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A Supersimple Nonlow Theory

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Abstract This paper presents an example of a supersimple nonlow theory and characterizes its independence relation.

1 Introduction Buechler introduced in [1] a subclass of simple theories called *low*. Every stable theory is low, and every supersimple theory having a finite bound of ranks of complete types is low. Buechler proved in [1] that in any low theory, Lascar strong type is the same as strong type. In Casanovas [3] an example of a simple nonlow theory appears. In this paper we show an example of a supersimple nonlow theory.

Recently, in [2] Buechler, Pillay, and Wagner extended Buechler's proof to the full class of supersimple theories. Namely, they proved that a supersimple theory eliminates hyperimaginaries. This fact implies that the notions of Lascar strong type and strong type coincide in a supersimple theory. But still we believe that knowing an example of a nonlow supersimple theory is useful. After Buechler's proof appeared, it was naturally asked, what is a nonlow (super)simple structure? The obvious candidates are the following.

Example 1.1 The model consists of a disjoint union of countable sets P_n ($n \in \omega \setminus \{0\}$), where each P_n is a disjoint union of countable sets U_n, V_n such that both U_n and V_n can be identified as $[\omega]^n = \{A \subseteq \omega : |A| = n\}$. Now there also is a binary relation $R(x, y)$ such that $R(a, b)$ if and only if $a \in U_n, b \in V_n$ for some $n \in \omega \setminus \{0\}$, and $a \cap b \neq \varnothing$ (when each of U_n and V_n is identified as $[\omega]^n$). It is easy to see that, for each $2 \leq k < \omega$, there is b_k such that $R(x, b_k)$ divides over $\varnothing$ with respect to k (as defined below), but not with respect to $k - 1$.

The theory T of the model has the strict order property, so T is not simple: the formula $R(x; y_1) \vee R(x; y_2)$ has the strict order property. For example, $R(x, \{1, \dots, n\}) \vee R(x, \{1, \dots, n\}) \subsetneq R(x, \{1, \dots, n\}) \vee R(x, \{n+1, 2, \dots, n\}) \subsetneq R(x, \{1, \dots, n\}) \vee R(x, \{n+1, n+2, 3, \dots, n\}) \cdots \subseteq R(x, \{1, \dots, n\}) \vee R(x, \{n+1, n+2, n+3, \dots, n+n\})$. Apply compactness.

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Example 1.2 The second model is the same as the first one in Example 1.1 up to that each P_n is a disjoint union of countable sets U_n, V_n. Now U_n and V_n are identified as $[\omega]^n$ and ω, respectively. Again there is a binary relation $R(x, y)$ such that $R(a, b)$ if and only if $a \in U_n, b \in V_n$ for some $n \in \omega \setminus \{0\}$, and $b \in a$. Similarly there is no fixed k such that whenever $R(x, c)$ divides over $\varnothing$, then it does with respect to k.

This time, the formula $R(x_1; y) \wedge R(x_2; y)$ has the strict order property: $R(\{1, \dots, n\}, y) \wedge R(\{1, \dots, n\}, y) \supsetneq R(\{1, \dots, n\}, y) \wedge R(\{1, \dots, n-1, n+1\}, y) \supsetneq R(\{1, \dots, n\}, y) \wedge R(\{1, \dots, n-2, n+1, n+2\}, y) \cdots \supsetneq R(x, \{1, \dots, n\}) \wedge R(x, \{n+1, n+2, n+3, \dots, n+n\})$. Again apply compactness.

After the previous types of constructions failed, the following was asked next: Is every (super)simple theory low? Our example here answers the question negatively. (Our construction is, in fact, a variation of Example 1.2) This says, in the meantime, Buechler-Pillay-Wagner's proof cannot be trivialized by simply showing a supersimple theory is low.

We assume the reader is familiar with the basic facts about simple theories as exposed in Kim [4] or [5] and Kim and Pillay [7]. Let T be a theory in language L and let $\varphi(x, y) \in L$. Recall that $\varphi(x, a)$ *divides over A with respect to a natural number* $k \geq 2$ if there are a_i, $(i < \omega)$ such that $\mathrm{tp}(a_i/A) = \mathrm{tp}(a/A)$ and $\{\varphi(x, a_i) : i < \omega\}$ is k-inconsistent. It is said that $\varphi(x, a)$ *divides over A* if it divides over A with respect to some k. Let α be an ordinal number. We say that $\varphi(x, y)$ *divides α times* if there are parameters $(b_i : i < \alpha)$ such that $\{\varphi(x, b_i) : i < \alpha\}$ is consistent and for every $i < \alpha$, $\varphi(x, b_i)$ divides over $\{b_j : j < i\}$. As remarked in [3], a formula has the tree property if and only if it divides ω_1 times. Hence a theory T is simple if and only if no formula divides ω_1 times in T.

We say that T is low if for every formula $\varphi(x, y) \in L$ there is a natural number n_φ such that $\varphi(x, y)$ does not divide n_φ times. This is equivalent to the original definition in [1] which is made in terms of some local rank. If a formula divides ω times in T, then T is not supersimple. In the example of a simple nonlow theory in [3] there is a formula which divides α times for every $\alpha < \omega_1$. Hence it is not supersimple.

In Section 2 we present the theory in our example and in Section 3 we prove its consistency and completeness. In Section 4 we show that it is nonlow and we check the supersimplicity of T by the method of counting types as developed in [3]. In Section 5 we characterize the notion of independence of T. This gives a second proof of supersimplicity according to the results in [7].

