

Strictly Primitive Recursive Realizability, II: Completeness with Respect to Iterated Reflection and a Primitive Recursive ω -Rule

ZLATAN DAMNJANOVIC

Abstract The notion of strictly primitive recursive realizability is further investigated, and the realizable prenex sentences, which coincide with primitive recursive truths of classical arithmetic, are characterized as precisely those provable in transfinite progressions $\{\text{PRA}(b) \mid b \in \underline{\mathbf{Q}}\}$ over a fragment $\text{PR}-(\Sigma_1^0\text{-IR})$ of intuitionistic arithmetic. The progressions are based on uniform reflection principles of bounded complexity iterated along initial segments of a primitive recursively formulated system $\underline{\mathbf{Q}}$ of notations for constructive ordinals. A semiformal system closed under a primitive recursively restricted ω -rule is described and proved equivalent to the transfinite progressions with respect to the prenex sentences.

1 Introduction In Damnjanovic [3] we introduced the notion of strictly primitive recursive realizability and proved with respect to it the soundness of a Gentzen-style formulation of the fragment $\text{PR}-(\Sigma_1^0\text{-IR})$ of intuitionistic arithmetic with the induction rule restricted to Σ_1^0 formulas and with terms and defining axioms for all primitive recursive (p.r.) functions. Like Kleene's recursive realizability, this notion is nonclassical in that there are classically true sentences that are neither realizable nor, in an appropriate sense, falsifiable, and there are realizable sentences that are classically false. For sentences in prenex normal form, however, our primitive recursive realizability coincides with primitive recursive truth in the sense of Kleene [8], and here we investigate this classical part of the realizability notion, seeking to characterize it independently in natural ways. In addition, we hope to develop methods that would facilitate the task of finding natural characterizations of the nonclassical part.

The methodology employed throughout the paper owes much to the early work of Feferman, who showed in [5] that transfinite progressions over the classical Peano

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Arithmetic (PA) based on uniform reflection principles iterated along initial segments of Kleene’s recursively formulated system O of ordinal notations give a complete representation of first-order arithmetical truth. (In §1 we describe in detail what distinguishes our approach from Feferman’s.) A crucial element in the argument of [5] is a proof that the progressions are closed under Shoenfield’s recursively restricted ω -rule, originally introduced in [13]. (The uniform reflection principles around which the transfinite progressions are built may be seen as attempts at *formalization* of the ω -rule.) The resulting characterization of arithmetical truth as “the closure of PA under iterated reflection,” or alternatively and equivalently, as “the closure of PA under the recursive ω -rule,” may be interpreted as providing a quasi-constructive justification of arithmetical truth as determinable *from below* by constructive processes. Our main result (Theorem 8.3) shows that these characterizations are analogously preserved when PA is replaced by the intuitionistic PR-(Σ_1^0 -IR), the transfinite progressions are appropriately defined as in §1, the ω -rule is restricted primitive recursively in a suitable fashion, and arithmetical truth is replaced by strictly primitive recursive realizability. This analogy strongly suggests that, from a broadly constructivist point of view, the concept of realizability explored here, which amounts to primitive recursive truth for prenex sentences, constitutes a natural notion. A parallel analogy relating transfinite progressions based on the intuitionistic version of PA—the Heyting Arithmetic (HA)—supplemented with terms for all p.r. functions and appropriate defining axioms, and Kleene’s recursive realizability, was established in Dragalin [4]. In view of the essential role that notations for constructive ordinals play in these characterizations, and considering that in general the problem of deciding whether or not a given integer represents such a notation is at least as difficult as that of deciding truth or realizability, it is doubtful that a *reduction* to more elementary concepts has been achieved in the process. Nonetheless, the fact that these relationships hold may be relevant for the philosophical debate about the extent of various constructivist conceptions of arithmetic. We do not discuss those issues here. A discussion of the bearing of restricted ω -rules on the characterization of finitism can be found in Ignjatovic [7].

Transfinite progressions based on iterated reflection, and primitive recursively restricted ω -rules, have been previously studied for different purposes in López-Escobar [10], and Schmerl [11] and [12]. In [10], such an ω -rule is added to true sentences of the form $t_1 = t_2$ and $t_1 \neq t_2$ for any closed terms t_1, t_2 , over intuitionistic logic, and the resulting system is proved equivalent to HA. In [11] a fine structure generated by reflection principles formulated in terms of partial truth definitions for arithmetic and iterated over the classical Primitive Recursive Arithmetic (PRA) along a fixed total p.r. well-ordering of integers is studied and, among other results, the well-known theorem of Kreisel and Levy to the effect that PA = “PRA+ uniform reflection schema for PRA” is derived. In [12] primitive recursively restricted ω -rules are used to study similarly defined transfinite progressions over PA. Our restricted ω -rule differs from those employed by these authors.

In §1 we define a transfinite progression of Gentzen-style proof systems $\text{PRA}(b)$ obtained by extending PR-(Σ_1^0 -IR) by suitably formulated reflection principles iterated along initial segments of a *primitive recursively* based system \underline{O} of notations for constructive ordinals. In §2 we prove the soundness of $\text{PRA}(b)$ for $b \in \underline{O}$ with respect to strictly primitive recursive realizability (Theorem 3.3). As a consequence,

we derive, in Theorem 3.4, that the provably recursive functions of $\text{PRA}(b)$ for any $b \in \underline{O}$ are the same as those of $\text{PR}-(\Sigma_1^0\text{-IR})$, namely, the p.r. functions. In §3 we establish, along with some other facts, the completeness of the transfinite progressions $\{\text{PRA}(b) | b \in \underline{O}\}$ with respect to prenex realizable sentences (Theorem 4.10). We develop these techniques further in §4 and proceed to introduce in §5 a semiformal system closed under a primitive recursively restricted ω -rule. In §6 we prove, in Theorem 7.1, that the transfinite progressions $\{\text{PRA}(b) | b \in \underline{O}\}$ are closed under the primitive recursive ω -rule under a certain hypothesis, and show that, on the other hand, all prenex realizable sentences are derivable by means of the ω -rule (Theorem 7.2). In the process we obtain another proof of the prenex completeness of $\{\text{PRA}(b) | b \in \underline{O}\}$ with respect to realizability, under the aforementioned hypothesis. This hypothesis is proved in §8, in Theorem 8.2, and we finally obtain in Theorem 8.3 our main result, a characterization of the realizable prenex sentences.

2 For present purposes it will be convenient to use a slightly different indexing of p.r. functions than the one we employed in [3].¹ By Kleene's Normal Form Theorem for recursive functions (see [8], §58), for each n there is a p.r. predicate T_{n+1}^* and a p.r. function U such that, for any n -place p.r. function φ ,

$$\begin{aligned} \varphi(\vec{x}_n) = y &\iff \exists z(T_{n+1}^*(e, \vec{x}_n, z) \ \& \ U(z) = y) \\ &\iff \forall z(T_{n+1}^*(e, \vec{x}_n, z) \ \& \ U(z) = y) \\ &\iff \varphi(\vec{x}_n) = U(\mu y T_{n+1}^*(e, \vec{x}_n, y)) \end{aligned}$$

holds for some integer e , a Kleene index of φ . This allows us to unofficially expand the language $\mathcal{L}(\text{PRA})$ as follows: for any formula $\varphi(x, y)$, we read ' $\varphi(x, [e](y))$ ' as an abbreviation for ' $\exists z \exists u (T_2^*(e, y, z) \ \& \ U(z) = u \ \& \ \varphi(x, u))$ '.² The Kleene indexing will be an essential element of our definition of transfinite progressions. Following Kleene [9], we use it to introduce a system \underline{O} of notations for ordinals.³

The principal tool in many arguments throughout the paper is the following analogue of Kleene's Recursion Theorem formulated for p.r. functions.

Theorem 2.1 (Primitive Recursion Theorem) *For any p.r. function $\varphi(y, \vec{x}_n)$ there is a p.r. function $\varphi^*(\vec{x}_n)$ with Kleene index b such that*

$$\varphi(b, \vec{x}_n) = \varphi^*(\vec{x}_n).$$

The proof (see [9]) shows how to obtain b primitive recursively from a given Kleene index of φ .

It was Turing who first proposed to counter incompleteness by extending a given intuitively correct formal system T of arithmetic with the sentence $\text{Con}(T)$ that expresses its consistency and imagining that the process continues indefinitely (see Turing [15]). It is easy to see that the process can be continued into the transfinite by showing that the finite stages can be described in a uniform primitive recursive way. However, the attempt to turn this intuitive idea into a formally precise definition encounters difficulties.⁴ Turing was aware of these complications but did not rigorously work out the details of the technique required for such a construction. This was done

by Feferman who showed how Gödel's technique for obtaining self-referential sentences can be applied to the task. We employ this general method and define the particular progressions we are interested in.

In [5] Feferman focused on the progressions based on the *uniform reflection principle* in which at the successor stages $\alpha + 1$ one adds all instances of the schema

$$\forall x \exists y \text{Prf}(b, y, \ulcorner \varphi(x) \urcorner) \rightarrow \forall x \varphi(x)$$

for $|b| = \alpha$, and the limit stages are simply defined as the unions of the preceding stages.⁵ Our procedure differs from Feferman's in several respects. One superficial difference is that we primarily work with Gentzen-style systems—such as the system PR-(Σ_1^0 -IR) studied in [3]—and so instead of the uniformized proof-predicate $\text{Prf}(x_1, x_2, x_3)$, we refer to the relation that holds between an ordinal notation m for α , a code s for a sequence $(s)_0, \dots, (s)_k$ of Gödel numbers of sequents and the Gödel number of a sequent $\Delta \vdash \varphi$ just in case s codes a derivation in the corresponding system with $\Delta \vdash \varphi$ as the endsequent. Secondly, the systems we consider, unlike Feferman's, are based on intuitionistic logic. Some other important differences are motivated by our desire to formulate a system of transfinite progressions that can be proved sound and complete with respect to strictly primitive recursive realizability. We use only indices of p.r. functions for ordinal notations, whereas Feferman relies on an indexing of all recursive functions. Furthermore, we do not distinguish between the successor and the limit stages: at each stage b , $b \in \underline{O}$, we add (roughly) all instances of the schema

$$\forall x \exists y \exists z (\text{Prf}(z, y, \ulcorner \varphi(x) \urcorner) \ \& \ z <^* b) \rightarrow \forall x \varphi(x).$$

This will enable us to effectively preserve *primitive recursive content* at limit stages. Finally, and perhaps most importantly, we formulate the reflection principles requiring a stronger condition to be satisfied for $\forall x \varphi(x)$ to be derivable at stage b than merely that proofs of all instances $\varphi(\underline{m})$ exist at stages $<_0 b$ and that they be enumerable by a function provably recursive at b : it is necessary that the proofs—in our case derivations—of the instances be all of *bounded complexity* in a specific sense to be explained shortly.

Let D be a PR-derivation (see [3], §4) with endsequent $\Gamma \vdash A$. Call a term t *operative in a derivation* D if t occurs as an instantiating term in a sequent of the form $\Gamma \vdash B(t)$ that results by an application of $(\forall E)$ in D or serves as the premise of an application of $(\exists I)$ in D . For each subderivation D^* of D we define *order of* D^* as follows. For any $n \geq 0$, D^* is of order n if it involves no applications of any of the rules (IND), $(\rightarrow E)$, $(\forall E)$, or $(\exists I)$. Assuming subderivations of order n have been defined, we say that D^* is of order $n + 1$ provided:

1. D_1 and D_2 are of order $\leq n$ if D^* is obtained from D_1 and D_2 by (IND) or by $(\rightarrow E)$;
2. D_1 is of order $\leq n$ if D^* is obtained by $(\forall E)$ from D_1 ; and
3. for any term t operative in D^* , $\lambda \vec{y}_k . t(\vec{y}_k) \in G_n$.

(The Grzegorzcyk classes G_n were defined in [3]). For all other rules of PR-(Σ_1^0 -IR), D^* is of order n if all the immediate subderivation(s) of D^* are of order $\leq n$. We write

$D : \Gamma \vdash_n A$ if D is a PR-derivation of order n , and $D : \vdash_n A$ if Γ is empty. Note that if D is of order n , then D is also of order m for any $m \geq n$; but not every PR-derivation is of arbitrarily low order.

