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A Model of $\hat{R}_3^2$ inside a Subexponential Time Resource

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Abstract Using nonstandard methods we construct a model of an induction scheme called $\hat{R}_3^2$ inside a "resource" of the form $\{M(a) : M$ is a Turing machine of code $\leq r$, and $M(a)$ is calculated in less than $2^{||a||^r}$ steps$\}$, where $|x|$ means the length of the binary expansion of x and a, r are nonstandard parameters in a model of S_3^1. As a consequence we obtain a model theoretic proof of a witnessing theorem for this theory by functions computable in time $2^{|n|^{O(1)}}$, a result first obtained by Buss, Krajíček, and Takeuti using proof theory.

1 Introduction In [2], Buss defined bounded arithmetic theory S_2 and fragments S_2^i. In an extended arithmetical language he defined a hierarchy of formulas Σ_i^b corresponding to Σ_i^p, that is, predicates in the ith level of the polynomial time hierarchy. For example, Σ_1^b formulas define NP predicates. Theory S_2^i is axiomatized by a finite set of open axioms for the symbols of the language plus a special schema of *length-induction* for Σ_i^b formulas. Thus $S_2^1 \subset S_2^2 \subset \ldots$ and $S_2 = \bigcup S_2^i$. It is stated that the Σ_{i+1}^b-recursive functions S_2^{i+1} can define are exactly those computable in polynomial time by a Turing machine using an oracle from the class Σ_i^p. It is then not a surprise if many important problems in complexity theory are closely related with the study of this hierarchy of theories. The main open question in bounded arithmetic is about the finite axiomatizability of S_2 (or of theory $I\Delta_0$, S_2 being a conservative extension of $I\Delta_0 + \Omega_1$ introduced in [11] by Wilkie and Paris). This is the same as whether or not the inclusions $S_2^i \subset S_2^{i+1}$ are strict, as each S_2^i is finitely axiomatizable (see [2]). Krajíček, Pudlák, and Takeuti showed in [8] that if S_2 is finitely axiomatizable then the polynomial hierarchy PH collapses. Buss [3] and, independently, Zambella [12], strengthened this by showing that S_2 is finitely axiomatizable if and only if it proves the collapse of PH. Most of this work has been done by using proof theoretical methods. Good introductory references for these topics are Buss [2], Hájek and Pudlák [6], and

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Krajíček [7]. We present here a model theoretic construction for bounded arithmetic theory $\hat{R}^2_3$, from which we derive a witnessing theorem for this theory by functions computable in time $2^{|n|^{O(1)}}$, a result first obtained by Buss, Krajíček, and Takeuti [5].

2 *Basic notions and results* We use Buss's notations (see [2]), working in the extended arithmetical language $L_3 = \{0, 1, +, \cdot, <, \lfloor x/2 \rfloor, |x|, \#_2, \#_3\}$, where $|x|$ is the length of the binary expansion of x, $x\#_2 y$ means $2^{|x|\cdot|y|}$ and $x\#_3 y$ stands for $2^{|x|\#_2|y|}$. Most of Buss's results in [2] were stated for theories in language L_2 without the $\#_3$ symbol (read *smash 3*). But, as he pointed out, they readily generalize to languages L_i including a function symbol $\#_i$ with the same rate of growing as function ω_{i-1} of [11] ($x\#_i y = 2^{|x|\#_{i-1}|y|}$), provided we substitute polynomial time by the corresponding S_i-time (also called $\#_i$-time in some texts). In particular, to language L_3 corresponds $2^{|n|^{O(1)}}$-time, to L_4 is $2^{2^{||n||^{O(1)}}}$-time, and so on. Quantifiers of the form $Qx \leq t$, where t is a term, are called bounded quantifiers. Those of the form $Qx \leq |t|$ are called sharply bounded quantifiers. Formulas with only sharply bounded quantifiers are called sharply bounded formulas. This class is noted Δ^b_0, Σ^b_0, or Π^b_0. For $i \geq 0$, Σ^b_{i+1} is the smallest class of formulas containing Σ^b_i, Π^b_i, and negations of Π^b_{i+1}, and closed by $\wedge$, $\vee$, sharply bounded quantifiers, and $\exists x \leq t$. Classes Π^b_i are defined analogously. A formula is said to be *strict* Σ^b_1 if it has the form $\exists y \leq t[\Delta^b_0]$. More generally, a formula is *strict* Σ^b_i if it has the form $\exists y \leq t[\textit{strict}\ \Pi^b_{i-1}]$. We denote by $\hat{\Sigma}^b_i$ the class of *strict* Σ^b_i formulas. The class $\hat{\Pi}^b_i$ is defined analogously. If T is any theory and $i \geq 1$, we say that Ψ is $\Delta^b_i(T)$ if $T \vdash (\Psi \equiv \Psi_1) \wedge (\Psi \equiv \Psi_2)$ for some $\Psi_1 \in \Sigma^b_i$ and $\Psi_2 \in \Pi^b_i$. By $\alpha(x)$-IND *up to* y we denote the formula

$$[\alpha(0) \wedge \forall x < y(\alpha(x) \Longrightarrow \alpha(x+1))] \implies \alpha(y)$$

and if Γ is a class of formulas and $m \in \mathbb{N}$, Γ-L$^{(m)}$IND denote the schema $\alpha(x)$-IND *up to* $|y|_m$ for α in Γ, where $|y|_m = |(|y|_{m-1})|$ and $|y|_0 = y$. In this article we are concerned with $m = 1, 2$ so we write LIND, LLIND and $||y||$ for L$^{(1)}$IND, L$^{(2)}$IND and $|y|_2$. BASIC$_3$ is a finite set of open axioms for the symbols of L_3 and S^i_3 is the theory BASIC$_3$ + Σ^b_i-LIND (originally it is defined by another induction schema called PIND, but these two axiomatizations are equivalent; see Buss and Ignjatović [4]). R^i_3 is the theory BASIC$_3$ + Σ^b_i-LLIND. By $\hat{S}^i_3$, $\hat{R}^i_3$ we denote the corresponding theories for strict formulas. We shall suppose that included in our language are some other useful primitives. These are known to be definable from L_3 with a little amount of induction, and its inclusion does not increase the strength of theories containing S^1_3, for example. In particular we suppose in L_3 the Cantor pairing function $\langle x, y\rangle$ and its projections $\langle z\rangle_1$, $\langle z\rangle_2$, as well as a binary function $y = (c)_x$ for *y is the xth element in the sequence coded by c*. In general, we will be able to code sequences of logarithmic length. By Σ^b_i-replacement we denote the schema

$$\forall x \leq |a| \exists y \leq b \Psi(x, y) \Longrightarrow \exists c \forall x \leq |a| \Psi(x, (c)_x)$$

for $\Psi \in \Sigma^b_i$. In fact c can be bounded by a term of L_3, so the conclusion is also Σ^b_i and, moreover, implies trivially the premise. Hence, this schema allows us to *push* *inside* sharply bounded quantifiers in Σ_i^b formulas. This, together with the possibility to merge two consecutive quantifiers of the same type into a single one using coding, permits us to put Σ_i^b formulas in the strict form. As $\hat{S}_3^i \vdash \Sigma_i^b$-replacement, we have that $\hat{S}_3^i \equiv S_3^i$. On the other hand, we have that $R_3^i \vdash \Sigma_i^b$-replacement (see Allen [1]), but it is not known if this holds for $\hat{R}_3^i$. Nevertheless, we can derive in $\hat{R}_3^i$ the $\hat{\Sigma}_{i-1}^b$-LIND axioms, thus proving that $\hat{R}_3^i \vdash S_3^{i-1}$. We note by S_3 the class of total functions computable in time $2^{|n|^{O(1)}}$. For an integer a we put $S_3(a) := \{f(a) : f \in S_3\}$ and we say that an L_3-structure K is S_3-closed if $S_3(a) \subset K$ for every $a \in K$. Let $C(e, T, x, y)$ mean *y is calculated from x in time T by $\{e\}$, the Turing machine coded by e*. Later we will see that this is definable in S_3^1. The aim of this article is to prove the following theorem.

Theorem 2.1 *Let M be a countable nonstandard model of S_3^1. Let $a, r \in M \backslash \mathbb{N}$ and suppose that $M \models \exists y(y = 2^{2^{||a||^r}})$. Let $R = \{y : M \models \exists e \leq r\, C(e, 2^{||a||^r}, a, y)\}$. There is an L_3-substructure K^* of M such that*

1. *$a \in K^*$;*
2. *K^* is S_3-closed, and so $K^* <_{\Delta_0^b} M$;*
3. *$K^* \subset R$;*
4. *$K^* \models \hat{R}_3^2$.*

As a consequence we get two known corollaries. Their proofs are classic; we give it for the sake of completeness.

Corollary 2.2 *Let $\varphi(x, y)$ be a Σ_1^b-formula and suppose that*

$$\hat{R}_3^2 \vdash \forall x \exists y\, \varphi(x, y)\ .$$

Then for some $f \in S_3$, $S_3^1 \vdash \forall x\, \varphi(x, f(x))$.

