formulas φ that do not involve bound 1-class variables:

$$\exists X^1 \forall x (x \in X^1 \longleftrightarrow \varphi(x))$$

wherein φ may contain first- or second-order free variables not displayed. Here X^1 is called the *extension* of φ. $\mathbf{U}_1(\mathbf{T})$ has as well the dyadic, triadic, tetradic, . . . , and analogues of the foregoing monadic apparatus. The *double* predicative second-order extension $\mathbf{U}_2(\mathbf{T})$ has, in addition, second-order variables $X^2, Y^2, Z^2, \ldots$ for certain further classes of individuals to be called 2-*classes*, with comprehension for formulas that (may involve bound 1-class variables but) do not involve bound 2-class variables, as well as the polyadic analogues thereof. The *triple, quadruple, quintuple*, . . . predicative second-order extensions $\mathbf{U}_3(\mathbf{T}), \mathbf{U}_4(\mathbf{T}), \mathbf{U}_5(\mathbf{T}), \ldots$ are similarly defined and the (full) predicative second-order extension $\mathbf{U}_\omega(\mathbf{T})$ of $\mathbf{T}$ is the union of the $\mathbf{U}_n(\mathbf{T})$ for all finite n.

The axiom of infinity comes in several variant versions. Dedekind's historic version is a second-order axiom asserting the existence of a set of individuals, a one-to-one function from that set to itself, and an individual in the set but not in the range of the function. Extending the function by defining it to be the identity outside its original domain, this version reduces to a version asserting the existence of a one-to-one function on the set of *all* individuals and an individual not in the range of that function. By giving the entities asserted to exist names, this existential second-order axiom can, like any other such axiom, be reduced to first-order form. It can then be divided into two conjuncts, one asserting that the named function is one-to-one and the other asserting that the named individual is not in its range. By using the suggestive names *successor* or $\mathbf{S}$ for the one-to-one function and *zero* or $\mathbf{0}$ for the individual not in its range, these two axioms can be made to assume a very familiar form.

(1) $\mathbf{0} \neq \mathbf{S}x$;
(2) $x \neq y \rightarrow \mathbf{S}x \neq \mathbf{S}y$.

Let $\mathbf{T}_0$ be the first-order theory with $\mathbf{0}$ and $\mathbf{S}$ as its nonlogical symbols and (1) and (2) above as its nonlogical axioms. $\mathbf{U}_\omega(\mathbf{T}_0)$ is then a variant version of the second-order part of the Russellian system. Working in $\mathbf{U}_\omega(\mathbf{T}_0)$, for every k there are monadic and polyadic second-order entities of level k, k-classes and k-relations. Call a class *inductive* if it contains $\mathbf{0}$ and is closed under $\mathbf{S}$, containing $\mathbf{S}x$ whenever it contains x. Then for each k there is also a notion of number, a $(k+1)$-number being an individual belonging to every inductive k-class.

For most of our work we will need only $\mathbf{U}_2(\mathbf{T}_0)$, which we call *predicative Dedekind arithmetic* or **PDA**. Working in **PDA**, we will use 'set', 'relation', 'class', and 'number' without further qualification to mean respectively, 1-class, 1-relation, 2-class, and 3-number. (Usually the contrast 'set' versus 'class' is used to mark a difference of order, but we have no need for terms to mark distinctions of order since we

are concerned only with second-order entities, whereas we do have a need for terms to distinguish the first two levels within the second-order.) With this terminology the obstacle encountered in trying to develop formal arithmetic within predicative logic is just this, that conditions mentioning the notion of number (and thus implicitly involving quantification over classes) do not determine classes, so that even if such a condition can be proved inductive, it cannot be concluded that it holds for all numbers. In the proofs to follow, every detour taken is in order to circumvent this obstacle.

The formal arithmetic $\mathbf{R}_3$ is a first-order theory having nonlogical symbols $\mathbf{0}$, $\mathbf{S}$, $+$, $*$, $\uparrow$, and having the nonlogical axioms 1 and 2 above, plus the following.

(3) $x \leq \mathbf{0} \iff x = \mathbf{0}$
(4) $x \leq \mathbf{S}y \iff x \leq y \vee x = \mathbf{S}y$
(5) $x + \mathbf{0} = x$
(6) $x + \mathbf{S}y = \mathbf{S}(x + y)$
(7) $x * \mathbf{0} = \mathbf{0}$
(8) $x * \mathbf{S}y = (x * y) + x$
(9) $x \uparrow \mathbf{0} = \mathbf{S0}$
(10) $x \uparrow \mathbf{S}y = (x \uparrow y) * x$

It also has the axiom scheme of *bounded induction*, according to which, for any φ with all quantifiers bounded, the following counts as an axiom:

$$\forall y\{[\varphi(\mathbf{0}, y) \wedge \forall x(\varphi(x, y) \rightarrow \varphi(\mathbf{S}x, y))] \rightarrow \forall x\varphi(x, y)\}$$

wherein there may be in place of y any finite number of variables.

$\mathbf{R}_3$ is at least as strong as elementary arithmetic as it is usually formulated in the literature. The part of $\mathbf{R}_3$ not involving $\uparrow$ will be called $\mathbf{R}_2$. Our main result will be that for every theorem of $\mathbf{R}_3$ it is provable in **PDA** that the theorem holds when variables are restricted to numbers. Section 3 treats (3) and (4), Section 4 treats (5) and (6), Section 5 treats (7) and (8), Section 6 treats bounded induction, and Section 7 treats (9) and (10). Section 8 considers two further axioms.

