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FREGE MEETS DEDEKIND: A NEOLOGICIST TREATMENT OF REAL ANALYSIS

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Abstract This paper uses neo-Fregean-style abstraction principles to develop the integers from the natural numbers (assuming Hume's Principle), the rational numbers from the integers, and the real numbers from the rationals. The first two are first-order abstractions that treat pairs of numbers:

(DIF) $\mathrm{INT}(a, b) = \mathrm{INT}(c, d) \equiv (a + d) = (b + c)$.

(QUOT) $\mathrm{Q}(m, n) = \mathrm{Q}(p, q) \equiv (n = 0 \;\&\; q = 0)$
$\vee\, (n \neq 0 \;\&\; q \neq 0 \;\&\; m \cdot q = n \cdot p)$.

The development of the real numbers is an adaption of the Dedekind program involving "cuts" of rational numbers. Let P be a property (of rational numbers) and r a rational number. Say that r is an upper bound of P, written $P \leq r$, if for any rational number s, if Ps then either $s < r$ or $s = r$. In other words, $P \leq r$ if r is greater than or equal to any rational number that P applies to. Consider the Cut Abstraction Principle:

(CP) $\forall P \forall Q(C(P) = C(Q) \equiv \forall r(P \leq r \equiv Q \leq r))$.

In other words, the cut of P is identical to the cut of Q if and only if P and Q share all of their upper bounds. The axioms of second-order real analysis can be derived from (CP), just as the axioms of second-order Peano Arithmetic can be derived from Hume's Principle. The paper raises some of the philosophical issues connected with the neo-Fregean program, using the above abstraction principles as case studies.

1. Philosophical and Technical Preliminaries

This work takes off from the ongoing neologicist development of arithmetic that began with Wright [30] and continues through many extensions, objections, and replies to objections. The basic plan is to develop branches of established mathematics using abstraction principles in the form

(ABS) $\forall a \forall b(\Sigma(a) = \Sigma(b) \equiv E(a, b))$,

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where a and b are variables of a given type (typically individual objects or properties), Σ is a higher-order operator denoting a function from items of the given type to objects in the range of the first-order variables, and E is an equivalence relation over items of the given type. In what follows, I will usually omit the initial universal quantifiers.

Frege ([14], [15]) himself employed three abstraction principles. One of them, used for illustration, comes from geometry: The direction of l_1 is identical to the direction of l_2 if and only if l_1 is parallel to l_2. Call this the *direction principle*. The second was dubbed $\mathrm{N}^=$ in [30] and is now called Hume's Principle:

$$(\mathrm{N}x : Fx = \mathrm{N}x : Gx) \equiv (F \approx G),$$

where $F \approx G$ is an abbreviation of the second-order statement that there is a one-to-one relation mapping the Fs onto the Gs. Hume's Principle states that the number of F is identical to the number of G if and only if F is equinumerous with G. Unlike the direction principle, the relevant variables F, G here are second order.

Let us call an abstraction principle *logical* if its right-hand side contains only logical terminology and operators which are themselves introduced via logical abstractions. Hume's Principle is logical and the direction principle is not. The development of real analysis below invokes only logical abstractions.

Frege's *Grundlagen* [14] contains the essentials of a derivation of the Peano postulates from Hume's Principle. This deduction, now called *Frege's Theorem*, reveals that Hume's Principle entails that there are infinitely many natural numbers. It is generally agreed that this is a powerful mathematical theorem. Who would have thought that so much could be derived from such a simple, obvious truth about cardinality?

The third example is the infamous Basic Law V:

$$(\mathrm{E}x : Fx = \mathrm{E}x : Gx) \equiv \forall x(Fx \equiv Gx).$$

Like Hume's Principle, Basic Law V is a second-order, logical abstraction, but unlike Hume's Principle, it is inconsistent.

An essential item on the neologicist agenda is to articulate principles that indicate which abstraction principles are legitimate and which are not (of which more later). For now, I simply assume that Hume's Principle is an acceptable abstraction principle yielding the natural numbers. My purpose is to present other (logical) abstraction principles that can be employed to develop a theory of the real numbers in much the same way that Hume's Principle yields a theory of the natural numbers. The crucial aspect of the treatment—where terminology for real numbers is introduced—roughly follows the development in Dedekind's celebrated *Stetigkeit und irrationale Zahlen* [6], but I formulate the relevant existence principles as Fregean abstractions rather than Dedekind-type structuralist principles.

Dedekind considered himself a logicist, giving an analysis of continuity in logical terms. He was out to refute the Kantian view that continuity is an intuitive notion underlying our perception of space and time. Dedekind argued that not only can we characterize continuity without invoking intuition, we have to. Continuity is not an intuitive notion, since intuition does not determine whether space is continuous.

Dedekind's own methodology invoked a rather different sort of abstraction. In contemporary terms, Dedekind gave a system that exemplified the target mathematical structure (arithmetic and analysis) and then abstracted the structure itself, as a "free creation" (see Shapiro [24], Chapter 5, §4, and the more scholarly sources cited there). This seems to be an instance of a traditional, perhaps Aristotelian, process where one abstracts a universal from one or more of its instances. Frege (e.g., [14], §13, p. 34, [16], p. 125) launched a sustained, bitter assault on abstraction procedures like these (see Shapiro [26], p. 67–68).

There are two different perspectives toward the neologicist quest. One is the orientation of the established mathematician, observing the neologicist project. She in interested in determining which mathematical structures have been recaptured by the neologicist and she inquires about the metatheoretic properties of the neologicist systems. Let us call this the *external perspective*. One prominent instance of this orientation is Boolos's proof that Hume's Principle is equiconsistent with second-order arithmetic as well as his result that Hume's Principle is satisfiable on every infinite domain [1]. From this perspective, the mathematician uses every tool at her disposal whether the neologicist is able to reconstruct it or not. The external goal is to see what structures the neologicist has produced. The perspective is important for making sure that the neologicist reconstructions merge smoothly with established mathematics—to avoid unnecessary charges of revisionism, for example—and for assessing the scope and limits of the neologicist program.

The other orientation tracked here is that of the neologicist herself. We focus on mathematical principles that can be stated and derived in a standard second-order logical deductive system augmented with various abstraction principles. Call this the *internal perspective*. The deductive system might be the one presented in Frege's *Begriffschrift* [11].

Concerning the philosophical agenda of neologicism, the internal perspective is certainly the more important one. At bottom, neologicism is an epistemological program. The neologicist is trying to provide what is, or what could be, an epistemic foundation of mathematics. She wants to show how the mathematician comes to know, or could come to know, propositions about abstract objects. It will not do to presuppose any established mathematics along the way, since questions would then be begged.

Frege's Theorem is a prime example of the internal perspective. Wright and Hale [34] argue that abstraction principles are (or are like) implicit definitions. One lays down truth conditions for some new vocabulary and, when successful, we introduce terms that denote abstract objects. Frege's Theorem thus shows how one can come to know the Peano axioms from an implicit definition of the number operator—assuming that Hume's Principle passes muster as an acceptable abstraction, of course.

Some of the abstraction principles employed here are not quite in the above form (ABS) since they operate with variables taken two at a time. To illustrate, consider a principle that introduces terminology for ordered pairs:

(PAIRS) $$\forall x \forall y \forall z \forall w(\pi(x, y) = \pi(z, w) \equiv E(x, y, z, w)),$$

where $E(x, y, z, w)$ is just $(x = z \mathbin{\&} y = w)$. The principle (PAIRS) differs from the form (ABS) since it has four bound variables rather than two, and the relation E on the right-hand side is not an equivalence since it is a four-place relation. Nevertheless, when the variables are taken two at a time, E has the properties corresponding to an equivalence relation. In particular, the relation is

reflexive:	$\forall x \forall y E(x, y, x, y)$ (i.e., $\forall x \forall y(x = x \mathbin{\&} y = y)$),
symmetric:	$\forall x \forall y \forall z \forall w(E(x, y, z, w) \rightarrow E(z, w, x, y))$,
transitive:	$\forall x \forall y \forall z \forall w \forall r \forall s((E(x, y, z, w) \mathbin{\&} E(z, w, r, s)) \rightarrow E(x, y, r, s))$.

Thus, (PAIRS) is the same kind of thing as an abstraction principle, with the variables taken two at a time. Hale [18] uses four-place abstractions in his own development of real analysis.

The principle (PAIRS) lies between first-order abstractions like the direction principle and second-order abstractions like Hume's Principle or Basic Law V. Like the direction principle, the bound variables in (PAIRS) are first order, but like Hume's Principle, (PAIRS) is not satisfiable on any finite domain with more than one element and so it is a principle of infinity. If the domain has size n, then we would need n^2 different ordered pairs. On an infinite domain of size κ, the satisfiability of the (PAIRS) principle is equivalent to $\kappa^2 = \kappa$ which is a consequence of the axiom of choice. So like Hume's Principle, if choice holds, then (PAIRS) is satisfiable on any infinite domain.

In what follows I do not make direct use of (PAIRS) but many of the abstraction principles invoked do operate on objects taken two at a time. As an alternative, of course, we could first invoke (PAIRS) and then employ only ordinary abstractions defined over pairs.

2. Integers and Rational Numbers

Our first chore is to define the integers via an abstraction over the differences between pairs of natural numbers:

(DIF) $\quad \mathrm{INT}(a, b) = \mathrm{INT}(c, d) \equiv (a + d) = (b + c).$

It is straightforward that the relevant analogues of reflexivity, symmetry, and transitivity hold for the equation on the right-hand side.

Next we define addition on the integers in the straightforward way:

$$\mathrm{INT}(a, b) + \mathrm{INT}(c, d) = \mathrm{INT}(a + c, b + d).$$

The definition is not circular, despite appearances. The '+' sign on the left-hand side represents addition on the integers while the '+' signs on the right-hand side represent addition on the natural numbers. It is straightforward to show (internally) that addition is well defined: if $\mathrm{INT}(a, b) = \mathrm{INT}(a', b')$ and $\mathrm{INT}(c, d) = \mathrm{INT}(c', d')$, then $\mathrm{INT}(a + c, b + d) = \mathrm{INT}(a' + c', b' + d')$. Moreover, addition on the integers is associative and commutative. There is an identity element on the integers, namely, $\mathrm{INT}(0, 0)$, and the integers are an abelian group.