We now define a transfinite progression of Gentzen-style proof systems $\text{PR}(b)$ based on an appropriately formulated uniform reflection principle. The technique is well known from [5], §3, so we omit the details. The first step is to formalize derivability in the sequent system $\text{PR}-(\Sigma_1^0\text{-IR})$.⁶ We let $\#E$ be the Gödel number of an expression ‘ E ’, and $\#\Delta$ the Gödel number of a finite sequence Δ of expressions. Let ‘ $x <^* y$ ’ be a fixed Σ_1^0 formula that defines the preordering $<^*$. Using Gödel’s *Self-Reference Lemma* we obtain a formula $\underline{\text{Der}}(x_1, x_2, x_3, x_4, x_5)$ expressing a p.r. relation Der with the following property: $\text{Der}(b, \#D, n, \#\Gamma, \#A)$ holds if and only if D is a $\text{PR}(b)$ -derivation of order n with endsequent $\Gamma \vdash A$, that is, a finite sequence of sequents such that each sequent in the sequence is either an axiom of $\text{PR}-(\Sigma_1^0\text{-IR})$, or a sequent of the form

$$(*) \quad \Gamma, \forall x \exists y (\underline{\text{Der}}_n((y)_0, (y)_1, \ulcorner \Delta(\dot{x}_k) \urcorner, \ulcorner \varphi(\dot{x}) \urcorner) \ \& \ (y)_0 <^* \underline{b}), \Delta \vdash \forall x \varphi(x)$$

for some $n > 0$, Γ , Δ and φ —which we write $\text{REF}_n(b, \Gamma, \Delta, \varphi)$ —or is derived from some preceding member(s) of the sequence by a single application of one of the rules of $\text{PR}-(\Sigma_1^0\text{-IR})$, and $\Gamma \vdash A$ is the endsequent of D . (We assume that in the process the definition of *order* of a derivation is extended to $\text{PR}(b)$ -derivations by stipulating in addition that a derivation that consists of a reflection axiom of the form $\text{REF}_n(b, \Gamma, \Delta, \varphi)$ for some Γ, Δ, φ , is of order m for any $m \geq n$.⁷ Also, for each n , we let $\underline{\text{Der}}_n(b, y, u, v) \equiv \underline{\text{Der}}(b, y, \underline{n}, u, v)$). We call the resulting system—which amounts to $\text{PR}-(\Sigma_1^0\text{-IR})$ plus the reflection principles $\text{REF}_n(b, \Gamma, \Delta, \varphi)$ for all $n > 0$, Γ, Δ , and φ — $\text{PR}(b)$.⁸ Sequents of the form $\text{REF}_n(b, \Gamma, \Delta, \varphi)$ are meant to express the reflection principles of bounded complexity central to our definition of transfinite progressions.

It is easily shown that, if $a <^* b$ provably in $\text{PR}-(\Sigma_1^0\text{-IR})$, then any sequent derivable in $\text{PR}(a)$ has a $\text{PR}(b)$ -derivation; moreover, it can be proved that, for any a, b , and each k ,

$$\begin{aligned} \text{PR}-(\Sigma_1^0\text{-IR}) \vdash \underline{\text{Der}}_m(\underline{a}, x, u, v) \ \& \ \underline{\text{Der}}_k(\underline{1}, y, \ulcorner x <^* \underline{a} \urcorner, \ulcorner x <^* \underline{b} \urcorner) \\ \rightarrow \exists z \underline{\text{Der}}_m(\underline{b}, z, u, v) \end{aligned}$$

for all but finitely many $m \geq k$. Finally, using a form of Primitive Recursion Theorem (cf. Lemma 2.3 of Feferman [6]), we may obtain a p.r. function χ such that, for any a, b

$$a <_0 b \implies \text{Der}_1(1, \chi(a, b), \ulcorner x <^* \underline{a} \urcorner, \ulcorner x <^* \underline{b} \urcorner).$$

3 Let t and n be integers, and A, B, C , sentences. We inductively define the relation ‘ $t \Vdash_n A$ ’:

$$\begin{aligned} t \Vdash_n A & \iff : t = 0 \text{ and } A \text{ is true, if } A \text{ is atomic;} \\ t \Vdash_n (B \ \& \ C) & \iff : (t)_0 \Vdash_n B \text{ and } (t)_1 \Vdash_n C; \end{aligned}$$

$$t \Vdash_n (B \vee C) \iff: (t)_0 = 0 \text{ and } (t)_1 \Vdash_n B, \text{ or } (t)_0 \neq 0 \text{ and } (t)_1 \Vdash_n C;$$

$$t \Vdash_n (B \rightarrow C) \iff: \text{In}(n, t) \text{ and } \forall j \geq n \text{ In}(j, e_{j+1}(t, \langle j \rangle)) \text{ and} \\ \forall j \geq n \forall b [b \Vdash_j B \implies e_{j+1}(e_{j+1}(t, \langle j \rangle), \langle b \rangle) \Vdash_j C];$$

$$t \Vdash_n \exists x B(x) \iff: \text{for some } m, (t)_1 \Vdash_n B(\underline{m}) \text{ and } (t)_0 = m;$$

$$t \Vdash_n \forall x B(x) \iff: \text{In}(n, t) \text{ and } \forall m [e_{n+1}(t, \langle m \rangle) \Vdash_n B(\underline{m})].$$

Here, $\text{In}(n, t)$ holds if t is an n -index and so a Kleene index of a function in the Grzegorzczak class G_n , and the functions $e_{j+1} \in G_{j+1}$ enumerate the classes G_j (see §2 of [3] for details). A is (*strictly primitive recursively*) *realizable* if $\exists t \exists n t \Vdash_n A$. If Γ is a sequence of sentences, we write $\vec{t}_m \Vdash_n \Gamma$ if $t_i \Vdash_n A_i$ for each A_i in Γ , $1 \leq i \leq m$.

In preparation for the proof of soundness with respect to strictly primitive recursive realizability of transfinite progressions defined in §1, we state, in a sharpened form, the principal result of [3], the Soundness Theorem for PR-(Σ_1^0 -IR). Detailed examination of the proof of Theorem 5.1 in [3] reveals the following.⁹

Theorem 3.1 *Let D be a PR-derivation and suppose that $D : \Gamma \vdash_p A$, where $\text{lth}(\Gamma) = m$ and all variables free in Γ or A are among \vec{x}_k . Then*

$$\forall q \exists n < q + p + 1 \exists f \in G_n \forall j > \max(q, n) \forall \vec{t}_m \forall \vec{n}_k [\vec{t}_m \Vdash_q \Gamma(\vec{n}_k) \\ \implies f(\vec{t}_m, \vec{n}_k) \Vdash_j A(\vec{n}_k)].$$

That is, every derivation in PR-(Σ_1^0 -IR) is (positively) q -fulfillable for all q in the sense of [3], §5, and order of the derivation allows us to place a bound on the complexity of an appropriate p.r. function f . Further analysis shows that witnesses to the q -fulfillability of D —an integer n and an n -index of a function $f \in G_n$ —may be obtained from $\#D$ and q uniformly by an elementary and an *almost elementary* function, respectively.

Theorem 3.2 *There is an elementary function Ψ and a function $\Phi \in G_0$ such that, if D is a PR-derivation and $D : \Gamma \vdash_p A$ (where Γ and A are as in Theorem 3.1), then $\forall q \Psi(\#D, q) < q + p + 1$ and*

$$\forall q \forall j \geq \max(q, \Psi(\#D, q)) + 1, \\ \forall \vec{t}_m \forall \vec{n}_k [\vec{t}_m \Vdash_q \Gamma(\vec{n}_k) \implies e_{\Psi(\#D, q)+1}(\Phi(\#D, q), \langle \vec{t}_m, \vec{n}_k \rangle) \Vdash_j A(\vec{n}_k)].^{10}$$

We now proceed to prove the soundness of PR(b) for all $b \in \underline{Q}$.

Theorem 3.3 *There is an elementary function Ψ^* and a function $\Phi^* \in G_0$ such that for any b , if $D : \Gamma \vdash_p A$ is a PR(b)-derivation and $b \in \underline{Q}$ (where Γ and A are as in Theorem 3.1), then $\forall q \Psi^*(b, \#D, q) < q + p + 1$ and*

$$\forall q \forall j \geq q + p + 1, \\ \forall \vec{t}_m \forall \vec{n}_k [\vec{t}_m \Vdash_q \Gamma(\vec{n}_k) \implies e_{\Psi^*(b, \#D, q)+1}(\Phi^*(b, \#D, q), \langle \vec{t}_m, \vec{n}_k \rangle) \Vdash_j A(\vec{n}_k)].$$

Proof: The argument is a tricky, self-referential one and so we go into some detail. We modify and extend the definitions of the functions Ψ and Φ from Theorem 3.2 to obtain definitions of Ψ^* and Φ^* . We define $\Psi^*(b, \#D, q)$ analogously to $\Psi(\#D, q)$ in all cases except when D consists of a reflection axiom of the form $\text{REF}_n(b, \Gamma, \Delta, \varphi)$

for some $n, \Gamma, \Delta, \varphi$, in which case we set $\Psi^*(b, \#D, q) =: q + n$. As for Φ^* , we first define a function $\Phi^+(a, b, \#D, q)$ by course-of-values recursion from elementary functions on analogy to $\Phi(\#D, q)$ in all cases except when D consists of a reflection axiom of the form $\text{REF}_n(b, \Gamma, \Delta, \varphi)$; then we let

$$\Phi^+(a, b, \#D, q) =: \Lambda \vec{x}_m z \vec{u}_j \vec{y}_k \cdot \beta(a, q, n, z, \langle u_j \rangle),$$

where $m = \text{lth}(\Gamma)$, $j = \text{lth}(\Delta)$, and all variables free in Γ, Δ , or φ are among \vec{x}_k (see [3], §2 for the $\Lambda \vec{x}_m$ notation); β is the elementary function such that, for any c , if c is a k -index of some p.r. function $\lambda x y v. \psi(x, y, v)$, then $\beta(c, q, n, z, u)$ is a $\max(k, q + n + 1)$ -index of the function

$$\lambda x. e_{q+n+1}(\psi((e_{q+1}(z, \langle x \rangle))_{0,0}, (e_{q+1}(z, \langle x \rangle))_{0,1}, q), u)$$

for any choice of q, n, z , and u . Then $\Phi^+ \in G_0$. By the primitive recursion theorem, there is an integer a^* that is a 0-index of the function $\Phi^* \in G_0$ such that $\Phi^*(b, \#D, q) = \Phi^+(a^*, b, \#D, q)$. Then, in particular, for each q, n, z , and \vec{u}_j , we have that $\beta(a^*, q, n, z, \langle \vec{u}_j \rangle)$ is a $q + n + 1$ -index of the function

$$\lambda x. e_{q+n+1}(\Phi^*((e_{q+1}(z, \langle x \rangle))_{0,0}, (e_{q+1}(z, \langle x \rangle))_{0,1}, q), \langle \vec{u}_j \rangle).$$

We now argue, by transfinite induction on $|b|$ for $b \in \underline{\mathbb{Q}}$, that the functions Ψ^* and Φ^* have the desired properties. For $|b| = 0$, the proof is essentially the same as that of Theorem 3.2. (We slightly modify the argument to obtain the present bound on j .) Assume, as the induction hypothesis, that the theorem holds for all $d <_0 b$ for some fixed b , to show the same for b . We proceed by induction on the length of $\text{PR}(b)$ -derivations. The sole new element is the case when D consists of a reflection axiom of the form $\text{REF}_n(b, \Gamma, \Delta, \varphi)$. (Hence, D is a $\text{PR}(b)$ derivation of order n). Then, for all q , $\Psi^*(b, \#D, q) = q + n$ by definition, so the first part of the theorem holds. Fix q and assume that $\vec{t}_m \Vdash_q \Gamma(\vec{n}_k)$,

$$t \Vdash_q \forall x \exists y (\underline{\text{Der}}_n((y)_0, (y)_1, \ulcorner \Delta(\vec{n}_k) \urcorner \varphi(x)(\vec{n}_k) \urcorner) \ \& \ (y)_0 <^* \underline{b})$$

and $\vec{u}_j \Vdash_q \Delta(\vec{n}_k)$. Then $\text{In}(q, t)$, and for each i ,

$$e_{q+1}(t, \langle i \rangle) \Vdash_q \exists y (\underline{\text{Der}}_n((y)_0, (y)_1, \ulcorner \Delta(\vec{n}_k) \urcorner \varphi(i)(\vec{n}_k) \urcorner) \ \& \ (y)_0 <^* \underline{b}).$$