(11) $x \Uparrow y = \mathbf{S0}$
(12) $x \Uparrow \mathbf{S}y = x \uparrow (x \Uparrow y)$

Certain laws derivable by induction from (1)–(10) will play a special role as auxiliaries in the construction: the reflexivity and counterinductiveness and transitivity of order, the associativity of addition, the distributivity of multiplication over addition, and the associativity of multiplication. We set these down now for future reference.

(13) $y \leq y$
(14) if $\mathbf{S}x \leq y$, then $x \leq y$
(15) if $x \leq y$ and $y \leq z$ then $x \leq z$
(16) $x + (y + z) = (x + y) + z$
(17) $x * (y + z) = x * y + x * z$
(18) $x * (y * z) = (x * y) * z$

3 Order Call a set *counterinductive* (for y) if it contains x whenever it contains $\mathbf{S}x$ (and it contains y). Define order by setting $x \leq y$ if and only if x is an element of every counterinductive set for y. We claim that (3), (4), and (13)–(15) all follow.

Indeed (13)–(15) are immediate or almost so on unpacking the definitions and so is the first of the following three items, which together would suffice to give (3) and (4):

(i) if $x \leq y$, then $x \leq \mathbf{S}y$;
(ii) if $x \leq \mathbf{0}$, then $x = \mathbf{0}$;
(iii) if $x \leq \mathbf{S}y$, then $x \leq y$ or $x = \mathbf{S}y$.

Then (ii) holds because $\{\mathbf{0}\}$, which is to say the extension of the formula $\rho(u)$ saying $u = \mathbf{0}$, is counterinductive for $\mathbf{0}$, using the fact that $\mathbf{0} \neq \mathbf{S}u$ for any u by axiom 1. And (iii) holds because if X^1 is a counterinductive set for y not containing x and $x \neq \mathbf{S}y$, then $X^1 \cup \{\mathbf{S}y\}$, which is to say the extension of the formula $\rho(u)$ saying $u \in X^1 \vee u = \mathbf{S}y$, is a counterinductive set for $\mathbf{S}y$ not containing x, using the fact that $\mathbf{S}y \neq \mathbf{S}u$ for any $u \neq y$ by axiom 2.

Call a class *closed downward* if x is a member of it whenever y is a member of it and $x \leq y$. The *contraction lemma* for order says that any inductive class A^2 has an inductive subclass B^2 that is closed downward. Indeed, it suffices to let B^2 be the class of all x such that for all $u \leq x$ we have $u \in A^2$. First, we have $\mathbf{0} \in B^2$ by (3) and the assumption that $\mathbf{0} \in A^2$. Further, if we have $x \in B^2$ so that $u \in A^2$ for all $u \leq x$, then we have $x \in A^2$ by (13), and $\mathbf{S}x \in A^2$ by the assumption that A^2 is closed under $\mathbf{S}$, and so by (4) we have $u \in A^2$ for all $u \leq \mathbf{S}x$, and so $\mathbf{S}x \in B^2$. Finally, if $y \in B^2$ so that $u \in A^2$ for every $u \leq y$, and if $x \leq y$, then for any $u \leq x$ we have $u \leq y$ by (15), and so $u \in A^2$ so that $x \in B^2$, completing the proof.

4 Sums Define a relation F^1 to be a *computation of the sum with* x (*of* y (*as* z)) if F^1 is a function from individuals to individuals (and y is in its domain (and $F^1(y) = z$)) and for all individuals u we have

(5a) $\mathbf{0}$ is in the domain of F^1 and $F^1(\mathbf{0}) = x$;
(6a) if $\mathbf{S}u$ is in the domain of F^1 then so is u, and we have $F^1(\mathbf{S}u) = \mathbf{S}F^1(u)$.

If there exists a unique individual z for which there exists a computation of the sum of x with y as z, call it $x + y$, which otherwise will be undefined. Call an individual y *summable* if $x + y$ is defined for all individuals x. The *definiteness lemma* for addition says that for all individuals x we have

(5b) $\mathbf{0}$ is summable and $x + \mathbf{0} = x$;
(6b) if y is summable then so is $\mathbf{S}y$, and we have $x + \mathbf{S}y = \mathbf{S}(x + y)$.

The proof is a more or less standard set-theoretic argument. Uniqueness is easy since (5a) specifies what $x + \mathbf{0}$ must be and (6a) specifies what $x + \mathbf{S}y$ must be given what $x + y$ is. For the existence part of (5b) it suffices to consider the function F^1 with domain $\{\mathbf{0}\}$ and $F^1(\mathbf{0}) = x$. This satisfies (5a) by definition and (6a) vacuously. For the existence part of (6b), consider a computation E^1 of the sum with x of y. If $\mathbf{S}y$

is already in the domain of E^1 we are done. Otherwise consider the function F^1 with domain $F^1 =$ domain $E^1 \cup \{\mathbf{S}y\}$ and with $F^1(u) = E^1(u)$ for $u \neq \mathbf{S}y$ and $F^1(\mathbf{S}y) = \mathbf{S}E^1(y)$. It suffices to show that F^1 is a computation of the sum with x of $\mathbf{S}y$, and for this (5a) and (6a) for $u \neq y$ are immediate from (5a) and (6a) for E^1, whereas (6a) for y holds by definition, completing the proof.