Corollary 2.3 *The theory $\hat{R}_3^2$ is $\forall\Sigma_1^b$-conservative over S_3^1.*

Proof of Corollary 2.2: As explained above we can suppose $\varphi \in \hat{\Sigma}_1^b$. Then, using coding to merge two consecutive existential quantifiers into a single one, we can assume that φ is Δ_0^b. Let a be a new constant symbol and let T be the theory

$$S_3^1 \cup \{\forall y(C(e, 2^{||a||^k}, a, y) \Longrightarrow \neg\varphi(a, y)) : e, k \in \mathbb{N}\}.$$

We claim that T is inconsistent. Suppose the contrary and let $T' = T \cup \{\forall y(C(e, 2^{||a||^k}, a, y) \Longrightarrow y < d) : e, k \in \mathbb{N}\}$, where d is another new constant symbol. Clearly T' is also consistent. Let M be a countable model for it. As d is a bound for $S_3(a)$, M must be nonstandard. We have for every $r_0 \in \mathbb{N}$

$$M \models \forall k \leq r_0 \forall e \leq k \forall y(C(e, 2^{||a||^k}, a, y) \Longrightarrow \neg\varphi(a, y)).$$

In particular,

$$M \models \forall k \leq r_0 \forall e \leq k \forall y \leq d(C(e, 2^{||a||^k}, a, y) \Longrightarrow \neg\varphi(a, y)).$$

As we will see later, this last formula is equivalent to a Π_1^b one in S_3^1, and $S_3^1 \vdash \Pi_1^b$-LIND. So by overspill it must be valid for some $r_0 \in M \backslash \mathbb{N}$. If a is interpreted by some standard integer then $S_3(a) = \mathbb{N}$ and thus, as $M \models T$, we would have for every $y \in \mathbb{N}$ $M \models \neg\varphi(a, y)$. By elementarity this formula holds in $\mathbb{N}$, hence $\mathbb{N} \models \forall y \neg\varphi(a, y)$. As $\mathbb{N}$ is obviously a model of $\hat{R}^2_3$, this contradicts the hypothesis of the theorem. So let us suppose $a \in M \backslash \mathbb{N}$ and let $r \leq r_0$ such that $M \models \exists y < d\ (y = 2^{2^{||a||^r}})$ (see Lemma 3.13). Then we have

$$M \models \forall e \leq r \forall y \leq d (C(e, 2^{||a||^r}, a, y) \Longrightarrow \neg\varphi(a, y)).$$

By definition of R we have $y < 2^{2^{||a||^r}} < d$ for every $y \in R$, and so the last equation reads

$$M \models \forall y \in R\, \neg\varphi(a, y).$$

By Theorem 2.1 there is an L_3-structure $K^* \subset M$ such that

1. $a \in K^*$;
2. K^* is S_3-closed;
3. $K^* \subset R$;
4. $K^* \models \hat{R}^2_3$.

By (1), (2), and (3) we have $K^* \models \forall y \neg\varphi(a, y)$, and by (4) $K^* \models \forall x \exists y\, \varphi(x, y)$. Thus we get a contradiction and the claim is proved. As T is inconsistent, by compactness there is some $n, e_0, \ldots, e_n, k_0, \ldots, k_n \in \mathbb{N}$ such that

$$S^1_3 \vdash \bigvee_{i=0}^{n} \exists y (C(e_i, 2^{||a||^{k_i}}, a, y) \wedge \varphi(a, y)).$$

By the theorem on constants

$$S^1_3 \vdash \forall x \bigvee_{i=1}^{n} \exists y (C(e_i, 2^{||x||^{k_i}}, x, y) \wedge \varphi(x, y)).$$

Let $f(x)$ be the result of the following search: for $i = 0$ to n we run $\{e_i\}$ on input x with clock $2^{||x||^{k_i}}$ looking for an output y satisfying $\varphi(x, y)$. Clearly $f \in S$ and by the last equation $S^1_3 \vdash \forall x \varphi(x, f(x))$. Hence the corollary is proved. □

Corollary 2.3 follows immediately.

Remark 2.4 Buss, Krajíček, and Takeuti [5] have shown a result stronger than this corollary: the theory R^2_3 is $\forall\Sigma^b_2$-conservative over S^1_3.

Remark 2.5 The proof of Buss's main theorem in [2], and those of Buss, Krajíček, and Takeuti in [5], uses proof theory methods. On the other side, Wilkie (in an unpublished manuscript) gave a proof of Buss's theorem in a model theoretic way, from which Pudlák gave a version in [6]. Another model theoretic proof is given by Zambella in [12].

Remark 2.6 Theorem 2.1 can be generalized as follows: if $M \models S^i_3$, $i > 1$, we can consider a larger resource R by giving the Turing machines access to oracles in the ith level of the S_3-time hierarchy. Then we can contruct a Δ^b_{i-1}-elementary L_3-substructure K^* of M which is a model of $\hat{R}^{i+1}_3$. The corresponding witnessing and conservation corollaries follow similarly as 2.2 and 2.3.

Remark 2.7 To drop the *strict* in Theorem 2.1 it would suffice to carry out the construction with formulas of the form $\forall x \leq |u| \exists y \leq t \forall z \leq s\, \psi$, $\psi \in \Delta^b_0$, instead of simply $\hat{\Sigma}^b_2$ formulas. The theory obtained in this way would prove Σ^b_2-replacement. But the inclusion of an extra quantifier, even a sharply bounded one, poses some problems. A solution for these could throw some light on how to treat the Σ^b_3 case without the use of oracles. Note parenthetically that we cannot use oracles if we want subexponential time witnessing theorems, and this makes it nontrivial to construct models for Σ^b_i induction axioms inside the corresponding resources.

Remark 2.8 Our proof is inspired by Wilkie's, but in addition it shows the possibility to use a nonstandard initial segment of Turing machine programs at the same time as an initial segment of computing times. We hope that this possibility will help to pass from Σ^b_2 to Σ^b_i formulas in the construction and the result of this article. In such a case, by extending the corollary one could obtain a proof of some recent results of Pollett [10], namely, that theory $\hat{T}^{i,i}_{i+1}$ has S_{i+1}-time witnessing functions for Σ^b_1 formulas. Here $\hat{T}^{i,i}_{i+1}$ is essentially the theory in the language L_{i+1}, including the $\#_{i+1}$ function symbol, with $\hat{\Sigma}^b_i$-L$^{(i)}$IND axioms, and S_{i+1}-time is the subexponential time corresponding to L_{i+1} (S_2-time is polynomial time, S_3-time is $2^{|n|^{O(1)}}$-time, etc.).

Remark 2.9 These results yield a hierarchy of theories $\hat{T}^{i,i}_{i+1}$ such that if $\hat{T}^{i,i}_{i+1}$ proves that a set X is NTIME(S_{i+1}) $\cap$ co-NTIME(S_{i+1}), then actually $X \in$ DTIME(S_{i+1}). Thus they are possible analogs of the P=NP $\cap$ co-NP problem, hence their interest: in view of the difficulty of P=NP $\cap$ co-NP it is important to have analogous problems which we can settle. In addition, a further study of the proof and model theory of $\hat{T}^{i,i}_{i+1}$ may yield lower bounds about the function which to a proof in $\hat{T}^{i,i}_{i+1}$ that X is NTIME(S_{i+1}) $\cap$ co-NTIME(S_{i+1}) associates an algorithm in DTIME(S_{i+1}) deciding X. Such lower bounds would shed precious light on NP $\cap$ co-NP. The reinforcement of model theory introduced here for the study of $\hat{T}^{i,i}_{i+1}$ should not be superfluous for such ambitious aims.

3 Proof of Theorem 2.1 In Section 3.1 we briefly explain how the proof goes. Section 3.2 presents some tools needed to work with Turing machines. Next we introduce the notions of sparse sequences and resources in 3.3, and finally we present construction of model K^* in Section 3.4.

3.1 Sketch of the proof Fix an enumeration of axioms θ-IND up to $||d||$ with parameters in M and θ running over $\hat{\Sigma}^b_2$ formulas. We construct K^* as the union of an increasing chain $(K_n)_{n<\omega}$. Let $K_0 = S_3(a) = \{ f(a) : f \in S_3\}$ and let θ_1-IND *up to* l_1 be the first axiom in the enumeration having its parameters in K_0. We want $K_1 \supset_{L_3} K_0$, K_1 S_3-closed and satisfying

$$\neg\theta_1(0) \vee \exists j < l_1[\theta_1(j) \wedge \neg\theta_1(j+1)] \vee \theta_1(l_1)$$

where $\theta_1(j) \equiv \exists y \leq t \forall z \leq s\, \psi(j, y, z)$. We can suppose $r < ||a||$ and $r = 2^{|r|-1}$. Let $(T_j)_{j \leq l_1+2}$ be a decreasing sequence such that $2^{||a||^r} \gg T_0 \gg T_1 \gg \cdots \gg T_{l_1+2} \gg 1$ (where $A \gg B$ means $A > B.2^{||a||^{O(1)}}$) and such that the T_j's are easy to calculate from

a and r (for example, $T_j = 2^{||a||^r - (j+1)||a||^{r/2}}$). For $j = 0, \ldots, l_1 + 2$ let $R_j(x) = \{\, y : C(e, T_j, x, y)\ \ for\ some\ e \leq r\}$. K_1 will be generated by an element a_1 obtained by running on input a the next program P (which depends on a code for $|r|$).

