Notre Dame Journal of Formal Logic
Volume 40, Number 1, Winter 1999

Rules and Arithmetics

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Abstract This paper is concerned with the *logical structure* of arithmetical theories. We survey results concerning logics and admissible rules of constructive arithmetical theories. We prove a new theorem: the admissible propositional rules of Heyting Arithmetic are the same as the admissible propositional rules of Intuitionistic Propositional Logic. We provide some further insights concerning predicate logical admissible rules for arithmetical theories.

1 Introduction Can we say anything interesting about the logical structure of constructive arithmetical theories? We might ask, for example, what the 'logic' of such a theory is. A question with an even more informative answer is, "What are the admissible rules of a given arithmetical theory?"

This paper is, in a sense, two papers in one. First, we survey results concerning logics and admissible rules of arithmetical theories. Second, we fill some gaps in our total picture.

1. We show that the propositional admissible rules of Heyting Arithmetic, HA, are the same ones as those of Intuitionistic Propositional Logic, IPC, itself. This characterization will follow from a general lemma.
2. In Subsection 3.3 we present a particularly simple proof that the predicate logical admissible rules of a wide range of constructive theories are complete Π^0_2.
3. In the appendix we provide some Orey-Hájek-Friedman-style characterizations of predicate logical admissibility for classical arithmetical theories.

The structure of the paper is as follows. In Section 2, we review what is known about the 'logics' of constructive theories. Specifically, we will be interested in the case where the logic of a theory is precisely IPC. Some of the results discussed here will be used as lemmas later in the paper. Section 3 will introduce the basics on admissible rules of arithmetical theories. Section 4 contains the proof of our result concerning the admissible rules of HA. Finally, in an appendix, we briefly consider what can be said about the predicate logical admissible rules of classical arithmetical theories.

Received April 16, 1998

1.1 Prerequisites The paper presupposes some knowledge of the Kripke semantics for constructive theories. See, for example, Smoryński [25] or Troelstra and van Dalen [28]. In Appendix A we employ some results concerning definable cuts, restricted proof predicates and the like. A good reference for the material in the appendix is Hájek and Pudlák [10]. See also Visser [35] and [36].

2 Theories and logics Let T be any theory formulated in either intuitionistic predicate logic or intuitionistic propositional logic. Let the language of T be $\mathcal{L}_T$. For $\mathcal{L}_{\mathsf{HA}}$, the usual language of arithmetic with 0, successor, plus, and times, we reserve the special name $\mathcal{R}$.

It is a natural question to ask ourselves: what are the schematic principles 'valid' in T? The answer to this question will depend on our notion of scheme. Do we mean scheme in the language of propositional logic, in the language of predicate logic, in a modal language as in provability logic?

Suppose T is a theory in classical logic. Then the *propositional schemes* valid in a consistent theory T with classical logic are, trivially, precisely the classical tautologies. The question becomes much more interesting if we consider classical theories and predicate logical schemes (see Yavorsky [42]), or if we enrich the propositional language with a modal predicate for *provability* (see Boolos [1] or Smoryński [26]).

If we consider constructive theories, already the purely propositional case has some interest. If a theory is 'purely constructive', one would surely expect the valid propositional schemes to be precisely the theorems of intuitionistic propositional logic IPC. This often turns out to be the case. However, the proofs are surprisingly nontrivial.

In this section we will survey what is known about propositional and predicate logics of arithmetical theories.

2.1 Propositional logics of theories Below I present the necessary definitions to speak a bit more precisely about substitutions, propositional schemes, and the like.

Let $\mathcal{P}$ be a *countable* set of propositional variables. The language $\mathcal{L}_{\mathsf{IPC},\mathcal{P}}$ is the language of IPC for the variables $\mathcal{P}$. We will denote IPC with this language by $\mathsf{IPC}(\mathcal{P})$. By our earlier convention, we have $\mathcal{L}_{\mathsf{IPC}(\mathcal{P})} = \mathcal{L}_{\mathsf{IPC},\mathcal{P}}$. A $\mathcal{P}$-scheme is simply a formula in $\mathcal{L}_{\mathsf{IPC},\mathcal{P}}$. A scheme is 'valid' in T *if all of its substitution instances are T-provable*. In most cases we will consider a finite set $\mathcal{P}$. We will use $\vec{p}, \vec{q}$ as the notation for such finite sets.

Let $\mathcal{L}$ be any language of propositional or of predicate logic. A $\mathcal{P}$-substitution σ for $\mathcal{L}$ is a function from $\mathcal{P}$ to the set of sentences of $\mathcal{L}$. The set of $\mathcal{P}$-substitutions for $\mathcal{L}$ will be called $\mathsf{sub}_{\mathcal{P},\mathcal{L}}$. In case $\mathcal{L} = \mathcal{L}_T$, we will also write $\mathsf{sub}_{\mathcal{P},T}$. We extend $\sigma \in \mathsf{sub}_{\mathcal{P},\mathcal{L}}$ in the usual way to $\mathcal{L}_{\mathsf{IPC},\mathcal{P}}$ by making it commute with the propositional connectives including $\top$ and $\bot$. We will use $\sigma(\varphi)$ for 'the extension of σ applied to φ'.

A $\mathcal{P}$-logic Λ is a set of $\mathcal{L}_{\mathsf{IPC},\mathcal{P}}$-formulas that extends the set of $\mathsf{IPC}(\mathcal{P})$-tautologies and is closed under modus ponens and under $\mathcal{P}$-substitutions for $\mathcal{L}_{\mathsf{IPC},\mathcal{P}}$.[1] So for $\sigma \in \mathsf{sub}_{\mathcal{P},\mathsf{IPC}(\mathcal{P})}$, we have $\varphi \in \Lambda \Longrightarrow \sigma(\varphi) \in \Lambda$. Here are some definitions.

Definition 2.1 Let $\varnothing \neq \mathcal{S} \subseteq \mathsf{sub}_{\mathcal{P},T}$. Define

$$\Lambda_{\mathcal{P},T}(\mathcal{S}) := \{\varphi \in \mathcal{L}_{\mathsf{IPC},\mathcal{P}} \mid \forall \sigma \in \mathcal{S}\ T \vdash \sigma(\varphi)\}$$

In case $\mathcal{S}$ is obtained by restricting the range of the substitutions to a class of formulas Θ, we will, *par abus de langage*, write $\Lambda_{\mathcal{P},T}(\Theta)$ for $\Lambda_{\mathcal{P},T}(\mathcal{S})$.

Definition 2.2 Let $\sigma \in \mathsf{sub}_{\mathcal{P},T}$. Omitting singleton brackets, we write

$$\Lambda_{\mathcal{P},T}(\sigma) := \Lambda_{\mathcal{P},T}(\{\sigma\})$$

We call $\Lambda_{\mathcal{P},T}(\sigma)$ *the exact $\mathcal{P}$-theory of σ for T*.

Definition 2.3 We will omit the set of substitutions, when we are considering *all substitutions* of the relevant kind

$$\Lambda_{\mathcal{P},T} := \Lambda_{\mathcal{P},T}(\mathsf{sub}_{\mathcal{P},T}).$$

It is easy to see that $\Lambda_{\mathcal{P},T}$ is a $\mathcal{P}$-logic and that for any $\mathcal{P}$-logic Λ, we have $\Lambda_{\mathcal{P},\Lambda} = \Lambda$. We will identify $\mathsf{IPC}(\mathcal{P})$ with $\Lambda_{\mathcal{P},\mathsf{IPC}}$. Note that $\Lambda_{\mathcal{P},T}(\mathcal{S})$ need not generally be a logic. It is easy to see that if $|\mathcal{P}| \leq |\mathcal{Q}|$, then $\Lambda_{\mathcal{P},\mathsf{IPC}(\mathcal{Q})} = \mathsf{IPC}(\mathcal{P})$. We show the following.

Theorem 2.4 *If $|\mathcal{P}| > |\mathcal{Q}|$, then $\Lambda_{\mathcal{P},\mathsf{IPC}(\mathcal{Q})} \supsetneq \mathsf{IPC}(\mathcal{P})$.*

Proof: Suppose $|\mathcal{P}| > |\mathcal{Q}|$. Remember that we assumed $\mathcal{P}$ and $\mathcal{Q}$ to be countable. So $\mathcal{Q}$ must be finite. Take $\vec{p} \subseteq \mathcal{P}$, with $|\mathcal{Q}| < |\vec{p}|$. Let C be the set of all conjunctions of formulas of the form p and $\neg p$, where for any $p \in \vec{p}$ precisely one of p, $\neg p$ is a conjunct. Take $\vartheta := \bigvee\{\neg\gamma \mid \gamma \in C\}$. Clearly, $\mathsf{IPC}(\mathcal{P}) \nvdash \vartheta$. Suppose $\sigma \in \mathsf{sub}_{\mathcal{P},\mathsf{IPC}(\mathcal{Q})}$. If we did have $\mathsf{IPC}(\mathcal{Q}) \nvdash \sigma(\vartheta)$, then there would be a *finite* rooted $\mathcal{Q}$-model $\mathcal{K}$, with root $\mathfrak{b}$ such that $\mathfrak{b} \nVdash \sigma(\vartheta)$. For every $\gamma \in C$, there would be a top node k above $\mathfrak{b}$ such that $k \Vdash \sigma(\gamma)$. Thus, there must be *at least* $2^{|\vec{p}|}$ top nodes with essentially different forcing relation. Since, $\mathcal{K}$ is a $\mathcal{Q}$-model there could be *at most* $2^{|\mathcal{Q}|}$ such nodes: a contradiction. So $\vartheta \in \Lambda_{\mathcal{P},\mathsf{IPC}(\mathcal{Q})}$. □

Note also that if T is any consistent classical theory, whether in propositional or in predicate logic, we have $\Lambda_{\mathcal{P},T} = \mathsf{CPC}(\mathcal{P})$. Here CPC is the classical propositional calculus.

2.2 Predicate logics of theories Let $\mathcal{L}$ be a language of predicate logic. Let T be a theory. An $\mathcal{L}$-scheme is simply a sentence in $\mathcal{L}$. A scheme is 'valid' in T *if all of its interpretations are T-provable*. An interpretation $\mathcal{M}$ assigns to a relation symbol R of $\mathcal{L}$ formulas of $\mathcal{L}_T$ with designated variables corresponding to the argument places of R. We usually assume that $\mathcal{M}(R)$ contains no other variables than those representing the argument places. In case $\mathcal{L}$ contains function symbols we treat $f(x_1, \ldots, x_n) = y$ as a relation symbol. $\mathcal{M}$ sends an arbitrary formula φ of $\mathcal{L}$ to the result of replacing all its relation symbols R by $\mathcal{M}(R)$, changing the variables representing the argument places into the variables following a given occurrence of R in φ. In case φ contains function symbols, we first apply the well-known procedure for reducing the nesting degree of function symbols to 1 and then run the procedure we just described. In case we eliminate function symbols, we demand that the interpreting theory verifies the translations of the statements expressing the fact that $f(x_1, \ldots, x_n) = y$ represents the graph of a function. Thus being an interpretation becomes dependent not only on the interpreting language, but on the interpreting theory. We call the class of interpretations of $\mathcal{L}$ in T, $\mathsf{int}_{\mathcal{L},T}$.

We often do not want simple interpretations but *relative interpretations*. A relative interpretation is like an interpretation with the following additional feature. There is an associated special formula $\delta(x)$ representing the domain of the interpretation. In relative interpretations we replace $\forall x\ \ldots$ by $\forall x(\delta(x) \to \ldots)$ and we replace $\exists x\ \ldots$ by $\exists x(\delta(x) \wedge \ldots)$. We demand that the *interpreting theory* proves $\exists x\ \delta(x)$. Thus, whether something is a relative interpretation or not will depend on the interpreting theory even in the absence of function symbols. We call the class of relative interpretations of $\mathcal{L}$ in T, $\mathsf{relint}_{\mathcal{L},T}$.

For more details on interpretations, see, for example, Tarski [27] or Visser [35] or [39]. Here are the relevant definitions.

Definition 2.5 Let $\varnothing \neq \mathcal{S} \subseteq \mathsf{relint}_{\mathcal{L},T}$. Define

$$\Lambda^{\mathsf{rel}}_{\mathcal{L},T}(\mathcal{S}) := \{\varphi \in \mathsf{sent}_{\mathcal{L}} \mid \forall \mathcal{M} \in \mathcal{S}\ \ T \vdash \mathcal{M}(\varphi)\}$$

In case $\mathcal{S}$ is obtained by restricting the range of the substitutions to a class of formulas Θ, we will, *par abus de langage*, write $\Lambda^{\mathsf{rel}}_{\mathcal{L},T}(\Theta)$ for $\Lambda^{\mathsf{rel}}_{\mathcal{L},T}(\mathcal{S})$.