2 Axioms of T

The language of our theory T has two binary relation symbols R, E, unary predicates Q^0, Q^1, and P_n, Q^0_n, Q^1_n, Q^2_n for every natural number $n \geq 1$ and, moreover, n-ary function symbols F_n for $n \geq 1$. The axioms are as follows.

1. The universe is the disjoint union of Q^0 and Q^1.
2. $R \subseteq Q^0 \times Q^1$.
3. E is an equivalence relation on the universe.
4. Each E-class is R-closed and has infinitely many elements in Q^0 and in Q^1.
5. Each P_n is an E-equivalence class.
6. P_n is the disjoint union of the infinite sets Q^0_n, Q^1_n, Q^2_n.
7. $Q^0_n \subseteq Q^0$ and $Q^1_n \cup Q^2_n \subseteq Q^1$.

8. $\forall x \in Q_n^0 \exists^{=n} u \in Q_n^1 \, R(x, u)$.
9. If $x, y \in Q_n^0$ and $\forall u \in Q_1^n (R(x, u) \longleftrightarrow R(y, u))$, then $x = y$.
10. If $u_1, \ldots, u_n \in Q_n^1$, then there exists $x \in Q_n^0$ such that $R(x, u_1) \wedge \cdots \wedge R(x, u_n)$.
11. If $u_1, \ldots, u_n \in Q_n^1$ are all different, $F_n(u_1, \ldots, u_n)$ is the unique $x \in Q_n^0$ such that $R(x, u_1) \wedge \cdots \wedge R(x, u_n)$. Otherwise $F_n(u_1, \ldots, u_n) = u_1$.
12. If A, B are finite disjoint subsets of Q_n^0, there exists a $v \in Q_n^2$ such that
$$\bigwedge_{x \in A} R(x, v) \wedge \bigwedge_{x \in B} \neg R(x, v).$$
13. If $u_1, \ldots, u_m$ are different (as sets) $(n-1)$-tuples, each one consisting of $n-1$ different elements of Q_n^1 and for each $i = 1, \ldots, m$, A_i, B_i are finite disjoint subsets of Q_n^2, then there exists a $u \in Q_n^1$ different from each point in $u_1, \ldots, u_m$ and such that for each $i = 1, \ldots, m$
$$\bigwedge_{v \in A_i} R(F_n(u_i, u), v) \wedge \bigwedge_{v \in B_i} \neg R(F_n(u_i, u), v).$$
14. If U is an E-equivalence class and A, B are finite disjoint subsets of $U \cap Q^1$ such that $|A \cap Q_n^1| < n$ for every n, then there exists a $x \in U \cap Q^0$ such that
$$\bigwedge_{u \in A} R(x, u) \wedge \bigwedge_{u \in B} \neg R(x, u).$$
15. If U is an E-equivalence class and A, B are finite disjoint subsets of $U \cap Q^0$, then there exists a $u \in U \cap Q^1$ such that
$$\bigwedge_{x \in A} R(x, u) \wedge \bigwedge_{x \in B} \neg R(x, u).$$

Remark 2.1

1. Axiom 14 can be expressed in a first-order language as follows: for every n, k: $\forall u \forall u_1, \ldots, u_n, v_1, \ldots, v_k \in Q^1$ different and such that $\bigwedge_{i=1}^n E(u, u_i) \wedge \bigwedge_{i=1}^k E(u, v_i)$ and $u_1, \ldots, u_n \notin (P_1 \cup \cdots \cup P_n)$, there exists a $x \in Q^0$ such that $E(u, x)$ and $\bigwedge_{i=1}^n R(x, u_i) \wedge \bigwedge_{i=1}^k \neg R(x, v_i)$.
2. From axiom 12 it follows that for any A, B disjoint subsets of Q_n^0 there are infinitely many $v \in Q_n^2$ such that
$$\bigwedge_{x \in A} R(x, v) \wedge \bigwedge_{x \in B} \neg R(x, v).$$
Similarly for axioms 13, 14, and 15.
3. If C is a set of $n-1$ elements of Q_n^1 and $a \in Q_n^1 \setminus C$, then for any two $(n-1)$-tuples c_1, c_2 enumerating C we have $F_n(c_1, a) = F_n(c_2, a)$. Hence we may use the notation $F_n(C, a)$ with the obvious meaning.

3 Consistency and completeness of T

Proposition 3.1 *T is consistent.*

Proof: Let T_n^0 be the theory of language R, P_n, Q_n^0, Q_n^1, Q_n^2, F_n whose axioms are $R \subseteq Q_n^0 \times (Q_n^1 \cup Q_n^2)$, $\forall x P_n(x)$ and (6) to (11). This theory is clearly consistent and it is preserved under unions of chains. It describes Q_n^2 as an arbitrary infinite set and Q_n^0 as the set $[Q_n^1]^n$ of all n-element subsets of Q_n^1, being R the inverse of membership and F_n the mapping taking n different elements $a_1, \dots, a_n$ to its set $\{a_1, \dots, a_n\}$. Let T_n be the extension of T_n^0 obtained by adding axioms 12 and 13. T_n is a complete theory (this follows from the proof of completeness of T) and it is the theory of all existentially closed models of T_n^0. The new axioms 12 and 13 indicate that R refines a bipartite random graph between Q_n^0 and Q_n^2. But there is at the same time a more subtle relation between elements of Q_n^1 and elements of Q_n^2: given m sets $A_1, \dots, A_m$ of $n-1$ elements of Q_n^1, each new element $a \in Q_n^1$ determines m sets of Q_n^2, namely, $A_i \cup \{a\}$ determines the set $\{b \in Q_n^2 : R(F_n(A_i, a), b)\}$.