We define multiplication on the integers thus:

$$\mathrm{INT}(a, b) \cdot \mathrm{INT}(c, d) = \mathrm{INT}(a \cdot c + b \cdot d, b \cdot c + a \cdot d).$$

It is tedious but straightforward to verify that this function is well defined, multiplication is associative and commutative, and multiplication distributes over addition. The integers form an integral domain.

The point here is that the relevant theorems are deductive consequences of Hume's Principle, the abstraction (DIF), and the other definitions, in a typical deductive system for second-order logic. In other words, the mathematical development can be carried out using only Fregean resources, and so it is internal to the neologicist program. The only open issue is whether the abstractions are kosher as implicit definitions.

There is, of course, a natural embedding of the natural numbers in the integers: if a is a natural number, then define $I(a)$, the *integer* a, to be $\mathrm{INT}(a, 0)$. It follows immediately from (DIF) that the embedding is one-to-one and is a homomorphism.

Can we simply *identify* the natural numbers with the corresponding integers? Can we say, for example, that the natural number 6 just *is* the integer $\mathrm{INT}(6, 0)$? Mathematicians typically talk that way, saying that the integers are an extension of the natural numbers. However, Russell claimed that we cannot make the identifications: "[The integer] $+m$ is under no circumstances capable of being identified with [the natural number] m . . . Indeed, $+m$ is every bit as distinct from m as $-m$ is" ([22], p. 64). The reason, of course, is that for Russell natural numbers and integers occur at different places in the type hierarchy. In Russell's system it is nonsense (not just false) to say that the integer $+m$ is the same as the natural number m. Statements of identity make sense only when applied to items of the same type. So despite Russell's rhetoric, he held that it is also nonsense to say that $+m$ is distinct from m.

In contrast, Frege and the neologicist take natural numbers and integers to be individual objects, all within the range of the first-order variables. In the Fregean and neologicist framework, the identity relation is unrestricted. So $6 = \mathrm{INT}(6, 0)$ is well formed, and so either $6 = \mathrm{INT}(6, 0)$ or $6 \neq \mathrm{INT}(6, 0)$. Which is it?

Students of logicism and neologicism will recognize this as an instance of the Caesar problem (see Hale and Wright [19]). The general problem is to select the criteria for determining whether the objects yielded by an abstraction principle are the same or distinct from objects not explicitly yielded by that abstraction principle. The instance here concerns the criteria for determining whether the values of one abstraction operator are the same or distinct from the values of another.

Having said that, I propose to avoid the issue here and speak of the function I as a *natural embedding*. If context makes it clear, I will sometimes use the term 'natural number' ambiguously to refer to both the natural numbers and the nonnegative integers, and I will use a numeral such as '6' to denote both the natural number 6 and the integer $\mathrm{INT}(6, 0)$.

We move on to the rational numbers. Here is another abstraction principle, giving *quotients*:

(QUOT) $$\mathrm{Q}(m, n) = \mathrm{Q}(p, q) \equiv (n = 0 \,\&\, q = 0) \vee (n \neq 0 \,\&\, q \neq 0 \,\&\, m \cdot q = n \cdot p),$$

where m, n, p, q are integers. We define a *rational number* to be a quotient $\mathrm{Q}(m, n)$, where $n \neq 0$ (i.e., $n \neq \mathrm{INT}(0, 0)$).

Once again, it is straightforward that the relevant analogues of reflexivity, symmetry, and transitivity hold for the equation on the right-hand side—due to the associative and commutative properties of multiplication on the integers. We define addition and multiplication thus:

$$\begin{aligned} \mathrm{Q}(m, n) + \mathrm{Q}(p, q) &= \mathrm{Q}(m \cdot q + p \cdot n, n \cdot q) \\ \mathrm{Q}(m, n) \cdot \mathrm{Q}(p, q) &= \mathrm{Q}(m \cdot p, n \cdot q). \end{aligned}$$

As is getting usual, it is straightforward but tedious to show that addition and multiplication are well defined on the rational numbers, that addition and multiplication are associative and commutative, and that multiplication distributes over addition. The additive identity is $\mathrm{Q}(0, 1)$ (i.e., $\mathrm{Q}(\mathrm{INT}(0, 0), \mathrm{INT}(1, 0))$) and the multiplicative identity is $\mathrm{Q}(1, 1)$. It can be established that the rational numbers are an ordered field. All of these results are deductive consequences of the various abstraction principles and the other definitions. That is, everything so far is internal.

Again, there is a natural embedding of the integers in the rational numbers. If m is an integer, then define $I(m) = \mathrm{Q}(m, 1)$. The embedding is one-to-one and preserves addition, multiplication, and the order. So the integers are isomorphic to a subset of the rational numbers. As with the natural numbers and integers, I duck this version of the Caesar problem, but I do sometimes speak ambiguously of some of the rational numbers as "integers", and I use terms like '0' and '-1' ambiguously to denote the indicated integers and rational numbers.

3. Enter Dedekind: The Real Numbers

If we start with a countable ontology and apply any abstraction principle that operates on pairs of first-order variables, then we will end up with at most countably many abstracts. So we cannot reconstruct the real numbers using the above techniques. Here we transform Dedekind's insight into a second-order abstraction principle via an equivalence relation on *properties* of rational numbers.

Let P be a property (of rational numbers) and r a rational number. Say that r is an *upper bound* of P, written $P \leq r$, if for any rational number s, if Ps then either $s < r$ or $s = r$. In other words, $P \leq r$ if r is greater than or equal to any rational number that P applies to. Consider the *Cut Abstraction principle*:

(CP) $\quad \forall P \forall Q(\mathrm{C}(P) = \mathrm{C}(Q) \equiv \forall r(P \leq r \equiv Q \leq r)).$

In words, the *cut* of P is identical to the cut of Q if and only if P and Q share all of their upper bounds.[1]

It is easy to establish that the relation on the right-hand side of (CP) is an equivalence. Notice that (CP) is (or abbreviates) a logical abstraction, in that all of the terminology on its right-hand side is either logical or an operator introduced in another logical abstraction (i.e., DIF, QUOT).

Define a property P to be *bounded* if there is a rational number r such that $P \leq r$. That is, P is bounded if there is a rational number that is greater than or equal to every number to which P applies. Define a property P to be *instantiated* if there is a rational number s such that Ps. We hereby define a *real number* to be a cut $\mathrm{C}(P)$ where P is bounded and instantiated. This, of course, is the analogue of the usual "definition" of a real number as the cut of a bounded, nonempty set of rationals.

As an aside, notice that the neologicist could also produce (a version of) real analysis by using Cauchy sequences instead of cuts: define a "sequence" to be a binary relation R between natural numbers and rational numbers such that for every natural number a there is exactly one rational number r such that Rar. Then introduce a *limit abstraction* on sequences as follows: $\mathrm{L}(R_1) = \mathrm{L}(R_2)$ if and only if for every rational number r, if $0 < r$ then there is a natural number a such that for every natural number $b > a$, and rational numbers s_1, s_2, if $R_1 bs_1$ and $R_2 bs_2$, then $-r < (s_1 - s_2) < r$. It is straightforward to define what it is for a sequence, so construed, to be *Cauchy*. Our neologicist could thus define "real number" to be "limit of a Cauchy sequence".

Returning to Dedekind, if $\mathrm{C}(P)$ and $\mathrm{C}(Q)$ are real numbers, then define $\mathrm{C}(P) < \mathrm{C}(Q)$ if $\mathrm{C}(P) \neq \mathrm{C}(\mathrm{Q})$ and for every rational number r, if $Q \leq r$ then $P \leq r$. It is straightforward to verify that this is well defined: if $\mathrm{C}(P) = \mathrm{C}(P')$, $\mathrm{C}(Q) = \mathrm{C}(Q')$, and $\mathrm{C}(P) < \mathrm{C}(Q)$ then $\mathrm{C}(P') < \mathrm{C}(Q')$. From excluded middle, we see that this relation is a linear order on the real numbers.

From an instance of the comprehension scheme, there is a property ZERO that holds of a rational number r if and only if $r < 0$. Define a real number $\mathrm{C}(P)$ to be *zero* if $\mathrm{C}(P) = \mathrm{C}(\mathrm{ZERO})$. There is only one such real number. Define a real number $\mathrm{C}(P)$ to be *positive* if there is a rational number $r > 0$ such that Pr (and so

C(ZERO) $<$ C(P)). Define a real number C(P) to be *negative* if there is a rational number r such that $r < 0$ and $P \leq r$. It is straightforward that for every real number C(P) exactly one of the following holds: C(P) is positive, C(P) is negative, or C(P) is zero.

If P and Q are properties of the rational numbers, then define $P + Q$ to be the property that holds of a rational r just in case r is less than the sum of a rational number of which P holds with a rational number of which Q holds: $(P + Q)r \equiv \exists x \exists y(Px \,\&\, Qy \,\&\, r < x + y)$. The existence of $P + Q$ follows from an instance of the comprehension scheme of the second-order language. If C(P) and C(Q) are real numbers, then so is C($P + Q$). We define addition on the real numbers:

$$\mathrm{C}(P) + \mathrm{C}(Q) = \mathrm{C}(P + Q).$$

Addition is well defined, and C(ZERO) is the additive identity.

If P is a property of rational numbers, then let $-P$ be the property that holds of a rational number r if and only if $P \leq -r$. That is $-Pr$ if and only if $-r$ is an upper bound for P. Notice that if $\mathrm{C}(P) = \mathrm{C}(P')$ then $-P$ is coextensive with $-P'$. If $P \leq r$ then $-P(-r)$. So if P is bounded then $-P$ is instantiated. Suppose that Ps. Then for any rational number r, if $-Pr$ then $P \leq -r$. So either $s = -r$ or $s < -r$. So either $r = -s$ or $r < -s$. So $-P \leq -s$. Thus, if P is instantiated, then $-P$ is bounded. Therefore, if C(P) is a real number then so is C($-P$).