The definiteness lemma implies that the class of summable individuals is inductive. Call x *additive* if x is summable and for all summable v the sum $v + x$ is summable, and moreover, for all individuals u, the sum $u + (v + x)$ is what it ought to be, namely, $(u + v) + x$. The *associativity lemma* for addition says that the class of additive individuals is inductive.

For the proof, recall the usual inductive proof of the associative law of addition 16 using axioms 5 and 6 and mathematical induction. That proof essentially consists of two series of equations, one to prove associativity with $\mathbf{0}$ as the third term and one to prove associativity with $\mathbf{S}x$ as the third term, assuming as induction hypothesis associativity with x as the third term:

$$u + (v + \mathbf{0}) = u + v = (u + v) + \mathbf{0};$$

$$\begin{aligned} u + (v + \mathbf{S}x) &= u + \mathbf{S}(v + x) \\ &= \mathbf{S}(u + (v + x)) \\ &= \mathbf{S}((u + v) + x) \\ &= (u + v) + \mathbf{S}x. \end{aligned}$$

Each equation is justified by (5) or (6) or the induction hypothesis. In the present context, (5b) is identical with (5) but (6b) is weaker than (6), being conditional on summability. To show the equations hold in the present context we need to check that summability holds where needed, using for this purpose our assumption that v is summable, and in the induction step, the assumption that x is additive. It is readily checked that under those assumptions $\mathbf{0}$, v, x, $v + x$, $\mathbf{S}x$, and $\mathbf{S}(v + x)$ are all summable and that the foregoing are all the terms whose summability is required for the above equations, thus completing the proof.

The *closure lemma* for addition says that if x and y are additive, so is $x + y$. For the proof, it suffices to check the following equations:

$$\begin{aligned} u + (v + (x + y)) &= u + ((v + x) + y) \\ &= (u + (v + x)) + y \\ &= ((u + v) + x) + y \\ &= (u + v) + (x + y). \end{aligned}$$

The contraction lemma for addition is a kind of generalization, saying that any inductive class A^2 has an inductive subclass B^2 (containing only additive individuals and) closed under $+$.

For the proof, since the intersection of any two inductive classes is inductive, it may be assumed without loss of generality that A^2 is a subclass of the class of additive individuals from the start, making the associative law available. It then suffices to let B^2 be the class of all $x \in A^2$ such that for all u, if $u \in A^2$ then $u + x \in A^2$. The proofs that B^2 (i) contains $\mathbf{0}$, (ii) is closed under $\mathbf{S}$, and (iii) is closed under $+$, are as follows.

Let $u \in A^2$ be given. For (i), $u + \mathbf{0} = u \in A^2$, and so $\mathbf{0} \in B^2$. For (ii), given $x \in B^2$ so that $u + x \in A^2$ we have $u + \mathbf{S}x = \mathbf{S}(u + x) \in A^2$ since A^2 is inductive, and so $\mathbf{S}x \in B^2$. For (iii), given $x \in B^2$ and $y \in B^2$ we have $u + (x + y) = (u + x) + y$, and $u + x \in A^2$ since $x \in B^2$, and then $(u + x) + y \in A^2$ since $y \in B^2$, and so $x + y \in B^2$. This completes the proof.

What our work to this point shows is that by restricting variables to additive individuals we get an interpretation of axioms 1 – 6 and 13 – 16. Henceforth for convenience we use 'individual' to mean '*additive* individual', so it may be said that individuals satisfy (1) – (6) and (13) – (16).

5 Products Define a relation F^1 to be a *computation of the product with* x (*of* y (*as* z)) if F^1 is a function from individuals to individuals (and y is in its domain (and $F^1(y) = z$)) and for all individuals u we have:

(7a) $\mathbf{0}$ is in the domain of F^1 and $F^1(\mathbf{0}) = \mathbf{S0}$;

(8a) if $\mathbf{S}u$ is in the domain of F^1 then so is u, and we have $F^1(\mathbf{S}u) = F^1(u) + x$.

If there exists a unique z for which there exists a computation of the product of x with y as z, call it $x * y$, which otherwise will be undefined. Call individual y *productive* if $x * y$ is defined for all individuals x.

The definiteness lemma for multiplication says that for all individuals x we have:

(7b) $\mathbf{0}$ is productive and $x * \mathbf{0} = \mathbf{0}$;

(8b) if y is productive then so is $\mathbf{S}y$, and we have $x * \mathbf{S}y = (x * y) + x$.

The proof is exactly like that of the corresponding lemma for addition.