Now if we fix a model M_n of each T_n and we define M as the disjoint union of all M_n, then with the obvious definition for E, Q^0, and Q^1, M is a model of T. □

Let A be a set in a model of T. Define $F_n^{-1}(A)$ as the set of all $a \in Q_n^1$ such that $F_n(a_1, \dots, a_{n-1}, a) \in A$ for some $a_1, \dots, a_{n-1} \in Q_n^1$ pairwise different and different from a and define $\mathrm{cl}_F(A)$ as the closure of A under each F_n and F_n^{-1}. This is independent of the choice of the model containing A since $\mathrm{cl}_F(A) \subseteq \mathrm{acl}(A)$. Clearly,

$$\mathrm{cl}_F(A) = A \cup \bigcup_{n \geq 1} F_n^{-1}(A) \cup F_n(A \cup F_n^{-1}(A))$$

and $\mathrm{cl}_F(A)$ is finite if A is finite. We will see that $\mathrm{cl}_F(A) = \mathrm{acl}(A)$.

Proposition 3.2 *T is complete.*

Proof: Call a set (in a model of this theory) *F-closed* if it is closed under cl_F. Let us work in ω-saturated models and let us consider all finite F-closed partial isomorphisms, that is, all finite partial isomorphisms which have F-closed domain and range. We show how to extend a given finite F-closed partial isomorphism f by adding a new element a to the domain A of f. This proves completeness. There are different cases to be considered.

Case 1: $a \in Q_n^1$. Since $a \notin A$ and A is F-closed, clearly $\neg R(d, a)$ for all $d \in A$. Let $A_1, \dots, A_m$ be all subsets of $A \cap Q_n^1$ having $n-1$ elements and for $i = 1, \dots, m$ set $a_i = F_n(A_i, a)$. Then $a_1, \dots, a_m \in Q_n^0$. For $i = 1, \dots, m$ let $b_1^i, \dots, b_{k_i}^i$ be all the elements b in $A \cap Q_n^2$ such that $R(a_i, b)$ and let $c_1^i, \dots, c_{h_i}^i$ be all the elements c in $A \cap Q_n^2$ such that $\neg R(a_i, c)$. By axiom 13, the theory proves $\exists x \varphi(x)$ where $\varphi(x)$ is the formula

$$Q_n^1(x) \wedge \bigwedge_{d \in A} x \neq f(d) \wedge \bigwedge_{i=1}^{m} \Big(\bigwedge_{j=1}^{k_i} R(F_n(f(A_i), x),\ f(b_j^i)) \wedge$$
$$\bigwedge\nolimits_{j=1}^{h_i} \neg R(F_n(f(A_i), x),\ f(c_j^i))\Big).$$

We take as $f(a)$ a realization of $\varphi(x)$. To make $f \cup \{(a, f(a))\}$ F-closed we have to add now $F_n(A_i, a)$, $(i = 1, \dots, m)$. Obviously we can do it taking as values $F_n(f(A_i), f(a))$, $(i = 1, \dots, m)$.

Case 2: $a \in Q^0_n$. Let $a_1, \dots, a_n$ be the different elements in Q^1_n such that $a = F_n(a_1, \dots, a_n)$. We know how to add $a_1, \dots, a_n$ to the domain of f. Hence we can get an F-closed $f' \supseteq f$ such that $a_1, \dots, a_n \in \operatorname{dom} f'$. But this implies $a \in \operatorname{dom} f'$.

Case 3: $a \in Q^2_n$. Let $b_1, \dots, b_m$ be all the elements in $A \cap Q^0_n$ such that $R(b_i, a)$ and let $c_1, \dots, c_k$ be all the elements in $A \cap Q^0_n$ such that $\neg R(c_i, a)$. By axiom 12 the theory proves $\exists x \varphi(x)$ for

$$\varphi(x) := Q^2_n(x) \wedge \bigwedge_{d \in A} x \neq f(d) \wedge \bigwedge_{i=1}^{m} R(f(b_i), x) \wedge \bigwedge_{i=1}^{k} \neg R(f(c_i), x).$$

Now we define $f(a)$ as a realization of $\varphi(x)$. In this case $f \cup \{(a, f(a))\}$ is F-closed.

Case 4: $a \notin \bigcup_n P_n$ and $E(a, a')$ for some $a' \in A$. Assume $a \in Q^0$. Let $b_1, \dots, b_m$ be all the elements in A such that $E(a, b_i)$ and $R(a, b_i)$ and let $c_1, \dots, c_k$ be all the elements in A such that $E(a, c_i)$ and $\neg R(a, c_i)$. Since $b_1, \dots, b_m \notin \bigcup_n P_n$, by axiom 14 the theory proves $\exists x \varphi(x)$ for

$$\varphi(x) := Q^0(x) \wedge E(x, f(a')) \wedge \bigwedge_{d \in A} x \neq f(d) \wedge \bigwedge_{i=1}^{m} (R(x, f(b_i)) \wedge$$
$$\textstyle\bigwedge_{i=1}^{k} \neg R(x, f(c_i))).$$

We take as $f(a)$ a realization of $\varphi(x)$. Again $f \cup \{(a, f(a))\}$ is F-closed. The case $a \in Q^1$ is analogous, by axiom 15.