Definition 2.6 We will omit the set of relative interpretations, when we are considering *all interpretations* of the relevant kind, $\Lambda^{\mathsf{rel}}_{\mathcal{L},T} := \Lambda^{\mathsf{rel}}_{\mathcal{L},T}(\mathsf{relint}_{\mathcal{P},T})$. It is easy to see that the unrelativized interpretations can be viewed as a subclass of the relativized interpretations. When we consider unrelativized interpretations, we simply drop the superscript rel. So, $\Lambda_{\mathcal{L},T} := \Lambda^{\mathsf{rel}}_{\mathcal{L},T}(\mathsf{int}_{\mathcal{P},T})$.

It is clear that, when we view $\mathcal{P}$ and $\mathcal{L}$ as *signatures*, our definitions for propositional logic are simply special cases of the ones for predicate logic. Here are a few further convenient notations.

Notation 2.7

1. $\mathcal{M} : T \rhd \varphi \ :\Longleftrightarrow\ T, \mathcal{M} \vdash \varphi \ :\Longleftrightarrow\ T \vdash \mathcal{M}(\varphi)$,
2. $T \rhd \varphi \ :\Longleftrightarrow\ \exists \mathcal{M} \in \mathsf{relint}_{\mathcal{L},T}\ \ T, \mathcal{M} \rhd \varphi$.
   We say that φ is relatively interpretable in T or that T interprets φ.

We note in passing that Tarski's notion of *weak interpretability* is reducible to $\Lambda^{\mathsf{rel}}_{\mathcal{L},T}$. A sentence φ of $\mathcal{L}$ is weakly interpretable in T if there is a relative $\mathcal{L}, T$-interpretation $\mathcal{M}$ such that $T + \mathcal{M}(\varphi)$ is consistent. We easily see that φ is weakly interpretable in T if and only if $\Lambda^{\mathsf{rel}}_{\mathcal{L},T} + \varphi$ is consistent. If we consider a *classical* theory T we can regain $\Lambda^{\mathsf{rel}}_{\mathcal{L},T}$ from the φ that are weakly interpretable in T. The notion of weak interpretability is important because of the following theorem. Let Q be Robinson's Arithmetic.

Theorem 2.8 (Tarski) *If Q is weakly interpretable in T, that is, if $\Lambda^{\mathsf{rel}}_{\mathcal{R},T} + \mathsf{Q}$ is consistent, then T is undecidable.*

Tarski uses the theorem in his proof of the undecidability of Group Theory (see [27]). Note that it follows that for decidable theories, such as the theory of Abelean Groups, we have $\Lambda^{\mathsf{rel}}_{\mathcal{R},T} \vdash \neg\mathsf{Q}$. For results concerning the $\Lambda_{\mathcal{L},T}$ for classical theories T, the reader is referred to Rybakov [24] and to Yavorsky [42]. See also Appendix A of the present paper. Here are three of Yavorsky's results.

1. $\Lambda_{\mathcal{L},\mathsf{PA}} = \mathsf{CQC}(\mathcal{L})$. Here $\mathsf{CQC}(\mathcal{L})$ is classical predicate logic for the language $\mathcal{L}$.[2]
2. $\Lambda_{\mathcal{L},\mathsf{GROUP}^+_{\mathsf{class}}} = \mathsf{CQC}(\mathcal{L})$. Here $\mathsf{GROUP}^+_{\mathsf{class}}$ is the classical theory of groups with one extra constant.
3. $\Lambda_{\mathcal{L},\mathsf{Pre}} \neq \mathsf{CQC}(\mathcal{L})$. Here Pre is classical Presburger Arithmetic.

2.3 A brief history of de Jongh's Theorem We present a brief survey of the development of our present knowledge of constructive arithmetical theories and classes of substitutions that give us precisely constructive logic.

1969 De Jongh shows in an unpublished paper that $\Lambda_{\mathcal{P},\mathsf{HA}} = \mathsf{IPC}(\mathcal{P})$. He uses substitutions of formulas of a complicated form. In fact he proves a much stronger result, viz., that the logic of relative interpretations in HA is Intuitionistic Predicate Logic, in other words, $\Lambda^{\mathsf{rel}}_{\mathcal{L},\mathsf{HA}} = \mathsf{IQC}(\mathcal{L})$. See the extended abstract [3]. De Jongh's argument uses an ingenious combination of Kripke models and realizability.

1973 Friedman in his paper [6] shows that $\Lambda_{\mathcal{P},\mathsf{HA}}(\Pi_2) = \mathsf{IPC}(\mathcal{P})$. In fact, Friedman provides a *single* substitution σ mapping $\mathcal{P}$ to Π_2-sentences such that $\Lambda_{\mathcal{P},\mathsf{HA}}(\sigma) = \mathsf{IPC}(\mathcal{P})$. We will say that IPC is uniformly complete for Π_2-substitutions in HA. Uniform Completeness tells us, in this case, that the free Heyting Algebra on countably many generators can be embedded in the Lindenbaum Algebra of HA. Friedman employs slash-theoretic methods.

1973 Smoryński strengthens and extends de Jongh's work in a number of respects in his very readable paper [25]. To state his results we need a few definitions. We write $\mathsf{D}\Pi_1$ for the set of disjunctions of Π_1-sentences, $\mathsf{Boole}(\Sigma_1)$ for Boolean combinations of Σ_1-sentences. MP is Markov's Principle, $\mathsf{RFN}_{\mathsf{HA}}$ is the formalized uniform reflection principle for HA, $\mathsf{TI}(\prec)$ is the transfinite induction scheme for a primitive recursive well-ordering $\prec$. We have

$$\Lambda_{\mathcal{P},T} = \Lambda_{\mathcal{P},T}(\Sigma_1) = \Lambda_{\mathcal{P},T}(\mathsf{D}\Pi_1) = \mathsf{IPC}(\mathcal{P}),$$

for the following theories T: HA, $\mathsf{HA}+\mathsf{RFN}(\mathsf{HA})$, $\mathsf{HA}+\mathsf{TI}(\prec)$. We have $\Lambda_{\mathcal{P},\mathsf{HA}+\mathsf{MP}}(\mathsf{Boole}(\Sigma_1)) = \mathsf{IPC}(\mathcal{P})$. Smoryński uses Kripke models in combination with the Gödel-Rosser-Mostowski-Kripke-Myhill Theorem to prove his results.

1975 Leivant in his Ph.D. thesis [12] shows that the predicate logic of interpretations of predicate logic in HA is precisely intuitionistic predicate logic. Leivant's method is proof theoretical. In fact, Leivant shows that one can use as interpretation a fixed sequence of Π_2-predicates. So Leivant proves that $\Lambda_{\mathcal{L},\mathsf{HA}}(\mathcal{M}) = \mathsf{IQC}(\mathcal{L})$, for some Π_2-interpretation $\mathcal{M}$. Leivant's results yield another proof of Friedman's results described above.

1976 De Jongh and Smoryński in their paper [4] show that $\Lambda_{\vec{p},\mathsf{HAS}}(\Sigma_1) = \mathsf{IPC}(\vec{p})$. They also show that there is a $\sigma : \mathcal{P} \to \Pi_2$, such that $\Lambda_{\mathcal{P},\mathsf{HAS}}(\sigma) = \mathsf{IPC}(\mathcal{P})$.

1981 Gavrilenko in [7] shows that $\Lambda_{\vec{p},\mathsf{HA+ECT}_0}(\Sigma_1) = \mathsf{IPC}(\vec{p})$. Here ECT_0 is Extended Church's Thesis. Gavrilenko proves this result as a corollary of the similar result of Smoryński for HA.

1981 Visser in his Ph.D. thesis [32] provides an alternative proof of de Jongh's Theorem for HA, HA+DNS, HA+ECT_0 for Σ_1-substitutions adapting the method of Solovay's proof of the arithmetical completeness of Löb's logic for substitutions in PA. Here DNS is the principle Double Negation shift. In fact, his proof extends to these theories with appropriate reflection principles or transfinite induction over primitive recursive well-orderings added.

1985 In his [34], Visser provides an alternative proof of de Jongh's uniform completeness theorem employing a single Σ_1-substitution. The proof is verifiable in HA+Con(HA). (Note that de Jongh's Theorem *implies* Con(HA), so the result is, in a sense, optimal.) The proof uses the NNIL-algorithm, an algorithm that is used to characterize the admissible rules for Σ_1-substitutions. See below.

1991 Van Oosten in his paper [31] provides a more perspicuous version of de Jongh's semantical proof of de Jongh's Theorem for (nonrelativized) interpretations of predicate logic. Van Oosten uses Beth models and realizability. See also [30].

1996 Using the methods developed by Visser in [33] and by de Jongh and Visser in [5] one can prove uniform completeness with respect to Σ_1-substitutions for HA+ECT_0, HA+ECT_0+RFN(HA+ECT_0), HA+TI$(\prec)$+ECT_0.

Open Question 2.9 Here are some open questions in this area.

What is the predicate logic of HA+MP?
What is the predicate logic of $\mathsf{HA} + \mathsf{ECT}_0$?

We end this section by providing a necessary condition for arithmetical theories to satisfy de Jongh's Theorem.[3] Consider a theory T. Suppose $\mathcal{N} \in \mathsf{int}_{\mathcal{R},T}$. Suppose we have

1. $T, \mathcal{N} \vdash \mathsf{iEA}$, where iEA is the intuitionistic version of Elementary Arithmetic, also known as $\mathsf{i}I\Delta_0 + \mathsf{Exp}$;
2. T is *locally essentially reflexive* with respect to $\mathcal{N}$. This means that T proves the full sentential reflection principle for $\mathsf{IQC}(\mathcal{L}_T)$, where provability is formalized 'in $\mathcal{N}$'; in other words, for any sentence φ of $\mathcal{L}_T$, $T \vdash \mathcal{N}(\Box_{\mathsf{IQC}(\mathcal{L}_T)}\varphi) \to \varphi$.

All extensions of HA in $\mathcal{R}$ are locally essentially reflexive. Let Q be the single axiom of (the intuitionistic variant of) Robinson's Arithmetic.

Theorem 2.10 *Let T be as above. Suppose $\Lambda_{\mathcal{R},T} = \mathsf{IQC}(\mathcal{R})$. Then T is Σ^0_1-sound with respect to $\mathcal{N}$. Moreover, T is closed under the Primitive Recursive Markov's Rule with respect to $\mathcal{N}$, that is, for any Σ^0_1-sentence σ, $T, \mathcal{N} \vdash \neg\neg\sigma \Longrightarrow T, \mathcal{N} \vdash \sigma$. Our two claims together are, clearly, equivalent to the following principle: for any*

Σ^0_1*-sentence* σ, $T, \mathcal{N} \vdash \neg\neg\sigma \Longrightarrow \mathbb{N} \models \sigma$.

Proof: Suppose $T, \mathcal{N} \vdash \neg\neg\sigma$. Then, $T, \mathcal{N} \vdash \neg\neg\Box_{\mathsf{IQC}(\mathcal{R})}(\mathsf{Q} \to \sigma)$, since iEA proves Σ-completeness for Q. Consider any $\mathcal{K} \in \mathsf{relint}_{\mathcal{R},T}$. Given the fact that we just have finitely many function symbols in $\mathcal{R}$, we only need a finite subtheory of T to verify the fact that $\mathcal{K}$ is an interpretation. Suppose φ axiomatizes such a finite subtheory. We find $T, \mathcal{N} \vdash \neg\neg\Box_{\mathsf{IQC}(\mathcal{L}_T)}(\varphi \to \mathcal{K}(\mathsf{Q} \to \sigma))$. Since T is locally essentially reflexive with respect to $\mathcal{N}$, we find $T \vdash \neg\neg\mathcal{K}(\mathsf{Q} \to \sigma)$. Hence, $T \vdash \mathcal{K}(\mathsf{Q} \to \neg\neg\sigma)$. Since $\mathcal{K}$ was arbitrary, we find $(\mathsf{Q} \to \neg\neg\sigma) \in \Lambda_{\mathcal{R},T}$. So, by our assumption, $\mathsf{IQC}(\mathcal{R}) \vdash \mathsf{Q} \to \neg\neg\sigma$. Since Q is classically true, we may conclude that $\mathbb{N} \models \sigma$. $\Box$

2.4 Markov's Principle and Church's Thesis In this subsection, we briefly consider cases, where the logics of a theory are *not* precisely intuitionistic logic. We have seen that $\Lambda_{\mathcal{P},\mathsf{HA+MP}} = \Lambda_{\mathcal{P},\mathsf{HA+ECT}_0} = \mathsf{IPC}(\mathcal{P})$. Remarkably, $\Lambda_{\mathcal{P},\mathsf{HA+MP+ECT}_0}$, for $|\mathcal{P}| > 1$, turns out to be a proper extension of $\mathsf{IPC}(\mathcal{P})$.

Consider the formulas χ and ρ which are defined as follows.