The definiteness lemma implies that the class of productive individuals is inductive. Call x *summably productive* if x is productive and for all productive v the sum $v + x$ is productive and moreover, for any individual u, the product $u * (v + x)$ is what it ought to be, namely, $u * v + u * x$. The *distributivity lemma* says that the class of summably productive x is inductive. For the proof, it suffices to check the equations used in the usual derivation of the distributive law 17:

$$u * (v + \mathbf{0}) = u * v = u * v + \mathbf{0} = u * v + u * \mathbf{0};$$

$$\begin{aligned} u * (v + \mathbf{S}x) &= u * \mathbf{S}(v + x) \\ &= u * (v + x) + u \\ &= (u * v + u * x) + u \\ &= u * v + (u * x + u) \\ &= u * v + u * \mathbf{S}x. \end{aligned}$$

The *additive closure lemma* for multiplication says that if x and y are summably productive, then $x + y$ is summably productive. For the proof, it suffices to check the following equations:

$$\begin{aligned} u * (v + (x + y)) &= u * ((v + x) + y) \\ &= (u * (v + x)) + u * y \\ &= (u * v + u * x) + u * y \\ &= u * v + (u * x + u * y) \\ &= u * v + u * (x + y). \end{aligned}$$

Call x a *multiplicative* if x is summably productive and for all summably productive v, the product $v * x$ is summably productive and, moreover, for any individual u the product $u * (v * x)$ is what it ought to be, namely, $(u * v) * x$. The associativity lemma for multiplication says that the class of multiplicative individuals is inductive. For the proof, it suffices to check the equations used in the usual derivation of the associative law 18 for multiplication:

$$u * (v * \mathbf{0}) = u * \mathbf{0} = \mathbf{0} = (u * v) * \mathbf{0};$$

$$\begin{aligned} u * (v * \mathbf{S}x) &= u * (v * x + v) \\ &= u * (v * x) + u * v \\ &= (u * v) * x + u * v \\ &= (u * v) * \mathbf{S}x. \end{aligned}$$

The closure lemma for multiplication says that if x and y are multiplicative, then $x + y$ and $x * y$ are multiplicative. For the proof of closure under $+$ it suffices to check the following equations:

$$\begin{aligned} u * (v * (x + y)) &= u * (v * x + v * y) \\ &= u * (v * x) + u * (v * y) \\ &= (u * v) * x + (u * v) * y \\ &= (u * v) * (x + y). \end{aligned}$$

The proof of closure under $*$ is exactly like the proof of the corresponding lemma for addition. The contraction lemma for multiplication is a kind of generalization, saying that any inductive class A^2 has an inductive subclass B^2 (containing only multiplicative individuals and) closed under $+$ and $*$.

For the proof, since the intersection of any two inductive classes is inductive, it may be assumed without loss of generality that A^2 is a subclass of the class of multiplicative individuals from the start, making the distributive and associative laws available. By the contraction lemma for addition it may be assumed without loss of generality that A^2 is closed under $+$. It then suffices to let B^2 be the class of all $x \in A^2$ such that for all u, if $u \in A^2$, then $u * x \in A^2$. The proofs that B^2 (i) contains $\mathbf{0}$, (ii) is closed under $\mathbf{S}$, and (iii) is closed under $+$, are as follows. Let $u \in A^2$ be given. For (i), $u * \mathbf{0} = \mathbf{0} \in A^2$ since A^2 is inductive, and so $\mathbf{0} \in B^2$. For (ii), given $x \in B^2$ and $u * x \in A^2$ we have $u * \mathbf{S}x = u * x + u \in A^2$ since A^2 is closed under $+$, and so $\mathbf{S}x \in B^2$. For (iii), given $x \in B^2$ and $y \in B^2$, we have $u * x \in A^2$ since $x \in B^2$, and then $u * y \in A^2$ since $y \in B^2$, and so $u * (x + y) = u * x + u * y \in A^2$ since A^2 is closed under $+$, so that $x * y \in B^2$. The proof that B^2 (iv) is closed under $*$ is exactly

like the corresponding step in the proof of the contraction lemma for addition. This completes the proof.

What our work to this point shows is that by restricting variables to multiplicative individuals, we get an interpretation of axioms 1 – 8 and 13 – 18. Henceforth, for convenience, we use 'individual' to mean 'multiplicative individual', so it may be said that individuals satisfy (1) – (8) and (13) – (18).[6]

6 Bounded induction Now recall that a number is an individual belonging to every inductive class. The closure lemma for numbers states that the numbers (1) contain $\mathbf{0}$, (2) are closed under $\mathbf{S}$, (3) are closed downward, (4) are closed under $+$, and (5) are closed under $*$. Here (1) and (2) are immediate from the definition, whereas (3) – (5) are almost immediate from the contraction lemmas. (For instance, given that x is a number and $u \leq x$, to show that u is a number it suffices to consider any inductive class A^2 and show $u \in A^2$. By the contraction lemma, there is an inductive subclass B^2 of A^2 that is closed downward. Since x is a number and B^2 is inductive, $x \in B^2$. Then since $u \leq x$ and B^2 is closed downward, $u \in B^2$. And then, since B^2 is a subclass of A^2, $u \in A^2$.)

The closure lemma guarantees that all axioms 1 – 8 of $\mathbf{R}_2$ (along with the auxiliaries 13 – 18) hold when quantifiers are restricted to numbers. To show that all axioms (and hence all theorems) of $\mathbf{R}_3$ hold when quantifiers are restricted to numbers, two things remain to be done: (1) to define exponentiation and show that numbers are closed under it and satisfy axioms 9 and 10; and (2) to show that the bounded induction axioms hold when quantifiers are restricted to numbers. These tasks may be taken up in either order and we will first undertake task 2. More precisely, we will show in this section that every bounded induction axiom of $\mathbf{R}_2$ holds when quantifiers are restricted to numbers and we will do so by a proof that will extend automatically to the additional bounded induction axioms of $\mathbf{R}_3$ (mentioning exponentiation), as soon as we accomplish task 1 in the next section.