We now show that if C(P) is a real number, then C($-P$) is its additive inverse. If $(P + -P)r$ then there are rationals s_1, s_2 such that Ps_1, $-Ps_2$ and $r < (s_1 + s_2)$. So $P \leq -s_2$. So either $s_1 = -s_2$ or $s_1 < -s_2$. So either $s_1 + -s_2 = 0$ or $s_1 + -s_2 < 0$. Therefore $r < 0$. For the converse, suppose that $r < 0$. Then $0 < -r$. Pick rationals s_1 and s_2 such that Ps_1, $P \leq s_2$, and $s_2 - s_1 < -r$. We have that $-P(-s_2)$ and $r < s_1 + (-s_2)$. So $(P + -P)r$. Therefore, $(P + -P)r$ if and only if $r < 0$. So C($P + -P$) is C(ZERO). Thus, the real numbers are an abelian group under addition.

The next item on the agenda is multiplication on the real numbers. I fear that matters get more tedious, mostly because we are dealing with the ordering of products of positive and negative rational numbers. Dedekind himself gave a reasonably strict account of the addition of real numbers and then added,

> Just as addition is defined, so can the other operations of the so-called elementary arithmetic be defined, viz., the formation of differences, products, quotients, powers, roots, logarithms, and in this way we arrive at real proofs of theorems . . . which to the best of my knowledge have never been established before. The excessive length that is to be feared in the definitions of the more complicated operations is partly inherent in the nature of the subject but can for the most part be avoided. ([6], §6)

However, Dedekind provided hardly any detail on how to avoid the "excessive length," beyond a few remarks about continuity.

If P and Q are properties of rational numbers, then define $P \cdot Q$ to be the property that holds of a rational number r if and only if

$$\begin{aligned}
&\exists s \exists t(Ps \,\&\, Qt \,\&\, 0 < s \,\&\, 0 < t \,\&\, r < s \cdot t)\\
&\quad \vee\ \exists s \exists t(P \leq s \,\&\, Q \leq t \,\&\, (s < 0 \vee s = 0) \,\&\, (t < 0 \vee t = 0) \,\&\, r < s \cdot t)\\
&\quad \vee\ P \leq 0 \,\&\, \exists t(Qt \,\&\, 0 < t)\\
&\qquad\qquad \&\, \forall u \forall v((P \leq u \,\&\, Qv \,\&\, (u < 0 \vee u = 0) \,\&\, 0 < v) \to r < u \cdot v))\\
&\quad \vee\ Q \leq 0 \,\&\, \exists t(Pt \,\&\, 0 < t)\\
&\qquad\qquad \&\, \forall u \forall v((Q \leq u \,\&\, Pv \,\&\, (u < 0 \vee u = 0) \,\&\, 0 < v) \to r < u \cdot v)).
\end{aligned}$$

The first disjunct is for the cases where $\mathrm{C}(P)$ and $\mathrm{C}(Q)$ are both positive; the second disjunct is for the cases where neither $\mathrm{C}(P)$ nor $\mathrm{C}(Q)$ is positive; the third disjunct is for the cases where $\mathrm{C}(P)$ is not positive and $\mathrm{C}(Q)$ is positive; and the last disjunct is for the remaining cases where $\mathrm{C}(P)$ is positive and $\mathrm{C}(Q)$ is not positive.

Using classical logic, it is tedious but straightforward to verify that multiplication is well defined and is a function on the real numbers. Define a property ONE that holds of a rational number r if and only if $r < 1$. It is straightforward to verify that for any real number $\mathrm{C}(P)$, $\mathrm{C}(P) \cdot \mathrm{C}(\mathrm{ONE}) = \mathrm{C}(\mathrm{P})$, and so C(ONE) is the multiplicative identity.

If P is a property of rational numbers, then let P^{-1} be the property that holds of a rational number r if and only if

$$\exists s \exists t \exists u(Ps \;\&\; 0 < s \;\&\; P \leq t \;\&\; t \cdot u = 1 \;\&\; r < u) \\ \vee \exists t \exists u(P \leq t \;\&\; t < 0 \;\&\; t \cdot u = 1 \;\&\; r < u).$$

Notice that if P is bounded and instantiated by a positive rational number, then (by the first disjunct), P^{-1} is instantiated. Moreover, if Ps and $0 < s$, then P^{-1} is bounded by s^{-1}. Similarly, if P is bounded with a negative rational number, then (by the second disjunct) P^{-1} is instantiated. In this case also, if Ps, then P^{-1} is bounded by s^{-1}. So if $\mathrm{C}(P)$ is a real number other than zero, then P^{-1} is instantiated and bounded, and so $\mathrm{C}(P^{-1})$ is a real number. It is straightforward to verify that if $\mathrm{C}(P)$ is a real number other than zero, then $\mathrm{C}(P) \cdot \mathrm{C}(P^{-1}) = \mathrm{C}(\mathrm{ONE})$. So the real numbers are a field. Since the positive real numbers are closed under addition and multiplication, the real numbers are an ordered field. We have all this internal to the neologicist framework.

It is straightforward to embed the rational numbers in the real numbers. If r is a rational number, then let P_r be the property that holds of a rational number q if and only if $q < r$. Clearly, P_r is instantiated and bounded. Let $I(r)$ be the corresponding cut $\mathrm{C}(P_r)$. Notice that $I(0) = \mathrm{C}(\mathrm{ZERO})$ and $I(1) = \mathrm{C}(\mathrm{ONE})$. The embedding I is one-to-one and preserves addition, multiplication, and the "less-than" relation. Therefore the rational numbers are isomorphic to a subset of the real numbers. Again, I neither assert nor deny that I is the identity mapping, but I will still speak of some of the real numbers as "rational," noting the (possible) ambiguity, and I will let a term such as '.5' ambiguously denote the indicated rational number and the corresponding real number.

The usual proof that the square root of 2 is irrational can be carried out in the indicated neologicist deductive system. Let Q be the property that holds of a rational number q just in case there is a rational number r such that $r \cdot r < 2$ and $q < r$. Then Q is instantiated and bounded, and then there is no rational number s such that $\mathrm{C}(Q) = \mathrm{C}(P_s)$.

The remaining axiom for the real numbers is the *completeness principle*, stating that if a nonempty set S (or property) of real numbers is bounded from above, then S has a least upper bound. It has a straightforward formulation as a second-order sentence:

$$\forall X\{(\exists y Xy \;\&\; \exists x \forall y(Xy \rightarrow (y < x \vee y = x))) \rightarrow \exists x[\forall y(Xy \rightarrow \\ (y < x \vee y = x)) \;\&\; \forall z(\forall y(Xy \rightarrow (y < z \vee y = z)) \rightarrow (x < z \vee x = z))]\}.$$

Here the second-order variable X ranges over all properties (or sets) of *real* numbers.

Standard reasoning establishes that the completeness principle holds for the real numbers presented here. We present this argument with a little detail in order to show that the completeness principle is derivable from the above abstraction principles and other definitions in a typical second-order deductive system. That is, the completeness principle is derivable internally. Proceeding informally, let Π be a property (or set) of real numbers and assume that Π is nonempty and bounded from above. That is, there is a real number $\mathrm{C}(A)$ such that $\Pi(\mathrm{C}(A))$ and there is a real number $\mathrm{C}(B)$ such that for any real number $\mathrm{C}(P)$, if $\Pi(\mathrm{C}(P))$ then either $\mathrm{C}(P) = \mathrm{C}(B)$ or $\mathrm{C}(P) < \mathrm{C}(B)$. We need to show that Π has a least upper bound. Define a property Q that holds of a given *rational* number r if and only if there is a real number $\mathrm{C}(P)$ such that $\Pi(\mathrm{C}(P))$ and Pr. That is, Qr holds if and only if r instantiates a real number of which Π holds.[2]

We first show that $\mathrm{C}(Q)$ is a real number (i.e., that Q is instantiated and bounded). We have that $\Pi(\mathrm{C}(A))$. Since $\mathrm{C}(A)$ is a real number, there is a rational number q such that Aq. Thus Qq and so Q is instantiated. Since $\mathrm{C}(B)$ is a real number, B is bounded. Let $B \leq s$. Suppose that Qq. Then there is a real number $\mathrm{C}(P)$ such that $\Pi(\mathrm{C}(P))$ and Pq. Since $\mathrm{C}(B)$ is an upper bound for Π, either $\mathrm{C}(P) = \mathrm{C}(B)$ or $\mathrm{C}(P) < \mathrm{C}(B)$. So we have that either $q < s$ or $q = s$. So $Q \leq s$ and Q is bounded. Thus, $\mathrm{C}(Q)$ is a real number.

Next we show that $\mathrm{C}(Q)$ is an upper bound for Π. Suppose that $\Pi(\mathrm{C}(P))$. For any rational number r, if Pr then Qr; so every bound of Q is a bound of P. Thus, either $\mathrm{C}(P) = \mathrm{C}(Q)$ or $\mathrm{C}(P) < \mathrm{C}(Q)$. Finally, we show that Q is the least upper bound for Π. So suppose that $\mathrm{C}(S)$ is an upper bound for Π. And suppose that $S \leq t$. We have to show that $Q \leq t$. Suppose that Qr. Then there is a real number $\mathrm{C}(P)$ such that $\Pi(\mathrm{C}(P))$ and Pr. Since $\mathrm{C}(S)$ is an upper bound for Π, we have that $\mathrm{C}(P)$ is less than or equal to $\mathrm{C}(S)$. So $P \leq t$, and thus r is less than or equal to t. So we have that $Q \leq t$. Therefore, either $\mathrm{C}(Q) = \mathrm{C}(S)$ or $\mathrm{C}(Q) < \mathrm{C}(S)$.

In sum, we can derive from the various abstraction principles and other definitions that the real numbers, as presented here, constitute a complete, ordered field. This completes the analogue of Frege's Theorem. Of course, unlike Frege's own derivation, there is not much originality here. Thanks to Dedekind, I knew what I was looking for. Still, we have an *internal* derivation of the axioms of real analysis, from the various abstraction principles and explicit definitions.

Recall that the axiomatization of second-order analysis is categorical. Thus, from the external perspective of the classical mathematician, the neologicist has reconstructed an instance of the familiar real-number structure. The real numbers, as presented here, are isomorphic to the continuum, as traditionally understood, and in particular, there are uncountably many real numbers.