1. $\chi := (\neg p \vee \neg q)$,
2. $\rho := [(\neg\neg\chi \to \chi) \to (\neg\neg\chi \vee \neg\chi)] \to (\neg\neg\chi \vee \neg\chi)$.

Clearly, ρ is $\mathsf{IPC}(p, q)$-invalid. We use $\mathbf{r}$ for *Kleene realizability*. In his classical paper [23], Rose showed that $\exists e\ \forall\sigma{\in}\mathsf{sub}_{\mathcal{P},\mathsf{HA}}\ \mathbb{N} \models e\,\mathbf{r}\,\sigma(\rho)$. Thus, Rose refuted a conjecture of Kleene that a propositional formula is IPC-provable if all its arithmetical instances are (truly and classically) realizable. Note the remarkable fact that one and the same realizer realizes all instances! Inspecting the proof one sees that only a small part of classical logic is involved in the verification of realizability, Markov's Principle. See McCarthy [13] for a detailed analysis. Thus we obtain

$$\exists e\ \forall\sigma{\in}\mathsf{sub}_{\mathcal{P},\mathsf{HA}}\ \mathsf{HA} + \mathsf{MP} \vdash e\,\mathbf{r}\,\sigma(\rho).$$

Hence, a fortiori, $\rho \in \Lambda_{\{p,q\},\mathsf{HA+MP+ECT}_0}$.

Open Question 2.11 The precise characterization of any of the following sets is an open problem.

1. $\Lambda_{\mathcal{P},\mathsf{HA+MP+ECT}_0}$,
2. $\{\varphi{\in}\mathcal{L}_{\mathsf{IPC},\mathcal{P}} \mid \exists e\ \forall\sigma{\in}\mathsf{sub}_{\mathcal{P},\mathsf{HA}}\ \mathsf{HA} \vdash e\,\mathbf{r}\,\sigma(\varphi)\}$,
3. $\{\varphi{\in}\mathcal{L}_{\mathsf{IPC},\mathcal{P}} \mid \forall\sigma{\in}\mathsf{sub}_{\mathcal{P},\mathsf{HA}}\ \exists e\ \mathbb{N} \models e\,\mathbf{r}\,\sigma(\varphi)\}$,
4. $\{\varphi{\in}\mathcal{L}_{\mathsf{IPC},\mathcal{P}} \mid \exists e\ \forall\sigma{\in}\mathsf{sub}_{\mathcal{P},\mathsf{HA}}\ \mathbb{N} \models e\,\mathbf{r}\,\sigma(\varphi)\}$.

One could well imagine that it would be possible to prove the sets (1) and (2) to be equal without having a characterization. Similarly for (3) and (4).

The situation for substitutions in predicate logic is even more spectacular. In a series of papers, [16], [17], [18], Plisko shows that the set of uniformly realizable principles of predicate logic is complete Π^1_1. In other words he shows that, for an appropriate $\mathcal{L}$, $\{\varphi{\in}\mathsf{sent}_{\mathcal{L}} \mid \exists e\ \forall\mathcal{K} \in \mathsf{int}_{\mathcal{L},\mathsf{Th}(\mathbb{N})}\ \mathbb{N} \models e\,\mathbf{r}\,\mathcal{K}(\varphi)\}$ is complete Π^1_1. ([19] provides a related result for modified realizability.) In two subsequent papers [20] and [21], Plisko shows that $\Lambda_{\mathcal{L},\mathsf{HA+MP+ECT}_0}$ is complete Π^0_2.[4]

2.5 Exactness and extension An exact theory is the theory of a single interpretation. If $U = \Lambda_{\mathcal{P},T}(\sigma)$, or, analogously, for the predicate logical case, $U = \Lambda_{\mathcal{L},T}(\mathcal{M})$, one says that σ or $\mathcal{M}$ is a *faithful interpretation* of U in T. For example, the Beltrami-Poincaré interpretation of hyperbolic geometry in Euclidean geometry is faithful; the usual interpretation of PA in ZF is not.

We will show that exact theories inherit a salient property, viz. the extension property, from their interpreting theories. To set the stage, we first introduce the idea of E-preservation. Consider a class $\mathbb{K}$ of models (of whatever kind). Let R be a binary relation on $\mathbb{K}$. A class Γ of formulas (with a semantics in $\mathbb{K}$) is E-*preserved* (in $\mathbb{K}$) by R if, for all $\mathcal{K} \in \mathbb{K}$, whenever $\mathcal{K} \Vdash \Gamma$, then, *for some* $\mathcal{N}$ with $\mathcal{K}\, R\, \mathcal{N}$, $\mathcal{N} \Vdash \Gamma$. In a similar way we can define A-preservation, by demanding that then, *for all* $\mathcal{N}$ with $\mathcal{K}\, R\, \mathcal{N}$, $\mathcal{N} \Vdash \Gamma$.

Here is an example of a characterization of a class of formulas employing E-preservation, a characterization of formula classes with the disjunction property, due to de Jongh (see his [2]).

Theorem 2.12 (de Jongh) *Let $\mathbb{K}$ be the class of (not necessarily rooted) Kripke $\mathcal{P}$-models. Define:*

> *$\mathcal{K} \prec \mathcal{N}$ if and only if $\mathcal{K}$ is a generated (i.e., upward closed) submodel of $\mathcal{N}$ and $\mathcal{N}$ is rooted.*

Suppose $\Gamma \subseteq \mathcal{L}_{\mathsf{IPC},\mathcal{P}}$. Then Γ has the disjunction property if and only if Γ is E-preserved by $\prec$.

Let $\mathcal{L}$ be a language of either intuitionistic propositional logic or of intuitionistic predicate logic. Let $\mathbb{K}$ be the class of Kripke $\mathcal{L}$-models. We define

> $\mathcal{K} \lessdot \mathcal{N}$ if and only if $\mathcal{N}$ is rooted and $\mathcal{K}$ is the result of omitting the root of $\mathcal{N}$.

We say that $\Gamma \subseteq \mathcal{L}$ has the *extension property* if and only if Γ is E-preserved by $\lessdot$. Alternatively, we say that Γ is extendible. So Γ has the extension property if any (nonrooted) model of Γ can be extended with a new root preserving the validity of Γ. We start with a triviality.

Theorem 2.13 *Every extendible theory has the disjunction property.*

The theorem is immediate by Theorem 2.12 and the fact that $\lessdot$ is a subrelation of $\prec$. The next theorem establishes a connection between exactness and extendibility.

Theorem 2.14 *Let T be an extendible theory with language $\mathcal{L}_T$ and let $\sigma \in \mathsf{sub}_{\mathcal{P},T}$. Then, $\Lambda_{\mathcal{P},T}(\sigma)$ has the extension property.*

Proof: Say $\mathcal{E} := \Lambda_{\mathcal{P},T}(\sigma)$. Consider any nonrooted Kripke $\mathcal{P}$-model $\mathcal{K}$. Suppose that $\mathcal{K} \Vdash \mathcal{E}$. Let $\Theta := \mathsf{Th}(\mathcal{K})$. Clearly, $\mathcal{E} \subseteq \Theta$. Consider $\psi \in \mathcal{L}_{\mathsf{IPC},\mathcal{P}} \setminus \Theta$. We claim that $\sigma(\Theta) \nvdash_T \sigma(\psi)$.

Suppose that $\sigma(\Theta) \vdash_T \sigma(\psi)$. Then, for some $\theta \in \Theta$, $T \vdash \sigma(\theta \to \psi)$. Hence, $(\theta \to \psi) \in \mathcal{E}$. It follows that $\Theta \vdash \psi$, quod non.

We can find a nonrooted Kripke model $\mathcal{M}$ of T such that

$$\sigma^{-1}(\mathsf{Th}(\mathcal{M})) := \{\varphi \in \mathcal{L}_{\mathsf{IPC},\mathcal{P}} \mid \mathcal{M} \Vdash \sigma(\varphi)\} = \Theta.$$

Since T has the extension property, we can extend $\mathcal{M}$ to a new model $\mathcal{M}^+$ satisfying T by adding a new bottom $\mathfrak{b}$. We extend $\mathcal{K}$ to $\mathcal{K}^+$ with a new bottom $\mathfrak{c}$, stipulating that $\mathfrak{c} \Vdash p :\Longleftrightarrow \mathfrak{b} \Vdash \sigma(p)$. We show that $\mathfrak{c} \Vdash \varphi :\Longleftrightarrow \mathfrak{b} \Vdash \sigma(\varphi)$ by induction on $\mathcal{L}_{\mathsf{IPC},\vec{p}}$.

The cases of atoms, conjunction, and disjunction are trivial. Suppose $\mathfrak{c} \Vdash \psi \to \chi$. It follows that $\mathcal{K} \Vdash \psi \to \chi$ and, hence, $(\psi \to \chi) \in \Theta$. Consider $m \geq \mathfrak{b}$. Suppose $m \Vdash \sigma(\psi)$. In case m is in $\mathcal{M}$, we have, by the fact that $\mathcal{M} \Vdash \sigma(\Theta)$, $m \Vdash \sigma(\psi \to \chi)$ and, hence, $m \Vdash \sigma(\chi)$. In case $m = \mathfrak{b}$, we have, by the Induction Hypothesis, $\mathfrak{c} \Vdash \psi$. So, by assumption, $\mathfrak{c} \Vdash \chi$ and, hence, again by the Induction Hypothesis, $m = \mathfrak{b} \Vdash \sigma(\chi)$. The converse is similar. $\square$

Consider $\sigma \in \mathsf{sub}_{\vec{p},\mathsf{IPC}(\mathcal{P})}$. De Jongh and Visser prove the following theorem (see [5]).

Theorem 2.15 *$\Lambda_{\vec{p},\mathsf{IPC}(\mathcal{P})}(\sigma)$ is finitely axiomatizable.*

The proof uses Pitts' Uniform Interpolation Theorem. (See Pitts [15], Ghilardi and Zawadowski [9], and Visser [38].) *Par abus de langage*, we call an axiom of $\Lambda_{\vec{p},\mathsf{IPC}(\mathcal{P})}(\sigma)$: ε_σ. Note that ε_σ is only determined up to provable equivalence. We call a formula ε axiomatizing some $\Lambda_{\vec{p},\mathsf{IPC}(\mathcal{P})}(\tau)$ *a $\vec{p}, \mathcal{P}$-exact formula*. The set of $\vec{p}, \mathcal{P}$-exact formulas is $\mathsf{exact}_{\vec{p},\mathcal{P}}$.

Ghilardi proved that for substitutions in IPC we have a converse of Theorem 2.14 (see his [8]).

Theorem 2.16 (Ghilardi) *Suppose that $\varepsilon \in \mathcal{L}_{\mathsf{IPC},\vec{p}}$ has the extension property. Then, for some $\sigma \in \mathsf{sub}_{\vec{p},\mathsf{IPC}(\vec{p})}$, we have $\varepsilon = \varepsilon_\sigma$.*

Note that Ghilardi's Theorem, as stated here, implies that if $\varepsilon \in \mathcal{L}_{\mathsf{IPC},\vec{q}}$ and ε is $\vec{p}, \mathcal{P}$-exact, then ε is $\vec{q}, \vec{q}$-exact. Ghilardi's Theorem will be used as a lemma in the characterization of the admissible rules of HA.

3 Admissible rules

3.1 Finitary admissible rules Let T be a theory and let $\mathcal{S} \subseteq \mathsf{sub}_{\mathcal{P},T}$. A $\langle \mathcal{P}, T, \mathcal{S}\rangle$-admissible rule is a pair of $\mathcal{L}_{\mathsf{IPC},\mathcal{P}}$-formulas $\langle \varphi, \psi\rangle$ such that, for all $\sigma \in \mathcal{S}$, $T \vdash \sigma(\varphi) \Longrightarrow T \vdash \sigma(\psi)$. We say that $\langle \varphi, \psi\rangle$ is $\mathcal{P}, T$-admissible if it is $\langle \mathcal{P}, T, \mathsf{sub}_{\mathcal{P},T}\rangle$-admissible.

Definition 3.1 $\mathcal{A}_{\mathcal{P},T}(\mathcal{S})$ is the set of $\langle \mathcal{P}, T, \mathcal{S}\rangle$-admissible rules.
$\mathcal{A}_{\mathcal{P},T} = \mathcal{A}_{\mathcal{P},T}(\mathsf{sub}_{\mathcal{P},T})$.

Definition 3.2 $\varphi \mathrel{|\!\!\sim}^{\mathcal{S}}_{\mathcal{P},T} \psi :\Longleftrightarrow \langle \varphi, \psi\rangle \in \mathcal{A}_{\mathcal{P},T}(\mathcal{S})$.
$\varphi \mathrel{|\!\!\sim}_{\mathcal{P},T} \psi :\Longleftrightarrow \langle \varphi, \psi\rangle \in \mathcal{A}_{\mathcal{P},T}$.