To commence, an important consequence of the closure lemma is the *absoluteness lemma* which says that for any formula φ of the language with $\mathbf{0}$, $\mathbf{S}$, $\leq$, $+$, and $*$ having all quantifiers bounded, and for any particular number u one has that $\varphi(u)$ holds if and only if $\varphi(u)$ holds *when all its quantifiers are restricted to numbers*, and herein there may be any finite number of variables in place of u. For example, if x is a number, then the following are equivalent:

$$\exists w \leq x(x = \mathbf{S}w);$$
$$\exists w \leq x(w \text{ is a number and } \wedge\, x = \mathbf{S}w).$$

This is immediate: the restriction 'w is a number' is already implied by the restriction '$w \leq x$'.

To continue, the absoluteness lemma greatly simplifies the task of showing that a bounded induction axiom holds when quantifiers are restricted to numbers. To show that such an axiom, as displayed in Section 1, holds when quantifiers are restricted to numbers, it will suffice to consider any fixed number y for which the antecedent $\varphi(\mathbf{0}, y) \wedge \forall x(\varphi(x, y) \rightarrow \varphi(\mathbf{S}x, y))$ of the axiom holds when quantifiers are restricted to numbers and to show that the consequent $\forall x\varphi(x, y)$ of the axiom holds when quantifiers are restricted to numbers. By the absoluteness lemma, the antecedent holding

when quantifiers are restricted to numbers amounts simply to $\varphi(\mathbf{0}, y)$ holding and to $\varphi(x, y) \to \varphi(\mathbf{S}x, y)$ holding for all numbers x, whereas the consequent holding when quantifiers are restricted to numbers amounts simply to $\varphi(x, y)$ holding for all numbers x. So it will suffice to consider a number y such that $\varphi(\mathbf{0}, y)$ holds and $\varphi(u, y) \to \varphi(\mathbf{S}u, y)$ holds for every number u, and to consider any number v, and to show that $\varphi(v, y)$ holds.

To conclude, we show this as follows. Consider the class A^2 of all individuals u such that $u \leq v \to \varphi(u, y)$ holds. By (13) it will suffice to show that v belongs to A^2 and since v is a number, for this it will suffice to show that A^2 is inductive. As to A^2 containing $\mathbf{0}$, $\varphi(\mathbf{0}, y)$ is given. As to A^2 being closed under $\mathbf{S}$, it suffices to show that if we have $u \leq v \to \varphi(u, y)$ then we have $\mathbf{S}u \leq v \to \varphi(\mathbf{S}u, y)$ or equivalently, that if we have $u \leq v \to \varphi(u, y)$ and $\mathbf{S}u \leq v$, then we have $\varphi(\mathbf{S}u, y)$. And indeed, given $u \leq v \to \varphi(u, y)$ and $\mathbf{S}u \leq v$, by (14) we get $u \leq v$ and, hence, $\varphi(u, y)$. From the downward closure of numbers and $u \leq v$, it follows that u is a number, and we are given that $\varphi(u, y) \to \varphi(\mathbf{S}u, y)$ holds for numbers, so we get $\varphi(\mathbf{S}u, y)$ as required to complete the proof.

7 Powers As already indicated, though we have chosen to treat bounded induction before exponentiation, these topics could have been taken in either order and the treatment of exponentiation does not require the results of the preceding section. Rather, we take as our starting point in this section the situation at the very end of Section 5, where it was said that individuals satisfy (1)–(8) and (13)–(18).

Define a relation F^1 to be a *computation of the power of x (to y (as z))* if F_1 is a function from individuals to individuals (and y is in its domain (and $F^1(y) = z$)) and for all individuals u we have

(9a) $\mathbf{0}$ is in the domain of F^1 and $F^1(\mathbf{0}) = \mathbf{S0}$;

(10a) if $\mathbf{S}u$ is in the domain of F^1, then so is u and we have $F^1(\mathbf{S}u) = F^1(u) * x$.

If there exists a unique z for which there exists a computation of the power of x to y as z, call it $x \uparrow y$, which otherwise will be undefined. Call an individual y *powerful* if $x \uparrow y$ is defined for all individuals x.

The definiteness lemma for exponentiation says that for all individuals x we have

(9b) $\mathbf{0}$ is powerful and $x \uparrow \mathbf{0} = \mathbf{S0}$;

(10b) if y is powerful, then so is $\mathbf{S}y$ and we have $x \uparrow \mathbf{S}y = (x \uparrow y) * x$.

The proof is exactly like that of the corresponding lemmas for addition and multiplication. The definiteness lemma implies that the class of powerful x is inductive.

The method used to prove the contraction lemmas for addition and multiplication is not applicable to exponentiation, since that operation is not associative. Still, without making use of the notions of class or number, a *weak* contraction lemma for exponentiation can be proved, saying that every inductive class A^2 has an inductive subclass B^2 (containing only powerful individuals and) closed under $+$ and $*$ such that for all x and y in B^2, $x \uparrow y$ is in A^2.