1. Compute $r = 2^{|r|-1}$.
2. Compute the parameters of θ_1-IND *up to* l_1 and T_0 from the input a.
3. Put $j := 0$, $y_{-1} := 0$.
4. Compute T_{j+1}.
5. Look for $y_j \in R_j(\langle j, a, y_{j-1}\rangle)$, $y_j \leq t$, such that for every $z \in R_{j+1}(\langle j + 1, a, y_j\rangle)$ such that $z \leq s$, $M \models \psi(j, y_j, z)$.
6. If there is no such y_j, stop the machine with output $a_1 = \langle j, a, y_{j-1}\rangle$.
7. If y_j is found and $j < l_1$, then put $j := j + 1$ and go to 4.
8. If y_{l_1} is found, stop the machine with output $a_1 = \langle l_1 + 1, a, y_{l_1}\rangle$.

Let $a_1 = \langle J_1 + 1, a, y_{J_1}\rangle$ and suppose, for example, $0 \leq J_1 < l_1$. Then we have

1. for every $z \in R_{J_1+1}(a_1)$ such that $z \leq s$, $M \models \psi(J_1, y_{J_1}, z)$;
2. for every $y \in R_{J_1+1}(a_1)$ such that $y \leq t$, there is some $z \in R_{J_1+2}(\langle J_1 + 2, a, y\rangle)$ such that $z \leq s$ and $M \models \neg\psi(J_1 + 1, y, z)$.

So, in order to have $K_1 \models \theta_1(J_1) \wedge \neg\theta_1(J_1 + 1)$, we choose K_1 contained in $R_{J_1+1}(a_1)$ and allowing computations in time T_{J_1+2} :

$$K_1 = \{\ \{e\}(a_1) < 2^{2^{||a||^{O(1)}}} calculated\ in\ time < O(1).r^2.T_{J_1+2},\ e < |r|^{O(1)}\}\ .$$

It is easy to see that $K_0 \subset_{L_3} K_1$ and K_1 is S_3-closed. To prove that $K_1 \subset R$ we use the fact that P can be coded by some $p < |r|^{O(1)}$ and calculates a_1 in less than $r^2.T_0$ steps. Consider now θ_2-IND *up to* l_2, the next axiom in the enumeration having its parameters in K_1. We want $K_2 \supset_{L_3} K_1$ satisfying this axiom while preserving $\theta_1(J_1) \wedge \neg\theta_1(J_1 + 1)$. The new axiom will be satisfied by letting the construction of K_2 imitate that of K_1, replacing a, θ_1, l_1 by a_1, θ_2, l_2, and the sequence T_i by another sequence T'_i. As explained above, $\theta_1(J_1) \wedge \neg\theta_1(J_1 + 1)$ will be preserved if $K_2 \subset R_{J_1+1}(a_1)$ and K_2 allows computations in time T_{J_1+2}. In other words, the maximal computation times T'_i are chosen between T_{J_1+1} and T_{J_1+2} (for example, $T'_j = T_{J_1+1}/2^{(j+1)||a||^{r/4}}$ if $T_j = 2^{||a||^r - (j+1)||a||^{r/2}}$). In this way $T_{J_1+1} \gg T'_0 \gg T'_1 \gg \cdots \gg T'_{l_2+2} \gg T_{J_1+2}$. Let P' be a program similar to P, running on input a_1, with θ_2-IND *up to* l_2 and T'_i in place of θ_1-IND *up to* l_1 and T_i. Let $a_2 = \langle J_2 + 1, a_1, y_{J_2}\rangle$ be its output and $K_2 = \{\ \{e\}(a_2) < 2^{2^{||a||^{O(1)}}}$ calculated in time $< O(1).r^2.T'_{J_2+2}$, $e < |r|^{O(1)}\}$. Then we prove as above that $K_1 \subset_{L_3} K_2$, K_2 is S_3-closed, $K_2 \subset R$ and $K_2 \models \theta_1$-IND *up to* $l_1 \wedge \theta_2$-IND *up to* l_2. In this way we get $K_3, K_4, \ldots$ and putting $K^* = \bigcup_{n<\omega} K_n$ we have the desired model. □

3.2 Definability of Turing machine computations We call S_3 the set of total functions computable in time $2^{|n|^{O(1)}}$ in the standard structure $\mathbb{N}$. For a predicate X we say that $X \in S_3$ if its characteristic function belongs to S_3. Note that (the intended interpretation in $\mathbb{N}$ of) function symbols of L_3 are in S_3. In particular Δ_0^b predicates are decidable in time $2^{|n|^{O(1)}}$, therefore, S_3-closed substructures are Δ_0^b-elementary. This will be used thoroughly. Σ_i^b predicates correspond exactly to predicates in the ith level of the $2^{|n|^{O(1)}}$-time hierarchy. We present here some known facts saying roughly that in any model of S_3^1 these functions are definable and have the expected properties, and this will also hold for some nonstandard functions when $M \neq \mathbb{N}$. Proofs are omitted since they are tedious and contain no new ideas. For a reference see [2] and [6]. In order to formalize computations we consider deterministic k-tapes Turing machines, for a fixed $k \in \mathbb{N}$, and a natural coding of its programs and computations. If e is an index for a Turing machine, that is, a code for its program, we note by $\{e\}$ both the machine itself and the function it computes. By $e \in S_3$ we mean $\{e\} \in S_3$ and $e \in \mathbb{N}$.

Lemma 3.1 *For every standard Turing machine M there is a $\Delta_1^b(S_3^1)$ formula $Comp_M(c, x)$ expressing that c is the code of a computation of M on input x.*

In S_3^1 we can code sequences of logarithmic length and there are terms $t_k(x)$ standing for $2^{2^{||x||^k}}$. In consequence we get

Lemma 3.2 *Every predicate in S_3 is Δ_1^b definable in S_3^1.*

Lemma 3.3 *For every standard Turing machine M*

$$S_3^1 \vdash \forall v \forall x \exists ! c (Comp_M(c, x) \wedge lh(c) = |v|)$$

where $lh(c)$ is the length of the computation coded by c.

If $M \models S_3^1$ and $log(M) := \{|y| : y \in M\}$, this lemma will allow us to define computations in time T provided $T \in log(M)$. In particular, as $2^{||a||^k} \in log(M)$ for every $k \in \mathbb{N}$, we have

Lemma 3.4 *Every function in S_3 is provably Δ_1^b (total) in S_3^1.*

Remark 3.5 By Buss's theorem (the version for S_3^1) every function provably Σ_1^b in S_3^1 is in S_3 (see [2]). As a consequence every $\Delta_1^b(S_3^1)$ predicate is decidable in time $2^{|n|^{O(1)}}$.

Now using Lemma 3.4 we can define a restricted version of a universal Turing machine which will nevertheless be able to simulate all functions in S_3.

Lemma 3.6 *There is a $\Delta_1^b(S_3^1)$ formula $U(e, v, x, y)$ expressing that e is the code of a (probably nonstandard) Turing machine and $\{e\}$ calculates y from x in less than $|v|$ steps.*

Lemma 3.7 *There is a $\Delta_1^b(S_3^1)$ formula $exp(x, y, z)$ expressing that $x^y = z$.*

We shall assume some properties of this definition. In particular $S_3^1 \vdash y = t_k(x) \Longleftrightarrow y = 2^{2^{||a||^k}}$, for every $k \in \mathbb{N}$. Moreover, we assume that for every term $t(\bar{x})$ in L_3, if $\varphi(\bar{x}, y)$ is the Δ_1^b definition of the corresponding function in S_3, then $S_3^1 \vdash y = t(\bar{x}) \Longleftrightarrow \varphi(\bar{x}, y)$.

Definition 3.8 $C(e, T, x, y)$ is the $\exists\Delta_1^b$ formula $\exists v(|v| = T \ \wedge \ U(e, v, x, y))$. It means that the Turing machine $\{e\}$ running on input x stops with output y before T steps.

Lemma 3.9 *There is $k_0 \in \mathbb{N}$ such that*

1. $S^1_3 \vdash \forall e, e' \, \exists e'' < (e.e')^{k_0} \, \forall x \, (\{e\}(\langle e', x\rangle) = \{e''\}(x))$;
2. $S^1_3 \vdash \forall e, e' \, \exists e'' < (e.e')^{k_0} \, \forall T, T', x, y, z, d$
$(T, T', T + T' < |d| \wedge C(e, T, x, y) \wedge C(e', T', y, z) \Longrightarrow C(e'', T + T', x, z))$.

Remark 3.10 Condition 1 will help us to estimate the code of a Turing machine. For example, suppose that X is a multiplicative closed cut in a model of S^1_3 and M a Turing machine. If M can be viewed as a standard program with some extra inputs $p_1, \ldots, p_n \in X$, $n \in \mathbb{N}$, then by (1) M can be coded by some $p \in X$.

Remark 3.11 By condition 2, if $e, e' \in X$ are Turing machine codes, then the composite function $\{e\} \circ \{e'\}$, if defined, has a code $e'' \in X$.

3.3 Sparse sequences, resources, and basic structures

Notation 3.12 Let M be a nonstandard model of S^1_3 and F a function from $\mathbb{N}$ to M. We put

1. $A > F(O(1))$ iff $A > F(n)$ for every $n \in \mathbb{N}$;
2. $F(O(1)) > B$ iff $F(n) > B$ for some $n \in \mathbb{N}$.

Even in a nonstandard model we keep $O(1)$ running over standard constants.