I presume that this is welcome confirmation of the present version of the neologicist program. Nevertheless, the last bit of information, concerning the size and structure of the real numbers, comes from the "outside." It relies on the theorem of set theory that the real numbers are uncountable (i.e., Cantor's Theorem) and it relies on the well-known fact that all complete, ordered fields are isomorphic. As noted above, the neologicist is trying to capture as much of traditional mathematics as possible, from the "inside," as it were. So the neologicist does not want to rely on an external set-theoretic metatheory in order to establish the claims. But at least externally, we know that the neologicist has hit (a structure isomorphic to) the target—assuming, of course, that all of the invoked abstraction principles are acceptable.

Fortunately, the relevant cardinality claim can be established internally. Within a pure second-order language augmented with the abstraction operators, one can formulate the statement that the neologicist's real numbers are uncountable, and one can derive this statement in a typical deductive system: First, define a binary relation R to be a *real-counter* if for each natural number n there is exactly one real number $\mathrm{C}(P)$ such that $Rn\mathrm{C}(P)$. That is, R is a real-counter if it establishes a function from the natural numbers to the real numbers. The real numbers are uncountable if and only if there is no real-counter that has every real number in its "range." A diagonal argument establishes this.

Theorem 3.1 *In a standard deductive system for second-order logic, one can deduce the following from the indicated abstraction principles and explicit definitions: for every real-counter R, there is a real number $\mathrm{C}(Q)$ such that for every natural number n, it is not the case that $Rn\mathrm{C}(Q)$.*

Proof sketch: The following is a sketch of a derivation within a typical second-order deductive system. It is a reproduction of a diagonal argument. Suppose that R is a real-counter. To fix notation, for each natural number n, let $\mathrm{C}(P_n)$ be the unique real number such that $Rn\mathrm{C}(P_n)$. So the real numbers "counted" by R are $\mathrm{C}(P_0), \mathrm{C}(P_1), \ldots$. We now define a relation S between natural numbers and rational numbers. The relation S is to be a function, in the sense that for each natural number n, there is exactly one rational number r such that Snr. So we write $S(n) = r$ for Snr.

We can proceed by recursion, thanks to Dedekind's and Frege's techniques for converting definitions by recursion into explicit definitions in the second-order language (using an instance of the comprehension scheme). If $P_0 \leq 1$ then let $S(0) = 2$; otherwise let $S(0) = 0$. If $P_1 \leq (S(0) + .1)$ then let $S(1) = S(0) + .2$; otherwise let $S(1) = S(0)$. If $P_2 \leq (S(1) + .01)$ then let $S(2) = S(1) + .02$; otherwise let $S(2) = S(1)$. In general, suppose that $S(n)$ has been defined. If $P_{n+1} \leq (S(n) + 10^{-(n+1)})$ then let $S(n+1) = S(n) + 2 \cdot 10^{-(n+1)}$; otherwise let $S(n+1) = S(n)$.

Now define Q to be the property that holds of a rational number r if and only if there is a natural number n such that $S(n) = r$. Clearly Q is instantiated since either $Q0$ or $Q2$. One can show by induction that for each natural number n, $S(n) < 4 - 10^{-n}$. A fortiori, for each natural number n, $S(n) < 4$, and so Q is bounded. Thus, $\mathrm{C}(Q)$ is a real number.

All that remains is to show that $\mathrm{C}(Q)$ is not in the "range" of R. This amounts to showing that for each natural number n, $\mathrm{C}(Q) \neq \mathrm{C}(P_n)$. We proceed by induction. Recall that if $P_0 \leq 1$ then $S(0) = 2$. In this case, we have that $Q2$, and so it is not the case that $Q \leq 1$ and so $\mathrm{C}(Q) \neq \mathrm{C}(P_0)$. If it is not the case that $P_0 \leq 1$ then $S(0) = 0$. In this case, we show by induction that for each natural number n, if $1 < n$ then $S(n) < .5 - 10^{-n}$. So $Q \leq .5$ and so $Q \leq 1$. Thus, $\mathrm{C}(Q) \neq \mathrm{C}(P_0)$. The induction step is similar. If $P_{n+1} \leq (S(n) + 10^{-(n+1)})$ then $S(n+1) = S(n) + 2 \cdot 10^{-(n+1)}$. In this case we have $Q(S(n) + 2 \cdot 10^{-(n+1)})$ and so it is not the case that $Q \leq (S(n) + 10^{-(n+1)})$, and $\mathrm{C}(Q) \neq \mathrm{C}(P_{n+1})$. If it is not the case that $P_{n+1} \leq (S(n) + 10^{-(n+1)})$ then $S(n+1) = S(n)$. Then as above we show by induction that for all m, $S(m) < (S(n) + 10^{-(n+1)})$. So $Q \leq (S(n) + 10^{-(n+1)})$, and hence $\mathrm{C}(Q) \neq \mathrm{C}(P_{n+1})$. □

The above result is an internal version of Cantor's Theorem. It indicates that the cut principle (CP) has increased the size of the ontology. Starting with the countably infinite domain of rational numbers, it produces uncountably many "cuts."

If the neologicist expands to third-order language, then she can state and prove—internally—that the real numbers are "equinumerous" with the properties of natural numbers. That is, one can show that there is a "one to one" relation from the real numbers onto equivalence classes of properties of natural numbers, under coextensiveness. This corresponds to the set-theoretic theorem that there are as many real numbers as sets of natural numbers.

It is widely agreed that model theory, and metamathematics generally, are foreign to the Fregean program.[3] However, Frege was able to use his logical system to recapitulate something sufficiently resembling metatheory (see his later lectures on geometry [12], [13], translated in [16]). With characteristic rigor, Frege anticipated a technique now attributed to Ramsey [21]: one replaces an axiomatization with an *explicit definition* of a second-level concept, that is, a relation on relations. We can do the same here. In the neologicist language, we can formulate a (third-order) formula which corresponds to the statement that a given sequence of properties, objects, functions, and relations is a complete-ordered field (i.e., a model of real analysis). The neologicist can prove that the real numbers, as defined above, together with the given functions and relations, satisfy this formula. Moreover, the neologicist can prove that any two sequences that satisfy this formula are isomorphic. That is, the neologicist can show, internally, that the real numbers are a complete ordered field, and she can show that any two complete ordered fields are isomorphic. This completes the internal development.

I noted above that Boolos [1] showed that Hume's Principle is equiconsistent with second-order arithmetic. Similar techniques can be used to establish that Hume's Principle, (DIF), (QUOT), and (CP) together are equiconsistent with second-order real analysis. This theorem is "external" to the neologicist framework itself in that the result is proved in the background set theory. Unlike the above categoricity result, the requisite background model theory has not been fully recaptured in the neologicist framework. One sticking point is the Boolos result that Hume's Principle is satisfiable on any infinite domain. In its full generality, this result uses the axiom of choice (in particular, that any set is equinumerous with a cardinal in the aleph-series). It turns out that this use of choice is necessary since there are models of Zermelo-Fraenkel set theory in which Hume's Principle is not satisfiable on the continuum. However, in order to capture real analysis, the neologicist need not invoke the full power of Hume's Principle, since the only "cardinalities" used to develop real analysis are natural numbers. So for the purposes of developing real analysis, the neologicist may get by with a restricted version of Hume's Principle (see Heck [20]). The issues involving consistency and choice are rather subtle, and I do not venture a conjecture as to whether the equiconsistency can be formulated and proved internally.

4. The Acceptability of Abstractions I: Conservation

One important, outstanding philosophical issue concerns the extent to which (CP), and the other abstraction principles invoked above, are acceptable neologicist principles. The tragic example of Basic Law V reminds us that not every abstraction principle can serve as an epistemic foundation for a mathematical theory. The neologicist must articulate and defend criteria that distinguish the legitimate abstraction principles from their syntactically similar pretenders. The response to this "bad company" objection remains an ongoing project on the agenda (see, for example, Boolos [3], Wright [31], and Weir [29]). Here I test (CP), and related abstraction principles, against some of the ideas put forth in the literature, and I suggest refinements on those criteria in light of the present framework.

One glaring difference between Basic Law V and Hume's Principle is that the latter is consistent while the former is not. Consistency is surely necessary for an abstraction principle to be acceptable but it is not sufficient. Boolos [3] has pointed out there are consistent abstraction principles that have no infinite models. One such is the *Nuisance principle*, presented in [31], which is satisfiable on any finite domain but not on any infinite domain. If we assume that Hume's Principle is an acceptable abstraction, then the Nuisance principle is not acceptable (and vice versa). The Nuisance principle cannot be satisfied on any domain that includes the natural numbers. Hume's Principle cannot be satisfied on any domain that satisfies the Nuisance principle.

One natural suggestion is that a legitimate abstraction principle should be a *conservative extension* of any theory to which it is added. Formally, let A be an abstraction principle and let T a theory whose language does not contain the operator introduced by A. Then A is conservative over T if for any sentence Φ in the language of T, Φ is a consequence of $T + A$ only if Φ is a consequence of T alone. That is, the addition of A to the theory T does not have any consequences in the old language that were not already consequences of the old theory.

Suppose that A is conservative over every base theory and suppose that Φ contains no nonlogical terminology. Then Φ is a consequence of A only if Φ is logically true. Although this would be a nice feature for a view that calls itself logicist, the requirement is too strong (if neologicism is to have any chance of success). Let INF be a second-order statement, with no nonlogical terminology, that entails that the universe is Dedekind-infinite (see Shapiro [23], p. 100). Let T be any theory that does not entail that the universe is Dedekind-infinite. Then Hume's Principle entails INF but, by hypothesis, T itself does not. So Hume's Principle is not a conservative extension of any consistent theory that does not already entail the existence of infinitely many objects.

Wright ([31], pp. 230–39) points out that this violation of conservativeness is due solely to the existence of the natural numbers and has nothing to do with the items recognized to be in the ontology of the base theory T. He thus proposes a modification of the conservativeness requirement: an acceptable abstraction principle A should not have any consequences other than what follows from the existence of the abstract objects yielded by A. That is, a legitimate abstraction principle should have no new consequences concerning any objects already recognized to be in the ontology of the base theory. Of course, Basic Law V violates this conservativeness requirement—big time. If T is consistent, then it plus Basic Law V has lots of consequences about the ontology of T that do not follow from T alone.