For the proof, since the intersection of any two inductive classes is inductive, it may be assumed without loss of generality that A^2 is a subclass of the class of powerful individuals from the start. By the contraction lemma for multiplication, it may

be assumed without loss of generality that A^2 is closed under $+$ and $*$. Now let B^2 be the class of all $v \in A^2$ such that for all $u \in A^2$ we have $u \uparrow v \in A^2$. We claim B^2 is inductive. As to B^2 containing $\mathbf{0}$, for any $u \in A^2$, by (9b) and the inductiveness of A^2 we have $u \uparrow \mathbf{0} = \mathbf{S0} \in A^2$. As to B^2 being closed under $\mathbf{S}$, if $v \in B^2$, so that for any $u \in A^2$ we have $u \uparrow v \in A^2$, then for any $u \in A^2$, by (10b) and the closure of A^2 under $*$, we have $u \uparrow \mathbf{S}v = (u \uparrow v) * u \in A^2$, and hence $\mathbf{S}u \in B^2$, proving the claim. If B^2 is not itself closed under $+$ and $*$, it can be replaced by an inductive subclass that is by the contraction lemma for multiplication, to complete the proof.

It is almost immediate from the weak contraction lemma already proved that *numbers* are closed under $\uparrow$. (Indeed, if x and y are numbers it suffices to prove that $x \uparrow y \in A^2$ for any inductive class A^2. Let B^2 be as in the weak contraction lemma. We have $x \in B^2$ and $y \in B^2$ by inductiveness of B^2, since x and y are numbers, so $x \uparrow y \in A^2$ as required.) This was the missing piece needed to show that all theorems of $\mathbf{R}_3$ can be proved in **PDA** to hold when quantifiers are restricted to numbers.

8 Superpowers To treat super-exponentiation, we employ a different method, yielding a weaker result. As to predicative logic, we pass from $\mathbf{PDA} = \mathbf{U}_2(\mathbf{T}_0)$ to $\mathbf{U}_\omega(\mathbf{T}_0)$, where we must distinguish various levels of numbers. As to formal arithmetic, we consider a theory $\mathbf{R}_4^\circ$ in a language without quantifiers (free variables in theorems being tacitly understood as universally quantified). $\mathbf{R}_4^\circ$ has the nonlogical axioms 1 – 12 (besides the logical axioms for identity which may be taken to be reflexivity, symmetry, transitivity, and substitution of equals for equals in atomic formulas). A proof in $\mathbf{R}_4^\circ$ is a sequence of formulas each of which is either an axiom or follows from earlier ones by truth-functional logic or by substitution of terms for variables or by a new *rule of inference* for bounded induction, allowing the inference from the premises $\varphi(\mathbf{0}, y)$ and $\varphi(x, y) \rightarrow \varphi(\mathbf{S}x, y)$ to the conclusion $\varphi(x, y)$, for all formulas φ of the quantifier-free language. The part of $\mathbf{R}_4^\circ$ not involving $\Uparrow$ may be called $\mathbf{R}_3^\circ$.

As a first step, define a relation F^1 to be a *computation of the superpower of x (to y (as z))* if F^1 is a function from 3-numbers to 3-numbers (and y is in its domain (and $F^1(y) = z$)) and for all 3-numbers u we have

(11a) $\mathbf{0}$ is in the domain of F^1 and $F^1(\mathbf{0}) = \mathbf{S0}$;

(12a) if $\mathbf{S}u$ is in the domain of F^1 then so is u, and we have $F^1(\mathbf{S}u) = x \uparrow F^1(u)$.

If there exists a unique z for which there exists a computation of the superpower of x to the y as z, call it $x \Uparrow y$, which otherwise will be undefined. Call a 3-number y *superproductive* if $x \Uparrow y$ is defined for all 3-numbers x. The definiteness lemma for super-exponentiation says that for all 3-numbers x the following hold.

(11b) $\mathbf{0}$ is superproductive and $x \Uparrow \mathbf{0} = \mathbf{S0}$;

(12b) if y is superproductive then so is $\mathbf{S}y$, and we have $x \Uparrow \mathbf{S}y = x \uparrow (x \Uparrow y)$.

The proof is much the same as that of the corresponding lemmas for other arithmetic operations.

What will now be shown is that $\mathbf{R}_4^\circ$ is interpretable in $\mathbf{U}_\omega(\mathbf{T}_0)$ in the following very weak sense: for every finite set Θ of theorems of $\mathbf{R}_4^\circ$, or what comes to the same thing—since the conjunction of any finite set of theorems is itself a theorem—for any single theorem θ of $\mathbf{R}_4^\circ$, there is a k_θ such that for all $n \geq k_\theta$ the theorem θ holds when the variables are taken to range over n-numbers. This condition is to be understood as requiring that the values of all terms in θ be defined when the values of the variables in θ are taken to be n-numbers, but not that the values of these terms themselves be n-numbers. Note that, weak as it is, interpretability in this sense does imply the consistency of $\mathbf{R}_4^\circ$ relative to $\mathbf{U}_\omega(\mathbf{T}_0)$: it shows that $\mathbf{0} \neq \mathbf{0}$ cannot be a theorem of the former unless it is a theorem of the latter. In fact, it guarantees that any result not mentioning $\Uparrow$ and provable in $\mathbf{R}_4^\circ$ is provable in $\mathbf{U}_\omega(\mathbf{T}_0)$ to hold for n-numbers for all sufficiently large n.[7]

The proof is by induction on the length of the proof of θ in $\mathbf{R}_4^\circ$ with five cases to be distinguished. First, consider the case where θ is any axiom but (11) or (12). The results of the previous sections literally say that the 3-numbers are closed under addition, multiplication, and exponentiation and satisfy all theorems of $\mathbf{R}_3$, but the proofs of the previous sections actually show that this holds for k-numbers for any $k \geq 3$. So in the case under consideration we may take $k_\theta = 3$.