Lemma 3.13 *Let M be a nonstandard model of S^1_3 and let $a, d \in M\backslash\mathbb{N}$ such that $S_3(a)$ is bounded by d. There is some $r \in M\backslash\mathbb{N}$ such that following properties hold in M:*

1. $\exists y < d \; (y = 2^{2^{||a||^r}})$.
2. *r is a power of 2, and so $r = 2^{|r|-1}$.*
3. $r < ||a||$.

Moreover, r can be chosen smaller than any given $r_0 \in M\backslash\mathbb{N}$.

Proof: We know that for every $k \in \mathbb{N}$, $t_k(a) \in S_3(a)$ and $t_k(a) = 2^{2^{||a||^k}}$ in M. Thus we have for every $r_1 \in \mathbb{N}$, $M \models \forall k \leq |r_1|(\exists y < d \; y = 2^{2^{||a||^k}})$. This formula is Σ^b_1 in M and so by overspill it is true for some $r_1 \in M\backslash\mathbb{N}$. Now let $r_2 \in M\backslash\mathbb{N}$ such that $r_2 < |r_1|$ and $r_2 < ||a||$, and put $r = 2^{|r_2|-1}$. Then we have $r \in M\backslash\mathbb{N}$, r is a power of 2, as $|r_2| = |r|$, and finally $r \leq r_2 < ||a||$. □

Remark 3.14 In fact we have proved $M \models \forall x \leq r \, \exists y < d \; (y = 2^{2^{||a||^x}})$.

Remark 3.15 By (1) of Lemma 3.13 we have $[0, 2^{||a||^r}] \subset log(M)$ and then, by Lemma 3.3, computations in time $T \leq 2^{||a||^r}$ are definable in M.

Remark 3.16 We want r to be computable from some Turing machine of code $< |r|^{O(1)}$. That is why we impose condition 2 (see (3) of Lemma 3.22).

Remark 3.17 We want also $2^{||a||^r} \in S_3(\langle a, r\rangle)$. For this $r < ||a||^{O(1)}$ would suffice, we put $r < ||a||$ for simplicity. In this way $2^{||a||^r}$ is calculated from $\langle a, r\rangle$ by the function $\langle x, y\rangle \longmapsto 2^{||x||^{min(y,||x||)}}$ which is clearly in S_3.

Definition 3.18 Let M be a model of S^1_3, $A, B, l, \alpha \in M$, $(T_j)_{j\leq l}$ a sequence in M and F a function from $\mathbb{N}$ to M. Suppose $A > B$.

1. The sequence $(T_j)_{j\leq l}$ is *between* A and B if $(T_j)_{j\leq l}$ is decreasing and $A > (T_j)_{j\leq l} > B$.
2. The sequence $(T_j)_{j\leq l}$ between A and B is *generated* by α if for some $e \in S_3$
   (a) $T_0 = \{e\}(\langle\alpha, A\rangle)$;
   (b) $T_{j+1} = \{e\}(\langle\alpha, T_j\rangle),\ \ j < l$.
3. The sequence $(T_j)_{j\leq l}$ between A and B is $F(O(1))$-*sparse* if
   (a) $A > F(O(1)).T_0$;
   (b) $T_j > F(O(1)).T_{j+1},\ \ j < l$;
   (c) $T_l > F(O(1)).B$.

Lemma 3.19 *Let M, a, r be as in Lemma 3.13. Let $A, B, \alpha \in M$ and suppose that $2^{||a||^r} \geq A > B$, $a \in S_3(\alpha)$, $(T_j)_{j\leq l}$ is a sequence between A and B generated by α, and $l < 2^{||a||^{O(1)}}$. Then for some $e \in S_3$ we have $T_j = \{e\}(\langle j, \alpha, A\rangle),\ \ j \leq l$.*

Proof: Let $e' \in S_3$ such that $T_0 = \{e'\}(\langle\alpha, A\rangle)$ and $T_{j+1} = \{e'\}(\langle\alpha, T_j\rangle),\ \ j < l$. Let $k \in \mathbb{N}$ such that $l < 2^{||a||^k}$ and consider the standard Turing machine which on input $\langle j, \alpha, A\rangle$ calculates a from α, then $2^{||a||^k}$ (k is coded in its program); next it compares j and $2^{||a||^k}$ and if $j < 2^{||a||^k}$ it computes $\{e'\}^{(j+1)}(\langle\alpha, A\rangle)$. It runs in time $2^{|n|^{O(1)}}$ as $e' \in S_3$ and we iterate this function at most $2^{||a||^k}$ times (note that $2^{||a||^k} < 2^{||\alpha||^{O(1)}}$ as $a \in S_3(\alpha)$). Finally, we have that it calculates T_j when $j \leq l$. This can be proved by induction on l as $l \in log(M)$ and the condition considered is Δ^b_1. □

Lemma 3.20 *Let M, a, r be as in Lemma 3.13. Let $A, B, l \in M$ and suppose that $2^{||a||^r} \geq A > 2^{||a||^{O(1)}}.B$ and $l < ||a||^{O(1)}$. There is a $2^{||a||^{O(1)}}$-sparse sequence $(T_j)_{j\leq l}$ between A and B generated by $\langle a, \rho\rangle$ for some $\rho \in M\backslash\mathbb{N}$. Moreover, ρ can be chosen smaller than any given nonstandard integer in M.*

Proof: We have for every $k \in \mathbb{N}$, $M \models \exists y \leq a(y = 2^{||a||^k} \wedge A > y.B)$. By overspill this formula is true for some $\rho \in M\backslash\mathbb{N}$, and we can choose it as small as we want. Take $\rho < ||a||$ and consider the function

$$f(x, y, z) = msp(x, ||y||^{min(\lfloor z/2\rfloor, ||y||)})$$

where $msp(u, v)$ stands for $\lfloor u/2^v\rfloor$ when $v \leq |u|$ (msp is for *most significant part*; see [2]). Then clearly $f \in S_3$ and so is g defined by $g(u, x) = f(x, \langle u\rangle_1, \langle u\rangle_2)$. Put

$$T_0 = g(\langle a, \rho\rangle, A) \ \text{ and } \ T_{j+1} = g(\langle a, \rho\rangle, T_j), \ \text{ for } j < l.$$

Then we have $T_0 = \lfloor A/2^{||a||^{\lfloor\rho/2\rfloor}}\rfloor$ and for $j < l$, $T_{j+1} = \lfloor T_j/2^{||a||^{\lfloor\rho/2\rfloor}}\rfloor$. It is then clear than $(T_j)_{j\leq l}$ is $2^{||a||^{O(1)}}$-sparse, between A and B and generated by $\langle a, \rho\rangle$. □

Definition 3.21 Let M be a model of S^1_3 and let $a, r, T, c \in M$.

1. We use $R(r, T, c)$ to denote the subset $\{y \in M : \exists e \leq r\, C(e, T, c, y)\}$. We call these definable sets *resources*.

2. The basic L_3-structures we will consider are of the form

$$\{y \in M : \exists k \in \mathbb{N}\ \exists e < |r|^k\ (y < 2^{2^{||a||^k}} \wedge C(e, k.T, c, y))\}$$

We write $K(a, r, T, c)$ as an abbreviation for the expression above.

Lemma 3.22 *Let M, a, r be as in Lemma 3.13. Let $c, T \in M$ satisfy $2^{||a||^r} > O(1).T$ and let $K = K(a, r, T, c)$. Then K has the following closure property.*

1. *If $y \in K$ and $T' < O(1).T$, then $K(a, r, T', y) \subset K$.*

Moreover, if $T > 2^{||a||^{O(1)}}$ then

2. *K is S_3-closed;*
3. *$[0, |r|^{O(1)}[\cup \{r\} \subset K$.*

Proof: (1) Let $T' < O(1).T$, $k \in K$, $e < |r|^k$, such that $C(e, k.T, c, y)$. If $z \in K(a, r, T', y)$ then for some $k' \in \mathbb{N}$, $z < 2^{2^{||a||^{k'}}}$ and $C(e', k'.T', y, z)$ for some $e' < |r|^{k'}$. We have that $k.T + k'.T' < O(1).T < 2^{||a||^r}$, hence by (2) of Lemma 3.9 there is some $k'' \in \mathbb{N}$, k'' sufficiently large and some $e'' < |r|^{k''}$ such that $C(e'', k''.T, c, z)$, that is, $z \in K$. (2) If $T > 2^{||a||^{O(1)}}$ and $z \in S_3(y)$ for some $y \in K$, then since $y < 2^{2^{||a||^{O(1)}}}$ we have that $z < 2^{2^{||a||^{O(1)}}}$ and $C(e, T', y, z)$ for some $e \in \mathbb{N}$ and $T' < 2^{||a||^{O(1)}} < T$. Hence $z \in K$ and K is S_3-closed. (3) If $p \leq |r|^{O(1)}$ there is some $e \leq |r|^{O(1)}$ such that $\forall x(\{e\}(x) = p)$ and $C(e, |p|, x, p)$ ($\{e\}$ is just a Turing machine that writes p regardless of the input; its program can be coded by some $e < |p|^{O(1)}$). As $|p| < 2^{||a||^{O(1)}} < T$ we have that $p \in K$. In particular $|r| \in K$. Now, r can be calculated from $|r|$ easily by a standard Turing machine in S_3 because $r = 2^{|r|-1}$. Hence, by (2), $r \in K$. □

Remark 3.23 We will consider only structures $K(a, r, T, c)$ with $T > 2^{||a||^{O(1)}}$. By Lemma 3.9 (2) we are guaranteed these structures will naturally be L_3-substructures of M and moreover, they will be Δ_0^b-elementary. In particular the $BASIC_3$ axioms will hold.