Note that $\Lambda_{\mathcal{P},T}(\mathcal{S})$ is completely determined by $\mathcal{A}_{\mathcal{P},T}(\mathcal{S})$, since $\varphi \in \Lambda_{\mathcal{P},T}(\mathcal{S})$ if and only if $\langle \top, \varphi\rangle \in \mathcal{A}_{\mathcal{P},T}(\mathcal{S})$. We define one more set of rules, the *implications* of a theory.

Definition 3.3 $\mathfrak{I}_\Lambda := \{\langle \varphi, \psi\rangle \mid (\varphi \to \psi) \in \Lambda\}$.

It is well known that, for example, $\mathcal{A}_{\{p,q,r\},\mathsf{IPC}(p,q,r)} \neq \Im_{\mathsf{IPC}(p,q,r)}$, since, for example,

$$\neg p \to (q \vee r) \mathrel{|\!\sim}_{\{p,q,r\},\mathsf{IPC}(p,q,r)} (\neg p \to q) \vee (\neg p \to r),$$

but $\neg p \to (q \vee r) \nvdash_{\mathsf{IPC}(p,q,r)} (\neg p \to q) \vee (\neg p \to r)$.

Here is the simplest possible result on admissible rules.

Theorem 3.4 *The implications of the logic of a theory T are admissible for T. Moreover, every admissible rule of T is also admissible for the logic of T. To be precise, let $\Lambda := \Lambda_{\mathcal{P},T}$. We have $\Im_\Lambda \subseteq \mathcal{A}_{\mathcal{P},T} \subseteq \mathcal{A}_{\mathcal{P},\Lambda}$.*

Proof: The first inclusion is obvious. Suppose that $\varphi \mathrel{|\!\sim}_{\mathcal{P},T} \psi$. We prove $\varphi \mathrel{|\!\sim}_{\mathcal{P},\Lambda} \psi$. Consider any $\sigma \in \mathsf{sub}_{\mathcal{P},\mathsf{IPC}(\mathcal{P})}$ and suppose $\Lambda \vdash \sigma(\varphi)$. It follows that, for all $\tau \in \mathsf{sub}_{\mathcal{P},T}$, $T \vdash (\tau \circ \sigma)(\varphi)$. Ergo, for all $\tau \in \mathsf{sub}_{\mathcal{P},T}$, $T \vdash (\tau \circ \sigma)(\psi)$. We may conclude $\Lambda \vdash \sigma(\psi)$. □

In this paper we will show that $\mathcal{A}_{\vec{p},\mathsf{HA}} = \mathcal{A}_{\vec{p},\mathsf{IPC}}$, where $\Lambda_{\vec{p},\mathsf{HA}} = \mathsf{IPC}(\vec{p})$. This shows that the 'upperbound' in Theorem 3.4 can be assumed. In [5], de Jongh and Visser show that $\mathcal{A}_{\vec{p},\mathsf{HA}^*} = \Im_{\mathsf{IPC}(\vec{p})}$, where $\Lambda_{\vec{p},\mathsf{HA}^*} = \mathsf{IPC}(\vec{p})$. Here HA^* is the theory $\mathsf{HA} + \{\psi \to \Box_{\mathsf{HA}^*}\psi \mid \psi \in \mathcal{R}\}$, the minimal extension of HA that believes that what is true is provable in *it*. This theory is studied in [33], [34], [37] (a rewrite of [34]), and [5]. The result shows that the "lowerbound" in Theorem 3.4 can be assumed.

We end this section by a brief survey of some theorems about admissible rules.

Theorem 3.5 (Rybakov) *The admissible rules of IPC are decidable. In other words, for any $\vec{p}$, $\vec{q}$, $\mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{q})}$ is decidable.*

For the proof we refer the reader to Rybakov [24].

Theorem 3.6 (Ghilardi) *The embedding $\vdash_{\mathsf{IPC}(\vec{p})} \hookrightarrow \mathrel{|\!\sim}_{\vec{p},\mathsf{IPC}(\vec{p})}$ has a left adjoint, say $(.)^\#$. So, $\varphi^\# \vdash_{\mathsf{IPC}(\vec{p})} \psi \iff \varphi \mathrel{|\!\sim}_{\vec{p},\mathsf{IPC}(\vec{p})} \psi$. $(A)^\#$ is the 'projective approximation' of A. $(.)^\#$ is a disjunction of $\mathcal{L}_{\mathsf{IPC},\vec{p}}$-formulas with the extension property. $(.)^\#$ is computable.*

Note that $(.)^\#$ considered as an operation of the free Heyting algebra on generators $\vec{p}$ is an interior operation. This operation is fully determined by its fixed points. These fixed points are precisely given by the disjunctions of formulas with the extension property. Ghilardi's Theorem provides a new proof of Rybakov's Theorem 3.5. For the proof the reader is referred to Ghilardi [8].

In the next few theorems, we present some results on Σ_1-substitutions over HA. Why are Σ_1-substitutions interesting? One motivation is the fact that they play an important role in the study of the provability logic of HA. The characterization of the closed fragment of HA in [34] and [37] essentially uses the results described below. The material has some clear analogies to the results described above on substitutions over IPC.

A $\mathsf{NNIL}(\mathcal{P})$-formula is a $\mathcal{P}$-formula with no nestings of implications to the left. We take $\neg p$ to be an abbreviation of $(p \to \bot)$. So $(p \to (q \vee \neg q))$ and $\neg p$ are NNIL-formulas, and $((p \to q) \to q)$ is not a NNIL-formula.

Theorem 3.7 (van Benthem, Visser) *The $\mathsf{NNIL}(\mathcal{P})$-formulas are precisely the $\mathcal{L}(\mathcal{P})$-formulas A-preserved under taking sub-Kripke models (modulo provable* *equivalence). Here a submodel is a full submodel given by an arbitrary subset of the nodes.*

For the proof we refer the reader to [34], [37], and Visser et al. [40]. Note that the theorem makes the NNIL-formulas the analogue of universal formulas in ordinary model theory. For more on the analogy, see [40].

Theorem 3.8 (Visser) *Let $\sigma \in \mathsf{sub}_{\vec{p},\Sigma_1}$. Then, $\Lambda_{\vec{p},\mathsf{HA}}(\sigma)$ is finitely axiomatizable, say by ν_σ. The ν_σ, for $\sigma \in \mathsf{sub}_{\vec{p},\Sigma_1}$, are precisely the $\mathsf{NNIL}(\vec{p})$-formulas with the disjunction property.*

So 'NNIL-formula with the disjunction property' is analogous to 'formula with the extension property' in the case of substitutions over IPC. For the proof see [34] and [37].

Theorem 3.9 (Visser) *The embedding*

$$\vdash_{\mathsf{IPC}(\vec{p})} \hookrightarrow \;\; \mathrel{|\!\!\sim}^{\Sigma_1}_{\vec{p},\mathsf{HA}}$$

has a left adjoint, say $(.)^*$. So, $\varphi^* \vdash_{\mathsf{IPC}(\vec{p})} \psi \iff \varphi \mathrel{|\!\!\sim}^{\Sigma_1}_{\vec{p},\mathsf{HA}} \psi$. $(.)^*$ is a $\mathsf{NNIL}(\vec{p})$-formula. $(.)^*$ is computable.

$(.)^*$ is completely determined by its fixed points, which are precisely given by the NNIL-formulas. For the proof see [34] and [37].

Finally, we remark that the theorem on HA^*, saying that the admissible rules for arbitrary arithmetical substitutions over HA^* are precisely the implications of IPC, also fits the pattern exhibited above; here the left adjoint simply is the identity and the formula class is $\mathcal{L}_{\mathsf{IPC},\vec{p}}$.

theory	substit.	logic	adm. rules	adjoint	form. class
$\mathsf{IPC}(\vec{p})$	$\vec{p} \to \mathcal{L}_{\mathsf{IPC},\vec{p}}$	$\mathsf{IPC}(\vec{p})$	$\mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{p})}$	$(.)^\#$	$\mathsf{D}(\mathsf{extens}_{\vec{p}})$
HA	$\vec{p} \to \mathcal{R}$	$\mathsf{IPC}(\vec{p})$	$\mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{p})}$	$(.)^\#$	$\mathsf{D}(\mathsf{extens}_{\vec{p}})$
HA	$\vec{p} \to \Sigma_1$	$\mathsf{IPC}(\vec{p})$	$\mathcal{A}_{\vec{p},\mathsf{HA}}(\Sigma_1)$	$(.)^*$	$\mathsf{NNIL}(\vec{p})$
$\mathsf{HA+ECT}_0$	$\vec{p} \to \mathcal{R}$	$\mathsf{IPC}(\vec{p})$	?	?	?
HA+MP	$\vec{p} \to \mathcal{R}$	$\mathsf{IPC}(\vec{p})$	?	?	?
$\mathsf{HA+MP+ECT}_0$	$\vec{p} \to \mathcal{R}$	?	?	?	?
HA^*	$\vec{p} \to \mathcal{R}$	$\mathsf{IPC}(\vec{p})$	$\mathfrak{I}_{\mathsf{IPC}(\vec{p})}$	$\mathsf{id}_{\mathsf{IPC}(\vec{p})}$	$\mathcal{L}_{\mathsf{IPC},\vec{p}}$
PA	$\vec{p} \to \mathcal{R}$	$\mathsf{CPC}(\vec{p})$	$\mathfrak{I}_{\mathsf{CPC}(\vec{p})}$	$\mathsf{id}_{\mathsf{CPC}(\vec{p})}$	$\mathcal{L}_{\mathsf{IPC},\vec{p}}$

3.2 Infinitary admissible rules In this subsection, we give an example to the effect that the *infinitary* propositional admissible rules of IPC and HA differ.[5] Let T be a theory and let $\mathcal{S} \subseteq \mathsf{sub}_{\mathcal{P},T}$. A $\langle \mathcal{P}, T, \mathcal{S}\rangle$-admissible infinitary rule is a pair $\langle \Gamma, \psi\rangle$, where $\Gamma \subseteq \mathcal{L}_{\mathsf{IPC},\mathcal{P}}$ and $\psi \in \mathcal{L}_{\mathsf{IPC},\mathcal{P}}$, such that, for all $\sigma \in \mathcal{S}$, $T \vdash \sigma(\Gamma) \Longrightarrow T \vdash \sigma(\psi)$. Here $\sigma(\Gamma) = \{\sigma(\gamma) \mid \gamma \in \Gamma\}$. We say that $\langle \Gamma, \psi\rangle$ is $\mathcal{P}, T$-admissible if it is $\langle \mathcal{P}, T, \mathsf{sub}_{\mathcal{P},T}\rangle$-admissible.

Definition 3.10 $\mathcal{A}^\infty_{\mathcal{P},T}(\mathcal{S})$ is the set of $\langle \mathcal{P}, T, \mathcal{S}\rangle$-admissible infinitary rules. $\mathcal{A}^\infty_{\mathcal{P},T} = \mathcal{A}^\infty_{\mathcal{P},T}(\mathsf{sub}_{\mathcal{P},T})$.

Definition 3.11 $\Gamma \mathrel{|\!\!\sim}^{\mathcal{S},\infty}_{\mathcal{P},T} \psi :\Longleftrightarrow \langle \Gamma, \psi\rangle \in \mathcal{A}^\infty_{\mathcal{P},T}(\mathcal{S})$.
$\Gamma \mathrel{|\!\!\sim}^{\infty}_{\mathcal{P},T} \psi :\Longleftrightarrow \langle \Gamma, \psi\rangle \in \mathcal{A}^\infty_{\mathcal{P},T}$.

We compare $\vdash\!\sim^{\infty}_{\mathcal{P},T}$ to validity for finite models. Suppose $\Gamma \subseteq \mathcal{L}_{\mathsf{IPC},\vec{p}}$, $\psi \in \mathcal{L}_{\mathsf{IPC},\vec{p}}$.

Definition 3.12 $\Gamma \models^{\mathsf{fin}}_{\vec{p}} \psi :\Longleftrightarrow$ for all finite Kripke $\vec{p}$-models $\mathcal{K}$, $\mathcal{K} \Vdash \Gamma \Longrightarrow \mathcal{K} \Vdash \psi$.

Theorem 3.13 $\Gamma \models^{\mathsf{fin}}_{\vec{p}} \psi \Longrightarrow \Gamma \vdash\!\sim^{\infty}_{\vec{p},\mathsf{IPC}(\vec{p})} \psi$.