For each i , let $r_i =: e_{q+1}(t, \langle i \rangle)$. Then

$$(r_i)_1 \Vdash_q (\underline{\text{Der}}_n((r_i)_{0,0}, (r_i)_{0,1}, \ulcorner \Delta(\vec{n}_k) \urcorner \varphi(i)(\vec{n}_k) \urcorner) \ \& \ (r_i)_{0,0} <^* \underline{b}).$$

The latter is expressible as a Σ_1^0 sentence, hence is true if realizable. Then for each i , $(r_i)_{0,1}$ is the Gödel number of a derivation $D_i : \Delta(\vec{n}_k) \vdash_n \varphi(i)(\vec{n}_k)$ in $\text{PR}(b_i)$, where $(b_i)_0 = (r_i)_{0,0}$ and $b_i <^* b$. But then for each i , $b_i \in \underline{\mathbb{Q}}$ and $|b_i| < |b|$. From the induction hypothesis and the assumption that $\vec{u}_j \Vdash_q \Delta(\vec{n}_k)$, it follows that, for each i ,

$$\exists f_i \in G_{\Psi^*(b_i, \#D_i, q)} \subseteq G_{q+n+1} \forall p \geq q + n + 1$$

$$f_i(\vec{u}_j) \Vdash_p \varphi(i)(\vec{n}_k) \text{ where } f_i \text{ has index } \Phi^*(b_i, \#D_i, q).$$

In particular, we have that, for each i ,

$$e_{q+n+1}(\Phi^*(b_i, \#D_i, q), \langle \vec{u}_j \rangle) \Vdash_{q+n+1} \varphi(i)(\vec{n}_k),$$

that is, for each i ,

$$e_{q+n+1}(\Phi^*((e_{q+1}(t, \langle i \rangle))_{0,0}, (e_{q+1}(t, \langle i \rangle))_{0,1}, q), \langle \vec{u}_j \rangle) \Vdash_{q+n+1} \varphi(i)(\vec{n}_k).$$

But this means that, for each i ,

$$e_{q+n+2}(\beta(a^*, q, n, t, \langle \vec{u}_j \rangle), \langle i \rangle) \Vdash_{q+n+1} \varphi(i)(\vec{n}_k),$$

and therefore,

$$\beta(a^*, q, n, t, \langle \vec{u}_j \rangle) \Vdash_{q+n+1} \forall x \varphi(x)(\vec{n}_k).$$

By the definition of Φ^* , and given that $\beta \in G_0$, we have that

$$e_1(\Phi^*(b, \#D, q), \langle \vec{t}_m, t, \vec{u}_j, \vec{n}_k \rangle) \Vdash_{q+n+1} \forall x \varphi(x)(\vec{n}_k).$$

But then, for all $j \geq q + n + 1$,

$$e_{\Psi^*(b, \#D, q)+1}(\Phi^*(b, \#D, q), \langle \vec{t}_m, t, \vec{u}_j, \vec{n}_k \rangle) \Vdash_j \forall x \varphi(x)(\vec{n}_k),$$

as required. \square

As a corollary to the soundness theorem, we have the following.

Theorem 3.4 *Suppose $b \in \underline{\mathcal{Q}}$, and D is a PR(b)-derivation of $\Gamma \vdash A$, where Γ is empty and A is a sentence of \mathcal{L}' . Then*

1. *for some a and m , $a \Vdash_m A$;*
2. *if A is of the form $\forall x \exists y B(x, y)$, where the formula $B(x, y)$ has x, y , as sole free variables, then there is a p.r. function such that for all m , $B(\underline{m}, \underline{f(m)})$ is true;*
3. *if A is a prenex formula of the form $\forall x_1 \exists y_1, \dots, \forall x_n \exists y_n B(\vec{x}_n, \vec{y}_n)$ with no two consecutive quantifiers of the same kind and $B(\vec{x}_n, \vec{y}_n)$ is a p.r. predicate, then there are p.r. functions f_1, \dots, f_n such that for all \vec{m}_n , $B(\underline{\vec{m}}_n, \underline{f_1(m_1)}, \dots, \underline{f_n(m_n)})$.*

This strengthens Theorems 7.1 and 7.2 of [3]. We may conclude that the provably recursive and the definable functions of PR(b) for all $b \in \underline{\mathcal{Q}}$ are the same as those of PR-(Σ_1^0 -IR)—which = PR(b) for $b = 1$ —namely, precisely the p.r. functions (cf. [3], §6).

4 In preparation for the proof of completeness of the transfinite progressions with respect to strictly primitive recursive realizability, we state some well-known meta-mathematical facts in a form suitable for our purposes.

Lemma 4.1 *Let \underline{f} be an n -ary primitive function symbol of \mathcal{L}' representing a p.r. function f . Then there is a p.r. function ψ depending on f , such that*

(i) *for some k ,*

$$\forall \vec{m}_n \forall m [f(\vec{m}_n) = m \implies \text{Der}_k(1, \psi(\vec{m}_n), 1, \ulcorner \underline{f}(\vec{m}_n) = \underline{m} \urcorner)],$$

and furthermore,

$$(ii) \text{ for some } k, \text{PR}-(\Sigma_1^0\text{-IR}) \vdash \underline{f}(\vec{x}_n) = y \rightarrow \underline{\text{Der}}_k(\underline{1}, \psi(\vec{x}_n), \underline{1}, \ulcorner \underline{f}(\vec{x}_n) = y \urcorner).$$

Part (ii) can be established by formalizing the proof of (i), which is standard (cf. e.g., [14], pp. 23–25); $\underline{1}$ codes the empty sequence. Call the formulas of \mathcal{L}' with no unbounded quantifiers and no occurrences of \rightarrow or \neg , PR-formulas. From 4.1(ii) it is possible to prove a form of demonstrable PR-completeness for PR- $(\Sigma_1^0\text{-IR})$ (see, e.g., Chapter 0 of [14]).

Lemma 4.2 *Let $\varphi(\vec{x}_n)$ be a PR-formula. Then, for some k ,*

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \varphi(\vec{x}_n) \rightarrow \exists z \underline{\text{Der}}_k(\underline{1}, z, \underline{1}, \ulcorner \varphi(\vec{x}_n) \urcorner).$$

This result extends to Σ_1^0 formulas of \mathcal{L}' , that is, those of the form $\exists x\varphi(x)$ where $\varphi(x)$ is a PR-formula.

Recall from [3], §5, that the derived rule (Cut) of PR- $(\Sigma_1^0\text{-IR})$ does not result in an increase of the order of PR-derivations. This fact can be formally proved in PR- $(\Sigma_1^0\text{-IR})$. (We omit the proof.)

Lemma 4.3 *For each m ,*

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \underline{\text{Der}}_m(a, x, \underline{1}, u) \ \& \ \underline{\text{Der}}_m(a, y, \langle u \rangle * \langle v \rangle, w) \rightarrow \exists z \underline{\text{Der}}_m(a, z, v, w).$$

Lemma 4.4 *For each b and $m > 0$, there is a sentence $F_m, (\underline{b})$ of \mathcal{L}' such that*

- (i) *if PR(b) is consistent, then $F_m(\underline{b})$ has no PR(b)-derivation of order $j \leq m$, and*
- (ii) *for all but finitely many m , there is a PR(2^b)-derivation of $F_m(\underline{b})$ derivation of order m .*

Proof: Recall from §1 the PR-formula $\underline{\text{Der}}(x_1, x_2, x_3, x_4, x_5)$ and for each fixed b and m consider the formula

$$\forall z \sim \underline{\text{Der}}_m(\underline{b}, z, \underline{1}, x_1)$$

with x_1 as its sole free variable.¹¹ By Gödel's Self-Reference Lemma there is a sentence $F_m(\underline{b})$ with Gödel number $f_m(\underline{b})$ such that the sequents

$$\forall z \sim \underline{\text{Der}}_m(\underline{b}, z, \underline{1}, \underline{f_m(\underline{b})}) \vdash F_m(\underline{b})$$

and

$$F_m(\underline{b}) \vdash \forall z \sim \underline{\text{Der}}_m(\underline{b}, z, \underline{1}, \underline{f_m(\underline{b})})$$

are both derivable in PR- $(\Sigma_1^0\text{-IR})$, as are the sequents

$$F_m(\underline{b}) \vdash \sim \underline{\text{Der}}_m(\underline{b}, \underline{d}, \underline{1}, \underline{f_m(\underline{b})})$$

for each d . All these PR-derivations are of order k for some fixed k . (Same k works for all m and d). Let $g_m^b(d) = \#(\sim \underline{\text{Der}}_m(\underline{b}, \underline{d}, \underline{1}, \underline{f_m(\underline{b})}))$. Then (i) follows by 4.1(i)

and the fact that $\text{PR}-(\Sigma_1^0\text{-IR}) \subseteq \text{PR}(b)$. On the other hand, note that, by 4.2, for each m ,

$$\sim \underline{\text{Der}}_m(\underline{b}, x, \underline{1}, \underline{f_m(b)}) \rightarrow \exists z \underline{\text{Der}}_n(\underline{b}, z, \underline{1}, \lceil \sim \underline{\text{Der}}_m(\underline{b}, \dot{x}, \underline{1}, \underline{f_m(b)}) \rceil) \quad (1)$$

is derivable in $\text{PR}-(\Sigma_1^0\text{-IR})$ for some n independent of the choice of m . Moreover, there is a p.r. function h such that, for each m ,

$$\underline{\text{Der}}_m(\underline{b}, x, \underline{1}, \underline{f_m(b)}) \ \& \ \underline{\text{Der}}_k(\underline{b}, y, \underline{f_m(b)}, g_m^b(x)) \rightarrow \underline{\text{Der}}_{\max(m,k)}(\underline{b}, h(x, y), \underline{1}, g_m^b(x))$$

is also derivable in $\text{PR}-(\Sigma_1^0\text{-IR})$. But then

$$\underline{\text{Der}}_m(\underline{b}, x, \underline{1}, \underline{f_m(b)}) \rightarrow \exists z \underline{\text{Der}}_{\max(m,k)}(\underline{b}, z, \underline{1}, \lceil \sim \underline{\text{Der}}_m(\underline{b}, \dot{x}, \underline{1}, \underline{f_m(b)}) \rceil) \quad (2)$$

is derivable in $\text{PR}-(\Sigma_1^0\text{-IR})$ for each m . Since

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \underline{\text{Der}}_{m_1}(x, z, u, v) \rightarrow \underline{\text{Der}}_{m_2}(x, z, u, v)$$

whenever $m_1 \leq m_2$, we have that the variants of (1) and (2) with n and $\max(m, k)$ replaced by $\max(\max(m, k), n)$ are both derivable in $\text{PR}-(\Sigma_1^0\text{-IR})$, for each m . Now, for each m ,

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \underline{\text{Der}}_m(\underline{b}, x, \underline{1}, \underline{f_m(b)}) \vee \sim \underline{\text{Der}}_m(\underline{b}, x, \underline{1}, \underline{f_m(b)})$$

since $\underline{\text{Der}}_m(\underline{b}, x, \underline{1}, \underline{f_m(b)})$ is a PR-formula. But then

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \exists z \underline{\text{Der}}_{\max(\max(m,k),n)}(\underline{b}, z, \underline{1}, \lceil \sim \underline{\text{Der}}_m(\underline{b}, \dot{x}, \underline{1}, \underline{f_m(b)}) \rceil),$$

and finally, for each m ,

$$\exists y (\underline{\text{Der}}_{\max(\max(m,k),n)}((y)_0, (y)_1, \underline{1}, \lceil \sim \underline{\text{Der}}_m(\underline{b}, \dot{x}, \underline{1}, \underline{f_m(b)}) \rceil) \ \& \ (y)_0 <^* \underline{2}^b)$$

has a PR-derivation of order q . (Again, same q works for all m). For Δ empty and $\varphi_m(z) \equiv \sim \underline{\text{Der}}_m(\underline{b}, z, \underline{1}, \underline{f_m(b)})$, we thus have that, for any Γ , $\Gamma \vdash \forall z \varphi_m(z)$ may be derived in $\text{PR}-(\Sigma_1^0\text{-IR}) + \text{REF}_{\max(\max(m,k),n)}(2^b, \Gamma, \Delta, \varphi_m, (z))$ by a derivation of order $\max(q, \max(\max(m, k), n))$. So $\text{PR}(2^b) \vdash \forall z \sim \underline{\text{Der}}_m(\underline{b}, z, \underline{1}, \underline{f_m(b)})$ and it follows that $F_m(\underline{b})$ is derivable in $\text{PR}(2^b)$ by a derivation of the same order. Hence, for $m \geq \max(k, n, q)$, there is a $\text{PR}(2^b)$ -derivation of $F_m(\underline{b})$ of order m . \square

Next, we modify an argument of Turing in [15] that will be crucial in establishing the completeness of the hierarchy $\{\text{PRA}(a) \mid a \in \underline{\mathbb{Q}}\}$ for true Π_1^0 statements. Let $\underline{\mathbb{Q}}(\omega) =: \{d \in \underline{\mathbb{Q}} \mid |d| = \omega\}$, and let $v(x)$ be the superexponential function $v(0) =: 1$ and $v(m+1) =: 2^{v(m)}$. (Then for each m , $v(m) \in \underline{\mathbb{Q}}$ and $|v(m)| = m$).