Second, consider the case where θ is one of the axioms 11 or 12. The definiteness lemma says that the 3-class of all superproductive y is inductive. So it contains all 4-numbers. The same proof shows that for any $k \geq 3$ and any $(k+1)$-numbers x and y, $x \Uparrow y$ is defined and is a k-number. So in the case under consideration we may take $k_\theta = 4$.

Third, consider the case where θ is inferred by truth-functional logic from various θ_i for which appropriate k_i has already been found. In this case we may take k_θ to be the maximum of the k_i.

Fourth, consider the case where θ is inferred by substitution of terms for variables from some ψ for which an appropriate k_ψ has already been found. Note that, given functions that for all k take $(k+m)$-numbers to k-numbers, and given functions that for all k take $(k+n)$-numbers to k-numbers, composition of the former functions with the latter functions yields functions that take $(k+p)$-numbers to k-numbers, where $p = m + n$. Since $+$, $*$, and $\uparrow$ carry k-numbers to k-numbers and $\Uparrow$ carries $(k+1)$-numbers to k-numbers, this guarantees that for any term $t(u, v, \ldots)$ built up from variables $u, v, \ldots$ using the symbols $\mathbf{0}$, $\mathbf{S}$, $+$, $*$, $\uparrow$, and $\Uparrow$, there is a p_t such that for all k the value of the term t is defined and a k-number whenever the values of the variables are $(k+p_t)$-numbers. So in the case under consideration we may let p_θ be the maximum of the p_t for t a term occurring in θ, and then let $k_\theta = k_\psi + p_\theta$.

Fifth, consider the case where $\theta = \theta(x, y)$ is inferred by induction from

$$\psi = \theta(\mathbf{0}, y) \wedge (\theta(x, y) \rightarrow \theta(\mathbf{S}x, y))$$

for which an appropriate k_ψ has been found. It follows that for any fixed k_ψ-number y the $(k_\psi + 1)$-class of all k_ψ-numbers x such that $\theta(x, y)$ holds is inductive and so contains all $(k_\psi + 1)$-numbers. The same proof shows it contains all $(k+1)$-numbers for any $k \geq k_\psi$. So in the case under consideration we may take $k_\theta = k_\psi + 1$. This completes the proof.

Essentially the same method can be applied to $\mathbf{R}_3^\circ$ to show that it can be interpreted in a similar very weak sense in $\mathbf{U}_1(\mathbf{T}_0)$.[8] The method is also applicable in other situations of related interest.[9]

NOTES

1. Heck [8] considers a Fregean system in which only the second and not the first of Russell's two changes to the system of the *Grundgesetze* is made, namely, a predicative extension of the first-order fragment of the *Grundgesetze* as studied in T. Parsons [16]. In this first-order fragment, a version of the axiom of infinity is already provable and does not have to be assumed, and a pairing function is definable, so that in considering extensions only monadic higher-order entities need to be assumed. Heck shows that in the context he considers, the basic laws of addition and multiplication can be derived using just *one* level of predicative second-order entities. Because his Fregean system is stronger than the Russellian, this result does not transfer directly. But it does suggest a *goal* that a development of the basic laws of addition and multiplication within a Russellian system might hope to achieve, and that indeed is achievable, as can be seen from the development below.
2. Shoenfield states the theorem for a particular $\mathbf{T}$, namely, Zermelo-Frankel set theory, where $\mathbf{T}^+$ boils down to Gödel-Bernays set theory, but his proof, which is formulated in terms of the epsilon-symbol theorem—though easily reformulable in terms of the cut-elimination theorem—is more widely applicable. Shoenfield himself notes that his proof gives a primitive recursive function F for which it is provable that if there were any derivation of a contradiction in $\mathbf{T}^+$ with Gödel number $\leq x$, there would be a derivation of a contradiction in $\mathbf{T}$ with Gödel number $\leq F(x)$. To prove consistency for a fragment of the Russellian system with a fixed finite number m of types of higher-order entities, it would be necessary to consider the mth function in the sequence defined by $F^0(x) = x$ and $F^{k+1}(x) = F(F^k(x))$. To prove consistency for the Russellian system as a whole, it would be necessary to proceed by induction and consider the function F^* defined by the recursion $F^*(\mathbf{0}, x) = x, F^*(\mathbf{S}y, x) = F(F^*(y, x))$. Whichever $\mathbf{\Gamma}_n$ supplies F will supply each F^m and be uninterpretable in any fragment of the Russellian system involving only finitely many types, whereas $\mathbf{\Gamma}_{n+1}$ will supply F^* and be uninterpretable in the Russellian system as a whole. It appears that one may take $n = 4$ which would make our positive results below best possible, since as Kripke already noted, the only conspicuous nonelementary step in the argument is the use of cut elimination, which involves a super-exponential function but nothing worse.
3. Thus this confirms the estimate conjectured in the preceding note. Other claims announced without proof in [10] pertain to transfinite iteration in systems of the kind there addressed. Predicative analysis as in [2] iterates the hierarchy of levels of second-order entities into the transfinite, subject to a restriction of *autonomy*, roughly to the effect that extension of the system of types up to an ordinal ξ is permitted *provided* the ordinal ξ is describable in as much of the system of types as we already have. If claims such as those in the cited abstract could be substantiated for a Russell-style as opposed to a Church-style system and subject to a restriction of autonomy, the whole of classical arithmetic would arguably become *deducible* in a predicativist framework, where heretofore it has simply been *assumed*.
4. This work obviously leaves open a number of questions about the exact strength of the Russellian system and natural subsystems thereof as compared with various well-known weak systems of arithmetic from the literature, though an unusually helpful report by an anonymous referee has, besides providing some important references to the more recent literature and other useful comments, sketched proofs of some interesting partial results.