Proof: Suppose $\Gamma \models^{\mathsf{fin}}_{\vec{p}} \psi$. Consider any $\sigma \in \mathsf{sub}_{\vec{p},\mathsf{IPC}(\vec{p})}$ and suppose $\mathsf{IPC}(\vec{p}) \vdash \sigma(\Gamma)$. We have to show, $\mathsf{IPC}(\vec{p}) \vdash \sigma(\psi)$. By Theorem 2.15, it is sufficient to show $\varepsilon_\sigma \vdash_{\mathsf{IPC}(\vec{p})} \psi$. Consider any finite Kripke $\vec{p}$-model $\mathcal{K}$ and suppose $\mathcal{K} \Vdash \varepsilon_\sigma$. Since, by Theorem 2.15, $\varepsilon_\sigma \vdash \Gamma$, we find $\mathcal{K} \Vdash \Gamma$. By assumption, $\mathcal{K} \Vdash \psi$. Since, for finite premise sets, we have Kripke completeness with finite models, we may conclude $\varepsilon_\sigma \vdash_{\mathsf{IPC}(\vec{p})} \psi$. Ergo, $\mathsf{IPC}(\vec{p}) \vdash \sigma(\psi)$. □

Consider two propositional variables p, q. Let

$$\Theta := \{\chi \to q \mid \chi \in \mathcal{L}_{\mathsf{IPC},p} \text{ and } \mathsf{IPC}(p) \nvdash \chi\}.$$

It is easy to see that $\Theta \models^{\mathsf{fin}}_{p,q} q$. Hence, by Theorem 3.13, $\Theta \vdash\!\sim^{\infty}_{\vec{p},\mathsf{IPC}(\vec{p})} q$. We now apply the following lemma due to de Jongh and Visser (see [5]).

Lemma 3.14 (de Jongh and Visser) *There is an arithmetical sentence $\mho$ with the following property. Suppose that $\Delta \subseteq \mathcal{L}_{\mathsf{IPC},\mathcal{P}}$ is recursively enumerable and has the disjunction property. Then there is a $\sigma \in \mathsf{sub}_{\mathcal{P},\Sigma_1}$ with $\Delta \vdash_{\mathsf{IPC}(\mathcal{P})} \varphi \Longleftrightarrow \mathsf{HA} + \mho \vdash \sigma(\varphi)$.*

Clearly, Θ is recursive. Moreover, a simple Kripke model argument shows that Θ has the disjunction property. Let σ be as given in Lemma 3.14. Since $\Theta \nvdash_{\mathsf{IPC}(p,q)} q$, we have $\mathsf{HA} + \mho \vdash \sigma(\Theta)$ and $\mathsf{HA} + \mho \nvdash \sigma(q)$. Consider τ with $\tau(p) = \sigma(p)$ and $\tau(q) := (\mho \to \sigma(q))$. By elementary propositional reasoning, we find $\mathsf{HA} \vdash \tau(\Theta)$ and $\mathsf{HA} \nvdash \tau(q)$.

3.3 Admissible rules in the predicate logical language To get our discussion off the ground, we need to fix a basic arithmetical theory. In this section, we take as our theory iEA. iEA is the constructive version of EA, elementary arithmetic also known as $I\Delta_0 + \mathsf{Exp}$. The theory consists of intuitionistic predicate logic, the usual universal axioms for successor, plus and times, Δ_0-induction, and an axiom expressing the totality of exponentiation. iEA is finitely axiomatizable. We will use 'E' to denote a single axiom axiomatizing iEA.[6]

We present some results about admissible rules for arithmetical theories. Here is an example of a principle that holds *for any RE theory T*, whether it contains any arithmetic or not.

♣ $\mathsf{E} \wedge \mathsf{Con}(T) \vdash\!\sim^{\mathsf{rel}}_{\mathcal{R}} \bot$.

In fact, ♣ is just a reformulation of the *Second Incompleteness Theorem*.

We show that for a wide class of constructive theories T with a modicum of arithmetic we have that, for a suitable $\mathcal{L}$, $\mathcal{A}_{\mathcal{L},T}$ is complete Π_2. Consider an RE theory T. Suppose $\mathcal{N}$ is a relative interpretation of iEA in T with domain ν. We remind the reader of Friedman's amazing theorem that the disjunction property implies the existence property.

Theorem 3.15 (Friedman) *Let T be as described above. Suppose T has the disjunction property. Then, T has the $\mathcal{N}$-numerical existence property, that is, for any $\varphi(x)$, with only x free, if $T \vdash \exists x(\nu(x) \wedge \varphi(x))$, then, for some natural number n, $T \vdash \exists x(\,\nu(x) \wedge \mathcal{N}(x = \underline{n}) \wedge \varphi(x)\,)$. (We could also write $T \vdash \varphi(\underline{n})$, as long as we keep in mind that we are dealing with the numeral according to $\mathcal{N}$.)*

Proof: The proof is, word for word, Friedman's original proof, just keeping an eye open to see whether everything can be done using just iEA. Note that we need things like the provable decidability of the proof-predicate and the presence of the Σ_1-minimum principle. □

The business of the interpretation helps us to apply Friedman's theorem, for example, to a theory like iZF in which the numerical language is only present via interpretation. Since, the numerical existence property in its turn implies the disjunction property, Friedman's theorem tells us that the numerical existence property is 'invariant', that is, independent of the choice of the interpretation of iEA.

Here is an alternative formulation of Friedman's theorem. Let us extend $\mathcal{R}$ to a language $\mathcal{L}$ by adding a unary predicate symbol P. Let $\mathcal{N}[P := \varphi]$ extend $\mathcal{N}$ by interpreting P by φ. We demand that φ has at most x free. Suppose T has the disjunction property. Then, we have

$$T, \mathcal{N}[P := \varphi] \vdash \exists x\ Px \Longrightarrow \exists n \in \omega\ \ T, \mathcal{N}[P := \varphi] \vdash P\underline{n}.$$

We can now prove our theorem.

Theorem 3.16 *Let T be as described above. Suppose T has the disjunction property. Then, $\mathcal{A}^{\mathsf{rel}}_{\mathcal{L},T}$ is complete Π_2.*

Proof: Let e be an index of a partial recursive function. We show how to reduce the problem of the totality of $\lambda n.\{e\}n$ to $\mathcal{A}^{\mathsf{rel}}_{\mathcal{L},T}$. We claim,

$$\forall n\ \{e\}n\downarrow\ \iff\ (\,\mathsf{E} \wedge \exists x\, Px \mathrel{\vdash^{\mathsf{rel}}_{\mathcal{L},T}} \exists x\,(\,Px \wedge \exists y\ T(\underline{e}, x, y)\,)\,).$$

Here $T(u, v, w)$ stands for Kleene's T-predicate. We verify our claim.

($\Longleftarrow$) Assume the right-hand side of the claim. Consider any natural number n. Clearly,

$$T, \mathcal{N}[P := (x = \underline{n})] \vdash \mathsf{E} \wedge \exists x\, Px.$$

(Here $\underline{n}$ is the $\mathcal{N}$-numeral.) Ergo, by assumption,

$$T, \mathcal{N}[P := (x = \underline{n})] \vdash \exists x\,(\,Px \wedge \exists y\ T(\underline{e}, x, y)\,).$$

In other words, $T, \mathcal{N} \vdash \exists y\ T(\underline{e}, \underline{n}, y)$. Since T is consistent and has the numerical existence property, T satisfies Σ_1-reflection. We may conclude that $\{e\}n\downarrow$.

($\Longrightarrow$) Assume $\forall n\ \{e\}n\downarrow$. Consider any relative interpretation $\mathcal{M}$ and suppose $T, \mathcal{M} \vdash \mathsf{E} \wedge \exists x\, Px$. By Friedman's Theorem, for some n, $T, \mathcal{M} \vdash P\underline{n}$. By assumption, $\{e\}n\downarrow$. Hence, by Σ_1-completeness, $T, \mathcal{M} \vdash \exists y\ T(\underline{e}, \underline{n}, y)$. We may conclude $T, \mathcal{M} \vdash \exists x\,(\,Px \wedge \exists y\ T(\underline{e}, x, y)\,)$. □

Certainly not all arithmetical theories give rise to Π_2-complete sets of admissible rules. For example, $\mathcal{A}_{\mathcal{L},\mathsf{PA}}$ is not complete Π_2. This is immediate from the characterization of $\mathcal{A}_{\mathcal{L},\mathsf{PA}}$ given in Appendix A.

4 What extendability means for admissibility In this section we will show that the admissible rules of HA are the same as the admissible rules of IPC. This result follows from the main lemma of this section.

4.1 The main lemma

Lemma 4.1 *Suppose T has the extension property and suppose $\Lambda_{\vec{p},T} = \mathsf{IPC}(\vec{p})$. Then, the $\vec{p}$-admissible rules of T are the same as those of $\mathsf{IPC}(\vec{p})$. In other words, we have $\mathcal{A}_{\vec{p},T} = \mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{p})}$.*

We will prove the main lemma from two lemmas. These lemmas are stated and proved in the next two subsubsections.

4.1.1 The disjunction property The lemma of this subsubsection tells us that certain restrictions of sets of formulas with the disjunction property inherit the disjunction property.

Lemma 4.2 *Let $\Delta \subseteq \mathcal{L}_{\mathsf{IPC},\mathcal{P}}$ be any deductively closed propositional theory with the disjunction property. Let X be any adequate set, that is, let X be closed under subformulas. $\Delta_X := \Delta \cap X$. Then Δ_X has the disjunction property.*

Proof: Suppose $\Delta_X \vdash \varphi_1 \vee \varphi_2$ and $\Delta_X \nvdash \varphi_i$ for $i = 1, 2$. For $i = 1, 2$, let $\mathcal{K}_i$ be a $\mathcal{P}$-model such that $\mathcal{K}_i \Vdash \Delta_X$ and $\mathcal{K}_i \nVdash \varphi_i$. We can construct a model $\mathcal{K}_3$ such that $\Delta = \mathsf{Th}(\mathcal{K}_3)$. Let $\mathcal{K}$ be the disjoint union of the $\mathcal{K}_i$ for $i = 1, 2, 3$. Clearly $\mathcal{K} \Vdash \Delta_X$ and $\mathcal{K} \nVdash \varphi_i$, for $i = 1, 2$. We construct a new model $\mathcal{K}^+$ by adding a new root $\mathfrak{b}$ under the $\mathcal{K}$. Put $\mathfrak{b} \Vdash p :\Longleftrightarrow p \in \Delta_X$. We show by induction on X: $\psi \in \Delta_X \Longrightarrow \mathfrak{b} \Vdash \psi$. The cases of atoms and conjunction are trivial.

Suppose $(\nu \vee \rho) \in \Delta_X$. Then $\nu \in \Delta$ or $\rho \in \Delta$. Suppose, for example, $\nu \in \Delta$. Since $\nu \in X$, it follows that $\nu \in \Delta_X$. Hence, by the Induction Hypothesis, $\mathfrak{b} \Vdash \nu$ and thus, $\mathfrak{b} \Vdash \nu \vee \rho$.

Suppose $(\nu \to \rho) \in \Delta_X$. Consider any node k and suppose $k \Vdash \nu$. If $k \in \mathcal{K}$, we are done, since $\mathcal{K} \Vdash \nu \to \rho$. If $k = \mathfrak{b}$, we have, by persistence, $\mathcal{K} \Vdash \nu$ and hence $\mathcal{K} \Vdash \rho$. By the Induction Hypothesis $\mathfrak{b} \Vdash \rho$.

We find $\mathfrak{b} \Vdash \Delta_X$, $\mathfrak{b} \nVdash \varphi_1 \vee \varphi_2$: a contradiction. $\square$

4.1.2 e-Compactness We prove a kind of compactness result. We state the lemma in the infinitary version, where in fact we will use only the finitary one.

Definition 4.3

1. Let $\mathsf{extens}_{\mathcal{P}}$ be the set of $\mathcal{L}_{\mathsf{IPC},\mathcal{P}}$-formulas with the extension property.
2. A set of $\mathcal{L}_{\mathsf{IPC},\mathcal{P}}$-formulas Γ is *e-compact* if
   $\Gamma \vdash \varphi \implies \exists \varepsilon {\in} \mathsf{extens}_{\mathcal{P}}\ \Gamma \vdash \varepsilon$ and $\varepsilon \vdash \varphi$.
3. $I_n(\vec{q})$ is the set of $\mathcal{L}_{\mathsf{IPC},\vec{q}}$-formulas of which the nesting degree of implications is smaller than or equal to n.

Theorem 4.4 Γ *has the extension property if and only if* Γ *is e-compact.*

Proof: Suppose Γ has the extension property. We show that Γ is e-compact. Consider $\vec{q} \subseteq \mathcal{P}$. Define $\Gamma_n(\vec{q}) := \{\gamma \in I_n(\vec{q}) \mid \Gamma \vdash \gamma\}$. Clearly, $\Gamma_n(\vec{q})$ is finitely axiomatizable. Moreover, whenever $\Gamma \vdash \varphi$ there are n and $\vec{q} \subseteq \mathcal{P}$ such that $\Gamma_n(\vec{q}) \vdash \varphi$. Without loss of generality we may restrict ourselves to $n \geq 1$. So it is sufficient to show that $\Gamma_n(\vec{q})$ has the extension property, for $n \geq 1$.