Lemma 4.5 (Turing's Lemma) *There is a p.r. function E such that for any PR-formula $A(x)$ of \mathcal{L}' ,*

$$\forall x A(x) \text{ is true} \iff E(\#A) \in \underline{\mathbb{Q}}(\omega).$$

Proof: Let $R(x)$ be the p.r. predicate expressed by $A(x)$. Define a p.r. function φ such that

$$\varphi(e, x) =: \begin{cases} \nu(x) & \text{if } \forall y \leq x R(y) \\ 2^{3 \cdot 5^e} & \text{if } \exists y \leq x \neg R(y) \end{cases}$$

Applying the primitive recursion theorem we may obtain primitive recursively a Kleene index e^* of the p.r. function φ^* such that

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \varphi^*(x) = \varphi(\underline{e}^*, x).$$

Let $d = 3 \cdot 5^{e^*}$. Assume $R(m)$ for all m . Then $[e^*](m) = \nu(m)$ for each m , and $d \in \underline{\mathcal{Q}}$ with $|d| = \omega$. If, on the other hand, for some $m \neg R(m)$, let m^* be the least such m . Then

$$\begin{aligned} \text{for all } k < m^*, & \quad [e^*](k) = \nu(k), \text{ and} \\ \text{for all } k \geq m^*, & \quad [e^*](k) = 2^d. \end{aligned}$$

But then $d \notin \underline{\mathcal{Q}}$ because φ^* is not increasing in $<^*$. The proof is complete if we let $E(\#A) = 3 \cdot 5^{e^*}$. \square

Remark 4.6 Note that the proof actually establishes that

$$\forall x A(x) \text{ is true} \iff E(\#A) \in \underline{\mathcal{Q}}^*(\omega)$$

where $\underline{\mathcal{Q}}^*(\omega) =: \{3 \cdot 5^e \mid e \text{ is a Kleene index of } \lambda x. \nu(x)\}$.

We now proceed to establish completeness for true Π_1^0 sentences. The essential idea of the proof is due to Turing, but the considerations about order of derivations and the fact that the systems $\text{PR}(b)$ are based on intuitionistic logic are crucial for our purposes.

Theorem 4.7 *Let $A(x)$ be a PR-formula of \mathcal{L}' . Then*

$$\forall x A(x) \text{ is true} \implies \exists d \in \underline{\mathcal{Q}}[\text{PR}(2^d) \vdash \forall x A(x) \text{ and } |d| = \omega].$$

Proof: Let $R(x)$ be the p.r. predicate expressed by $A(x)$. We argue indirectly. Assume that $\neg R(n)$ for some n . Then $d \notin \underline{\mathcal{Q}}$, where $d = 3 \cdot 5^{e^*}$ and e^* is as in the proof of Turing's Lemma. Nonetheless, the p.r. relation $\text{Der}(d, x, y, u, v)$ is defined. Since $y <^* [\underline{e}^*](x) \vdash y <^* \underline{3 \cdot 5^{e^*}}$ is derivable from the definition of $<^*$ by a PR-derivation of order 1, we have that, for each k and any $m > 0$, Γ, Δ , and φ , $\text{REF}_m([\underline{e}^*](k), \Gamma, \Delta, \varphi)$ has a $\text{PR}(d)$ -derivation of order m . By the choice of e^* it follows that $\text{REF}_m(\underline{2^d}, \Gamma, \Delta, \varphi)$ is derivable in $\text{PR}(d)$ by a derivation of order m , for all but finitely many $m \geq 1$ and any Γ, Δ , and φ . Then $\text{PR}(2^d) \subseteq \text{PR}(d)$, and from Lemma 4.4(ii) we have that

$$\exists n \neg R(n) \implies \text{for some PR}(d)\text{-derivation } D, D : \vdash_m F_m(\underline{d}) \quad (3)$$

for all but finitely many m .

On the other hand, since $(\sim \underline{\text{Der}}_m(\underline{d}, z, \underline{1}, \underline{f}_m(\underline{d})) \rightarrow \neg \underline{\text{Der}}_m(\underline{d}, z, \underline{1}, \underline{f}_m(\underline{d})))$ is derivable in $\text{PR}-(\Sigma_1^0\text{-IR})$, we have, by the choice of $F_m(\underline{d})$, that

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash F_m(\underline{d}) \rightarrow \forall z \neg \underline{\text{Der}}_m(\underline{d}, z, \underline{1}, \underline{f}_m(\underline{d}))$$

for each m , whence

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash F_m(\underline{d}) \rightarrow \neg \exists z \underline{\text{Der}}_m(\underline{d}, z, \underline{1}, \underline{f}_m(\underline{d})).$$

The formalized version of (3) is the \mathcal{L}' formula

$$\exists x \neg A(x) \rightarrow \exists z \underline{\text{Der}}_m(\underline{d}, z, \underline{1}, \underline{f}_m(\underline{d})). \quad (4)$$

If (4) is derivable in $\text{PR}-(\Sigma_1^0\text{-IR})$ by a derivation of a fixed order p for all but finitely many m , then so will be

$$\neg \exists z \underline{\text{Der}}_m(\underline{d}, z, \underline{1}, \underline{f}_m(\underline{d})) \rightarrow \forall x A(x)$$

since $A(x)$ is a PR-formula. But then there is a PR-derivation of a fixed order q of the sequent $F_m(\underline{d}) \vdash \forall x A(x)$ for all but finitely many m . From 4.4(ii) it then follows that

$$\text{PR}(2^d) \vdash \forall x A(x)$$

by a derivation of order m , for all but finitely many m . To complete the proof of the theorem, it remains to show that there is a PR-derivation of (4) of a fixed order p , for all but finitely many m . First observe that, from the proof of Turing's Lemma and the choice of e^* , we can obtain a PR-derivation of

$$\exists x \sim A(x) \vdash \exists x [e^*](x) = \underline{2}^d. \quad (5)$$

But then we may obtain a PR-derivation D of the sequent $\exists x \sim A(x), y <^* \underline{2}^d \vdash y <^* \underline{d}$, and, using 4.1, a PR-derivation of the formula

$$\underline{\text{Der}}_{k_1}(\underline{1}, \ulcorner \#D \urcorner, \langle \ulcorner \exists x \sim A(x) \urcorner \rangle * \langle \ulcorner y <^* \underline{2}^d \urcorner \rangle, \ulcorner y <^* \underline{d} \urcorner) \quad (6)$$

for some k_1 . Secondly, by demonstrable Σ_1^0 -completeness of $\text{PR}-(\Sigma_1^0\text{-IR})$, we have that

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \exists x \sim A(x) \rightarrow \exists z \underline{\text{Der}}_{k_2}(\underline{1}, z, \underline{1}, \ulcorner \exists x \sim A(x) \urcorner). \quad (7)$$

From (6) and (7) and Lemma 4.3 we may then derive

$$\exists x \neg A(x) \rightarrow \exists z \underline{\text{Der}}_k(\underline{1}, z, \underline{1}, \ulcorner y <^* \underline{2}^d \urcorner, \ulcorner y <^* \underline{d} \urcorner)$$

in $\text{PR}-(\Sigma_1^0\text{-IR})$, for $k = \max(k_1, k_2)$.¹² But then

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \exists x \neg A(x) \rightarrow (\exists y \underline{\text{Der}}_m(\underline{2}^d, y, u, v) \rightarrow \exists z \underline{\text{Der}}_m(\underline{d}, z, u, v))$$

for all but finitely many m ; in particular,

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \exists x \neg A(x) \rightarrow (\exists y \underline{\text{Der}}_m(\underline{2}^d, y, \underline{1}, \underline{f}_m(\underline{d})) \rightarrow \exists z \underline{\text{Der}}_m(\underline{d}, z, \underline{1}, \underline{f}_m(\underline{d})))$$

From 4.4(ii) and 4.1 we have that $\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \exists y \underline{\text{Der}}_m(\underline{2}^d, y, \underline{1}, \underline{f}_m(\underline{d}))$, whence finally, $\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \exists x \neg A(x) \rightarrow \exists z \underline{\text{Der}}_m(\underline{d}, z, \underline{1}, \underline{f}_m(\underline{d}))$ as required. \square

Remark 4.8 Notice that the proof that $\text{PRA}(2^d) \vdash \forall x A(x)$ does not really depend on the hypothesis that $\forall x A(x)$ is true. For, if $\exists x \neg A(x)$, then by (4) we have that, for all but finitely many m , $\text{PRA}(d) \vdash F_m(\underline{d})$ by a derivation of order m , which implies that $\text{PR}(d)$ is inconsistent. Since in general $\text{PRA}(a) \subseteq \text{PRA}(2^a)$, it would follow that $\text{PRA}(2^d)$ is inconsistent and hence, trivially, $\text{PRA}(2^d) \vdash \forall x A(x)$. The difference is that if $\forall x A(x)$ is true, we have that $2^d \in \underline{\mathcal{Q}}$.

Given the properties of our realizability semantics, completeness for sentences of the form $\forall x A(x)$ where $A(x)$ is a PR-formula suffices to establish completeness for all realizable prenex sentences of \mathcal{L}' .

Theorem 4.9 *Let A be a Π_n^0 sentence of \mathcal{L}' for $n \geq 0$. Then*

$$\exists e \exists m e \Vdash_m A \implies \exists b \in \underline{\mathcal{Q}}, \text{PRA}(b) \vdash A \text{ and } |b| = \omega + 1.$$

Proof: We may assume without loss of generality that $n = 2(k + 1)$. Suppose $e \Vdash_m A$. From Theorem 3.2 of [3] we have that A is PR-true, that is, there are p.r. functions f_0, \dots, f_k such that for any m_0, \dots, m_k , $B(\underline{m}_0, \underline{f}_0(\underline{m}_0), \underline{m}_1, \underline{f}_1(\underline{m}_0, \underline{m}_1), \dots, \underline{m}_k, \underline{f}_k(\underline{m}_0, \dots, \underline{m}_k))$ holds, where $A \equiv \forall x_0 \exists y_0, \dots, \forall x_k \exists y_k B(x_0, y_0, \dots, x_k, y_k)$ and $B(x_1, \dots, x_n)$ is a PR-formula of \mathcal{L}' . Then we also have that $\forall x C(x)$ where

$$C(x) \equiv B((x)_0, f_0((x)_0), (x)_1, \dots, (x)_k, f_k((x)_0, \dots, (x)_k))$$

is a true Π_1^0 sentence of \mathcal{L}' . Note that $\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \forall x C(x) \rightarrow A$ by logic. By Theorem 4.7, $\text{PRA}(b) \vdash \forall x C(x)$ for some $b \in \underline{\mathcal{Q}}$ such that $|b| = \omega + 1$. But then $\text{PRA}(b) \vdash A$, as required. \square

We may then derive the following theorem.