Although it is consistent, the Nuisance principle also violates this requirement. Recall that this particular abstraction is not satisfiable in any infinite domain. Suppose that we add the Nuisance principle to a consistent theory about rock stars. It follows, in the combined theory, that there are only finitely many *rock stars*. So unlike Hume's Principle, the Nuisance principle has consequences concerning the ontology of the base theory. Perhaps it is plausible enough that there are only finitely many rock stars, but this may not have been a consequence of our prior theory about rock stars. Invoking an abstraction principle should not by itself tell us how many rock stars there are.

Let κ be a cardinal number. It seems that a legitimate abstraction principle should not entail that there are at most κ-many things (or exactly κ-many things) unless the prior theory already entails this for the objects recognized to be in its ontology.

Wright [31] provides a first approximation to a rigorous formulation of the revised conservativeness requirement. Suppose that A is an abstraction principle and let Sx be a predicate "true of exactly the referents of the" newly introduced terms. In the case of Hume's Principle, Sx states that x is a cardinal number (i.e., $\exists F(x = Ny : Fy)$). If Φ is a sentence in the language of the base theory, then let Φ^{Σ} be the result of restricting the quantifiers[4] in Φ to $\neg S$. So in the case of Hume's Principle, Φ^{Σ} states that Φ holds of the nonnumbers. Let T be any theory. The conservativeness requirement is that for any sentence Φ in the language of T, $A + T$ entails Φ^{Σ} only if T entails Φ. In other words, if the combined theory entails something about the nonabstracts, then that must be a consequence of the base theory alone.

This formulation is not quite right, for two reasons. First, let Φ be the sentence 'Tony Blair is more intelligent than George W. Bush'. Since Φ has no quantifiers, Φ^{Σ} is just Φ. Let U be the theory with the single axiom: 'If the universe is Dedekind-infinite, then Tony Blair is more intelligent than George W. Bush': $\{(\mathrm{INF} \rightarrow \Phi)\}$. Then U plus Hume's Principle entails Φ^{Σ}. However, U itself does not entail Φ. So Hume's Principle does not meet the letter of the present articulation of the conservativeness requirement.

What went wrong? The intuitive idea behind the requirement is that abstraction principles should have no consequences concerning the "old" objects, the items not explicitly yielded by that very abstraction principle and not explicitly recognized to be in the range of the first-order variables of the base theory. But as the base theory U is formulated, its quantifiers (i.e., the quantifiers in INF) are not explicitly restricted to nonabstracts. This suggests the following formulation of the requirement, as the next approximation:[5] for any sentence Φ in the language of T, T^{Σ} plus the abstraction principle entails Φ^{Σ} only if T entails Φ.

This handles the above counterexample. Recall that the theory is 'If the universe is Dedekind-infinite, then Tony Blair is more intelligent than George Bush'. So T^{Σ} is 'If the nonabstracts are Dedekind-infinite, then Tony Blair is more intelligent than George Bush'. Hume's Principle has no untoward consequences concerning *that* proposition.

The conservativeness requirement needs a little more tweaking. Like Wright's original formulation of conservativeness, this last, corrected formulation makes the most sense when T is a theory about *concrete* objects. Assuming that no abstract objects are concrete, we can be sure that none of the items in the intended range of the quantifiers of the base theory T include the objects yielded by the abstraction principle A. So in this case, it is appropriate to restrict the quantifiers of the base theory to nonabstracts since, presumably, the base theory is solely about nonabstracts. But this is not the most general case. In the above treatment, for example, we introduce (DIF), (QUOT), and (CP) on theories that are already about abstract objects—the natural numbers, the integers, and the rational numbers, respectively. In each case, the quantifiers of the *base* theories range over abstract objects. Recall that we leave it open whether some of the "introduced" abstracts are already in the range of the quantifiers of the base theory. For example, we leave it open whether the real number 2 is identical to or distinct from the rational number 2, the integer 2, and the natural number 2. Depending on how these instances of the Caesar problem are resolved, it may not be correct to restrict the quantifiers of the base theory to the items not "introduced" by the abstraction principle in question, for some of those items may already be in the range of the quantifiers of the base theory.

Instead, when one adds an abstraction principle A to a base theory T, she should restrict the quantifiers of T to whatever range it had previously, explicitly leaving it open whether there is any overlap between that range and the abstracts yielded by A. Formally, let O be a monadic predicate that is not in the language of the abstraction principle A or the base theory T. Intuitively, the extension of O is to be the intended range of the quantifiers of the base theory—the class of objects its variables are supposed to range over. If Φ is a formula, then let Φ^O be the result of restricting the quantifiers in Φ to O. Our final formulation of conservativeness is this: for any sentence Φ in the language of T, $T^O + A$ entails Φ^O only if T entails Φ.

Since O is a new predicate, there are no formal constraints on its extension. So if $T^O + A$ entails Φ^O then it does so no matter how the Caesar question is resolved. If the neologicist has a general solution to the Caesar question (see [19]), we could further tweak the requirement so that the extension of O is the exact ontology of the base theory T, according to the Caesar resolution. The stronger, general requirement will do for present purposes.

We are not finished articulating the conservativeness requirement. There is still an interesting and important issue concerning how logical consequence is to be understood in this context. There is, first, a deductive approach. Say that an abstraction principle A is *deductively conservative* over a base theory A if

> for any sentence Φ in the language of T, if Φ^O can be deduced from $T^O + A$, then Φ can be deduced from T alone.

Unfortunately, the relevant results are not forthcoming for this notion of conservativeness, on standard deductive systems for second-order logic (see [23], Chapter 3).[6]

Theorem 4.1 *The cut principle* (CP) *is* not *deductively conservative over its own base theory (Hume's Principle together with the* (DIF), (QUOT)*, and the explicit definitions).*

Proof sketch: Recall that Frege's Theorem is a derivation of the axioms of second-order Peano Arithmetic from Hume's Principle (plus explicit definitions). Let G be a standard Gödel sentence for second-order PA, so that G is true of the natural numbers, but G cannot be derived from the axioms of second-order Peano Arithmetic. Boolos's [1] argument shows that Hume's Principle (plus the definitions) is conservative over second-order Peano Arithmetic. So G cannot be derived from Hume's Principle. The abstractions (DIF) and (QUOT) used to introduce the integers and the rational numbers are conservative over Hume's Principle, since one can define a model of those structures in the natural numbers (with a pair function). So G cannot be derived from the base theory of (CP). However, we saw above that (CP) entails the axioms of second-order real analysis. The latter is equivalent to *third-order* Peano Arithmetic, and is not deductively conservative over second-order Peano Arithmetic. In particular, from (CP) one can define a truth predicate for second-order Peano Arithmetic, and prove the Gödel sentence G^O for that theory. □

I submit that if neologicism is to have any chance of success, then deductive conservativeness is the wrong requirement. Whether the program in this paper or the one in [18] succeeds or not, at some point the neologicist is going to (try to) introduce the real numbers from abstraction principles and derive the axioms of second-order real analysis from those. The resulting theory will thus fail to be deductively conservative over Hume's Principle. In general, strong theories are not deductively conservative over weaker ones. If the neologicist wants to develop theories as strong as classical real analysis, she must eschew deductive conservativeness.

It is common in mathematics to learn more about a mathematical structure by embedding it in a richer one. In the case at hand, reference to the "new" abstracts—the real numbers—allows us to define properties (or sets) of natural numbers that cannot be defined in the language of arithmetic. Application of the induction principle to these properties yields the new theorems.

I maintain, however, that there is something right about conservativeness. One option would be for the neologicist to formulate a more subtle deductive notion that restricts the instances of comprehension to be used in the derivations—more tweaking. For example, she might insist that all *predicative* consequences of the combined theory be provable from the original theory. Besides looking ad hoc, however, this Lakatosian monster-barring misses the point.

The resolution is to invoke a notion of logical consequence according to which the new theorems of $T^O + A$ are still entailed by the base theory T (alone). Since second-order logic is not complete (see [23], Chapter 4), model-theoretic consequence does not match deductive consequence. Say that an abstraction principle A is *model-theoretically conservative* over a base theory T if

> for any sentence Φ in the language of T, if Φ^O is true in every model of $T^O + A$, then Φ is true in every model of T.

Suppose that A is model-theoretically conservative but not deductively conservative over a base theory T. Then by adding the abstraction principle, we can derive new theorems in the language of T, but these new theorems are in fact logical (model-theoretic) consequences of T alone. One might argue that the abstraction principle A allows us to see that these new theorems are in fact logical consequences of T alone.

There is an interesting model-theoretic property that bears on this matter. Let P be an interpretation of the combined language of an abstraction A and base theory T, and let d be a subset of the domain of P that is closed under the functions of T. Define the d-restriction of P to be the interpretation whose domain is d in which the extensions of the nonlogical terminology of T are the restrictions of their extensions in P. Let M be an interpretation of the base theory T. Say that P is an *extension* of M, written $M \preceq P$, if there is a subset d of the domain of P such that M is isomorphic to the d-restriction of P. In other words, P is an extension of M if M is isomorphic to a submodel of P.

Define an abstraction principle A to be *uniformly compatible* with a base theory T if for each model M of T there is a model P of A such that $M \preceq P$. Uniform compatibility is a nice feature for a proposed abstraction principle to enjoy: if A is uniformly compatible with T then every model of T can be extended to a model of the abstraction A, possibly by adding elements to the domain of discourse. The new elements, of course, are the abstracts, or some of the abstracts (depending on how the Caesar issue is resolved). This seems to be the main idea behind the neologicist program. Uniform compatibility is sufficient for model-theoretic conservativeness.

Theorem 4.2 *If A is uniformly compatible with T then A is model-theoretically conservative over T.*

Proof: Suppose that a sentence Φ is false in some model M of T. Let P be a model of A such that $M \preceq P$. Then M is isomorphic to a submodel of P. Let the extension of O be the domain of this submodel. So P satisfies $T^O + A$. Since Φ is false in M, Φ^O is false in P. □

The desired results are now forthcoming, sometimes for rather trivial and unilluminating reasons.