5. An earlier draft of this paper by the first author alone proved that elementary arithmetic can be interpreted in a weak sense in the Russellian system. The second author suggested that by using the methods of [6], elementary arithmetic could be interpreted in the ordinary sense in a small fragment of the Russellian system. This suggestion was then verified by both authors. The historical information given comes largely from the second author, and the treatment of super-exponentiation in the closing section from the first author.

6. Our work thus far and the fairly routine extension thereof that would be needed to obtain the interpretability of **Q** , makes no essential use of the notion of class, and could have been carried out in $\mathbf{U}_1(\mathbf{T}_0)$. For whereas the contraction lemmas were stated as theorems about classes, associating to any class A^2 a subclass B^2, they could easily be restated as metatheorems about formulas, associating to any formula α a stronger formula β.

7. The result can be extended to a system allowing bounded recursion—or essentially equivalently, bounded induction—as does super-elementary arithmetic $\mathbf{\Gamma}_4$ as it is usually formulated in the literature. Details are omitted here because the positive result is so weak. Indeed, it may even be regarded as negative, since what the construction of this section actually shows is that given the existence of a class closed under all the fundamental operations up to the mth, there follows the existence of an interpretation in $\mathbf{U}_\omega(\mathbf{T}_0)$ in the very weak sense for a theory of the fundamental operations up to the $(m+1)^{\text{st}}$. If the possibility of the latter is excluded (say by consistency proof considerations, as in an earlier note) then the possibility of the former is excluded as well. (It thus appears that in the most natural sense of the phrase super-exponentiation is *not* provably total in the Russellian system.)

8. The idea would be to show that for every theorem $\mathbf{R}_3$ there is a formula $\kappa_\varphi(u)$ such that it is provable in $\mathbf{U}_1(\mathbf{T}_0)$ that κ_φ is inductive and that $\kappa_\varphi(x) \wedge \kappa_\varphi(y) \to \varphi(x, y)$, wherein it is required that the values of the terms appearing in φ are defined for x and y satisfying κ_φ but not that the values of such terms themselves satisfy κ_φ. The proof is by induction on the length of the proof of φ. The proof of the weak contraction lemma for exponentiation associates to any inductive formula α a stronger formula $\alpha^\dagger = \beta$ such that powers of members of class B determined by β are members of the class A determined by α. In the case where φ is obtained by substitution in θ, for κ_φ we may take $\kappa_\theta^{\dagger\dagger\cdots\dagger}$, where the number of iterations of $\dagger$ matches the depth of nesting of $\uparrow$ in the terms substituted in θ to yield φ. In the case where φ is obtained by induction from ψ, we may take $\kappa_\varphi = \kappa_\psi \wedge \varphi$. Remarks similar to the preceding note apply.

9. Notably the following: Gödel's first incompleteness theorem holds for the rudimentary arithmetic **Q** or any other theory at least as strong. Indeed, the system **Q** was introduced precisely in order to prove such a result. Textbook presentations tend to leave the impression that **Q** is a very weak system and that one would have to go to a much stronger system to obtain a formalization of the proof of the first incompleteness theorem, which is the crucial step in proving the second incompleteness theorem. This impression is reinforced by Bezboruah and Shepherdson [1], which proves the second incompleteness theorem for **Q** by a method which the cited authors emphasize is un-Gödelian and inapplicable to stronger systems. But it has long been known to specialists that the formalization of the proof of the first incompleteness theorem can in fact be carried out in a comparatively weak system, namely, elementary arithmetic $\mathbf{\Gamma}_3$. Indeed, the system $\mathbf{\Gamma}_3$ was introduced precisely in order to prove such a result. Moreover, it is now known to specialists that **Q** is not such a very weak system after all, in that many ostensibly stronger systems are interpretable in it, including systems involving a certain amount of exponentiation. Though $\mathbf{\Gamma}_3$ does not seem to be among the systems for which interpretability has been explicitly stated in the published literature, by essentially the same

method as in the preceding note, the results on interpretability (in a very weak sense) in $\mathbf{U}_1(\mathbf{T}_0)$ indicated there can be extended to interpretability in $\mathbf{Q}$.

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