Theorem 4.10 *Let A be a prenex sentence of \mathcal{L}' . If A is strictly primitive recursively realizable, then $\text{PRA}(b) \vdash A$ for some $b \in \underline{\mathcal{Q}}$ such that $|b| = \omega + 1$.*

Remark 4.11 From the proofs of 4.7 and Theorem 3.2 in [3] it can be shown that $b \in \underline{\mathcal{Q}}$ is primitive recursively obtainable from $\#A$.

From the soundness and completeness theorems, 3.4.1 and 4.10, we then have

Theorem 4.12 *Let A be any prenex sentence of \mathcal{L}' . Then*

$$\begin{aligned} \exists e \exists n e \Vdash_n A &\iff A \text{ is PR-true} \iff A \in \cup_{b \in \underline{\mathcal{Q}}} \{\text{PRA}(b) \mid |b| \leq \omega + 1\} \\ &\iff A \in \cup_{b \in \underline{\mathcal{Q}}} \{\text{PRA}(b)\}. \end{aligned}$$

We recall the notion of strictly primitive recursive falsifiability introduced in [3], §3.

Theorem 4.13 *Let A be any prenex sentence of \mathcal{L}' . Then $\exists e \exists n e \dashv\vdash_n A \implies \exists b \in \underline{\mathcal{Q}}, \text{PRA}(b) \vdash \neg A$ and $|b| = \omega + 1$.*

Proof: From Theorems 3.2 and 3.3 of [3] we have that

$$\exists e \exists n e \dashv\vdash_n A \iff A \text{ is PR-false} \iff \sim A \text{ is PR-true} \iff \exists e \exists n e \Vdash_n \sim A.$$

Then, by 4.10, the hypothesis implies that $\text{PRA}(b) \vdash \sim A$ for some $b \in \underline{\mathcal{Q}}$ such that $|b| = \omega + 1$, whence $\text{PRA}(b) \vdash \neg A$. \square

Let $S_R =: \{\#A \mid A \text{ is a prenex sentence and } \exists e \exists n e \Vdash_n A\}$, and let $S_F =: \{\#A \mid A \text{ is a prenex sentence and } \exists e \exists n e \dashv\vdash_n A\}$. Let ' \leq_1^{pr} ' stand for 1–1 reducibility by a p.r. function, and let ' $A \equiv_1^{pr} B$ ' abbreviate ' $A \leq_1^{pr} B$ and $B \leq_1^{pr} A$ '. Then we also have

Theorem 4.14

- (a) The sets $\underline{Q}(\omega)$ and $\underline{Q}^*(\omega)$ are Π_1^0 complete.
- (b) $\underline{Q}^*(\omega) \leq_1^{pr} S_R$.
- (c) $S_R \equiv_1^{pr} S_F$.
- (d) The sets S_R and S_F are both Π_1^0 complete.

We omit the proof.

5 Here we introduce some auxiliary machinery for use later on. First, observe that from the definition of the Kleene indexing of p.r. functions, one may easily obtain an elementary function such that for any a, b, e ,

$$\text{for all } n, [\theta(a, b, e)](n) = [a](b, [e](n)).$$

(Cf. [6], p. 105.) In fact, we also have that, for each a, b , and e , $\text{PR}-(\Sigma_1^0\text{-IR}) \vdash [a](\underline{b}, [e](x)) = [\theta(\underline{a}, \underline{b}, \underline{e})](x)$. From the definition of $<^*$ one may then derive

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash \forall x ([a](\underline{b}, [e](x)) <^* 3 \cdot 5^{\theta(\underline{a}, \underline{b}, \underline{e})}) \quad (8)$$

Furthermore, by [9] there is a p.r. function $+^*$ with Kleene index a^* such that for any x, y, e ,

$$\begin{aligned} x +^* 1 &= x && \text{if } x \neq 0 \\ x +^* 2^y &= 2^{x+^*y} && \text{if } y \neq 0 \\ x +^* 3 \cdot 5^e &= 3 \cdot 5^{\theta(a^*, x, e)} \end{aligned}$$

We then have, for any a, b, e if $a \in \underline{Q}$ and $b \in \underline{Q}$, that

- (i) $a +^* b \in \underline{Q}$,
- (ii) $|a +^* b| = |a| + |b|$, and
- (iii) $b \neq 1 \implies a <_0 a +^* b$.

First we establish the following “relativized” version of Turing’s Lemma (4.5).

Lemma 5.1 *There is a p.r. function E_0 such that for any $b \in \underline{Q}$, and any PR-formula $A(x)$ of \mathcal{L}' ,*

- (i) $\forall x A(x) \text{ is true} \iff E_0(b, \#A) \in \underline{Q}$,
- (ii) $E_0(b, \#A) \in \underline{Q} \implies |E_0(b, \#A)| = |b| + \omega$.

A form of this result stated in terms of Kleene’s recursively based system of ordinal notations O was originally proved in [5], p. 288. Our proof is completely analogous and we omit it. We remark that the argument extends straightforwardly to sentences of the form $\forall x_1, \dots, \forall x_m A(x_1, \dots, x_m)$, where $A(x_1, \dots, x_m)$ is a PR-formula of \mathcal{L}' that expresses an m -place p.r. relation R .

The “relativized” Turing’s Lemma allows us to prove a “relativized” version of Theorem 4.7.¹³

Theorem 5.2 *There are binary p.r. functions E_1, E_2 such that for any b and any PR-formula $A(x)$ of \mathcal{L}' , if $\forall xA(x)$ is true, then*

- (i) $b \in \underline{\mathbf{Q}} \implies E_1(b, \#A) \in \underline{\mathbf{Q}}$.
- (ii) $b \in \underline{\mathbf{Q}} \implies b <_0 E_1(b, \#A)$ and $|E_1(b, \#A)| = |b| + \omega + 1$,
- (iii) $\text{PRA}(E_1(b, \#A)) \vdash \forall xA(x)$ by a derivation with Gödel number $E_2(b, \#A)$.

Proof: We argue indirectly as in the proof of Theorem 4.7. Let $R(x)$ be the p.r. predicate expressed by $A(x)$ and assume that $\neg R(m)$ for some m . Then $E_0(b, \#A) \notin \underline{\mathbf{Q}}$ by 5.1. We consider the sequent system $\text{PRA}(d)$ for $d = E_0(b, \#A)$. We then argue, with the aid of Lemma 5.1, exactly as in the proof of 4.7, that $\text{PRA}(2^d) \vdash \forall xA(x)$. A p.r. function E_2 that gives the Gödel number of a $\text{PRA}(2^d)$ -derivation of $\forall xA(x)$ with the desired property can be obtained from a detailed analysis of that proof since the formal proof of (4) in $\text{PR}(\Sigma_1^0\text{-IR})$ is primitive recursively uniform in b and $\#A$. Now, if $\forall xA(x)$ is true, it follows from 5.1 that provided $b \in \underline{\mathbf{Q}}, d \in \underline{\mathbf{Q}}$ where $d = 3 \cdot 5^{E'(b, \#A)} = E_0(b, \#A)$, and furthermore, $|d| = |b| + \omega$. Thus, if we let $E_1(b, \#A) =: 2^d$ we have, under the hypothesis that $b \in \underline{\mathbf{Q}}$ and $\forall xA(x)$ is true, that $E_1(b, \#A) \in \underline{\mathbf{Q}}, b <_0 E_1(b, \#A)$ and $|E_1(b, \#A)| = |b| + \omega + 1$, which establishes parts (i) and (ii) of the theorem. \square

Theorem 5.2 extends to sentences of the form $\forall x_1, \dots, \forall x_m A(x_1, \dots, x_m)$, where $A(x_1, \dots, x_m)$ is a PR-formula of \mathcal{L}' expressing an m -place p.r. relation R .

Let $\forall x D_n(\underline{a}, \underline{e}, \underline{m}, \underline{g}, \underline{b}, x)$ abbreviate a Π_1^0 sentence that says that

$$\forall x \text{Der}_n([e](a, [m](x)), [a]([m](x)), \#\Gamma, \#B(x)),$$

where $g = \#\Gamma$ and $b = \#\forall x B(x)$ with x not free in Γ . Let $d(n, a, e, m, g, b)$ be the Gödel number of $D_n(\underline{a}, \underline{e}, \underline{m}, \underline{g}, \underline{b}, x)$. The following lemma is best understood in the context of the proof of Theorem 7.1.

Lemma 5.3 *There is a p.r. function E_3 such that, for $g = \#\Gamma$ and $b = \#\forall x B(x)$ where x is not free in Γ , if $\forall x D_n(\underline{a}, \underline{e}, \underline{m}, \underline{g}, \underline{b}, x)$ is true and $3 \cdot 5^{\theta(e, a, m)} \in \underline{\mathbf{Q}}$, then for any $t \in \underline{\mathbf{Q}}, \Gamma \vdash \forall x B(x)$ is derivable in $\text{PRA}(E_1(3 \cdot 5^{\theta(e, a, m)}, d(n, a, e, m, g, b)) +^* t)$ by a derivation with Gödel number $E_3(n, a, e, m, g, b, t)$.*

Proof: From the hypothesis we have, by 5.2(iii), that

$$\forall x \underline{\text{Der}}_n([\underline{e}](\underline{a}, [\underline{m}](x)), [\underline{a}]([\underline{m}](x)), \ulcorner \Gamma \urcorner, \ulcorner B(\dot{x}) \urcorner)$$

is derivable in $\text{PRA}(E_1(1, d(n, a, e, m, g, b)))$. Then so is

$$\forall x \exists y (\underline{\text{Der}}_n((y)_0, (y)_1, \ulcorner \Gamma \urcorner, \ulcorner B(\dot{x}) \urcorner) \ \& \ (y)_0 = [\underline{e}](\underline{a}, [\underline{m}](x)))$$

But then by (8) the same holds of

$$\forall x \exists y (\underline{\text{Der}}_n((y)_0, (y)_1, \ulcorner \Gamma \urcorner, \ulcorner B(\dot{x}) \urcorner) \ \& \ (y)_0 <^* 3 \cdot 5^{\theta(e, a, m)}) \quad (9)$$

Now, since $3 \cdot 5^{\theta(e, a, m)} \in \underline{\mathbf{Q}}$ by hypothesis, we have by Theorem 5.2(ii) and Σ_1^0 -completeness of $\text{PR}(\Sigma_1^0\text{-IR})$ that

$$\text{PR}(\Sigma_1^0\text{-IR}) \vdash 3 \cdot 5^{\theta(e, a, m)} <^* E_1(3 \cdot 5^{\theta(e, a, m)}, d(\underline{n}, \underline{a}, \underline{e}, \underline{m}, \underline{g}, \underline{b})) +^* \underline{t} \quad (10)$$

since $d(n, a, e, m, g, b)$ is the Gödel number of a true sentence. From (9) and (10) and the remarks at the end of §1 it follows that

$$\forall x \exists y (\underline{\text{Der}}_n((y)_0, (y)_1, \ulcorner \Gamma \urcorner, \ulcorner B(x) \urcorner) \& \\ (y)_0 <^* E_1(3 \cdot 5^{\theta(e,a,m)}, d(\underline{n}, \underline{a}, \underline{e}, \underline{m}, \underline{g}, \underline{b})) +^* \underline{t})$$

is derivable in $\text{PRA}(E_1(1, d(n, a, e, m, g, b)))$, and since by Σ_1^0 -completeness of $\text{PR}-(\Sigma_1^0\text{-IR})$ we have that

$$\text{PR}-(\Sigma_1^0\text{-IR}) \vdash E_1(\underline{1}, \ulcorner A \urcorner) <^* E_1(\underline{c}, \ulcorner A \urcorner) \quad (11)$$

for any $c \in \underline{\mathbb{Q}}$ and any true sentence A of the form $\forall x_1, \dots, \forall x_m C(x_1, \dots, x_m)$ where $m \geq 1$ and $C(x_1, \dots, x_m)$ expresses a p.r. predicate, it follows that the above sentence is derivable in $\text{PRA}(E_1(3 \cdot 5^{\theta(e,a,m)}, d(n, a, e, m, g, b)) +^* t)$. But then $\Gamma \vdash \forall x B(x)$ is derivable in $\text{PRA}(E_1(3 \cdot 5^{\theta(e,a,m)}, d(n, a, e, m, g, b)) +^* t)$, as required. Since the derivations in (10) and (11) can be obtained uniformly primitive recursively from n, a, e, m, g, b, t and $A \equiv \forall x D_n(\underline{a}, \underline{e}, \underline{m}, \underline{g}, \underline{b}, x)$ and $c = 3 \cdot 5^{\theta(e,a,m)} +^* t$, we may obtain a p.r. function E_3 depending on n, a, e, m, g, b , and t with the above stated property. \square

6 Here we formulate a primitive recursively restricted ω -rule in the style of Shoenfield (cf. [13]) and describe a semiformal system of ω -derivations closed under the rule. Shoenfield's original recursive ω -rule, added to the deductive apparatus of PA , suffices to derive all first-order arithmetical truths. An important element in our version of the rule is a complexity restriction that mimics the one employed in connection with the reflection principles $\text{REF}_n(b, \Gamma, \Delta, \varphi)$ and having to do with the rules of $\text{PR}-(\Sigma_1^0\text{-IR})$ and the rank in the Grzegorzczuk hierarchy of the p.r. functions expressed by instantiating terms.