Consider any nonrooted $\mathcal{P}$-model $\mathcal{K}$ with $\mathcal{K} \Vdash \Gamma_n(\vec{q})$. Let

$$\Theta := \mathsf{Th}_{\vec{q},n-1}(\mathcal{K}) := \mathsf{Th}(\mathcal{K}) \cap I_{n-1}(\vec{q}).$$

We show that, for $\psi \in I_{n-1}(\vec{q})$, $\Gamma, \Theta \vdash \psi \Longrightarrow \Theta \vdash \psi$, in other words, that $\Gamma \cup \Theta$ is $I_{n-1}(\vec{q})$-conservative over Θ.

Suppose $\Gamma, \Theta \vdash \psi$. Then, for some $\theta \in \Theta$, $\Gamma \vdash (\theta \to \psi)$. We have, clearly, that $(\theta \to \psi) \in I_n(\vec{q})$, and hence, $(\theta \to \psi) \in \Gamma_n(\vec{q})$. We find $\mathcal{K} \Vdash \theta$, since $\theta \in \Theta$, and $\mathcal{K} \Vdash (\theta \to \psi)$, since $(\theta \to \psi) \in \Gamma_n(\vec{q})$. So $\mathcal{K} \Vdash \psi$. Moreover, $\psi \in I_{n-1}(\vec{q})$, so $\psi \in \Theta$.

Consider any $\mathcal{P}$-model $\mathcal{M}$ such that $\mathsf{Th}(\mathcal{M}) = \mathsf{dc}(\Gamma \cup \Theta)$. (Here dc stands for deductive closure.) Note that

$$\mathsf{Th}_{\vec{q},n-1}(\mathcal{K}) = \Theta = \mathsf{Th}_{\vec{q},n-1}(\mathcal{M}).$$

Let $\mathcal{M}^*$ be a downward extension of $\mathcal{M}$ with a new bottom $\mathfrak{b}$ such that $\mathcal{M}^* \Vdash \Gamma$. We extend $\mathcal{K}$ to $\mathcal{K}^*$ by adding a new bottom $\mathfrak{c}$, with $\mathfrak{c} \Vdash p :\Longleftrightarrow \mathfrak{b} \Vdash p$.

It is easy to show by induction that

$$\text{for any } \psi \in I_{n-1}(\vec{q}), \quad \mathfrak{c} \Vdash \psi :\Longleftrightarrow \mathfrak{b} \Vdash \psi.$$

We show that $\mathfrak{c} \Vdash \Gamma_n(\vec{q})$. Γ has the extension property and, hence, the disjunction property. So, by Lemma 4.2, $\Gamma_n(\vec{q})$ has the disjunction property. Our proof is by induction on $\gamma \in \Gamma_n(\vec{q})$. The cases of atoms and conjunction are trivial. The case of disjunction is immediate by the fact that $\Gamma_n(\vec{q})$ has the disjunction property. Suppose that $\gamma = (\nu \to \rho) \in \Gamma_n(\vec{q})$. We want to show that $\mathfrak{c} \Vdash (\nu \to \rho)$. Clearly $\mathfrak{b} \Vdash (\nu \to \rho)$ and $\nu, \rho \in I_{n-1}(\vec{q})$. Consider any $k \geq \mathfrak{c}$ and suppose $k \Vdash \nu$. To show $k \Vdash \rho$: in case $k \neq \mathfrak{c}$, we are done by the fact that k is in $\mathcal{K}$ and $\mathcal{K} \Vdash \Gamma_n(\vec{q})$. Suppose $k = \mathfrak{c}$. Then it follows that $\mathfrak{b} \Vdash \nu$ and hence, $\mathfrak{b} \Vdash \rho$ and thus, $\mathfrak{c} \Vdash \rho$.

We prove the converse. Suppose Γ is e-compact. It is our standing assumption that $\mathcal{P}$ is countable. Say, $\mathcal{P} = \{p_1, p_2, \ldots\}$. Let $\vec{p}_i := \{p_1, \ldots, p_i\}$. Take $\gamma_0 := \top$ and let γ_{n+1} be the formula with the extension property such that

1. $\Gamma \vdash \gamma_{n+1}$,
2. $\gamma_{n+1} \vdash \gamma_n \wedge \bigwedge \Gamma_n(\vec{p}_n)$,
3. γ_{n+1} is the first in a suitable enumeration of formulas satisfying (1), (2).

We prove by induction that γ_n is defined and that $\Gamma \vdash \gamma_n$. It is immediate that Γ is axiomatized by the γ_n. Consider any $\mathcal{P}$-model $\mathcal{K}$ of Γ. For each n we can add a new root $\mathfrak{b}_n$ to $\mathcal{K}$ such that $\mathfrak{b}_n \Vdash \gamma_n$. Let Tree be the set of 0,1 sequences α such that $\alpha \in \mathsf{Tree}$ if and only if, for infinitely many n, for all $i < \mathsf{length}(\alpha)$, $\mathfrak{b}_n \Vdash p_i \Longleftrightarrow \alpha_i = 1$. It is easy to see that Tree has an infinite path π. Add a new root $\mathfrak{c}$ to $\mathcal{K}$, setting $\mathfrak{c} \Vdash p_i \Longleftrightarrow \pi_i = 1$. It is immediate that $\mathfrak{c} \Vdash \Gamma$. $\square$

4.1.3 Proof of the main lemma Suppose T has the extension property and suppose $\Lambda_{\vec{p},T} = \mathsf{IPC}(\vec{p})$. We will show that the admissible rules of T are the same as those of IPC. In other words, we have $\mathcal{A}_{\vec{p},T} = \mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{p})}$.

Proof: Theorem 3.4 tells us that $\mathcal{A}_{\vec{p},T} \subseteq \mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{p})}$. We prove the converse direction. Suppose that $\varphi \mathrel{|\!\!\sim}_{\vec{p},\mathsf{IPC}(\vec{p})} \psi$ and suppose, for $\sigma \in \mathsf{sub}_{\vec{p},T}$, that $T \vdash \sigma(\varphi)$. It follows that $\Lambda_{\vec{p},T}(\sigma) \vdash \varphi$. Since T has the extension property, we may conclude, by Theorem 2.14, that $\Lambda_{\vec{p},T}(\sigma)$ has the extension property. Hence, by Lemma 4.4, there is an $\varepsilon \in \mathsf{extens}_{\vec{p}}$ with $\Lambda_{\vec{p},T}(\sigma) \vdash \varepsilon$ and $\varepsilon \vdash \varphi$. By Theorem 2.16, we can find a $\tau \in \mathsf{sub}_{\vec{p},\mathsf{IPC}(\vec{p})}$, such that $\varepsilon = \varepsilon_\tau$. Ergo, $\mathsf{IPC}(\vec{p}) \vdash \tau(\varphi)$. By assumption, $\mathsf{IPC}(\vec{p}) \vdash \tau(\psi)$. Hence, $\varepsilon_\tau \vdash \psi$ and so, $\Lambda_{\vec{p},T}(\sigma) \vdash \psi$. We may conclude $T \vdash \sigma(\psi)$. □

Remark 4.5 There are a few alternative ways to set up the machinery leading to the main lemma. If you think about the extension property, it is easy to see that the forcing in the new bottom just depends on the atomic forcing in the new bottom and the *theory* of the original model. This shows that we can think of the extension property in a purely syntactical way. Thus we could set up things using slash theoretic methods rather than Kripke models. This alternative is very close to our present setup. My choice for Kripke models is purely a matter of taste.

A second alternative approach is just in the other direction: rather less than more syntactical. It is to use bounded bisimulations in the way Ghilardi uses them in [8]. This approach has the advantage of connecting to more theory. It is, perhaps in the end, more beautiful and, again perhaps, more open to generalization. However, it would take a bit more work to set it up.

4.2 Applications of the main lemma

4.2.1 Intended consequences It was our intention in proving the main lemma to characterize the admissible rules of HA. Here is the argument. Every Kripke Model of HA is extendible by ω preserving the validity of HA in the model. Adding root ω is called *Smoryński's operation*. So, HA has the extension property. We already know that HA satisfies de Jongh's Theorem. By the main lemma, we may conclude that the admissible rules of HA are the admissible rules of IPC.

Note that if $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathsf{sub}_{\mathcal{P},\mathcal{L}}$, then $\mathcal{A}_{\mathcal{P},T}(\mathcal{T}) \subseteq \mathcal{A}_{\mathcal{P},T}(\mathcal{S})$. Since we have de Jongh's Theorem for Σ_1-substitutions, it is clear that we should be able to restrict the substitutions leading to the characterization of the admissible rules. Inspection of the proof gives us: $\mathcal{A}_{\vec{p},\mathsf{HA}}(\mathsf{Boole}(\Sigma_1)) = \mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{p})}$.

The use of Boolean combinations is essential here. It is easy to see that $\mathcal{A}_{\vec{p},\mathsf{HA}}(\Sigma_1)$ strictly extends $\mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{p})}$, since $\neg\neg p \mathrel{|\!\!\sim}^{\Sigma_1}_{\vec{p},\mathsf{HA}} p$ (Markov's Rule), but *not* $\neg\neg p \mathrel{|\!\!\sim}_{p,\mathsf{IPC}} p$ (substitute $(p \vee \neg p)$ for p).

The same considerations show that the admissible rules of $\mathsf{HA{+}RFN(HA)}$, $\mathsf{HA{+}TI}(\prec)$, and $\mathsf{HA{+}DNS}$ are the same as those of IPC. Since $\Lambda_{\{p,q\},\mathsf{HA+MP+ECT_0}}$ strictly extends $\mathsf{IPC}(p,q)$, clearly, the admissible rules of $\mathsf{HA} + \mathsf{MP} + \mathsf{ECT_0}$ are not those of $\mathsf{IPC}(p,q)$.

Open Question 4.6

What are the admissible rules of $\mathsf{HA} + \mathsf{ECT}_0$?
What are the admissible rules of $\mathsf{HA} + \mathsf{MP}$?

4.2.2 Random applications In this subsubsection we provide a few more or less randomly chosen examples: the theory of groups and the theory of fields. For information about these theories, see, for example, van Dalen and Troelstra [29].

The basic theory of apartness APP is given by the following axioms.

ID	The usual axioms for identity
AP1	$\neg x \# y \Longleftrightarrow x = y$
AP2	$x \# y \to y \# x$
AP3	$x \# y \to (x \# z \vee y \# z)$

Note that **AP2** follows from the other axioms. If the language has an *n*-ary function symbol f, we will often insist that the corresponding function is *strictly extensional*:

SE(f) $\quad f(x_1, \ldots, x_n) \# f(x'_1, \ldots, x'_n) \to \bigvee_{i=1}^{n} x_i \# x'_i$

The constructive theory of groups with apartness $\mathsf{GROUP_{ap}}$ is formulated in the language with symbols $=, \#, \cdot, ^{-1}, e$. Its axioms are the apartness axioms plus the usual universal axioms of group theory and finally, two axioms expressing the strict extensionality of $\cdot$ and $^{-1}$.

We show that $\Lambda_{\mathcal{P},\mathsf{GROUP_{ap}}} = \mathsf{IPC}(\mathcal{P})$. Consider any formula $\varphi \in \mathcal{L}_{\mathsf{IPC},\mathcal{P}}$ such that $\mathsf{IPC}(\mathcal{P}) \nvdash \varphi$. Suppose that the propositional variables of φ are among $\vec{p} = \{p_0, \ldots, p_{n-1}\}$. Let $\mathcal{K}$ be a rooted $\vec{p}$-countermodel to φ.

We convert $\mathcal{K}$ to a model $\mathcal{K}'$ for $\mathsf{GROUP_{ap}}$. First, $\mathcal{K}'$ has the same ordering as $\mathcal{K}$. Let π be an injective mapping of $n = \{0, \ldots, n-1\}$ to the prime numbers. Let Z_i be the additive group of the integers modulo i. Z_1 is the trivial group. Define a mapping $\nu : K \times n \to \omega$ as follows.

$$\nu(k,i) := \begin{cases} \pi(i) & \text{if } k \Vdash p_i \\ 1 & \text{otherwise.} \end{cases}$$

We assign to the node k the group $\prod_{i=0}^{n-1} Z_{\nu(k,i)}$. We stipulate that in a given node two elements are apart whenever they differ. The further details are obvious.

Definition 4.7 $\quad \sigma(p_i) := \exists x\, (x^{\pi(i)} = 1 \wedge \bigwedge_{j=1}^{\pi(i)-1} x^j \# 1)$

It is easy to see that $k \Vdash_{\mathcal{K}} \psi \Longleftrightarrow k \Vdash_{\mathcal{K}'} \sigma(\psi)$. Ergo, $\mathcal{K}' \nVdash \sigma(\varphi)$.