Remark 3.24 In connection with Lemma 3.20, condition 3 will be useful to generate $2^{||a||^{O(1)}}$-sparse sequences, any *small* nonstandard integer being available in K.

Lemma 3.25 *Let M, a, r be as in Lemma 3.13. Let $c, c', T_2, T, T_{c'} \in M$ and let $K = K(a, r, T_2, c)$, $K' = K(a, r, T', c')$. Suppose that*

1. *$c \in K'$;*
2. *$2^{||a||^r} > O(1).T'$;*
3. *$T' \geq T_2$.*

Then $K \subset K'$.

Proof: Let $z \in K$. Then $z < 2^{2^{||a||^{O(1)}}}$ and $C(e, k.T_2, c, z)$ for some $k \in \mathbb{N}$ and $e < |r|^k$. But $k.T_2 < O(1).T' < 2^{||a||^r}$ and $c \in K'$, hence, by Lemma 3.22, $z \in K'$. □

Lemma 3.26 *Let M, a, r be as in Lemma 3.13. Let $c, c', T_1, T', T_{c'} \in M$ and let $K' = K(a, r, T', c')$. Suppose that*

1. $C(p, T_{c'}, c, c')$ *for some* $p < |r|^{O(1)}$;
2. $2^{||a||^r} \geq T_1 > T_{c'} + O(1).T'$.

Then $K' \subset R(r, T_1, c)$.

Proof: Let $y \in K'$ and let $k \in \mathbb{N}$, $e < |r|^k$ such that $C(e, k.T', c', y)$. We have that $C(p, T_{c'}, c, c')$ for some $p < |r|^{O(1)}$ and $T_{c'} + k.T' < T_1 \leq 2^{||a||^r}$. By (2) of Lemma 3.9 there is some $e' < |r|^{O(1)} < r$ such that $C(e', T_1, c, y)$, hence $y \in R(r, T_1, c)$. □

3.4 Constructing a model of $\hat{R}^2_3$ Let M, a, r be as in Lemma 3.13. Let R denote the resource $R(r, 2^{||a||^r}, a)$. We call it the main resource. The aim of this section is to construct inside it a model K^* of $\hat{R}^2_3$ containing a. This model will be constructed as the union of an increasing chain $(K_n)_{n\in\mathbb{N}}$, each K_n satisfying a new instance of $\hat{\Sigma}^b_2$-LLIND while preserving those satisfied previously. First we prove the key lemma which will help us to pass from K_n to K_{n+1}.

Lemma 3.27 *Let* M, a, r *be as in Lemma 3.13. Let* $c, T_1, T_2 \in M\backslash\mathbb{N}$ *and* $K = K(a, r, T_2, c)$. *Let* $b_0, \ldots, b_m \in K$, $l \in log(log(K))$, $\psi(j, y, z, \bar{b})$ *a* Δ^b_0 *formula with parameters* $\bar{b}$ *and let* $\theta(j, \bar{b})$ *be the formula* $\exists y \leq t \forall z \leq s\, \psi(j, y, z, \bar{b})$, *where* $t = t(j, \bar{b})$, $s = s(j, y, \bar{b})$ *are* L_3*-terms (parameters* $\bar{b}$ *will frequently be omitted). Suppose that*

(a) $a \in K$ *and* $c \in K(a, r, T_c, a)$ *for some* T_c *such that* $2^{||a||^r} > O(1).T_c$;
(b) $T_1 \in K$ *and* $2^{||a||^r} \geq T_1 > T_2 > 2^{||a||^{O(1)}}$;
(c) $(T_j)_{j\leq l+2}$ *is a* $||a||^{O(1)}$*-sparse sequence between* T_1 *and* T_2 *generated by* $\langle a, \rho\rangle$ *for some* $\rho \in K$.

Then there are integers $p, q, c', Y \in M$, $J \in M \cup \{-1\}$, *and an* L_3*-structure* K' *satisfying*

1. $p < |r|^{O(1)}$ *and* $C(p, r^2.T'_0, c, c')$;
2. $c' = \langle J+1, c, Y\rangle$, $-1 \leq J \leq l$ *and* $Y \leq t(J)$;
3. *If* $J \neq -1$ *then* $\forall z \in R(r, T'_{J+1}, c'), z \leq s(J, Y) \Longrightarrow \psi(J, Y, z)$;
4. $q < |r|^{O(1)}$ *and* $\forall y \exists z \leq s(J+1, y)\, C(q, r^2.T'_{J+2}, \langle c', y\rangle, z)$;
5. *If* $J \neq l$ *then* $\forall y \in R(r, T'_{J+1}, c')$, $y \leq t(J+1) \wedge z = \{q\}(\langle c', y\rangle) \Longrightarrow z \leq s(J+1, y) \wedge \neg\psi(J+1, y, z)$;
6. $K' = K(a, r, r^2.T'_{J+2}, c')$;
7. K' *is* S_3*-closed;*
8. $K \subset K' \subset R$;
9. $K' \subset R(r, T_1, c)$;
10. *If* $x \in K'$, $K(a, r, r^2.T_2, x) \subset K'$;
11. $K' \models BASIC_3 + \theta(j)$*-IND up to* l.

Proof: First note that $r \in K$ by Lemma 3.22 and integers $a, \bar{b}, l, T_1, \rho$ are in K by hypothesis. Hence we can obtain them all from c in time $O(1).T_2$ by means of some (possibly) nonstandard Turing machine of code $< |r|^{O(1)}$, and these integers are bounded by $2^{2^{||a||^{O(1)}}}$. The integer p will be the index of the Turing machine P that is working as follows on input c.

1. Compute $r, a, \bar{b}, l, T_1, \rho$ from c.
2. Compute T'_0 from a, ρ, T_1.
3. Put $j := 0$, $y_{-1} := 0$.
4. Compute T'_{j+1} from a, ρ, T'_j.
5. Look for $y_j \in R(r, T'_j, \langle j, c, y_{j-1}\rangle)$ such that
$$y_j \leq t \text{ and } \forall z \in R(r, T'_{j+1}, \langle j+1, c, y_j\rangle), z \leq s \Longrightarrow \psi(j, y_j, z).$$

(Searching in $R(r, T, x)$ is done by simulating no more than T steps in the computation of $\{e\}(x)$, if e is the code of a Turing machine and this for all values of e from 0 to r. Verification of a condition for every $z \in R(r, T, x)$ is done in a similar way.)

6. If there is no such y_j, stop the machine with output $P(c) = \langle j, c, y_{j-1}\rangle$.
7. If y_j is found and $j < l$, then put $j := j + 1$ and go to 4.
8. If y_l is found, stop the machine with output $P(c) = \langle l + 1, c, y_l\rangle$.

Let $\langle J + 1, c, Y\rangle$ be the output, that is, $Y = y_J$, and let us name it c'. Then (2) and (3) follow easily from the definition of P, once the existence of the computation is established. As explained above, to execute the first line the machine needs a standard number of programs of code $< |r|^{O(1)}$ (namely, $6 + m$ programs, as $\bar{b} = b_0, \ldots, b_m$). By (c) a unique standard function in S_3 suffices to obtain T'_0 from a, ρ, T_1 and T'_{j+1} from a, ρ, T'_j. Having r, T'_j, j, c, y_{j-1} we generate the elements of $R(r, T'_j, \langle j, c, y_{j-1}\rangle)$ by means of a standard program. Computation of the values of terms t, s and evaluation of Δ_0^b formulas is also done by standard programs in S_3. Thus P can be viewed as a standard Turing machine running on c with a standard number of extra inputs bounded by $|r|^{O(1)}$. By (1) of Lemma 3.9 we conclude that P can be coded by some $p < |r|^{O(1)}$. For the running time we have that $r, a, b_0, \ldots, b_m, l, T_1, \rho,$ are calculated in time $O(1).r^2.T_2$ from c. As $T_1, \rho \in K$ we have $T_1, \rho < 2^{2^{||a||^{O(1)}}}$ and then $T'_j < T_1 < 2^{2^{||a||^{O(1)}}}$ for every $j \leq l + 2$. By (c), $T'_0 \in S_3(\langle a, \rho, T_1\rangle)$ and $T'_{j+1} \in S_3(\langle a, \rho, T'_j\rangle)$ for $j \leq l + 1$, hence T'_j is obtained in time $2^{||a||^{O(1)}}$ for every j. It is known that simulating T'_j steps of the computation of $\{e\}$ can be done in time $O(1).|e|.T'_j$ by an universal program (see Papadimitriou [9], for example). As $e \leq r$ we can bound it by $|r|^2.T'_j$. We calculate the values of terms $t(j, \bar{b}), s(j, y, \bar{b})$ in time $2^{||a||^{O(1)}}$, as they correspond to functions in S_3 and its arguments are all bounded by $2^{2^{||a||^{O(1)}}}$. Deciding if $y_j \leq t$ is done in time $O(1).|t|$, thus less than $2^{||a||^{O(1)}}$ since $t < 2^{2^{||a||^{O(1)}}}$. The same is valid for $z \leq s$. Evaluation of $\psi(j, y_j, z, \bar{b})$ when $y_j \leq t$ and $z \leq s$ takes time $2^{||a||^{O(1)}}$ because ψ is Δ_0^b and $j, t, s, b_0, \ldots, b_m < 2^{2^{||a||^{O(1)}}}$. Thus, we have that c' is calculated in time T less than

$$O(1).T_2 + 2^{||a||^{O(1)}} + \sum_{j=0}^{l} [2^{||a||^{O(1)}} + r(|r|^2.T'_j + 2^{||a||^{O(1)}} + r(|r|^2.T'_{j+1} + 2^{||a||^{O(1)}}))] \,.$$