Theorem 4.3 *Hume's Principle is uniformly compatible with (and so model-theoretically conservative over) any theory T.*

Proof: Let M be a model of T. Suppose, first, that the domain of M is finite. Let the domain of P consist of the domain of M together with the natural numbers and the one additional set $\aleph_0$. Interpret the nonlogical terminology of T in P as it is in M. For each subset F of the domain of P, define $Nx : Fx$ to be the cardinality of F. It is straightforward to verify that P makes Hume's Principle true under this interpretation, and that $M \preceq P$. Now suppose that the domain of M is infinite. Then by a well-known result noted in [1], it is possible to interpret the $Nx : Fx$ operator on P to make Hume's Principle true.[7] So $M \preceq P$ (even without adding new elements). Thus, Hume's Principle is uniformly compatible with T. □

Theorem 4.4 (DIF) *is uniformly compatible with (and so model-theoretically conservative over) second-order Peano Arithmetic.* (QUOT) *is uniformly compatible with (and so model-theoretically conservative over) the second-order theory of the integers.* (CP) *is uniformly compatible with (and so model-theoretically conservative over) second-order rational analysis.*

Proof: The second-order theories of Peano Arithmetic, the integers, and the rational numbers are all categorical. Each theory has only one model, up to isomorphism. The foregoing treatment shows how to extend the standard model of each theory to a model of the respective abstraction principle. □

Model-theoretic conservativeness is not as illuminating as it might look. Since second-order Peano Arithmetic is categorical, it is semantically complete. For any sentence Φ in the language of Peano Arithmetic, either Φ is a model-theoretic consequence of the axioms or $\neg\Phi$ is a model-theoretic consequence of the axioms. Thus, every arithmetic truth is already a model-theoretic consequence of the theory. So the only way an abstraction principle could yield "new" arithmetic consequences would be for it to have no models that contain a model of second-order Peano Arithmetic. That is, the only way A could fail to be model-theoretically conservative over second-order Peano Arithmetic would be for A to not have any (Dedekind-)infinite models at all, in which case it is incompatible with arithmetic!

In general, let T be a semantically complete base theory. Then the only way a proposed abstraction principle A could fail to be model-theoretically compatible with T would be for $T^O + A$ to have no models at all. Similarly, let T be categorical. Then the only way a proposed abstraction A can fail to be uniformly compatible with T is for there to be *no* models of $T^O + A$ so that A is in fact logically incompatible with T.

Thus, for semantically complete base theories, model-theoretic conservativeness is not a discerning requirement; and for categorical base theories, uniform compatibility is not discerning. It just comes to joint satisfiability.

As they stand, model-theoretic conservativeness and uniform compatibility make direct reference to models of the various theories and thus presuppose a fairly substantial set theory (given that the languages are higher order). If the neologicist manages to reconstruct a sufficiently strong set theory, she can formulate the constraints internally and invoke them from that point onward. Those constraints might then serve as an after-the-fact check on the principles used to develop the set theory and the other mathematical theories that appeared along the way. For now, however, model-theoretic conservativeness and uniform compatibility are taken in the *external* perspective. We investigate them using whatever techniques are available. They serve as the mathematician's explication of an intuitive constraint that the neologicist places on abstraction principles.

To sum things up so far, the deductive articulation of conservativeness is available internally to the neologicist but is inappropriate since the powerful target theories (real analysis in this case) are not deductively conservative over the relatively weak base theories (arithmetic). The model-theoretic articulation is external. So how should the neologicist herself understand the conservativeness requirement internally? One option would be for the neologicist to simply leave the notion of logical consequence at an intuitive, pretheoretic level. To say that a conclusion Φ is entailed by a set of premises Γ is to say that it is not possible for the members of Γ to be true and Φ false, or that Φ is somehow implicit in the members of Γ.

To be sure, this "articulation" of conservativeness makes it harder for the neologicist to *prove* that a proposed abstraction principle is acceptable. But the neologicist is not without resources. For Φ to be entailed by Γ, it is necessary, but perhaps not sufficient, for Φ to be true in all set-theoretic models of Γ; and it is sufficient, but not necessary, that Φ be deducible from Γ.

If a proposed abstraction is in fact model-theoretically conservative over the relevant base theory or theories, the neologicist might take it as a working hypothesis that the abstraction principle is conservative, in the appropriate intuitive sense. She might adopt an attitude that the principle is acceptable until reason is shown otherwise, shifting the burden of proof to someone who wishes to challenge the principle. The remaining onus on the neologicist would be to deal with any violations of deductive conservativeness, perhaps on a case by case basis. Suppose, for example, that the neologicist assumes that the cut principle (CP) is conservative over Peano Arithmetic, in the relevant intuitive sense of "conservative." It follows that the standard Gödel sentence G for second-order Peano Arithmetic is a consequence of the base theory, second-order Peano Arithmetic. The neologicist would have to defend this conclusion and argue that the Gödel sentence is in fact implicit in the original theory, despite not being deducible from it.

5. The Acceptability of Abstractions II: Inflation

Boolos reiterates a view widely held since at least Kant (*Critique of Pure Reason*, B622–23) that nothing in the neighborhood of an analytic truth has ontological consequences:

> it was a central tenant of logical positivism that the truths of mathematics were analytic. Positivism was dead by 1960 and the more traditional view, that

> analytic truths cannot entail the existence either of particular objects or too many objects, has held sway ever since. ([3], p. 249–50)

The widely held view in question is that one cannot learn about what objects exist from meaning or conceptual analysis alone. Any logicist-style account that accepts the existence of mathematical objects must reject, or at least attenuate, this view. The neologicist claims that the existence of mathematical objects follows from abstraction principles (and logical truths). She might back off of the thesis that acceptable abstraction principles are analytic, or true solely in virtue of the meanings of the terms (as, for example, Wright [31] does), but she maintains that acceptable abstractions have a privileged epistemic status, something at least akin to implicit definitions. The opposing view is that we cannot know about the existence of objects in such an epistemically inexpensive manner.

Boolos's phrase "either of particular objects or too many objects" suggests a compromise. Perhaps an acceptable abstraction principle can entail the existence of some objects but not "too many" of them. We can then focus on the question of how many is too many. Cook [5] is a detailed study of the "inflationary" aspects of abstraction principles like the cut principle (CP). Much of this section is a response to that paper.

The discussion here (and in [5]) is, for the most part, external to the neologicist framework. We invoke a substantial set theory in order to compare the sizes of various models of various principles. Sometimes the unacceptability of a given abstraction principle is due to its inconsistency which, of course, is internal. But not always.

Consider, first, Basic Law V. Of course, this abstraction is inconsistent and so it has no models at all. Suppose that an intended model of our base theory has a domain of size κ. Then there are 2^{κ}-many extensions composed of those items. So Basic Law V implies the existence of more abstracts than objects in the base theory. This "inflation" does not stop there. Since the quantifiers of Basic Law V are unrestricted, it implies the existence of extensions of properties of the extensions of objects in the original domain. There are $2^{2^{\kappa}}$ of those. And Basic Law V entails the existence of extensions composed of *those* extensions; and on it goes. The problem, of course, is that this inflation does not stop.

Now consider Hume's Principle. Suppose that the intended interpretation of the base theory is finite, of size n. Then Hume's Principle entails the existence of $n + 1$ cardinal numbers—zero and one for each nonempty size of objects from the base theory (see [1]). So there is some mild inflation. Since the quantifiers in Hume's Principle are unrestricted, it implies the existence of numbers of *properties of those numbers*. There are $n + 2$ such cardinal numbers and so on. But, in a sense, this inflation does end. As above, the result of adding the natural numbers and $\aleph_0$ to the domain of the original model is a structure that satisfies Hume's Principle. There is no more inflation—at least not on that model.

The addition of Hume's Principle to any denumerably infinite domain does not inflate. In that context, Hume's Principle only entails the existence of countably many cardinal numbers, the same size as the domain we start with. Suppose that the intended domain of the base theory has cardinality $\aleph_\alpha$. Then the addition of Hume's Principle yields $\aleph_0 + |\alpha| \leq \aleph_\alpha$ cardinal numbers. So Hume's Principle does not inflate on this domain either. So under the axiom of choice, Hume's Principle does not inflate on any (Dedekind-)infinite set.

The following is a modification of Cook's useful framework for treating the inflationary aspects of abstraction principles. Let A be an abstraction principle and, as above, let O be a monadic predicate that does not occur in A (or in the base theory, if there is one). Let A^O be the result of restricting all of the quantifiers in A to O. For example, if B is Basic Law V, then B^O says that all properties of objects that have O have extensions. It does not entail that the properties of those extensions have extensions. In fact, B^O is satisfiable.

Let d be a set. Define a d-model of A to be a model of A^O in which the extension of O is d. So A^O helps measure the abstracts yielded by A on d. Let κ be a cardinal number. If $|d| = \kappa$, then every d-model of Basic Law V has at least 2^κ-members, and every d-model of Hume's Principle has at least $\kappa + 1$ members.

Say that an abstraction principle A is *κ-inflationary* if for every set d of size κ, the cardinality of the domain of every d-model of A is greater than κ. In other words, A is κ-inflationary, if starting with a domain of size κ, A yields more than κ-many abstracts on that domain (if it is satisfiable on that domain at all).[8] If κ is finite, then Hume's Principle is κ-inflationary, and if κ is infinite and well ordered then Hume's Principle is not κ-inflationary. It is independent of Zermelo-Fraenkel set theory whether Hume's Principle is inflationary on the continuum (see note 7).

Define an abstraction principle A to be *strictly noninflationary* if there is no κ such that A is κ-inflationary. So if A is strictly noninflationary, then for any domain d, A does not yield more than $|d|$ abstracts. This, I presume, is best. Say that an abstraction A is *boundedly inflationary* if there is some cardinal λ such that for all $\kappa > \lambda$, A is not κ-inflationary.[9] This is second-best. If A is boundedly inflationary, then if the starting domain is sufficiently large, then A does not inflate on it. Define A to be *unboundedly inflationary* if it is not boundedly inflationary, and say that A is *universally inflationary* if for every κ, A is κ-inflationary. This is worst.