$\mathsf{GROUP_{ap}}$ has the extension property, since we can always add the trivial group as root preserving $\mathsf{GROUP_{ap}}$. We may conclude that $\mathcal{A}_{\vec{p},\mathsf{GROUP_{ap}}} = \mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{p})}$.

The weak constructive theory of fields FIELD^- has a language with the following symbols: $=, \#, \cdot, +, -, 0, 1$. FIELD^- has as axioms the apartness axioms, the usual universal axioms of the theory of commutative rings, axioms expressing the strong extensionality of $+$ and $\cdot$, plus the following axioms.

INTEGRAL	$x \# 0 \wedge y \# 0 \Longrightarrow x \cdot y \# 0$
INVERSE	$x \# 0 \to \exists y \; x \cdot y = 1$

The full theory of fields FIELD is obtained by adding the following axiom to FIELD^-.

NONTRIV $0 \# 1$

Note that in FIELD the axiom integral becomes derivable from the other axioms. Note also that FIELD does not have the disjunction property, since, for example, $3 \# 0 \vee 3 \# 1$ is derivable. We obtain $\mathsf{FIELD}_{\mathsf{char}=0}$, the theory of fields of characteristic 0, by adding the following axioms to FIELD^-.

CHAR=0 $0 \# \underline{n+1}$, where $\underline{n+1} := \overbrace{1 + \cdots + 1}^{n+1}$

Note that in $\mathsf{FIELD}_{\mathsf{char}=0}$ we can derive of any two different elements of $\mathbb{Q}$ that they are apart. We can easily prove de Jongh's Theorem for $\mathsf{FIELD}_{\mathsf{char}=0}$. We simply proceed as in the case of groups, only now we assign to the node k the structure $\mathbb{Q}(\{\sqrt{\nu(k,i)} \mid i \in n\})$. We take $\sigma(p_i) := (\exists x\, x^2 = \underline{\pi(i)}\,)$. Note that we automatically obtain de Jongh's Theorem for FIELD^- and FIELD, too.

FIELD^- has the extension property. We can always add $\mathbb{Z}$ as root, preserving FIELD^-. Here we arrange it so that no two different elements of $\mathbb{Z}$ are apart at the root. FIELD does not have the extension property. $\mathsf{FIELD}_{\mathsf{char}=0}$, on the other hand, does have the extension property. We can always add $\mathbb{Q}$ as a root, preserving $\mathsf{FIELD}_{\mathsf{char}=0}$. Here we stipulate that whenever two rationals are different then they are apart at the root. We may conclude that $\mathcal{A}_{\vec{p},\mathsf{FIELD}^-} = \mathcal{A}_{\vec{p},\mathsf{FIELD}_{\mathsf{char}=0}} = \mathcal{A}_{\vec{p},\mathsf{IPC}(\vec{p})}$.

Open Question 4.8 Characterize $\mathcal{A}_{\vec{p},\mathsf{FIELD}}$.

Appendix

A The predicate logical admissible rules of PA In this appendix we provide characterizations for predicate logical admissibility in arithmetical theories in the style of the Orey-Hájek and the Friedman characterizations for interpretability. The appendix uses some machinery not presupposed in the rest of the paper. See, for example, [10], [35], and [36].

Let Q be Robinson's Arithmetic. We work with a slightly stronger theory Q^+ in which the methodology of definable cuts works smoothly. Q^+ is Q plus the axioms expressing that the usual ordering on the natural numbers is a linear ordering. It is well known that Q interprets Q^+. We will call T *arithmetical* or *an arithmetic* if Q^+ is interpretable in T.[7] We fix some notations and introduce some conventions.

Notation A.1

1. $\mathcal{R}$ is the arithmetical language, with $0, S, +, \cdot$.
2. We write Δ for the formalization of cut-free or tableaux provability. See, for example, Wilkie and Paris [41] for a description of tableaux provability. We write ∇ for cut-free or tableaux consistency, so ∇ is $\neg\Delta\neg$.
3. $\Box$ stands for ordinary provability and $\Diamond := \neg\Box\neg$, in other words, $\Diamond$ means ordinary consistency.
4. $\Box_n$ stands for provability with a proof in which all (nonlogical) axioms used have Gödel numbers smaller than n and in which only formulas occur of complexity smaller than n. $\Diamond_n := \neg\Box_n\neg$.
5. Unless in those cases where it is stipulated otherwise, our theories are RE.

In our first few theorems, we connect admissibility for substitutions in an arbitrary language with admissibility for substitutions in the arithmetical language.

Theorem A.2 *Let T be any classical theory. T could be complete Π^1_1, T could have just a propositional language with only 0-ary predicate symbols, or T could be even inconsistent! We have*

$$\varphi \mathrel{|\!\sim}^{\mathsf{rel}}_{\mathcal{L},T} \psi \implies (\mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi) \mathrel{|\!\sim}^{\mathsf{rel}}_{\mathcal{A},T} \Delta_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi).$$

Proof: Assume the antecedent of the theorem. Consider any $\mathcal{N}$ in $\mathsf{relint}_{\mathcal{L},T}$ and suppose $T, \mathcal{N} \vdash \mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi$. In the theory $\mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi$, we can construct an interpretation $\mathcal{K}$, such that

1. $\mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi, \mathcal{K} \vdash \varphi$,
2. $\mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}(\varphi \wedge \neg\psi), \mathcal{K} \vdash \varphi \wedge \neg\psi$.

One uses the formalized model-construction for tableaux in combination with the methodology of shortening cuts, developed by Solovay, Pudlák, and Wilkie and Paris. A detailed verification of the construction can be found in Kalsbeek [11].[8] One uses the definable cuts to compensate for the lack of induction. The disjunctive effect can be obtained, for example, by constructing two interpretations $\mathcal{K}_1$ and $\mathcal{K}_2$, corresponding to (1) respective to (2) first, and taking, for example,

$$\mathcal{K}(P) := (\,(\mathcal{K}_1(P) \wedge \Delta_{\mathsf{CQC}(\mathcal{L})}\neg(\varphi \wedge \neg\psi)) \vee (\mathcal{K}_2(P) \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}(\varphi \wedge \neg\psi))\,).$$

Taking $\mathcal{M} := \mathcal{N} \circ \mathcal{K}$, we find $T, \mathcal{M} \vdash \varphi$. Hence, by assumption, $T, \mathcal{M} \vdash \psi$. But then, $T, \mathcal{N} \vdash \neg(\mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}(\varphi \wedge \neg\psi))$. Since, by assumption, $T, \mathcal{N} \vdash \mathsf{Q}^+$, we may conclude: $T, \mathcal{N} \vdash \Delta_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi)$.[9] □

A theory T is *sequential* if there is an interpretation $\mathcal{N}$ of Q^+ for which we have a good theory sequences of all objects of the theory in T and where we can find elements of the sequences by projecting using the $\mathcal{N}$-numbers (see [10]). The relevant feature of sequential theories here is the possibility of constructing partial truth-predicates in such theories. This allows us to prove things like cut-free consistency of finite subtheory on a definable cut.

Theorem A.3 *Let T be a classical, sequential theory. Then,*

$$(\mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi) \mathrel{|\!\sim}^{\mathsf{rel}}_{\mathcal{A},T} \Delta_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi) \implies \varphi \mathrel{|\!\sim}^{\mathsf{rel}}_{\mathcal{L},T} \psi.$$

Proof: Suppose that $\mathcal{N}$ provides the numbers involved in the sequentiality of T. Assume the antecedent of the theorem. Suppose that $T, \mathcal{K} \vdash \varphi$, for some $\mathcal{K} \in \mathsf{relint}_{\mathcal{L},T}$. Since our theory is sequential, we can produce a definable $\mathcal{N}$-cut, I, such that

1. $T \vdash \mathcal{N}(\,I(\Delta_{\mathsf{CPC}(\mathcal{L})}\neg\varphi)\,) \to \mathcal{K}(\neg\varphi)$,
2. $T \vdash \mathcal{N}(\,I(\Delta_{\mathsf{CPC}(\mathcal{L})}(\varphi \to \psi))\,) \to \mathcal{K}(\varphi \to \psi)$.

The proof of this fact employs the construction of a partial truth predicate and a variant of the standard proof of the reflection principle, using the transition to a definable cut to compensate for the lack of induction. (See, for details, [10].) Let $\mathcal{M} := \mathcal{N} \circ I$. We have $T, \mathcal{M} \vdash \mathsf{Q}^+$. From $T, \mathcal{K} \vdash \varphi$ and (1), we find that $T, \mathcal{M} \vdash \mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi$. Ergo, by assumption, $T, \mathcal{M} \vdash \Delta_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi)$. By (2) and the definition of $\mathcal{M}$, we get $T, \mathcal{K} \vdash \varphi \to \psi$. We may conclude that $T, \mathcal{K} \vdash \psi$. □

Combining Theorems A.2 and A.3, we state the following.

Theorem A.4 *Let T be a classical, sequential theory. Then,*

$$\varphi \mathrel{|\!\!\sim}^{\mathsf{rel}}_{\mathcal{L},T} \psi \iff (\mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi) \mathrel{|\!\!\sim}^{\mathsf{rel}}_{\mathcal{R},T} \Delta_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi).$$

Let $\mathsf{arith}_T := \{\mathcal{M} \in \mathsf{relint}_{\mathcal{R}} \mid T, \mathcal{M} \vdash \mathsf{Q}^+\}$. The characterization of the predicate logic of T provided by Theorem A.4 is as follows.

Corollary A.5 *Let T be a classical, sequential theory. We have*

1. $\psi \in \Lambda_{\mathcal{L},T} \iff \mathsf{Q}^+ \mathrel{|\!\!\sim}^{\mathsf{rel}}_{\mathcal{R},T} \Delta_{\mathsf{IPC}(\mathcal{L})}\psi \iff \Lambda_{\mathcal{R},T}(\mathsf{arith}_T) \vdash \Delta_{\mathsf{IPC}(\mathcal{L})}\psi$.
2. $T \rhd_{\mathcal{L}} \varphi \iff T \rhd_{\mathcal{R}} (\mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi)$.

Proof: (1) is immediate. We prove (2). In case T is inconsistent, we are immediately done. Suppose T is consistent. Then $T \rhd_{\mathcal{L}} \varphi \iff \neg(\varphi \mathrel{|\!\!\sim}^{\mathsf{rel}}_{\mathcal{L},T} \bot)$. The desired result is now immediate. □

A theory T is *weakly* Σ_1*-sound*, if, for all Σ_1-sentences σ, $\mathsf{Q}^+ \mathrel{|\!\!\sim}^{\mathsf{rel}}_{\mathcal{R},T} \sigma \Longrightarrow \mathbb{N} \models \sigma$, in other words, of $\Lambda_{\mathcal{R},T}(\mathsf{arith}_T) \cap \mathsf{sent}_{\Sigma_1} \subseteq \mathsf{Th}(\mathbb{N})$. Note that a weakly Σ_1-sound theory is automatically an arithmetic. A theory T is Σ_1, $\mathcal{N}$*-sound*, for $\mathcal{N} \in \mathsf{relint}_{\mathcal{R},T}$, if $T, \mathcal{N} \vdash Q^+$ and, for all $\sigma \in \mathsf{sent}_{\Sigma_1}$, $T, \mathcal{N} \vdash \sigma \Longrightarrow \mathbb{N} \models \sigma$. Finally, T is *strongly* Σ_1*-sound*, if T is Σ_1, $\mathcal{N}$-sound, for some $\mathcal{N}$.

Theorem A.6 *Let T be a sequential theory that is weakly Σ_1-sound. Then,*

$$\varphi \mathrel{|\!\!\sim}^{\mathsf{rel}}_{\mathcal{L},T} \psi \iff (T \rhd_{\mathcal{L}} \varphi \Longrightarrow \mathsf{CQC}(\mathcal{L}) \vdash \varphi \to \psi)$$

Proof: Suppose (a) $\varphi \mathrel{|\!\!\sim}^{\mathsf{rel}}_{\mathcal{L},T} \psi$ and $T \rhd_{\mathcal{L}} \varphi$. By Corollary A.5, we can find an $\mathcal{N} \in \mathsf{relint}_{\mathcal{R},T}$, such that (b) $T, \mathcal{N} \vdash \mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi$. By weak Σ_1-soundness, it is sufficient to show that, for every $\mathcal{M} \in \mathsf{arith}_T$, $T, \mathcal{M} \vdash \Delta_{\mathcal{L},T}(\varphi \to \psi)$. Consider any $\mathcal{M} \in \mathsf{arith}_T$. We can find definable cuts I and $\mathcal{J}$ of respective $\mathcal{N}$ and $\mathcal{M}$ that are T-provably isomorphic. (See Pudlák [22] or [10] or [36].) By downward persistence of Π_1-sentences, we find $T, \mathcal{N} \circ I \vdash \mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}\mathcal{L}}\varphi$. By isomorphism, we obtain $T, \mathcal{M} \circ \mathcal{J} \vdash \mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}\mathcal{L}}\varphi$. Applying Theorem A.4 to (a) and (b), we get $T, \mathcal{M} \circ \mathcal{J} \vdash \Delta_{\mathcal{L},T}(\varphi \to \psi)$. By upward persistence of Σ_1-sentences, we find $T, \mathcal{M} \vdash \Delta_{\mathcal{L},T}(\varphi \to \psi)$. The converse is trivial. □

An alternative formulation of our theorem is $\mathcal{A}_{\mathcal{L},T} = (\mathsf{INT}^c_{\mathcal{L},T} \times \mathsf{sent}_{\mathcal{L}}) \cup I_{\mathsf{CQC}(\mathcal{L})}$. Here $\mathsf{INT}^c_{\mathcal{L},T} = \mathsf{sent}_{\mathcal{L}} \setminus \mathsf{INT}_{\mathcal{L},T}$.