Case 5: $a \notin \bigcup_n P_n$ and $\neg E(a, a')$ for every $a' \in A$. Assume $a \in Q^0$. Since E has infinitely many classes and every E-class has infinitely many elements in Q^0, the following is consistent:

$$p(x) := \{Q^0(x)\} \cup \{\neg P_n(x) : n \geq 1\} \cup \{\neg E(x, f(d)) : d \in A\}.$$

We take as $f(a)$ a realization of $p(x)$. The case $a \in Q^1$ is analogous. $\square$

4 *T* *is supersimple and nonlow*

Lemma 4.1

1. $\operatorname{acl}(A) = \operatorname{cl}_F(A)$
2. *Any partial isomorphism between algebraically closed sets is elementary.*
3. *Assume* $A = \operatorname{acl}(A)$ *and* $a, b \notin \bigcup_n Q^0_n$. *If* a, b *have the same atomic type over* A, *then they have the same type over* A.
4. *If* A *is finite,* $\operatorname{acl}(A)$ *is finite.*

Proof: (4) follows from (1) and (2) follows from (1) and the proof of Proposition 3.2 since we may assume that the algebraically closed sets are finite and then the partial isomorphism belongs to the family which we use to prove completeness. By looking at the proof of Proposition 3.2 one can also see that if $A = \mathrm{cl}_F(A)$ is finite and $a, b \notin \bigcup_n Q_n^0$ have the same atomic type over A, then the mapping which is the identity on A and takes a to b can be extended to a finite F-closed partial isomorphism. Therefore, it belongs to the collection considered in the proof of Proposition 3.2 and it is elementary. Hence (3) follows from (1) too. To conclude the proof of (1) we have to show that $\mathrm{acl}(A) \subseteq \mathrm{cl}_F(A)$. We may assume A is finite. Let $a \notin \mathrm{cl}_F(A)$. In case $a \notin \bigcup_n Q_n^0$, looking at the axioms, we easily see that there are infinitely many objects with the same (atomic) type over $\mathrm{cl}_F(A)$ as a. Hence $a \notin \mathrm{acl}(A)$. Now assume $a \in Q_n^0$. Choose $b \in Q_n^1 \setminus \mathrm{cl}_F(A)$ such that $R(a, b)$. By what we have proven we know that $b \notin \mathrm{acl}(A)$. Let b^i $(i < \omega)$ be different conjugates of b over A. Let $b_1, \ldots, b_n$ with $b = b_1$ the different elements in Q_n^1 with $F_n(b_1, \ldots, b_n) = a$ and choose $b_1^i, \ldots, b_n^i$ such that $b^i = b_1^i$ and $\mathrm{tp}(b_1, \ldots, b_n/A) = \mathrm{tp}(b_1^i, \ldots, b_n^i/A)$. Obviously if $a_i = F_n(b_1^i, \ldots, b_n^i)$, then $\{a_i : i \in \omega\}$ is infinite and $\mathrm{tp}(a_i/A) = \mathrm{tp}(a/A)$. This shows that $a \notin \mathrm{acl}(A)$. □

We can take as a definition of supersimplicity the nonexistence of a sequence of formulas $\varphi_i(x, y_i) \in L$, $(i < \omega)$ and parameters b_i, $(i < \omega)$ such that $\{\varphi_i(x, b_i) : i < \omega\}$ is consistent and for every $i < \omega$, $\varphi_i(x, b_i)$ divides over $\{b_j : j < i\}$. As shown in [4], Remark II.2.18, if T is not supersimple there exists such sequence with the additional condition that x is a single variable. By the same arguments, if $x = x_1, \ldots, x_n$ and for some fixed formula $\theta(x)$, $\varphi_i(x, y_i) \vdash \theta(x_1) \wedge \cdots \wedge \theta(x_n)$, we can obtain the sequence $\psi_i(x_j, z_i)$ $(i < \omega)$ in one variable x_j with the additional property that $\psi_i(x_j, z_i) \vdash \theta(x_j)$. This will be used in what follows.

In [3] it is shown how to decide if a theory is supersimple by counting types. For κ, λ infinite cardinal numbers, define $\mathrm{NT}(\kappa, \lambda)$ as the supremum of the cardinalities $|P|$ of families P which consist of pairwise incompatible partial types of size $\leq \kappa$ over a set of cardinality λ. As shown in [3], a theory T is supersimple if and only if $\mathrm{NT}(\kappa, \lambda) \leq 2^{|T|+\kappa} + \lambda$ for all κ, λ with $\kappa \leq \lambda$. As remarked in [3], by the same reason as above we may restrict ourselves to types in one variable. In fact if there is a big family P (a family of cardinality $> 2^{|T|+\kappa} + \lambda$) of incompatible partial types $p = p(x_1, \ldots, x_n)$ of size κ over a fixed set of cardinality λ and for each $p \in P$ it holds $p(x_1, \ldots, x_n) \vdash \theta(x_1) \wedge \cdots \wedge \theta(x_n)$, then there is also a big family Q of incompatible partial types of size κ in one variable $q = q(y)$ with parameters in a fixed set of cardinality λ and such that for each $q \in Q$, $q(y) \vdash \theta(y)$. We apply this procedure of counting types to our theory T.

Proposition 4.2 *T is supersimple.*

Proof: Let $\kappa \leq \lambda$. We show that for any set A of cardinality $\leq \lambda$ there are at most $2^\kappa + \lambda$ pairwise incompatible partial 1-types of size κ over A. Without loss of generality, A is algebraically closed. By Lemma 4.1 in many cases we have only to look at the atomic part of the types. Let P be a family of incompatible partial 1-types over A of size $\leq \kappa$. We may assume that for each $p \in P$, $p \in S(A_p)$ for an algebraically closed set $A_p \subseteq A$ of cardinality κ. Since there are only countably many types over the empty set, we may assume that all types in P have the same restriction to the empty set and since there are only λ many algebraic types over A, we may also assume that no type in P is algebraic. Write $A = \{a_i : i < \lambda\}$.