Basic Law V, of course, is universally inflationary—worst. If we assume the axiom of choice, then Hume's Principle is boundedly inflationary—second-best. Shapiro and Weir [25] show that Boolos's [2] New V (called "VE" in [31]) is unboundedly inflationary—it inflates on all singular cardinals.

Recall that since the quantifiers of many of the abstraction principles under consideration are unrestricted, their range includes the abstracts yielded by that very principle. In such cases, at least, we are more interested in A itself than in the restricted A^O. Notice that A is *not* κ-inflationary if and only there is a model of A^O in which the extension of O and the domain of the model both have cardinality κ. If κ is finite, then the extension of O would have to be the whole domain, in which case the model of A^O is also a model of A. Assuming choice, we can establish something similar in general: A is not κ-inflationary if and only if A itself (and not just A^O) has a model of size κ.

So if A is not κ-inflationary, then it is consistent with A for the universe to be of size κ exactly. If A is strictly noninflationary then for every cardinal κ, A has a model of size κ. If A is boundedly inflationary, then there is some cardinal λ such that for all $\kappa > \lambda$, A has a model of size κ. So if A is unboundedly inflationary then for every cardinal λ there is a $\kappa > \lambda$ such that A has no model of size κ.

Say that A is *unboundedly satisfiable* if, for every cardinal λ, there is a $\kappa > \lambda$ such that A has a model of size κ. Notice that if A is unboundedly satisfiable, then (assuming choice) we can turn any set into a model of A by adding more elements: for every set d, there is a model of A whose domain contains d. In the best cases, the "new" elements will be the new abstracts.

Shapiro and Weir [25] show that if the generalized continuum hypothesis is true, then New V is satisfiable at all regular cardinals, and so it is unboundedly satisfiable. However, it is independent of Zermelo-Fraenkel set theory (with choice) whether New V is actually unboundedly satisfiable. It might not have any uncountable models at all.

So, again, how much inflation is too much? Cook [5] argues that only strictly non-inflationary and boundedly inflationary abstraction principles should be acceptable to a neologicist. Let us examine the arguments, since they go to the heart of the goals of neologicism. Concerning unbounded inflation, Cook writes:

> The neo-logicist is claiming that the abstraction principles implicitly define, or at least ground our use of mathematical concepts and theories. Definitions of the abstract objects of mathematics, even implicit ones, ought to determine a unique group of objects which necessarily fall under the definition. If this 'defining' abstraction principle is *unboundedly inflationary*, however, then the neo-logicist has failed in his task.

Assume, for example, that an abstraction principle A is unboundedly inflationary, and suppose that M is a model of both A and the background theory T. Let κ be the cardinality of the domain of M and let γ be the smallest cardinal greater than κ such that A is γ-inflationary. Cook continues:

> had there been γ objects in the universe, there would have, by [A], been more than γ (and thus more than κ) abstracts. But then the original abstracts are not all of the objects whose identity conditions are given by [A]. This process can be repeated indefinitely (and transfinitely), so that we never have all the objects that fall under the purview of [A]. In other words, if [A] is *unboundedly inflationary* then it fails to secure a definite collection of objects as the domain of its abstraction operator, but instead gives us different abstracts relative to how many objects exist.

In a note, Cook adds that "an adequate definition should determine a unique extension independently of the existence of any other objects."

Some abstraction principles do characterize a unique domain of objects, at least up to isomorphism. The present cut abstraction principle (CP), for example, yields all and only the real numbers plus two extra abstracts. Another example would be the restriction of Hume's Principle to finite concepts (see [20]). This yields all and only (an isomorphic copy of) the natural numbers.

Cook is correct that *if* an abstraction principle A is intended to characterize a unique structure (such as the natural numbers or the real numbers), then it should not be unboundedly inflationary. In this case, A should yield the required objects and no others. It should not inflate on any domain that is as large as or larger than the requisite structure.

However, it is not true that every legitimate abstraction principle determines "a definite collection of objects" as the range of the defined operator. Some principles do yield "different abstracts relative to how many objects exist." Consider Hume's Principle. Thanks to Frege's Theorem, it implies the existence of the natural numbers and the cardinality of the natural numbers (i.e., $\aleph_0$). But what of other cardinalities? Since Hume's Principle has countable models, it does not, by itself, entail that the cardinality of the continuum exists. But Hume's Principle does entail that *if* there is a property that holds of continuum-many objects, then the cardinality of the continuum exists. So, for example, Hume's Principle and (CP) together entail that the cardinality of the continuum exists. In general, which cardinal numbers exist depends on how many objects there are. I, at least, do not see this as a problem with Hume's Principle as an abstraction. Some acceptable abstractions are open-ended in the sense that the abstracts they yield depend on the ontology of the background theory. Increasing the ontology might increase the abstracts.

A different problem with unboundedly inflationary abstraction principles is that they may conflict with each other. Weir [29] formulates a pair of "distraction" principles B, B', such that B and B' are each unboundedly satisfiable, but are mutually inconsistent. Suppose that the background theory has a model of size κ_0. To extend this to satisfy B, we add $\kappa_1 > \kappa_0$ abstracts. But this new model does not satisfy B'. To satisfy B', we add $\kappa_2 > \kappa_1$ abstracts. But once we add these abstracts to satisfy B', we no longer satisfy B. To (re)satisfy B, we have to add $\kappa_3 > \kappa_2$ more abstracts. But then we no longer satisfy B'. In short, the conjunction B & B' is *universally* inflationary.

Faced with such a pair of abstractions, the neologicist must find a principled way to choose among them. Or else she can play it safe and reject any unboundedly inflationary abstraction principle and require that all acceptable abstractions be boundedly inflationary. Then, once we are satisfied that the universe is sufficiently large, the abstraction will be satisfied no matter how much larger we go on to recognize the universe to be.

Let us turn to Cook's treatment of universally inflationary principles—those that inflate on every cardinality. Of course, if an abstraction A is inconsistent, then it is unacceptable. Suppose that A is consistent, but universally inflationary. Let b be a set and $\kappa = |b|$. Since A is κ-inflationary, A cannot be satisfied on b. Since b is arbitrary, A has no models whose domain is a *set* (with a cardinality). As Cook puts it, A "will be satisfied (if satisfied at all) only by a structure that is at least the size of proper class." This, he says, is problematic, since proper classes are "extremely badly behaved." The idea is that if A can be satisfied only on a proper class, then it yields a proper class of abstracts (so to speak). Thus, the abstraction takes "us far from the epistemically innocent implicit definitions that the neologicists argue acceptable abstraction ought to provide."

In sum, Cook's claim is that the "generation" of a proper class of abstracts is incompatible with the epistemic goals of neologicism. Notice that this judgment comes from the external perspective. Internally, the neologicist claims that we can come to know about the existence of some objects through deduction from principles in the neighborhood of implicit definitions or analytic truths. Externally, we use the set-theoretic metatheory, accepted already by the established mathematician, to show that a certain abstraction principle yields more objects than there are members of any element of the set-theoretic hierarchy. Cook seems to hold that a proper class of abstracts is indeed "too many" objects to obtain this way. As noted above, for neologicism to have a chance, we have to temper the widely-held view that definitions, or principles much like definitions, have no ontological consequences. Cook's claim, in effect, is that enough is enough. He presupposes that from the external perspective of an advocate of Zermelo-Fraenkel set theory, the abstracts must constitute (or be equinumerous with) a *set*.

But the fact is that the objects of mathematics do not constitute a set, for well-known reasons. So Cook's thesis entails that neologicism must fall short of its grand goal of providing an epistemic foundation for *all* of mathematics. A neologicist set theory and a neologicist theory of ordinals and cardinals is out of the question. Thus, the neologicist must rest content with an account of arithmetic, real and complex analysis, and perhaps a little more. The main (external) question that remains is just how big the neologicist's ontology can be. What is the cardinality of the objects yielded by all acceptable abstractions together? Presumably, it will be $\aleph_\alpha$ for some ordinal α. If the neologicist wants to avoid demanding revisions to established mathematics, she must provide some other epistemic foundation for those branches of mathematics—such as set theory, ordinal theory, and cardinal theory—whose ontology is not a set.

For what it is worth, I believe that Cook's view begs the question against the neologicist quest. So far as I know, no argument has been given that the objects yielded by an abstraction principle must constitute a definite, set-sized totality. The neologicist thesis is that an acceptable abstraction is akin to an implicit definition, providing an epistemic foundation of the theory of the objects it yields. There is no requirement that the objects be limited in any way, or that they constitute a definite totality. Perhaps further discussion of this should await either specific arguments concerning the limits of abstraction principles or the presentation of particular candidate principles that do yield a proper class of abstracts. We briefly revisit the issue at the end of the next section.

6. The Acceptability of Cut Abstraction

I now turn to the inflation and satisfiability of the abstraction principles presented here: (DIF), (QUOT), and (CP). Unlike Basic Law V and Hume's Principle, the quantifiers in all three of these principles are restricted. Since the right-hand side of the difference principle (DIF) explicitly invokes addition on the natural numbers, (DIF) entails the existence of a difference-abstract for each pair of natural numbers, but that is all. It says nothing about "differences" of (pairs of) other objects, and in particular, it does not yield difference-abstracts for (pairs of) difference-abstracts. Similarly, the quotient principle (QUOT) yields a ratio for each pair of integers, but nothing else. And (CP) yields a cut for each property of rational numbers, but nothing else.

Since there are only countably many integers, the difference principle is satisfiable on any domain that contains the natural numbers (using standard coding techniques if necessary). In a sense, (DIF) is universally satisfiable in that it is satisfiable on any domain *over which it is defined*. And so it does not inflate on any such domain. Since there are only countably many rational numbers, the quotient principle (QUOT) is satisfiable on any domain that contains the integers, that is, on any domain on which it is defined.

Since there are continuum-many distinct cuts, (CP) inflates on the rational numbers, but that is the end of its inflation. So (CP) is boundedly inflationary, in that it is satisfiable on any domain that is at least the size of the continuum and contains the rational numbers. Concerning inflation and satisfiability, the neologicist cannot do any better than this. If she hopes to recapture real analysis, she will need principles that yield continuum-many abstracts. The cut principle does that, and no more.