Remembering that $T'_j > T_2 > 2^{||a||^{O(1)}}$ we get that

$$T < \sum_{j=0}^{l} r[|r|^2.T'_j + r(|r|^2+1)T'_{j+1}].$$

But $r(|r|^2+1).T'_{j+1} < T'_j$ since $r < ||a||$ and $(T'_j)_{j \le l+2}$ is $||a||^{O(1)}$-sparse, thus

$$T < r(|r|^2+1).\sum_{j=0}^{l} T'_j < r(|r|^2+1)(T'_0 + l.T'_1) .$$

Now, $l.T'_1 < T'_0$ because $l < ||a||^{O(1)}$ and $(T'_j)_{j\le l+2}$ is $||a||^{O(1)}$-sparse. So we conclude that c' is calculated in time

$$T < 2r(|r|^2+1).T'_0 < r^2.T'_0.$$

Finally note that $r^2.T'_0 \in log(M)$ since $r^2.T'_0 < T_1 \le 2^{||a||^r}$ and $2^{||a||^r} \in log(M)$ by Lemma 3.13. Therefore we have $\exists w(|w| = r^2.T'_0 \wedge U(p, w, c, c'))$, that is, $C(p, r^2.T'_0, c, c')$ and (1) is proved.

The required integer q will be the index of the Turing machine Q working as follows on input $\langle c', y\rangle$.

1. Compute $J+2, c$ from c'.
2. Compute $r, a, b_0, \dots, b_m, T_1, \rho$ from c.
3. Compute $t = t(J+1, \bar{b})$ from $J+2, b_0, \dots, b_m$.
4. Compute T'_{J+2} from $J+2, a, \rho, T_1$.
5. If $y \le t$, compute $s = s(J+1, y, \bar{b})$ and look for $z \in R(r, T'_{J+2}, \langle J+2, c, y\rangle)$ such that $z \le s \wedge \neg\psi(J+1, y, z)$. Else, stop the machine with output 0.
6. If such a z is found, stop the machine with output z. Else, stop it with output 0.

As $c' = \langle J+1, c, Y\rangle$ we can obtain $J+2$ and c from c' by means of two standard functions in S_3. Integers $r, a, b_0, \dots, b_m, T_1, l$ can be calculated from c using a standard number of functions of code $< |r|^{O(1)}$ since they belong to K as we explained above. The values of terms t, s are calculated by standard functions in S_3. By Lemma 3.19 and hypothesis (c), T'_{J+2} is obtained from $J+2, a, \rho, T_1$ by means of a standard function in S_3. The computations of line 5 require only a standard program, analogously for line 5 of program P. In the same way as we did for P, we conclude that Q can be coded by some $q < |r|^{O(1)}$.

For its running time first note that $c < 2^{2^{||a||^{O(1)}}}$ since $c \in K(a, r, T_c, a)$ by hypothesis (a). We have also $t, l < 2^{2^{||a||^{O(1)}}}$, hence $Y < t < 2^{2^{||a||^{O(1)}}}$ and $J+1 \le l+1 < 2^{2^{||a||^{O(1)}}}$. Thus we get that $c' = \langle J+1, c, Y\rangle < 2^{2^{||a||^{O(1)}}}$. As $J+2, c \in S_3(c')$, computations on line 1 are done in time $2^{||a||^{O(1)}}$. Integers in line 2 are in K, hence they are calculated in time $O(1).T_2$ from c. The value of t is calculated in time $2^{||a||^{O(1)}}$ as for program P. We obtain T'_{J+2} in time $2^{||a||^{O(1)}}$ as $T'_{J+2} \in S_3(\langle J+2, a, \rho, T_1\rangle)$ and $J+2, a, \rho, T_1 < 2^{2^{||a||^{O(1)}}}$. Deciding if $y \le t$ takes time $2^{||a||^{O(1)}}$ and when this inequality holds the value of s is calculated in time $2^{||a||^{O(1)}}$ since $y \le t < 2^{2^{||a||^{O(1)}}}$ and the other arguments of s are also bounded by $2^{2^{||a||^{O(1)}}}$.

Searching for z in $R(r, T'_{J+2}, \langle J+2, c, y\rangle)$ verifying the condition in line 5 is done in time less than $r(|r|^2.T'_{J+2} + 2^{||a||^{O(1)}})$. Thus, $Q(\langle c', y\rangle)$ is calculated in time less than $2^{||a||^{O(1)}} + O(1).T_2 + r(|r|^2.T'_{J+2} + 2^{||a||^{O(1)}})$. Since $T'_{J+2} > T_2 > 2^{||a||^{O(1)}}$, we can conclude that $Q(\langle c', y\rangle)$ is calculated in time less than $r^2.T'_{J+2}$. Thus if $z = Q(\langle c', y\rangle)$ then $C(q, r^2.T'_{J+2}, \langle c', y\rangle, z)$ and it is clear that $z \leq s(J+1, y)$ in all cases. This shows (4).

To see (5) suppose $J < l$. As $c' = \langle J+1, c, Y\rangle$ and $Y = y_J$, $J < l$ means that the program P did not find the y_{J+1} it looked for. In other words this says that $\forall y \in R(r, T'_{J+1}, \langle J+1, c, Y\rangle)$ such that $y \leq t(J+1)$, there is some $z \in R(r, T'_{J+2}, \langle J+2, c, Y\rangle)$ satisfying $z \leq s(J+1, y) \wedge \neg\psi(J+1, y, z)$. Then, the program Q will eventually find this z and so (5) holds.

Now let $K' = K(a, r, r^2.T'_{J+2}, c')$. We have $O(1).r^2.T'_{J+2} > r^2.T_2 > 2^{||a||^{O(1)}}$, so (7) and (10) follow from Lemma 3.22. By (2), $c \in S_3(c')$, and by (7) $S_3(c') \subset K'$, so $c' \in K'$. Also $2^{||a||^r} > O(1).T_1 > O(1).r^2.T'_{J+2}$ since $(T_j)_{j\leq l+2}$ is $||a||^{O(1)}$-sparse and $r < ||a||$, and clearly $r^2.T'_{J+2} > T_2$ because $(T_j)_{j\leq l+2}$ is between T_1 and T_2. We can then apply Lemma 3.25 to conclude that $K \subset K'$.

Now we use Lemma 3.26 to prove (9) and $K' \subset R$. We have $C(p, r^2.T'_0, c, c')$ and $p < |r|^{O(1)}$ by (1), and $2^{||a||^r} \geq T_1 > O(1).r^2.T'_0 > r^2.T'_0 + O(1).r^2.T'_{J+2}$, thus by Lemma 3.26 $K' \subset R(r, T_1, c)$ and (9) is proved. By (a) there is some $k \in \mathbb{N}$ and $e < |r|^k$ such that $C(e, k.T_c, a, c)$. By (1), $C(p, r^2.T'_0, c, c')$ and $p < |r|^{O(1)}$. Then by (2) of Lemma 3.9 there is some $e' < |r|^{O(1)}$ such that $C(e', k.T_c + r^2.T'_0, a, c')$. We have $2^{||a||^r} > k.T_c + T_1$ since $2^{||a||^r} > O(1).T_c$ and $2^{||a||^r} > O(1).T_1$ by hypothesis. As indicated above $T_1 > r^2.T'_0 + O(1).r^2.T'_{J+2}$, thus we get that $2^{||a||^r} > k.T_c + r^2.T'_0 + O(1).r^2.T'_{J+2}$ which implies by Lemma 3.26 that $K' \subset R(r, 2^{||a||^r}, a)$, that is, $K' \subset R$ and (8) is proved. By (7) $K' \prec_{\Delta_0^b} M$ and so $K' \models BASIC_3$. Now we use the previous points to get two easy consequences implying (11). Remember that $-1 \leq J \leq l$.

Fact 3.28 *If* $0 \leq J \leq l$ *then* $K' \models \theta(J)$.

Proof: First note that $J, Y \in S_3(c') \subset K'$ by (2) and (7), and also $K' \subset R(r, T'_{J+1}, c')$, since $K' = K(a, r, r^2.T'_{J+2}, c')$ and $T'_{J+1} > r^2.T'_{J+2}$. Let $z \in K'$, $z \leq s(J, Y)$. Then $z \in R(r, T'_{j+1}, c')$ and by (3) $M \models \psi(J, Y, z)$. We just noted that $K' \prec_{\Delta_0^b} M$, so $K' \models \psi(J, Y, z)$ and thus $K' \models \exists y \leq t(J) \forall z \leq s(J, y)\psi(J, y, z)$, that is, $K' \models \theta(J)$. □

Fact 3.29 *If* $-1 \leq J \leq l-1$ *then* $K' \models \neg\theta(J+1)$.