Assume first $p \vdash Q^1_n(x)$ for each $p \in P$. Each $p \in P$ is axiomatized by $\{Q^1_n(x)\} \cup \{\neg R(a, x) : a \in A_p\}$ and a set of formulas of the form $R(F_n(C, x), a)$ and $\neg R(F_n(D, x), b)$ where $a, b \in A_p$ and C, D are subsets of $n-1$ elements of A_p. We enumerate the set of all subsets of A of cardinality $n-1$, $[A]^{n-1} = \{C_j : j < \lambda\}$. Let $f_p : \{(i, j) \in \lambda \times \lambda : C_i \subseteq A_p \text{ and } a_j \in A_p\} \to 2$ be the mapping defined by: $f(i, j) = 0$ if and only if $R(F_n(C_i, x), a_j) \in p$. Each f_p belongs to the collection $\mathrm{Fn}(\lambda \times \lambda, 2, \kappa^+)$ of all partial mappings from a subset of $\lambda \times \lambda$ of power $< \kappa^+$ into $2 = \{0, 1\}$. If $p, q \in P$ are incompatible, the mappings f_p, f_q are incompatible too, that is, $f_p \cup f_q$ is not a function. Since $\mathrm{Fn}(\lambda \times \lambda, 2, \kappa^+)$ has the $(2^\kappa)^+$-chain condition (cf. [8], Lemma VII.6.10) and $\{f_p : p \in P\}$ is an antichain, we conclude that there are at most 2^κ such types in P.

Consider now the case $p \vdash Q^0_n(x)$ for each $p \in P$. Observe that there is a natural bijection between types $p(x)$ such that $p(x) \vdash Q^0_n(x)$ and sets $\{p_1(x_1), \ldots, p_n(x_n)\}$ of types $p_i(x_i)$ such that $p_i(x_i) \vdash Q^1_n(x_i)$, namely, given p choose $a \models p$, choose different $a_1 \ldots, a_n \in Q^1_n$ such that $a = F_n(a_1, \ldots, a_n)$ and define p_i as the type of a_i. The types in Q^0_n are incompatible if and only if the corresponding sets of types in Q^1_n are incompatible. By the remarks above, the bound for families of incompatible types with $p \vdash Q^1_n(x)$ is also a bound for families of incompatible types with $p \vdash Q^0_n(x)$.

In the third possible case we have $p \vdash Q^2_n(x)$ for each $p \in P$. Each such p is axiomatized by $\{Q^2_n(x)\} \cup \{x \neq a : a \in A_p\}$ and by a set of formulas of the form $R(a, x)$ and $\neg R(b, x)$ where $a, b \in Q^0_n \cap A_p$. Again by a chain condition argument we see that there are at most λ many such types.

Now assume $p \vdash \neg P_n(x)$ for each $p \in P$ and for each $n \geq 1$. Suppose that for all $p \in P$, $p \vdash Q^0(x)$. The case where $p \vdash Q^1(x)$ is similar, so we will not consider it. If p and q do not have E-representatives, that is, if for all $a \in A_p$, $p \vdash \neg E(a, x)$ and for all $a \in A_q$, $q \vdash \neg E(a, x)$, then p is compatible with q. Hence we may assume that for each $p \in P$ there is an $a_p \in A_p$ such that $p \vdash E(a_p, x)$. Since there are only λ many E-classes with representatives in A, it is enough to show that for each $a \in A$ there are at most 2^κ many types $p \in P$ such that $p \vdash E(a, x)$. Observe that R is a bipartite random graph between Q^0 and Q^1 in the E-class of a. This means that again a chain condition argument gives the result. $\square$

Proposition 4.3 *T is nonlow.*

Proof: We show that the formula $R(x, y)$ divides n times for any $n \in \omega$. Choose different $a_1, \ldots, a_n \in Q^1_n$. Clearly $\{R(x, a_i) : i = 1, \ldots, n\}$ is consistent. We claim that for every i, $R(x, a_i)$ divides over $A_i := \{a_j : j < i\}$ with respect to $n+1$. To witness it we take different b_j $(j < \omega)$ in $Q^1_n \setminus A_i$. Each b_j has the same type over A_i as a_i and $\{R(x, b_j) : j < \omega\}$ is $(n+1)$-inconsistent. $\square$

5 *Forking and independence in* T In this last section we characterize the independence relation of T, that is, nonforking in T. We define directly the relation $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ between sets A, B, C and we show that it satisfies all the required properties of independence in a simple theory. In fact the existence of such relation gives another proof of the simplicity of T. It is shown in [7] that any theory with an independence relation satisfying some basic properties must be simple and, moreover, this independence relation must be just nonforking. Since in our case it is clear that for each tuple a and each set B there exists a finite $C \subseteq B$ such that $a \mathop{\smile\!\!\!\!\!\!|}_C B$ this gives also a proof of supersimplicity. We will see that our independence relation satisfies the Independence Theorem over algebraically closed sets and not only over models. By the results in Kim [6] this implies that in our theory, Lascar strong type is the same as strong type.