Corollary A.7 *Let T be a sequential theory that is weakly Σ_1-sound. Then,* $\Lambda_{\mathcal{L},T} = \mathsf{CQC}(\mathcal{L})$.

Proof: Obvious. □

Corollary 5.3 of [36] tells us that a consistent finitely axiomatized sequential theory T is weakly Σ_1-sound. From Theorem 5.9 of [36], we can even show that such a T is strongly Σ_1-sound. The result is somewhat delicate in that the theorem may be verifiable in T itself: for some $\mathcal{M}, \mathcal{N} \in \mathsf{arith}_T$, and for all $\sigma \in \mathsf{sent}_{\Sigma_1}$,

$$T, \mathcal{M} \vdash (\Diamond_T \top \wedge \Box_T \mathcal{N}(\sigma)) \to \sigma.$$

Familiar Gödelean results do not yield a contradiction, but only the observation that $\mathcal{M}$ and $\mathcal{N}$ cannot be the same. The above results lead us immediately to the following corollary.

Corollary A.8 *Every consistent finitely axiomatized sequential theory T satisfies*

$$\varphi \mathrel{\vdash^{\mathsf{rel}}_{\mathcal{L},T}} \psi \iff (T \rhd_{\mathcal{L}} \varphi \Longrightarrow \mathsf{CQC}(\mathcal{L}) \vdash \varphi \to \psi).$$

Proof: If T is inconsistent, we are immediately done. If T is consistent, we may apply the results quoted above. □

Note that for the case of finitely axiomatized theories $(\mathsf{INT}^{c}_{\mathcal{L},T} \times \mathsf{sent}_{\mathcal{L}}) \cup I_{\mathsf{CQC}(\mathcal{L})}$ becomes the union of a Π_1-set with a Σ_1-set. Examples of theories to which the corollary may be applied are GB, ACA_0, $I\Sigma_n$, $I\Delta_0 + \mathsf{Exp}$, Q^+. To each of these theories we may add finitely many axioms without invalidating the result—as long as we preserve consistency.

The situation for theories satisfying full induction is rather different.

Theorem A.9 *Suppose there is an $\mathcal{N}$ with domain ν, such that $T, \mathcal{N} \vdash \mathsf{Q}^+$ and such that T proves full induction with respect to the whole language for the $\mathcal{N}$-numbers. So T proves*

$$[\exists x\, (\nu(x) \wedge \mathcal{N}(x = 0) \wedge \varphi(x)) \wedge$$
$$\forall x\, ((\nu(x) \wedge \varphi(x)) \to \exists y\, (\nu(y) \wedge \mathcal{N}(Sx = y) \wedge \varphi(y)))] \to \forall x\, (\nu(x) \to \varphi(x)).$$

Then we have

$$\varphi \mathrel{\vdash^{\mathsf{rel}}_{\mathcal{L},T}} \psi \iff (T, \mathcal{N} \vdash \Diamond_{\mathsf{CQC}(\mathcal{L})}\varphi \Longrightarrow T, \mathcal{N} \vdash \Box_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi)).$$

Proof: ($\Longrightarrow$) Assume the left-hand side of the theorem. Suppose $T, \mathcal{N} \vdash \Diamond_{\mathsf{CQC}(\mathcal{L})}\varphi$. If we have full induction, we can prove Supexp, the axiom stating the that the superexponentiation function is total. If we have superexponentiation, we can prove cut-elimination. Hence, $\Diamond$ will be provably equivalent to ∇. We may apply Theorem A.4 to obtain $T, \mathcal{N} \vdash \Delta_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi)$. Hence, a fortiori, $T, \mathcal{N} \vdash \Box_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi)$.

($\Longleftarrow$) Assume the right-hand side of the theorem. Consider any arithmetical interpretation $\mathcal{K}$. Suppose $T, \mathcal{K} \vdash \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi$. Since $\mathcal{N}$ satisfies full induction, the $\mathcal{N}$-numbers will be verifiably an initial segment of the $\mathcal{K}$-numbers. By downward persistence of Π_1-sentences, it follows that $T, \mathcal{N} \vdash \nabla_{\mathsf{CQC}(\mathcal{L})}\varphi$. Hence, $T, \mathcal{N} \vdash \Diamond_{\mathsf{CQC}(\mathcal{L})}\varphi$. By assumption, we get $T, \mathcal{N} \vdash \Box_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi)$. Hence, $T, \mathcal{N} \vdash \Delta_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi)$. By upward persistence of Σ_1-sentences, we find $T, \mathcal{K} \vdash \Delta_{\mathsf{CQC}(\mathcal{L})}(\varphi \to \psi)$. We may apply Theorem A.4, to obtain the desired conclusion. □

Note that the present theorem makes $\mathcal{A}_{\mathcal{L},T}$ a union of a Π_1-set and a Σ_1-set. Examples of theories to which the theorem can be applied are PA and ZF. Note that we cannot drop the '$T \vdash \ldots$' in the conclusion, since, for example, $\mathsf{Q}^+ \mathrel{\vdash^{\mathsf{rel}}_{\mathsf{PA}+\Box_{\mathsf{PA}}\bot}} \Box_{\mathsf{PA}}\bot$ and even $\Lambda_{\mathcal{R},\mathsf{PA}+\Box_{\mathsf{PA}}\bot} \vdash \mathsf{Q}^+ \to \Box_{\mathsf{PA}}\bot$.

Corollary A.10 *Suppose there is an $\mathcal{N}$ with domain ν, such that $T, \mathcal{N} \vdash \mathsf{Q}^+$ and such that T proves full induction with respect to the whole language for the $\mathcal{N}$-numbers. Then*

1. $\psi \in \Lambda_{\mathcal{L},T} \iff T, \mathcal{N} \vdash \Box_{\mathsf{CQC}(\mathcal{L})}\psi$,
2. $T \rhd_{\mathcal{L}} \varphi \iff T, \mathcal{N} \vdash \Diamond_{\mathcal{L},\mathsf{CQC}(\mathcal{L})}\varphi$.

We can strengthen the above result by considering reflexive and (locally) essentially reflexive theories. Consider a theory T and an $\mathcal{N} \in \mathsf{arith}_T$.

Definition A.11

1. T is $\mathcal{N}$-reflexive, if for all $n \in \omega$, $T, \mathcal{N} \vdash \Diamond_{T,n}\top$.
2. T is locally essentially $\mathcal{N}$-reflexive, if for all $n \in \omega$ and for all $\varphi \in \mathsf{sent}_{\mathcal{L}_T}$, $T \vdash \mathcal{N}(\Box_{T,n}\varphi) \to \varphi$.

Theorem A.12 *Suppose T is locally essentially $\mathcal{N}$-reflexive. We have*

$$\varphi \vdash^{\mathsf{rel}}_{\mathcal{L},T} \psi \iff \exists n{\in}\omega\, (\, T, \mathcal{N} \vdash \Diamond_{\mathsf{CQC}(\mathcal{L}),n}\varphi \Longrightarrow T, \mathcal{N} \vdash \Box_{\mathsf{CQC}(\mathcal{L}),n}(\varphi \to \psi)\,).$$

Proof: ($\Longrightarrow$) This part of the proof is fully analogous to the proof of Theorem A.2.

($\Longleftarrow$) Assume the right-hand side of the theorem. Let n be the promised number. Suppose that $\mathcal{K}$ is a relative $\mathcal{L}, T$-interpretation such that $T, \mathcal{K} \vdash \varphi$. It follows that, for some k, we have $T, \mathcal{N} \vdash \Box_{T,k}\mathcal{K}(\varphi)$.

Reason in $T, \mathcal{N}$. Suppose $\Box_{\mathsf{CQC}(\mathcal{L}),n}\neg\varphi$. Then certainly for an appropriate standard number m, $\Box_{T,m}\mathcal{K}(\neg\varphi)$. Taking $q := \max(k, m)$, we find $\Box_{T,q}\bot$. Quod non, by $\mathcal{N}$-reflexivity. We may conclude: $\Diamond_{\mathsf{CQC}(\mathcal{L}),n}\varphi$.

Leaving $T, \mathcal{N}$, we see that $T, \mathcal{N} \vdash \Diamond_{\mathsf{CQC}(\mathcal{L})}\varphi$. By our assumption, we find $T, \mathcal{N} \vdash \Box_{\mathsf{CQC}(\mathcal{L}),n}(\varphi \to \psi)$. Hence, for some r, we have $T, \mathcal{N} \vdash \Box_{T,r}\mathcal{K}(\varphi \to \psi)$. Combining this with $T, \mathcal{N} \vdash \Box_{T,k}(\mathcal{K}(\varphi))$, we find that $T, \mathcal{N} \vdash \Box_{T,s}\mathcal{K}(\psi)$, where $s = \max(n, r)$. By reflection, we obtain $T \vdash \mathcal{K}(\psi)$. □

Theorem A.12 substantially extends Theorem A.9, since *local* essential reflectiveness is much weaker than full induction. Our theorems still give no information about Primitive Recursive Arithmetic, PRA. PRA is reflexive and Σ_1-sound with respect to the identity interpretation. The following theorem does the trick.

Theorem A.13 *Suppose T is $\mathcal{N}$-reflexive and Σ_1, $\mathcal{N}$-sound. Then we have*

$$\varphi \vdash^{\mathsf{rel}}_{\mathcal{L},T} \psi \;:\iff\; \exists n{\in}\omega\, (\, T, \mathcal{N} \vdash \Diamond_{\mathsf{CQC}(\mathcal{L})}\varphi \Longrightarrow \mathsf{CQC}(\mathcal{L}) \vdash \varphi \to \psi\,).$$

Proof: The proof is a trivial variation of the proof of Theorem A.12. □

Acknowledgments The present paper is one spinoff of a project on substitutions in logic in which the author closely collaborates with Dick de Jongh. I thank Dick for his stimulation, for his discussions and comments, and for being a wonderful colleague. I thank Rosalie Iemhoff for her careful reading of the manuscript. The following people contributed at various stages of the manuscript: Lev Beklemishev, Dirk van Dalen, Jaap van Oosten, and Valentin Shehtman. Thank you all!

NOTES

1. I feel that this usage of *logic* is slightly perverse. The correct notion of logic should obviously explicitly contain the machinery for obtaining theorems. The current usage should be viewed as a convenient way of speaking *in the present context*.

2. In Appendix A we will prove a result that immediately implies this fact.
3. The argument is inspired by Gödel's observation that the completeness theorem for ordinary models of predicate logic constructively implies Markov's Principle.
4. Both qua content and qua methodology Plisko's result is similar to Vardanyan's result that the predicate provability logic of PA is complete Π^0_2. See [1] for an exposition and further references.
5. The example is an adaptation of Example 2.2 of [5].
6. I am sure that we can do better and work with a suitably large finite fragment of $iI\Delta_0 + \Omega_1$, the constructive version of Wilkie and Paris's $I\Delta_0 + \Omega_1$ [41].
7. Alternatively, we could demand that an appropriate weak set theory is interpretable. See Montagna and Mancini [14].
8. Another way to obtain the same result is as follows. First we prove that, for a suitable definable cut I, $\mathsf{Q}^+ \wedge \nabla_{\mathsf{CQC}(\mathcal{L})}\chi, I \vdash \Diamond_{\mathsf{CQC}(\mathcal{L}),n}\chi$. This uses the fact that cut-elimination for an n-proof is only multi-exponential. Then we construct a relative interpretation O, such that $\mathsf{Q}^+ \wedge \Diamond_{\mathsf{CQC}(\mathcal{L}),n}\chi, O \vdash \chi$. We obtain this O by the ordinary formalized Henkin construction applied to formulas of complexity below n. See [35]. Take $\mathcal{K} := I \circ O$.
9. We work with a version of tableaux provability in which the transformation from a tableaux proof of $\neg(\varphi \wedge \neg\psi)$ to a tableaux proof of $(\varphi \rightarrow \psi)$ is easy, perhaps even simply definitional.

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