Proof: Let $y \in K'$, $y \leq t(J+1)$ and let $z = \{q\}(\langle\langle c', y\rangle\rangle)$. We have $y \in R(r, T'_{J+1}, c')$, so by (5) $M \models z \leq s(J+1, y) \wedge \neg\psi(J+1, y, z)$. By Lemma 3.22 and (4), $z \in K'$, so by elementarity, $K' \models z \leq s(J+1, y) \wedge \neg\psi(J+1, y, z)$. We have proved $K' \models \forall y \leq t(J+1) \exists z \leq s(J+1, y) \neg\psi(J+1, y, z)$, that is, $K' \models \neg\theta(J+1)$. □

From Facts 3.28 and 3.29 we obtain $K' \models \neg\theta(0) \vee \exists j < l[\theta(j) \wedge \neg\theta(j+1)] \vee \theta(l)$, that is, $K' \models \theta(j)$-IND *up to* l . □

Now we are ready to construct the chain $(K_n)_{n\in\mathbb{N}}$. Starting from some K_0 (for practical reasons chosen different from the one used in the sketch of the proof), we inductively define K_n for $n \geq 1$, using the procedure of extension exhibited in Lemma 3.27. This is the content of the next lemma. First we define some useful notations for the rest of the section.

Notation 3.30 When M, a, r as in Lemma 3.13 are fixed, we write $A \gg B$ for $A > 2^{||a||^{O(1)}}.B$. If sequences $(T^i_j)_{j \leq l_i}$, $i = 0, 1, \ldots$ are defined, we note $R^i_j(x)$ the resource $R(r, T^i_j, x)$. By $\mathbf{(\bar{b})_i}$ we denote a set of parameters $b^i_0, \ldots, b^i_{m_i}$.

Lemma 3.31 *Let M, a, r be as in Lemma 3.13. Let $T^0_1, T^0_2 \in M$ such that $T^0_1 \in S_3(\langle a, r\rangle)$ and $2^{||a||^r} \geq T^0_1 \gg T^0_2 \gg 1$. Let $K_0 = K(a, r, T^0_2, a)$, $J_0 = 0$, $a_0 = a$. Let $n \in I\!N$, $n \geq 1$ and suppose we have n L_3-structures $K_0, \ldots, K_{n-1}$, a $\hat{\Sigma}^b_2$ formula $\theta_n(j) \equiv \exists y \leq t_n \forall z \leq s_n\, \psi_n(j, y, z)$, $\psi_n(j, y, z) \in \Delta^b_0$, with parameters $(\bar{b})_n \in K_{n-1}$, and some integer $l_n \in log(log(K_{n-1}))$. If $n = 1$ we have just K_0, θ_1 and l_1. If $n > 1$ suppose we have also for each $1 \leq i < n$:*

- *(a) integers $(\bar{b})_i, \rho_i \in K_{i-1}$, $l_i \in log(log(K_{i-1}))$;*
- *(b) a $\hat{\Sigma}^b_2$ formula $\theta_i(j) \equiv \exists y \leq t_i \forall z \leq s_i\, \psi_i(j, y, z)$ with parameters $(\bar{b})_i$, $\psi_i(j, y, z) \in \Delta^b_0$;*
- *(c) integers $p_i, q_i, a_i, Y_i \in M$, $J_i \in M \cup \{-1\}$;*
- *(d) a $2^{||a||^{O(1)}}$-sparse sequence $(T^i_j)_{j \leq l_i+2}$ between $T^{i-1}_{J_{i-1}+1}$ and $T^{i-1}_{J_{i-1}+2}$ generated by $\langle a, \rho_i\rangle$;*

satisfying (1)–(8) below.

1. *$p_i < |r|^{O(1)}$ and $C(p_i, r^2.T^i_0, a_{i-1}, a_i)$.*
2. *$a_i = \langle J_i + 1, a_{i-1}, Y_i\rangle$, $-1 \leq J_i \leq l_i$ and $Y_i \leq t_i(J_i)$.*
3. *If $J_i \neq -1$ then $\forall z \in R^i_{J_i+1}(a_i)$, $z \leq s_i(J_i, Y_i) \Longrightarrow \psi_i(J_i, Y_i, z)$.*
4. *$q_i < |r|^{O(1)}$ and $\forall y \exists z \leq s_i(J_i + 1, y) C(q_i, r^2.T^i_{J_i+2}, \langle a_i, y\rangle, z)$.*
5. *If $J_i \neq l_i$ then $\forall y \in R^i_{J_i+1}(a_i)$, $y \leq t_i(J_i + 1) \wedge z = \{q_i\}(\langle a_i, y\rangle) \Longrightarrow z \leq s_i(J_i + 1, y) \wedge \neg\psi_i(J_i + 1, y, z)$.*
6. *$K_i = K(a, r, r^2.T^i_{J_i+2}, a_i)$.*
7. *K_i is S_3-closed.*
8. *$K_{i-1} \subset K_i \subset R$.*

Then there is a $2^{||a||^{O(1)}}$-sparse sequence $(T^n_j)_{j \leq l_n+2}$ between $T^{n-1}_{J_{n-1}+1}$ and $T^{n-1}_{J_{n-1}+2}$ generated by $\langle a, \rho_n\rangle$ for some $\rho_n \in K_{n-1}$, integers $p_n, q_n, a_n, Y_n \in M$, $J_n \in M \cup \{-1\}$, and an L_3-structure K_n such that (1)–(8) hold for $i = n$ and

9. *$K_n \subset R^i_{J_i+1}(a_i)$, for $i = 0, \ldots, n$;*
10. *If $y \in K_n$ then $\{q_i\}(\langle a_i, y\rangle) \in K_n$, for $i = 1, \ldots, n$;*
11. *$K_n \models BASIC_3 +\ \theta_i(j)$-IND up to l_i, for $i = 1, \ldots, n$.*

Proof: Let $n \geq 1$. By hypothesis $T^{n-1}_{J_{n-1}+1} \gg T^{n-1}_{J_{n-1}+2}$ and from $l_n \in log(log(K_n))$ it follows that $l_n < ||a||^{O(1)}$. By recurrence on n we have that $2^{||a||^r} \geq T^{n-1}_{J_{n-1}+1}$. Thus by Lemma 3.20 there is a $2^{||a||^{O(1)}}$-sparse sequence $(T^n_j)_{j \leq l_n+2}$ between $T^{n-1}_{J_{n-1}+1}$ and $T^{n-1}_{J_{n-1}+2}$ generated by $\langle a, \rho_n\rangle$ for some small ρ_n. As $T^{n-1}_{J_{n-1}+2} \gg 1$ is easily proved by recurrence on n, we can use Lemma 3.22 to argue that ρ_n can be chosen in K_{n-1}. We want to apply Lemma 3.27 for $K = K_{n-1}$. So let us first check its hypotheses (a), (b), and (c). If $n = 1$ then $K_{n-1} = K_0 = K(a, r, T_2^0, a)$ and thus hypothesis (a) is trivially verified ($c = a$). We have $2^{||a||^r} \geq T_1^0 \gg T_2^0 \gg 1$ and hence by Lemma 3.22 K_0 is S_3-closed and $r \in K_0$. Thus $T_1^0 \in S_3(\langle a, r\rangle) \subset K_0$ and so, condition (b) is verified for $T_1 = T_1^0$ and $T_2 = T_2^0$. As the sequence $(T_j^1)_{j \leq l_1+2}$ is obviously $||a||^{O(1)}$-sparse, and $\rho_1 \in K_0$, we have (c) for $T'_j = T_j^1$, $l = l_1$ and $\rho = \rho_1$. If $n > 1$ we check the hypotheses of Lemma 3.27 for $c = a_{n-1}$, $T_1 = T^{n-1}_{J_{n-1}+1}$, $T_2 = r^2.T^{n-1}_{J_{n-1}+2}$, $K = K_{n-1}$, $\bar{b} = (\bar{b})_n$, $l = l_n$, $\theta = \theta_n$ and $T'_j = T_j^n$ for $j \leq l_n + 2$. First, we have $(\bar{b})_n \in K_{n-1}$ and $l_n \in log(log(K_{n-1}))$ by hypothesis.

Now we verify (a), (b), and (c):

(a) From $a_i = \langle J_i + 1, a_{i-1}, Y_i\rangle$ it follows that $a_{i-1} \in S_3(a_i)$, $i = 1, \ldots, n-1$. It follows also, by recurrence on i, that $a_i, J_i, Y_i < 2^{2^{||a||^{O(1)}}}$. In particular this implies $a_i \in K_i$ for every $i < n$. Composing functions in S_3 we get that $a = a_0 \in S_3(a_{n-1})$, and by (7) $S_3(a_{n-1}) \subset K_{n-1}$. Therefore $a \in K_{n-1}$. Now, we have that $a_{n-1} = \{p_{n-1}\}(a_{n-2}), \ldots, a_1 = \{p_1\}(a_0)$ and $p_i < |r|^{O(1)}$, $i = 1, \ldots, n-1$. By (2) of Lemma 3.9 there is some $e < |r|^{O(1)}$ such that $C(e, T, a_0, a_{n-1})$ for $T = \sum_{i=1}^{n-1} r^2.T_0^i$. But $\sum_{i=1}^{n-1} r^2.T_0^i < (n-1).r^2.T_0^1 \ll T_1^0 \ll 2^{||a||^r}$. Hence $2^{||a||^r} > O(1).T$ and $a_{n-1} \in K(a, r, T, a)$.