The initial definition of $A \mathop{\smile\!\!\!\!\!\!|}_B C$ is useful to check the basic properties. We will see that it is equivalent to $\mathrm{acl}^{\mathrm{eq}}(AC) \cap \mathrm{acl}^{\mathrm{eq}}(BC) \subseteq \mathrm{acl}^{\mathrm{eq}}(C)$. We start defining $\mathrm{cl}_E(A) = \mathrm{acl}(A) \cup \{[a]_E : a \in A \smallsetminus \bigcup_n P_n\}$ and

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff \mathrm{cl}_E(A) \cap \mathrm{cl}_E(BC) \subseteq \mathrm{cl}_E(C).$$

Remark 5.1

1. $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}_C BC$.
2. $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $\mathrm{acl}(A) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(C)} \mathrm{acl}(B)$.
3. In case $A \subseteq \bigcup_n P_n$, we have $\mathrm{cl}_E(A) = \mathrm{acl}(A)$ and, therefore, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $\mathrm{acl}(A) \cap \mathrm{acl}(BC) \subseteq \mathrm{acl}(C)$.
4. If $A \cap \bigcup_n P_n = \varnothing$, then $\mathrm{cl}_E(A) = A \cup (A/E)$ and, moreover, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $A \cap BC \subseteq C$ and $(A/E) \cap (BC/E) \subseteq C/E$.

The following properties are easy to check:

invariance under automorphisms: if $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $f(A) \mathop{\smile\!\!\!\!\!\!|}_{f(C)} f(B)$ for any automorphism f;

local character: for every tuple a and every set B there is a countable subset $C \subseteq B$ (even finite in our case) such that $a \mathop{\smile\!\!\!\!\!\!|}_C B$;

finite character: if a is a tuple and for all tuples b in B, $a \mathop{\smile\!\!\!\!\!\!|}_C b$, then $a \mathop{\smile\!\!\!\!\!\!|}_C B$;

monotonicity: if $A \subseteq B \subseteq C$ and $D \mathop{\smile\!\!\!\!\!\!|}_A C$, then $D \mathop{\smile\!\!\!\!\!\!|}_B C$;

transitivity: if $A \subseteq B \subseteq C$, $D \mathop{\smile\!\!\!\!\!\!|}_A B$ and $D \mathop{\smile\!\!\!\!\!\!|}_B C$, then $D \mathop{\smile\!\!\!\!\!\!|}_A C$.

It remains only to prove that $\mathop{\smile\!\!\!\!\!\!|}$ has the three following properties:

symmetry: if $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $B \mathop{\smile\!\!\!\!\!\!|}_C A$;

extension: for all sets $B \subseteq C$, for every tuple a there is a tuple a' such that $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$ and $a' \mathop{\smile\!\!\!\!\!\!|}_B C$;

the Independence Theorem over algebraically closed sets: for all tuples a, b and sets A, B, C such that $C \subseteq A \cap B$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$, $C = \mathrm{acl}(C)$, $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$, $a \mathop{\smile\!\!\!\!\!\!|}_C A$ and $b \mathop{\smile\!\!\!\!\!\!|}_C B$, there exists a tuple c such that $\mathrm{tp}(c/A) = \mathrm{tp}(a/A)$, $\mathrm{tp}(c/B) = \mathrm{tp}(b/B)$ and $c \mathop{\smile\!\!\!\!\!\!|}_C AB$.

We will prove that this is the case in the next lemmas.

Lemma 5.2

1. *If $a \notin \bigcup Q_n^0$ and $a \in \mathrm{acl}(A)$, then $a \in \mathrm{acl}(b)$ for some $b \in A$.*
2. *If $a \in Q_n^0$ and $a \in \mathrm{acl}(A)$, then there are $a_1, \dots, a_n \in A$ such that $a \in \mathrm{acl}(a_1, \dots, a_n)$.*
3. *Let $B_i = \mathrm{acl}(B_i)$ for any $i = 1, \dots, k$. If $a \in Q_n^0 \cap \mathrm{acl}(B_1 \cup \dots \cup B_k)$, then there are $a_1, \dots, a_n \in Q_n^1 \cap (B_1 \cup \dots \cup B_k)$ such that $a = F_n(a_1, \dots, a_n)$.*
4. *Let $B_i = \mathrm{acl}(B_i)$ for any $i = 1, \dots, k$. If $a \in \mathrm{acl}(B_1 \cup \dots \cup B_k) \smallsetminus \bigcup_n Q_n^0$, then $a \in B_1 \cup \dots \cup B_k$.*

Proof: (1) is clear and (4) follows from (1). To prove (2) and (3) we choose $a_1, \dots, a_n \in Q_n^1$ such that $a = F_n(a_1, \dots, a_n)$. In case $a \in \mathrm{acl}(A)$, then $a_1, \dots, a_n \in \mathrm{acl}(A)$ and by (1) there are $b_1, \dots, b_n \in A$ such that $a_i \in \mathrm{acl}(b_i)$. Hence $a \in \mathrm{acl}(b_1, \dots, b_n)$. Assume now $a \in \mathrm{acl}(B_1 \cup \dots \cup B_k)$. By (1) again, there are $b_1, \dots, b_n \in B_1 \cup \dots \cup B_k$ such that $a_i \in \mathrm{acl}(b_i)$. Since every B_j is algebraically closed, $a_i \in B_1 \cup \dots \cup B_k$. □