For the sake of brevity, we forgo giving the actual definition of the set of ω -derivations in favor of a sketch from which the definition can be easily reconstructed. We characterize the set of ω -derivations as the smallest set of integers satisfying the following conditions:¹⁴

- (i) if $\Gamma \vdash A$ is the axiom of $\text{PR}-(\Sigma_1^0\text{-IR})$, then for any d, n , $\langle 1, n, \#\Gamma, \#A, d \rangle$ is an ω -derivation of $\Gamma \vdash A$. Assume $d \geq 1$.
- (ii) if for some i, j , $(d)_i$ and $(d)_j$ are ω -derivations of $\Gamma \vdash A$ and $\Gamma \vdash B$, respectively, then $\langle 2, \max((d)_{i,1}, (d)_{j,1}), \#\Gamma, \#(A \& B), d, i, j \rangle$ is an ω -derivation of $\Gamma \vdash A \& B$;
- (iii) if for some i , $(d)_i$ is an ω -derivation of $\Gamma \vdash A \& B$, then $\langle 3, (d)_{i,1}, \#\Gamma, \#A, d, i \rangle$ is an ω -derivation of $\Gamma \vdash A$, and $\langle 4, (d)_{i,1}, \#\Gamma, \#A, d, i \rangle$ is an ω -derivation of $\Gamma \vdash B$.

We continue in this way and co-opt each one of the rules of $\text{PR}-(\Sigma_1^0\text{-IR})$, introducing in each case as an ω -derivation of a sequent $\Gamma \vdash A$ an integer that codes a sequence (in the case of 2-premise rules) of the form

$$\langle k, n, \#\Gamma, \#A, d, i, j \rangle,$$

where $k(2 \leq k \leq 16)$ indicates a particular rule of PR- $(\Sigma_1^0\text{-IR})$, d codes a finite sequence such that $(d)_i$ and $(d)_j$ are ω -derivations of sequents from which $\Gamma \vdash A$ follows by that rule and n depends on $(d)_{i,1}$ and $(d)_{j,1}$ in the same way in which order of a PR-derivation by which $\Gamma \vdash A$ is derived from the sequents ω -derived by $(d)_i$ and $(d)_j$ depends on orders of its immediate subderivations. Thus, for example, we have:

- (ix) if for some i, j , $(d)_i$ and $(d)_j$ are ω -derivations of $\Gamma \vdash A(\underline{Q})$ and $\Gamma, A(x) \vdash A(x')$, respectively, where $A(x)$ is a Σ_1^0 formula and x is not free in Γ , then $\langle 11, \max((d)_{i,1}, (d)_{j,1}) + 1, \#\Gamma, \#A(x), i, j \rangle$ is an ω -derivation of $\Gamma \vdash A(x)$.

The one exception to this pattern is this:

- (xv) if $B(x)$ is a formula of \mathcal{L}' and the variable x is not free in Γ , for each n , $[e](n)$ is an ω -derivation of $\Gamma \vdash B(\underline{n})$ such that $([e](n))_1 \leq m$, then, for any $d \geq 1$,

$$\langle 17, m, \#\Gamma, \#\forall x B(x), d, e \rangle$$

is an ω -derivation of $\Gamma \vdash \forall x B(x)$.

This completes the description of the inductive definition of the set of ω -derivations. We write $\text{Der}^\omega(m, n, \#\Gamma, \#A)$ just in case m is an ω -derivation of $\Gamma \vdash A$ and $(m)_1 = n$. Clearly, derivability in PR- $(\Sigma_1^0\text{-IR})$ implies ω -derivability: one may define by course-of-values recursion a p.r. function π where

$$\text{Der}_n(1, m, \#\Gamma, \#A) \implies \text{Der}^\omega(\pi(m), n, \#\Gamma, \#A)$$

for any Γ, A and any m, n .

To establish universal assertions about ω -derivations we need to be able to argue by induction on the complexity of ω -derivations. As mentioned earlier, one measure of complexity is explicitly built into the definition of ω -derivations. However, it essentially ignores the primitive recursive ω -rule. Another complexity measure that does take into account the ω -rule is the *height* of ω -derivations which we define so that, in particular, an application of the primitive recursive ω -rule, as described in (xv) above, results in an ω -derivation of $\Gamma \vdash \forall x B(x)$ of a higher degree of complexity than any of the ω -derivations of $\Gamma \vdash B(\underline{n})$ which serve as the ‘‘premises’’. For this purpose we assign an ordinal $\text{OD}(m)$ to an ω -derivation m as follows:

$$\text{OD}(m) = \begin{cases} (m)_4 & \text{if } (m)_0 = 1 \\ \Sigma_{0 \leq i \leq \text{pth}((m)_4)} \text{OD}((m)_{4,i}) + 1 & \text{if } 1 < (m)_0 < 17 \\ \lim_n \{ \text{OD}([(m)_5](n)) \} + \omega + 1 + (m)_4 & \text{if } (m)_0 = 17 \end{cases}$$

where $\lim_n \{ \text{OD}([(m)_5](n)) \}$ is the least limit ordinal $\lambda > \text{OD}([(m)_5](n))$ for all n .¹⁵

Let c and d be ω -derivations. We say that c is an *immediate subderivation* of d if and only if either $1 < (d)_0 < 17$ and $c = (d)_{4,i}$ where $i = (d)_k$ and $5 \leq k < \text{pth}(d)$, or $(d)_0 = 17$ and $c = [(d)_4](n)$ for some n . (Thus, e.g., referring back to the definition of ω -derivations, if $(d)_0 = 2$, both $(d)_{4,i}$ and $(d)_{4,j}$ are immediate subderivations of d and d has no immediate subderivations if $(d)_0 = 1$). Let d be any ω -derivation. We

define the set $SD(d)$ of *subderivations* of d to be the transitive closure of $\{d\}$ with respect to the relation of immediate subderivation. We may then establish, by induction on the generating relation of $SD(d)$, that for any ω -derivation d ,

$$c \in SD(d) \quad \text{and} \quad c \neq d \implies OD(c) < OD(d).$$

Clearly, if $c \in SD(d)$, then $SD(c) \subseteq SD(D)$. The notion of subderivations will help us define the concept of “proper” ω -derivation which will be instrumental in the proof of the main theorem. We say that an ω -derivation d is *proper* if for every subderivation c of d such that $(c)_0 = 17$ we have that for all n ,

$$OD([(c)_5](n)) < OD([(c)_5](n+1)).$$

In other words, an ω -derivation d is proper if for each subderivation c of d that results from an application of the ω -rule the ordinal height $OD(c_n)$ of its immediate subderivations $c_n = [(c)_5](n)$ strictly increases with n . Clearly, any subderivation of a proper ω -derivation is a proper ω -derivation.

We now define a p.r. function Od that will provide a measure of the ordinal height of ω -derivations but which will be of interest only in connection with proper ω -derivations. The definition is intimately connected with the proof of Theorem 7.1 and its significance is best understood in the context of that proof. We first define a 3-place p.r. function Od^* by the following course-of-values recursion.

$$Od^*(e, a, m) = \begin{cases} v((m)_4) & \text{if } (m)_0 = 1 \\ 2^{\sigma(\tilde{O}d^*(e, a, (m)_4), lth((m)_4))} & \text{if } 1 < (m)_0 < 17 \\ E_1(3 \cdot 5^{\theta(e, a, (m)_5)}, d((m)_1), a, e, \\ \quad (m)_5, (m)_2, (m)_3) +^* v((m)_4) & \text{if } (m)_0 = 17 \\ 0 & \text{otherwise} \end{cases}$$

where $\tilde{O}d^*(e, a, x)$ is the course-of-values function $\prod_{0 \leq j < lth(x)} p_j^{Od^*(e, a, (x)_j)}$ (p_j the j th prime), and σ is defined by the primitive recursion

$$\begin{cases} \sigma(y, 0) = 1 \\ \sigma(y, x+1) = \sigma(y, x) +^* (y)_x \end{cases} .$$

By the primitive recursion theorem there is a 2-place p.r. function Od with Kleene index o^* such that

$$Od(a, m) = Od^*(o^*, a, m) = [o^*](a, m).$$

We then have the following lemma.

Lemma 6.1 *Suppose d is a proper ω -derivation and e is an integer such that, for any $c \in SD(d)$ with $(c)_0 = 17$ and $(c)_1 = m$, $\forall x D_m(\underline{c}^*, \underline{e}, (\underline{c})_5, (\underline{c})_2, (\underline{c})_3, x)$ is a true Π_1^0 sentence. Then*

- (a) $Od(e, d) \in \underline{O}$ and $|Od(e, d)| = OD(d)$,
- (b) for any $b \in SD(d)$, $Od(e, b) <_0 Od(e, d)$.

Proof: (a) We argue by induction on the generating relation of the set of ω -derivations. We consider here only the case $(d)_0 = 17$. Then $A \equiv \forall xB(x)$ for some formula $B(x)$ where x is not free in Γ , and

$$d = \langle 17, m, \#\Gamma, \#\forall xB(x), (d)_4, b \rangle$$

where for each n , $[b](n)$ is a proper ω -derivation of $\Gamma \vdash B(\underline{n})$ such that $([b](n))_1 \leq m$. By the induction hypothesis, for all n ,

$$Od(e, [b], (n)) \in \underline{O} \text{ and } |Od(e, [b](n))| = OD([b](n)).$$

Since d is a proper ω -derivation, we have that, for all n ,

$$|Od(e, [b](n))| = OD([b](n)) < OD([b](n+1)) = |Od(e, [b](n+1))|.$$

Given that for all n , $Od(e, [b](n)) = [\theta(o^*, e, b)](n)$, this means that $\theta(o^*, e, b)$ is a Kleene index of an increasing function in $<_0$. Hence $3 \cdot 5^{\theta(o^*, e, b)} \in \underline{O}$, and by Theorem 5.2 and the hypothesis

$$b^* = E_1(3 \cdot 5^{\theta(o^*, e, b)}, d(m, e, o^*, b, \#\Gamma, \#A)) \in \underline{O} \text{ and } |b^*| = |3 \cdot 5^{\theta(o^*, e, b)}| + \omega + 1.$$

But then

$$|Od(e, d)| = |b^* +^* \nu((d)_4)| = \lim_n \{OD([b](n))\} + \omega + 1 + (d)_4 = OD(d)$$

as required. (b) follows from (a). \square

7 We shall now establish the closure of transfinite progressions $\{\text{PR}-(\Sigma_1^0\text{-IR})|b \in \underline{O}\}$ under the primitive recursive ω -rule, under the hypothesis that the applications of the ω -rule are restricted solely to proper ω -derivations.