(b) We prove by recurrence on n that $T^{n-1}_{J_{n-1}+1} \in K_{n-1}$. For $n = 1$ it was stated above. Suppose $T^{n-2}_{J_{n-2}+1} \in K_{n-2}$. By (8) $T^{n-2}_{J_{n-2}+1}$ is in K_{n-1} also, as well as ρ_{n-1}, a_{n-1} and, consequently, $J_{n-1} + 1$. By Lemma 3.19 $T^{n-1}_{J_{n-1}+1} \in S_3(\langle J_{n-1} + 1, a, \rho_{n-1}, T^{n-2}_{J_{n-2}+1}\rangle)$, hence $T^{n-1}_{J_{n-1}+1} \in K_{n-1}$ as K_{n-1} is S_3-closed. The rest follows from $2^{||a||^r} \geq T^{n-1}_{J_{n-1}+1} \gg T^{n-1}_{J_{n-1}+2} \gg 1$ which was remarked at the beginning of the proof.

(c) The sequence $(T_j^n)_{j \leq l_n+2}$ is obviously $||a||^{O(1)}$-sparse and is between $T^{n-1}_{J_{n-1}+1}$ and $r^2.T^{n-1}_{J_{n-1}+2}$ since $r < ||a||$.

Applying now Lemma 3.27 we get $p_n, q_n, a_n, Y_n \in M$, $J_n \in M \cup \{-1\}$ and an L_3-structure K_n satisfying already (1) – (8). Let us see (9) – (11).

(9) For $i = n$ it is clear by definition (6) of K_n and the fact that $T^n_{J_n+1} > O(1).r^2.T^n_{J_n+2}$. Consider the case $i < n$. We have that a_n can be calculated from a_i by composing successively $\{p_{i+1}\}, \ldots, \{p_n\}$, and the total computing time is bounded by $r^2.(T_0^{i+1} + \cdots + T_0^n) < (n-i).r^2.T_0^{i+1} \ll T^i_{J_i+1}$. By (2) of Lemma 3.9 we have $C(e, T, a_i, a_n)$ for some $e < |r|^{O(1)}$ and $T \ll T^i_{J_i+1}$. Since $T + O(1).T^n_{J_n+2} < T^i_{J_i+1} < 2^{||a||^r}$, we can apply Lemma 3.26 to conclude that $K_n \subset R^i_{J_i+1}(a_i)$.

(10) Let $1 \leq i \leq n$ and $y \in K_n$. Clearly $a_i \in K_n$ and then so is $\langle a_i, y\rangle$ since K_n is S_3-closed. If $z = \{q_i\}(\langle a_i, y\rangle)$ then by (4), $z \leq s_i(J_i + 1, y)$ and $C(q_i, r^2.T^i_{J_i+2}, \langle a_i, y\rangle, z)$. If $y \leq t_i(J_i + 1)$ then $s_i(J_i + 1, y) < 2^{2^{||a||^{O(1)}}}$, and when $y > t_i(J_i + 1)$ then $z = 0$ by definition of $\{q_i\}$. In all cases we have $z < 2^{2^{||a||^{O(1)}}}$. But $r^2.T^i_{J_i+2} < O(1).r^2.T^n_{J_n+2}$, since $T^i_{J_i+2} \leq T^n_{J_n+2}$ when $i \leq n$, so we can apply Lemma 3.22 to conclude that $z \in K_n$.

(11) This fact is a direct consequence of (3), (5), (8), (9), and (10). Surprisingly, it will not be used later and this is because our extensions preserve only Δ_0^b formulas. We will rather imitate its proof for a bigger model of the form $\bigcup K_n$

in the proof of Theorem 2.1 below. This is the reason we do not prove it here. □

Proof of Theorem 2.1: Arguing as in the proof of Lemma 3.13, there is $r_0 \in M \setminus \mathbb{N}$, $r_0 \leq r$ (and thus $2^{2^{||a||^{r_0}}}$ exists also), such that $r_0 = 2^{|r_0|-1}$ and $r_0 < ||a||$. As $R(a, r_0, 2^{||a||^{r_0}}) \subset R$, it suffices to prove the theorem for r_0. So we can assume $r = 2^{|r|-1}$ and $r < ||a||$ without losing generality. Let $T^0_1 = 2^{||a||^r}$ and let T^0_2 be such that $T^0_1 \gg T^0_2 \gg 1$ (any $2^{||a||^\rho}$ with $r > \rho > O(1)$, for example). As we remarked after Lemma 3.13, we have $2^{||a||^r} \in S_3(\langle a, r\rangle)$. Let $K_0 = K(a, r, T^0_2, a)$. Fix an enumeration with infinite repetitions of pairs $(\theta(j, \bar{b}), ||d||)$ where θ is a $\hat{\Sigma}^b_2$ formula and $\bar{b}, d$ are parameters in M. Consider the first pair in the enumeration with parameters in K_0 and name it $(\theta_1(j, (\bar{b})_1), l_1)$. Then $\theta_1(j) \equiv \exists y \leq t_1 \forall z \leq s_1 \psi_1(j, y, z)$, with ψ_1 a Δ^b_0 formula with parameters $(\bar{b})_1$, and we are in the case $n = 1$ of the hypothesis of Lemma 3.31. This gives us K_1. Suppose we have just obtained K_n from K_{n-1} using this lemma and let $(\theta_{n+1}(j, (\bar{b})_{n+1}), l_{n+1})$ be the first pair in the enumeration after (θ_n, l_n) having its parameters in K_n. Lemma 3.31 says that K_n satisfies also (1)–(8), thus we are again verifying its hypothesis and therefore we obtain K_{n+1}. In this way we get an increasing chain of L_3-structures $(K_n)_{n\in\mathbb{N}}$. At each step a new $\hat{\Sigma}^b_2$-LLIND axiom is satisfied while the preceding ones are preserved. But the chain is only Δ^b_0-elementary and hence preservation of these axioms under the union of the chain is not guaranteed since they are Δ^b_3-formulas. Rather, this preservation is a consequence of the specific way the models are built. In other words, we have not yet proved that $K^* := \bigcup_{n\in\mathbb{N}} K_n$ is a model of $\hat{\Sigma}^b_2$-LLIND . Instead, (a), (b), and (c) are inmediately verified and thus $K^* \prec_{\Delta^b_0} M$. Let $\theta(j)$ be a $\hat{\Sigma}^b_2$ formula with parameters $\bar{b} \in K^*$ and let $l \in log(log(K^*))$. Suppose that $(\theta(j), l)$ was considered when constructing K_n, that is, $\theta(j) \equiv \theta_n(j)$ is the formula $\exists y \leq t_n \forall z \leq s_n\, \psi_n(j, y, z)$, $\bar{b} = (\bar{b})_n$, $l = l_n$, with $(\bar{b})_n \in K_{n-1}$, $l_n \in log(log(K_{n-1}))$. Note that $a_n \in K^*$ and hence by (b) J_n and Y_n are also in K^*. Note too that $K^* \subset R^n_{J_n+1}(a_n)$ by (9) of Lemma 3.31. Remember that $-1 \leq J_n \leq l_n$.

Fact 3.32 *If* $0 \leq J_n \leq l_n$ *then* $K^* \models \theta_n(J_n)$.

Proof: Let $z \in K^*$ such that $z \leq s_n(J_n, Y_n)$. As we just remarked, $z \in R^n_{J_n+1}(a_n)$ so by (5) of Lemma 3.31 $M \models \psi_n(J_n, Y_n, z)$, and by (2) $Y_n \leq t_n(J_n)$. By Δ^b_0-elementarity $K^* \models \psi_n(J_n, Y_n, z)$. We have proved $K^* \models \exists y \leq t_n \forall z \leq s_n \psi_n(J_n, y, z)$, that is, $K^* \models \theta_n(J_n)$. □

Fact 3.33 *If* $-1 \leq J_n \leq l_n - 1$ *then* $K^* \models \neg\theta_n(J_n + 1)$.

Proof: Let $y \in K^*$ such that $y \leq t_n(J_n)$ and let $m \geq n$ such that $y \in K_m$. We have $a_n \in K_n \subset K_m$, so by (10) of Lemma 3.31 $\{q_n\}(\langle a_n, y\rangle) \in K_m$. By (9) $K_m \subseteq R^n_{J_n+1}(a_n)$, hence $y \in R^n_{J_n+1}(a_n)$ and by (5), if $z = \{q_n\}(\langle a_n, y\rangle)$ then $M \models z \leq s_n(J_n, y) \wedge \neg\psi_n(J_n + 1, y, z)$. Therefore we have that $z \in K^*$ and by Δ^b_0-elementarity $K^* \models z \leq s_n(J_n, y) \wedge \neg\psi_n(J_n + 1, y, z)$. Thus $K^* \models \forall y \leq t_n \exists z \leq s_n \neg\psi_n(J_n + 1, y, z)$, that is, $K^* \models \neg\theta_n(J_n + 1)$. □

From Facts 3.32 and 3.33, $K^* \models \neg\theta_n(0) \vee \exists j < l_n[\theta_n(j) \wedge \neg\theta_n(j+1)] \vee \theta_n(l_n)$, that is, $K^* \models \theta_n(j)$-IND *up to* l_n. Thus we have proved that $K^* \models \hat{\Sigma}_2^b$-LLIND. □

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