Lemma 5.3 *The symmetry property holds.*

Proof: Assume $A \mathop{\smile\!\!\!\!\!\!|}_C B$. We want to show $B \mathop{\smile\!\!\!\!\!\!|}_C A$. Without loss of generality, the sets A, B, C are all algebraically closed. Suppose that for some $a \in \mathrm{cl}_E(B) \cap \mathrm{cl}_E(AC)$, $a \notin \mathrm{cl}_E(C)$. Suppose first a is not an E-equivalence class. In case $a \notin \bigcup_n Q_n^0$ by Lemma 5.2 it is clear that $a \in \mathrm{acl}(AC) \smallsetminus C$ implies $a \in A$, so we obtain a contradiction. Assume $a \in Q_n^0$. By Lemma 5.2 there are different $a_1, \dots, a_n \in (A \cup C) \cap Q_n^1$ such that $F_n(a_1, \dots, a_n) = a$, say $a_1, \dots, a_i \in A$ and $a_{i+1}, \dots, a_n \in C$. Since $a \in B$, $a_1, \dots, a_i \in \mathrm{acl}(B) = B$. By the initial hypothesis $A \cap \mathrm{acl}(CB) \subseteq C$ and we see that $a_1, \dots, a_i \in C$. It follows that $a \in C$, a contradiction. Now assume $a = [b]_E$ for some $b \in (A \cup C) \smallsetminus \bigcup_n P_n$. Since $a \notin \mathrm{cl}_E(C)$, $b \notin C$. Hence $b \in A$ and $a \in \mathrm{cl}_E(A)$. Since $a \in \mathrm{cl}_E(BC)$, by the initial hypothesis $a \in \mathrm{cl}_E(C)$, a contradiction again. □

Observe that once we know that independence is symmetric, we can also characterize it as follows:

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff \mathrm{cl}_E(AC) \cap \mathrm{cl}_E(BC) \subseteq \mathrm{cl}_E(C).$$

Observe also that $\mathrm{cl}_E(A) \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$ and that if a is an E-equivalence class of an element outside $\bigcup_n P_n$ and $a \in \mathrm{acl}^{\mathrm{eq}}(A)$, then $a = [b]_E$ for some $b \in A$. This means that $\mathrm{acl}^{\mathrm{eq}}(AC) \cap \mathrm{acl}^{\mathrm{eq}}(BC) \subseteq \mathrm{acl}^{\mathrm{eq}}(C)$ implies $\mathrm{cl}_E(AC) \cap \mathrm{cl}_E(BC) \subseteq \mathrm{cl}_E(C)$ and hence $A \mathop{\smile\!\!\!\!\!\!|}_C B$. On the other hand the independence relation coming from nonforking in a simple theory has always the property that $A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $\mathrm{acl}^{\mathrm{eq}}(AC) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}^{\mathrm{eq}}(C)} \mathrm{acl}^{\mathrm{eq}}(BC)$ and hence $\mathrm{acl}^{\mathrm{eq}}(AC) \cap \mathrm{acl}^{\mathrm{eq}}(BC) \subseteq \mathrm{acl}^{\mathrm{eq}}(C)$. Therefore, after proving that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the extension property and the Independence Theorem we will have that

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff \mathrm{acl}^{\mathrm{eq}}(AC) \cap \mathrm{acl}^{\mathrm{eq}}(BC) \subseteq \mathrm{acl}^{\mathrm{eq}}(C).$$

Lemma 5.4 *If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $A \mathop{\smile\!\!\!\!\!\!|}_{CD} BD$.*

Proof: Again we may assume that A, B, C, D are algebraically closed. We assume $\mathrm{cl}_E(A) \cap \mathrm{cl}_E(BC) \subseteq \mathrm{cl}_E(C)$ and show $\mathrm{cl}_E(A) \cap \mathrm{cl}_E(BCD) \subseteq \mathrm{cl}_E(CD)$. Let $a \in \mathrm{cl}_E(A) \cap \mathrm{cl}_E(BCD)$. Suppose first a is not an equivalence class. In case $a \notin \bigcup_n Q_n^0$ it is clear by Lemma 5.2 that $a \in B \cup C \cup D$ and therefore that $a \in \mathrm{acl}(CD)$. Let us consider the case $a \in Q_n^0$. By Lemma 5.2 there are $a_1, \ldots, a_n \in (B \cup C \cup D) \cap Q_n^1$ different and such that $F_n(a_1, \ldots, a_n) = a$, say $a_1, \ldots, a_i \in B$, $a_{i+1}, \ldots, a_j \in C$ and $a_{j+1}, \ldots, a_n \in D$. Since $a_1, \ldots, a_n \in A$, we see that $a_1, \ldots, a_j \in C$. Hence $a_1, \ldots, a_n \in CD$ and therefore $a \in \mathrm{acl}(CD)$. Assume now $a = [b]_E$ for some $b \in (B \cup C \cup D) \smallsetminus \bigcup_n P_n$. In case $b \in D$ we are done. And in case $b \in B \cup C$ we get $a \in \mathrm{cl}_E(BC)$ and we may apply the hypothesis to obtain $a \in \mathrm{cl}_E(C)$. $\square$

Lemma 5.5 *The extension property holds.*

Proof: With Lemma 5.4, an easy induction on the length of the tuples shows that it is enough to check the extension property for elements, that is, for tuples of length 1. Assume a, B, C are given. We find an element a' such that $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$ and $a' \mathop{\smile\!\!\!\!\!\!\,|}_C B$. Without loss of generality, the sets B, C are algebraically closed and $a \notin C$. In the case $a \in \bigcup_n P_n$ it is enough to find $\mathrm{tp}(a'/C) = \mathrm{tp}(a/C)$ such that $\mathrm{acl}(a') \cap \mathrm{acl}(BC) \subseteq C$ and as it is well known this is always possible in any theory. Now assume $a \notin \bigcup_n P_n$. If $[a]_E \in \mathrm{cl}_E(C)$ we need to find $a' \notin BC$ with the same (atomic) type over C as a and this is possible since the types

$$p_i(x) := \{Q^i(x)\} \cup \{E(x, a)\} \cup \{x \neq b : b \in BC\}$$

for $i = 0, 1$ are consistent. In case $[a]_E \notin \mathrm{cl}_E(C)$ we take for a' a realization of one of the types

$$p_i(x) := \{Q^i(x)\} \cup \{\neg E(x, b) : b \in BC\} \cup \{\neg P_n(x) : n \geq 1\}.$$