Theorem 7.1 *Let d be a proper ω -derivation of $\Gamma \vdash A$. Then $\Gamma \vdash A$ is derivable in $\text{PRA}(b)$ for some $b \in \underline{O}$.*

Proof: We prove the theorem by establishing a somewhat stronger result: there is a p.r. function φ with Kleene index f such that for any $d, n \geq 1, \Gamma, A$,

$$\text{Der}^\omega(d, n, \#\Gamma, \#A) \implies \text{Der}_n(Od(f, d), \varphi(d), \#\Gamma, \#A).$$

Toward this, we first define a 2-place p.r. function φ^+ by course-of-values recursion. We let φ^+ satisfy a condition of the following form.

$$\varphi^+(e, x) = \begin{cases} \alpha((x)_2, (x)_3) & \text{if } (x)_0 = 1 \\ \beta(x, \varphi^+(e, \chi(x)_4), Od(e, \chi, ((x)_4))) & \text{if } 1 < (x)_0 < 17 \\ E_3((x)_1, e, o^*, (x)_5, (x)_2, (x)_3, \nu^*(x)_4) & \text{if } (x)_0 = 17 \\ 0 & \text{otherwise} \end{cases}$$

for appropriate p.r. functions α, β , and χ , where o^* is the Kleene index of the function Od obtained earlier and E_3 is the p.r. function from Lemma 5.3.¹⁶ We then apply the primitive recursion theorem to obtain a Kleene index f of a p.r. function φ such that

$$\varphi(x) = \varphi^+(f, x) \text{ and } \text{PR}-(\Sigma_1^0\text{-IR}) \vdash \varphi^+(\underline{f}, x) = \underline{[f]}(x),$$

and proceed to prove that the function φ provides the desired “translation” of any proper ω -derivation d into a derivation in $\text{PRA}(b)$ for an appropriate $b \in \underline{\mathbf{Q}}$. The argument is by induction on the generating relation of the set of ω -derivations. The case $(d)_0 = 1$ is obvious. For $(d)_0 \neq 1$, assume as the induction hypothesis that the claim holds for all ω -derivations $c \in \text{SD}(d)$. Since each such c is a proper ω -derivation and $(c)_1 \leq n$, this implies, in particular, that for any $c \in \text{SD}(d)$ such that $(c)_0 = 17$,

$$\forall x D_n(\underline{f}, \underline{o}^*, (\underline{c})_5, (\underline{c})_2, (\underline{c})_3, x)$$

is a true Π_1^0 sentence. Hence, by Lemma 6.1, we have that

$$\text{Od}(f, c) \in \underline{\mathbf{Q}} \quad \text{and} \quad |\text{Od}(f, c)| = \text{OD}(c)$$

for any such c . We proceed by cases depending on $(d)_0$. For $1 < (d)_0 < 17$, we consider the case $(d)_0 = 2$ as an illustration of the type of argument that applies in the other cases as well. If $(d)_0 = 2$ then $d = \langle 2, (d)_1, \#\Gamma, \#(A \& B), (d)_4, i, j \rangle$, where $(d)_1 \leq n$ and $o \leq i, j < \text{lth}((d)_4)$ and $(d)_{4,i}$ and $(d)_{4,j}$ are ω -derivations of $\Gamma \vdash A$ and $\Gamma \vdash B$, respectively. Then $(d)_{4,i}, (d)_{4,j} \in \text{SD}(d)$. From the induction hypothesis we then have that $\Gamma \vdash A$ has a $\text{PRA}(\text{Od}(f, (d)_{4,i}))$ -derivation of order n with Gödel number $\varphi((d)_{4,i})$, and $\Gamma \vdash B$ a $\text{PRA}(\text{Od}(f, (d)_{4,j}))$ -derivation of order n with Gödel number $\varphi((d)_{4,j})$. But $\text{Od}(f, (d)_{4,i}) <_0 \text{Od}(f, d)$ and $\text{Od}(f, (d)_{4,j}) <_0 \text{Od}(f, d)$. It follows that $\Gamma \vdash A$ and $\Gamma \vdash B$ have $\text{PR}(\text{Od}(f, d))$ -derivations of order n , which can be obtained uniformly primitive recursively in $\varphi((d)_{4,i}), \text{Od}(f, (d)_{4,i})$ and $\varphi((d)_{4,j}), \text{Od}(f, (d)_{4,j})$. But then $\varphi(d)$ is the easily obtained, via β and χ , $\text{PR}(\text{Od}(f, d))$ -derivation of $\Gamma \vdash A \& B$. Finally, we consider the case $(d)_0 = 17$. Then

$$d = \langle 17, n, \#\Gamma, \#\forall x B(x), (d)_4, (d)_5 \rangle$$

where for each m , $[(d)_m](m)$ is an ω -derivation of $\Gamma \vdash B(\underline{m})$ with $([(d)_5](m))_1 \leq n$. By the induction hypothesis, for each m , $\Gamma \vdash B(\underline{m})$ has a $\text{PRA}(\text{Od}(f, [(d)_5](m)))$ -derivation with Gödel number $\varphi([(d)_5](m))$ and of order $\leq n$. Recall that $\varphi(x) = [f](x)$. We then have that

$$\forall x \underline{\text{Der}}_n([\underline{o}^*], (\underline{f}, [(d)_5](x)), [\underline{f}]([(d)_5](x)), \ulcorner \Gamma \urcorner, \ulcorner B(\dot{x}) \urcorner)$$

can be expressed as a true Π_1^0 sentence. Therefore, by Theorem 5.2(iii),

$$\text{PRA}(E_1(1, d(n, f, o^*, (d)_5, (d)_2, (d)_3))) \vdash \forall x D_n(\underline{f}, \underline{o}^*, (\underline{d})_5, (\underline{d})_2, (\underline{d})_3, x).$$

Given that d is a proper ω -derivation, we have that

$$\text{for all } m, \text{Od}(f, [(d)_5](m)) <_0 \text{Od}(f, [(d)_5](m+1)).$$

Since also, for all m , $\text{Od}(f, [(d)_5](m)) = [\theta(o^*, f, (d)_5)](m)$, this means that $3 \cdot 5^{\theta(o^*, f, (d)_5)} \in \underline{\mathbf{Q}}$. But then, by Lemma 5.3, $\Gamma \vdash \forall x B(x)$ is derivable in $\text{PRA}(E_1(3 \cdot 5^{\theta(o^*, f, (d)_5)}, d(n, f, o^*, (d)_5, (d)_2, (d)_3)) +^* v((d)_4))$ by a derivation with Gödel number $E_3(n, f, o^*, (d)_5, (d)_2, (d)_3, v((d)_4))$. This is precisely what we need given the definitions of $\varphi(d)$ and $\text{Od}(f, d)$. \square

Next we show that all realizable prenex sentences are ω -derivable. Then (7.1) and (7.2) will give us another proof of the prenex completeness of the transfinite progressions $\{\text{PRA}(b) \mid b \in \underline{\mathcal{O}}\}$ with respect to strictly primitive recursive realizability.

Theorem 7.2 *Let A be a prenex sentence of \mathcal{L}' . Then*

$$\exists e \exists n e \Vdash_n A \implies \exists d \exists n \underline{\text{Der}}^\omega(d, n, 1, \#A).$$

Proof: Without loss of generality we may assume that $A \equiv \forall x_0 \exists y_0, \dots, \forall x_k \exists y_k B(x_0, y_0, \dots, x_k, y_k)$ where $B(x_0, y_0, \dots, x_k, y_k)$ is a PR-formula. Now A is PR-true if realizable (by Theorem 3.2 of [3]), so we have that $\forall x C(x)$ is true for some PR-formula $C(x)$ of \mathcal{L}' , by reasoning as in the proof of 4.9. By PR-completeness of PR- $(\Sigma_1^0\text{-IR})$, there is a p.r. function ψ and an integer k , depending on $C(x)$, such that for any m ,

$$C(\underline{m}) \text{ is true} \implies \text{Der}_k(1, \psi(m), 1, \#C(\underline{m})) \implies \text{Der}^\omega(\pi(\psi(m)), k, 1, \#C(\underline{m})).$$

Thus the realizability of A implies that $\langle 17, k, 1, \#\forall x C(x), 1, b \rangle$ is an ω -derivation of $\forall x C(x)$, provided $[b](m) = \pi(\psi(m))$ for all m . But this ω -derivation need not be proper. To obtain one that is, we let \oplus be the p.r. function such that, for any ω -derivations c, d , $c \oplus d$ is the ω -derivation that differs from d only in that $(c \oplus d)_4 = (d)_4 * \langle c \rangle$. We then define a function π^* by primitive recursion:

$$\begin{cases} \pi^*(0) = \pi(\psi(0)) \\ \pi^*(m+1) = \pi^*(m) \oplus \pi(\psi(m+1)) \end{cases}$$

Then, it is easily proved by induction on m , that for all m ,

$$\text{Der}^\omega(\pi^*(m), k, 1, \#C(\underline{m})) \text{ and } \text{OD}(\pi^*(m)) < \text{OD}(\pi^*(m+1)).$$

If $[b^*](m) = \pi^*(\psi(m))$ for all m , then $\langle 17, k, 1, \#\forall x C(x), 1, b^* \rangle$ is a proper ω -derivation of $\forall x C(x)$. Since PR- $(\Sigma_1^0\text{-IR}) \vdash \forall x C(x) \rightarrow A$, we also have that $\exists d \exists n \text{Der}^\omega(d, n, 1, \#(\forall x C(x) \rightarrow A))$ without any applications of the ω -rule, and so finally it follows that there exists a proper ω -derivation of A . \square

Corollary 7.3 *Let A be a prenex sentence of \mathcal{L}' . Then*

$$\exists e \exists n e \Vdash_n A \implies \exists b \in \underline{\text{OPRA}}(b) \vdash A.$$

8 It is time now to pay the debt incurred in Theorem 7.1 and prove that given any ω -derivation d of $\Gamma \vdash A$, there exists a proper ω -derivation d^* of $\Gamma \vdash A$. The thought is simple: given a subderivation c of d obtained as a result of an application of the ω -rule to its immediate subderivations $c_n = [(c)_4](n)$, we obtain a corresponding subderivation c^* of d^* by applying the ω -rule to derivations c_n^* which are ω -derivations of the same sequents as c_n but whose ordinal height $\text{OD}(c_n^*)$ is strictly increasing with n . The simplest way to achieve this would be to “paste together” c_n ’s successively so that $c_0^* = c_0$, $c_1^* = c_0$ plus c_1 , $c_2^* = c_1^*$ plus c_2 , and so on. However, this idea is not easily implemented primarily because the proof of the theorem requires us to obtain d^* uniformly primitive recursively in d .

In a particular case, we have already applied a “pasting together” procedure to the ω -derivation considered in the proof of 7.2. So we may assert

Lemma 8.1 *Let d be an ω -derivation of $\Gamma \vdash A$. For any ω -derivation c there is an ω -derivation $c \oplus d$ of $\Gamma \vdash A$ such that*

$$\text{OD}(c) < \text{OD}(c \oplus d)$$

Moreover, $c \oplus d$ can be obtained primitive recursively from c and d , and $c \oplus d$ is proper if d is.