$\square$

Lemma 5.6 *The Independence Theorem over algebraically closed sets holds.*

Proof: It is enough to prove it for elements, that is, for tuples of length one. The reason is that starting from this case we can easily prove by induction on the length that it holds for any *closed tuple* $a_1, \ldots, a_n$, that is, a tuple such that for every $i = 1, \ldots, n$, $\mathrm{acl}(a_i) \subseteq \{a_j : j \leq i\}$, and in our theory every tuple a can be extended to a such closed tuple a^* with $\mathrm{cl}_E(a) = \mathrm{cl}_E(a^*)$. Assume $C = \mathrm{acl}(C)$, $C \subseteq A \cap B$, $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$, $a \mathop{\smile\!\!\!\!\!\!\,|}_C A$, $b \mathop{\smile\!\!\!\!\!\!\,|}_C B$ and $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$. We have to find c such that $c \mathop{\smile\!\!\!\!\!\!\,|}_C AB$, $\mathrm{tp}(c/A) = \mathrm{tp}(a/A)$ and $\mathrm{tp}(c/B) = \mathrm{tp}(b/B)$. Without loss of generality, A, B are algebraically closed and $a, b \notin C$. It follows that $a \notin A$ and $b \notin B$. Moreover, $C = A \cap B$. There are different cases.

Case 1: $a \in Q_n^1$. Then $b \in Q_n^1$. Observe that for any $(n-1)$-tuple $d \in C \cap Q_n^1$ and any $c \in C \cap Q_n^2$ we have that $R(F_n(d, a), c)$ if and only if $R(F_n(d, b), c)$. Let I_A be the set of all pairs (d, c) where d is an $(n-1)$-tuple in A, $c \in A$ and $R(F_n(d, a), c)$ and let J_A be the set of all pairs (d, c) in A such that $\neg R(F_n(d, a), c)$. Define similarly I_B and J_B with b instead of a and B instead of A. It suffices to take as c a realization of the type

$$p(u) := \{Q_n^1(u)\} \cup \{u \neq d : d \in A \cup B\} \cup \{R(F_n(d,u),c) : (d,c) \in I_A \cup I_B\} \cup \{\neg R(F_n(d,u),c) : (d,c) \in J_A \cup J_B\}.$$

Case 2: $a \in Q_n^2$. Then $b \in Q_n^2$. Since $A \cap B = C$ and $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$, the following is consistent:

$$p(u) := \{Q_n^2(u)\} \cup \{R(d,u) : d \in A,\, R(d,a)\} \cup \{\neg R(d,u) : d \in A, \neg R(d,a)\} \cup \{R(d,u) : d \in B,\, R(d,b)\} \cup \{\neg R(d,u) : d \in B, \neg R(d,b)\} \cup \{u \neq d : d \in A \cup B\}.$$

We define c as a realization of this type.

Case 3: $a \in Q_n^0$. Then $b \in Q_n^0$. Let $a_1, \dots, a_n \in Q_n^1$ be such that $a = F_n(a_1, \dots, a_n)$ and let $b_1, \dots, b_n$ be such that $\mathrm{tp}(a, a_1, \dots, a_n/C) = \mathrm{tp}(b, b_1, \dots, b_n/C)$. Then $b = F_n(b_1, \dots, b_n)$. By iteration of Case 1 we find $c_1, \dots, c_n$ such that $\mathrm{tp}(c_1, \dots, c_n/A) = \mathrm{tp}(a_1, \dots, a_n/A)$, $\mathrm{tp}(c_1, \dots, c_n/B) = \mathrm{tp}(b_1, \dots, b_n/B)$ and $c_1, \dots, c_n \mathop{\smile\!\!\!\!\!\mid}_C AB$. We define $c = F_n(c_1, \dots, c_n)$.

Case 4: $a \notin \bigcup_n P_n$ and $[a]_E \in \mathrm{cl}_E(C)$. Then $b \notin \bigcup_n P_n$ and $E(a,b)$. Without loss of generality, $a, b \in Q^0$. Fix $c' \in C$ such that $[c']_E = [a]_E$. The following is consistent

$$\begin{aligned} p(x) := \{Q^0\} \cup \{E(x,c')\} \cup \{R(x,d) : d \in A,\, R(a,d)\} \cup \\ \{\neg R(x,d) : d \in A, \neg R(a,d)\} \cup \{R(x,d) : d \in B,\, R(b,d)\} \cup \\ \{\neg R(x,d) : d \in B, \neg R(b,d)\} \cup \{x \neq d : d \in A \cup B\}. \end{aligned}$$

We define c as a realization of this type.

Case 5: $a \notin \bigcup_n P_n$ and $[a]_E \notin \mathrm{cl}_E(C)$. Then $b \notin \bigcup_n P_n$ and since $a \mathop{\smile\!\!\!\!\!\mid}_C A$ and $b \mathop{\smile\!\!\!\!\!\mid}_C B$, we see that $[a]_E \notin \mathrm{cl}_E(A)$ and $[b]_E \notin \mathrm{cl}_E(B)$. We may assume $a, b \in Q^0$. We take as c a realization of

$$p(x) := \{Q^0(x)\} \cup \{\neg P_n(x) : n \geq 1\} \cup \{\neg E(x,d) : d \in A \cup B\}.$$

$\square$

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