Theorem 8.2 *Let d be an ω -derivation of $\Gamma \vdash A$. Then there exists a proper ω -derivation of $\Gamma \vdash A$.*

Proof: Let θ be the 2-place variant of the function $\theta(a, b, e)$ introduced in §4, omitting the parameter b . Let $\theta^+(e, d)$ be a Kleene index of the function $\lambda xy. y \oplus [\theta(e, (d)_5)](x)$. We easily obtain an elementary function ξ^* such that, for each fixed e, d , $\xi^*(e, d)$ is a Kleene index of the p.r. function $\psi_{e,d}$ satisfying the condition

$$\begin{cases} \psi_{e,d}(0) = 1 \\ \psi_{e,d}(x+1) = \psi_{e,d}(x) \oplus [\theta(e, (d)_5)](x) \end{cases} .$$

The functions $\psi_{e,d}$ play a crucial role in *pasting together* of subderivations c_n of d that appear “in” d as “premises” of an application of the ω -rule. Let ψ^* be the elementary function such that for any e, d ,

$$[\psi^*(e, d)](x) = [\xi^*(e, d)](x+1).$$

We define a 2-place p.r. function χ^+ by the course-of-values recursion:

$$\chi^+(e, d) = \begin{cases} d & \text{if } (d)_0 = 1 \\ \beta(\chi^+(e, (d)_4)) & \text{if } 1 < (d)_0 < 17 \\ \langle 17, (d)_1, (d)_2, (d)_3, (d)_4, \psi^*(e, d) \rangle & \text{if } (d)_0 = 17 \\ 0 & \text{otherwise} \end{cases} .$$

for an appropriate p.r. function β . We apply the primitive recursion theorem to obtain a Kleene index h of a p.r. function χ such that for any d ,

$$\chi(d) = \chi^+(h, d) = [h](d),$$

and proceed to show that the function χ determines the desired map so that, for any d, Γ, A ,

$$\text{Der}^\omega(d, n, \#\Gamma, \#A) \implies \text{Der}^\omega(\chi(d), n, \#\Gamma, \#A) \text{ and } \chi(d) \text{ is proper.}$$

The argument is by induction on the generating relation of the set of ω -derivations (or, alternatively, by transfinite induction on $\text{OD}(d)$). The claim holds trivially for $(d)_0 = 1$. For $1 < (d)_0 < 17$ we consider the case $(d)_0 = 2$ as an example. Then $d = \langle 2, n, \#\Gamma, \#(A \& B), (d)_4, i, j \rangle$ where $0 \leq i, j < \text{lth}((d)_4)$ and the immediate subderivations $(d)_{4,i}$ and $(d)_{4,j}$ of d are ω -derivations of $\Gamma \vdash A$ and $\Gamma \vdash B$, respectively. Then, by the induction hypothesis, $\chi((d)_{4,i})$ and $\chi((d)_{4,j})$ are proper ω -derivations of the same sequents, whence by the choice of the function β , we have

that $\chi(d)$ is a proper ω -derivation of $\Gamma \vdash A \ \& \ B$. The sole interesting case is when $(d)_0 = 17$. Then

$$d = \langle 17, n, \#\Gamma, \#\forall x B(x), (d)_4, (d)_5 \rangle$$

where, for each m , $[(d)_5](m)$ is an ω -derivation of $\Gamma \vdash B(\underline{m})$ and $([(d)_5](m))_1 \leq n$. By the induction hypothesis, for each m $\chi([(d)_5](m))$ is a proper ω -derivation of $\Gamma \vdash B(\underline{m})$ and $(\chi([(d)_5](m)))_1 \leq n$. Note that $\chi([(d)_5](m)) = [\theta(h, (d)_5)](m)$. We claim that for each m

$$\psi_{h,d}(m+1) \text{ is a proper } \omega\text{-derivation of } \Gamma \vdash B(\underline{m}), \text{ and} \quad (12)$$

$$\text{OD}(\psi_{h,d}(m)) < \text{OD}(\psi_{h,d}(m+1)) \quad (13)$$

It is easily seen that $(\psi_{h,d}(m+1))_1 \leq n$ for all m . Since for all m

$$\psi_{h,d}(m+1) = [\xi^*(h, d)](m+1) = [\psi^*(h, d)](m),$$

(12) and (13) will suffice for our purposes because then

$$\langle 17, n, \#\Gamma, \#\forall x B(x), (d)_4, \psi^*(h, d) \rangle,$$

which = $\chi(d)$, will be a proper ω -derivation of $\Gamma \vdash \forall x B(x)$, as required. Now (13) follows immediately from 8.1 and the definition of $\psi_{h,d}$. So to complete the proof of the theorem it remains to prove (12). We argue by induction on m . For $m = 0$, we have

$$\psi_{h,d}(1) = \psi_{h,d}(0) \oplus [\theta(h, (d)_5)](0) = [\theta(h, (d)_5)](0) = \chi[(d)_5](0).$$

Then the case $m = 0$ follows from the induction hypothesis. For the induction step, assume that $\psi_{h,d}(m+1)$ is a proper ω -derivation of $\Gamma \vdash B(\underline{m})$. Then $\psi_{h,d}(m+2) = \psi_{h,d}(m+1) \oplus [\theta(h, (d)_5)](m+1)$, whence, by 8.1 and the induction hypothesis, we have that $\psi_{h,d}(m+2)$ is a proper ω -derivation of $\Gamma \vdash B(\underline{m+1})$. \square

We now derive our main result, characterizing the strictly primitive recursively realizable prenex sentences of \mathcal{L}' .

Theorem 8.3 *For any prenex sentence A of \mathcal{L}' ,*

$$\exists e \exists m \ e \Vdash_m A \iff A \text{ is PR-true} \iff \exists d \in \text{OPRA}(d) \vdash A \iff \exists z \exists n \ \text{Der}^\omega(z, 1, \#A).$$

NOTES

1. In particular, we assume that unlimited primitive recursion is accommodated directly without regard to the complexity considerations related to the Grzegorzcyk hierarchy. (A similar indexing was originally described in [9], and so we call this one the *Kleene indexing*.)
2. We are implicitly referring to a recursive enumeration of n -place p.r. functions defined relative to the Kleene indexing as described in [9], p. 74, where we have the $\varphi(\vec{x}_n) = [e](\vec{x}_n)$ if e is a Kleene index of φ .

3. For definiteness we describe this in some detail. (Cf. also [6]). We first define an r.e. “pre-ordering” relation $<^*$ satisfying the condition

$$x <^* y \iff (x = 1 \ \& \ y \neq 1) \vee \exists z < y (y = 2^z \ \& \ z \neq 0 \ \& \ (x <^* z \vee x = z)) \vee \\ \exists n \exists z < y (y = 3 \cdot 5^z \ \& \ (x <^* [z](n) \vee x = [z](n))).$$

We say that a function f is *increasing in $<^*$* if for all n , $f(n) <^* f(n+1)$. Let \underline{O} be the smallest set X of integers satisfying the conditions: (1) $1 \in X$; (2) $b \in X \implies 2^b \in X$; and (3) if $[d](x) = f(x)$ where f is increasing in $<^*$ and $f(n) \in X$ for all n , then $3 \cdot 5^d \in X$. We set $x <_0 y \iff x, y \in \underline{O}$ and $x <^* y$. For each $b \in \underline{O}$ there is a 1-1 order-preserving map from an initial segment of the ordinals onto the ordered set $\underline{O}_b = \{x \mid x \leq_0 b\}$, in which the successor ordinals are mapped to integers of the form 2^c and limit ordinals to integers of the form $3 \cdot 5^d$. In fact, in general, if $b \in \underline{O}$ and $a <^* b$, then $a \in \underline{O}$. The theory of such ordinal notations is similar to that of Kleene’s recursively based O . In [9] it is shown that the class of ordinals represented in this way remains the same if condition 3 is weakened to allow d to be an index of a (total) *recursive* function increasing in (an appropriately defined) $<^*$.

4. These largely stem from the fact that we are unable to directly refer to ordinals in a purely arithmetical language, but must avail ourselves of the notations for ordinals such as those described above. See, e.g., [5], §3.
5. Here $\text{Prf}(x_1, x_2, x_3)$ is a formula expressing a p.r. predicate that describes the proof relation of a formal system at the stage α whenever the variable x_1 is replaced by the numeral for an ordinal notation for α . The notation $\ulcorner \varphi(\dot{x}) \urcorner$ denotes the Gödel number of the formula that results from substituting the numeral for a given integer x for the free occurrences of the variable x_1 in φ . (Thus, the variable x occurs free in $\ulcorner \varphi(\dot{x}) \urcorner$). The notation extends naturally to cover simultaneous substitution of several variables.
6. It is convenient to include among the primitives bounded quantifiers $\forall x \leq t, \exists x \leq t$, where t is a term not containing the variable x . We call the resulting language \mathcal{L}' . PR-(Σ_1^0 -IR), called PRA in [3], is formulated in the language $\mathcal{L}(\text{PRA})$ described in §3 of [3].
7. We assume that a Gödel numbering $\#$ of terms of \mathcal{L}' has been set up in such a way that a Kleene index \underline{t} of the function $\lambda \vec{y}_k. t(\vec{y}_k)$ expressed by a term $t(\vec{x}_k)$ is obtainable from $\#t(\vec{x}_k)$ by an elementary function. Then a Gödel numbering $\#$ of PR-derivations can be set up so that for any PR-derivation D , an integer r such that $\lambda \vec{y}_k. t(\vec{y}_k) \in G_r$ for any term t operative in D can be obtained by an elementary function from $\#D$.
8. When Γ is a singleton $\langle A \rangle$, we write ‘ $\#A$ ’ instead of the code for $\langle \#A \rangle$.
9. The definition of realizability and the proof are easily amended to reflect the fact that \mathcal{L}' includes the bounded quantifiers $\forall x \leq t, \exists x \leq t$ among its primitives. We do that in detail, in a different context, in [2]. Our notation differs slightly from that used in [3]: we write ‘ $t \Vdash_n A$ ’ instead of ‘ $t \Vdash_n \ulcorner A \urcorner$ ’.
10. In the course of defining Φ we make use of the elementary function ξ , where $\xi(n+1)$ is an $n+1$ -index of the enumeration e_{n+1} of G_n . The function ξ is obtained from the proofs—given in detail in [1]—of the facts stated in Theorems 1.1(d) and 2.1 of [3].

11. Here $\sim A$ is the \mathcal{L}' formula that results when all occurrences of ' $\forall x \leq t$ ', ' $\exists x \leq t$ ', ' \forall ', and ' $\&$ ' in A are replaced by occurrences of ' $\exists x \leq t$ ', ' $\forall x \leq t$ ', ' $\&$ ', and ' \forall ', respectively, and every atomic subformula of A of the form ' $t(\vec{x}_k) = \underline{0}$ ' where $t(\vec{x}_k)$ is an \mathcal{L}' term—all atomic subformulas may be assumed to be of this form—is replaced by ' $\bar{s}\bar{g}(t(\vec{x}_k)) = \underline{0}$ '. (We shall subsequently extend this notation to all prenex formulas of \mathcal{L}' by treating the unbounded quantifiers in the same way.) It is easily seen that $\bar{s}\bar{g}(x) = \underline{0} \rightarrow (x = \underline{0} \rightarrow \perp)$ is derivable in PR- $(\Sigma_1^0\text{-IR})$. That PR- $(\Sigma_1^0\text{-IR}) \vdash \sim\varphi \rightarrow \neg\varphi$ holds for all prenex formulas φ is proved by induction on the complexity of prenex formulas. (Classically, $\sim A$ and $\neg A$ are equivalent.)
12. Here k_1 depends on the order of the PR-derivation of (5) which may be obtained primitive recursively from e^* , which in turn, depends primitive recursively on $\#A(x)$. On the other hand, k_2 depends primitive recursively on $\#A(x)$.
13. Again, a form of this result in which primitive recursively based \underline{O} is replaced by Kleene's recursively based O and the starting point of the transfinite progression is the classical PA instead of our intuitionistic PR- $(\Sigma_1^0\text{-IR})$ was proved in [5], pp. 287–89.
14. Here Γ, Δ denote finite sequences of formulas, and A, B, C formulas of $\mathcal{L}(\text{PRA})$. Throughout we are assuming that $\text{lth}(d) > 0$, and that $i, j, < \text{lth}(d)$ where they are mentioned.
15. In the case $(m)_0 = 17$ it may seem more natural to let $\text{OD}(m) = \lim_n \{\text{OD}([(m)_5](n))\}$. The extra elements in our definition are needed to give a simpler proof of Theorem 7.1 and in the proof of Theorem 8.2.
16. For the sake of clarity and conciseness we do not explicitly define α, β, χ here. The definitions can be easily reconstructed from the sketch of the proof given below: e.g., β expresses a definition by cases in which the thirteen different types of ω -derivations d such that $1 < (d)_0 < 17$ are distinguished.

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School of Philosophy
University of Southern California
Los Angeles, CA 90089
email: zlatan@usc.edu