Perhaps we should not be sanguine. The main reason why (CP) does not inflate beyond the real numbers is that it only defines cuts for properties of rational numbers. But we can mimic the development of (CP) for any linear order[10] '$<$' defined on a set or class h. Let P be a property of items in h and suppose that $r \in h$. Say that r is an *upper bound* of P, written $P \preceq r$, if for any $s \in h$, if Ps then either $s \prec r$ or

$s = r$. In other words, $P \preceq r$ if r is greater than or equal to any object that P applies to (under the given linear order). Consider the following abstraction principle.

$$(h, \prec) - (\mathrm{CP}) \qquad \forall P \forall Q (\mathrm{C}(P) = \mathrm{C}(Q) \equiv \forall r (P \preceq r \equiv Q \preceq r)).$$

In strict analogy with (CP), the cut of P is identical to the cut of Q if and only if P and Q share all of their upper bounds (under $\prec$). The abstraction $(h, \prec) - (\mathrm{CP})$ might inflate in the sense that it may yield more cuts than members of h. If the cardinality of h is κ, then there can be as many as 2^κ cuts. But in light of the above, this is the extent of the inflation for this one principle.

There is, of course, a linear order on the real numbers which extends the linear order on the rational numbers (under the natural embedding of the rational numbers into the real numbers). The version of the cut principle formulated on the natural linear order for the real numbers does not inflate. It is a consequence of the completeness property of the real numbers that the "cuts" of nonempty, bounded properties of real numbers are isomorphic to the real numbers themselves: a pleasing, well-known result, and more good news on the inflation front.

Something similar holds for $(h, \prec) - (\mathrm{CP})$ in general. Let h' be the collection of cuts of nonempty, bounded properties. There is a natural embedding of h into h', and an extension of the linear order '$\prec$' on h to a linear order '$\prec'$' on h'. But there is no new inflation on *that* linear order. The cuts yielded by $(h', \prec') - (\mathrm{CP})$ are isomorphic to those yielded by $(h, \prec) - (\mathrm{CP})$.

So each of the various cut abstraction principles is at least relatively innocuous. Some of the cut-principles do inflate, but in each case, the inflation is contained. The problem, if there is one, is that the *totality* of cut abstraction principles together might generate too much inflation. It seems ad hoc to claim that the original (CP) is the only legitimate abstraction principle in this form. If (CP) is acceptable, then at least some of the others are. Perhaps they all are.

Cook formulates a generalization of the cut-abstraction principle used in [18] (a restricted version of Basic Law V). In the present context, the analogous principle is a single, second-order sentence asserting the existence of the cuts of every linear order:

$$\begin{array}{ll} (\mathrm{GCP}) & \forall H \forall R[\text{ if } R \text{ is a linear order on } H \text{ then} \\ & \forall P \forall Q[\forall x((Px \to Hx)\ \&\ \forall x(Qx \to Hx)) \to (\mathrm{C}(P,H,R) = \mathrm{C}(Q,H,R) \\ & \qquad \equiv \forall r(\forall x(Px \to (x = r \vee Rxr)) \equiv \forall x(Qx \to (x = r \vee Rxr))))]]. \end{array}$$

That is, for any property H and any relation R, if R is a linear order on the objects that have H, then if P and Q are subproperties of H, then the cut of P is identical to the cut of Q (relative to H and R) if and only if P and Q have the same upper bounds under R.

Again, the neologicist who accepts (CP) might be committed to the existence of the cuts on any linear order. The sentence (GCP) is a formulation of that commitment. The alternative, of course, is to articulate a principled distinction between the linear orders that have cuts and those that do not. Cook establishes an interesting result that bears on the inflation of cut principles.

Theorem 6.1 (Cook [5]) *Assume the axiom of choice in the metatheory. Let κ be an infinite cardinal number. There is a set h such that $|h| \leq \kappa$, and a linear order '$\prec$' on h such that $(h, \prec) - (\mathrm{CP})$ yields more than κ cuts.*

Proof: Let λ be the least cardinal such that $2^\lambda > \kappa$. Of course, $\lambda \leq \kappa$. Let h be the set of subsets of λ (as an ordinal) that are smaller than λ. So $h = \{b \subseteq \lambda : |b| < \lambda\}$. A straightforward calculation shows that $|h| \leq \kappa$ (or see [5]). Define the linear order as follows:

$$a \prec b \equiv \exists\alpha(\alpha \in b \;\&\; \alpha \notin a \;\&\; \forall\beta < \alpha(\beta \in a \equiv \beta \in b)).$$

In other words, $a \prec b$ if the first ordinal on which they differ is in b. The cuts yielded by $(h, \prec) - (\text{CP})$ are isomorphic to the subsets of λ: if $x \subseteq \lambda$, then x corresponds to the cut of the property of being an initial segment of x. So there are $2^\lambda > \kappa$ such cuts. □

The original cut principle (CP) yields continuum-many real numbers. By Theorem 6.1, there is a linear order on a subset of those real numbers, and the cut principle on that linear order yields more than continuum-many abstract objects. There is a linear order on a subset of *those* objects that yields even more abstract objects. And on it goes. The process can be carried into the transfinite: there is a linear order on (a subset of) the union of the objects yielded by the first, second, third, . . . of these abstractions, such that the cut principle on that big linear order yields more abstract objects than the totality of objects yielded by the previous principles. Is the neologicist committed to the thesis that we can know that all of these objects exist via principles that are a priori knowable, akin to implicit definitions?

It might be objected that one cannot exactly *define* these linear orders internal to the neologicist program, unless the background theory T includes a pretty substantial set theory. That is, the neologicist cannot define the various linear orders $(h, \prec)$ unless she has produced (internally) a set theory sufficient to manipulate sets of ordinals. If the neologicist manages that, then any worries about inflation should be focused on the set theory. The powerset axiom produces at least as much inflation as the various cut principles, possibly more (depending on the generalized continuum hypothesis).

So perhaps the neologicist is not committed to the generalized cut principle (GCP). At most, she is only committed to the acceptability of those cut principles $(h, \prec) - (\text{CP})$ in which the linear order is definable internally.[11] Neologicism is, after all, an epistemic program. The goal is to show how mathematical principles can become known with minimal epistemic presuppositions. A mathematical domain is brought into the fold if its axioms (or characterizing properties) can be derived from abstraction principles which are akin to implicit definitions, all but analytic. To accomplish that, the abstraction principles must be explicitly formulated in an acceptable language. In the present context, this entails that the linear order must be definable.

Nevertheless, if the original cut principle (CP) is acceptable, then one would think that for the neologicist, any other cut principle $(h, \prec) - (\text{CP})$ is acceptable at least in a relative sense. The idea is that *if* the objects in the domain h exist and are ordered as indicated, *then* the indicated cuts exist. It does not matter if the objects in h are themselves grasped through an abstraction principle.

The neologicist is a realist in ontology, holding that mathematical objects exist independently of the mathematician. If mathematical objects are not of our making, then why think that the universe is constrained by the limited expressive resources of human languages? Recall that, at present, we are in the external perspective, seeing how various abstraction principles mesh with accepted mathematics. So it is fair to see what happens if principles in the form $(h, \prec) - (\mathrm{CP})$ are added to various mathematical domains.

What of (GCP) itself? I do not know if (GCP) is consistent, but even if it is, it inflates a *lot*. It follows from Theorem 6.1 above that (GCP) cannot be satisfied on any set. It inflates universally. I do not know if (GCP) can be satisfied on a proper class, or on the set-theoretic hierarchy itself.

There may be a problem for linear orders defined on proper classes. Let Π be the class of all sets of ordinals and consider the corresponding variant of the linear order from Theorem 6.1, defined on Π:

$$a \prec b \equiv \exists \alpha(\alpha \in b \;\&\; \alpha \notin a \;\&\; \forall \beta < \alpha(\beta \in a \equiv \beta \in b)).$$

Again, $a \prec b$ if the first ordinal on which they differ is in b. Call the principle $(\Pi, \prec) - (\mathrm{CP})$.

Notice that in the context of second-order Zermelo-Fraenkel set theory, the generalized cut principle (GCP) entails $(\Pi, \prec) - (\mathrm{CP})$. I do not know whether $(\Pi, \prec) - (\mathrm{CP})$ is consistent, but it does go beyond Zermelo-Fraenkel set theory. There are as many cuts yielded by $(\Pi, \prec) - (\mathrm{CP})$ as *properties* (or classes) of ordinals: if P is a property of ordinals, then let P' be the property (of sets) of being an initial segment of P (under membership). The cut of P' under $(\Pi, \prec) - (\mathrm{CP})$ corresponds to P. So there are more cuts than ordinals. The above internal proof that the real numbers are uncountable can be extended to $(\Pi, \prec) - (\mathrm{CP})$. That is, we can show, internal to the neologicist framework, that the cuts yielded from $(\Pi, \prec) - (\mathrm{CP})$ cannot be mapped one-to-one into the ordinals. Externally, if Zermelo-Fraenkel set theory is the metatheory, then $(\Pi, \prec) - (\mathrm{CP})$ entails that there are more sets than ordinals. A fortiori, the cuts cannot be well ordered. This contradicts global choice.[12] This might be troublesome since many theorists, including Hilbert and Zermelo, hold that global choice is a logical truth (see [23], p. 106–8).

On the other hand, the fact that $(\Pi, \prec) - (\mathrm{CP})$ goes beyond established set theory may not be problematic in itself (provided that it is consistent). Neologicism might have contributions to make to set theory itself.

There is a recognized problem with applying unrestricted abstractions to what are, in effect, proper classes. Boolos [3] once pointed out that Hume's Principle entails that the property of being self-identical has a cardinal number. This would be the number of all objects whatsoever. Similarly, Hume's Principle entails that there is a number of all cardinal numbers, and in the context of a background set theory, Hume's Principle entails that there is a number of all sets and a number of all ordinals. Boolos notes that prima facie, this presents a conflict with ordinary Zermelo-Fraenkel set theory:

> [I]s there such a number as [the number of all objects whatsoever?] According to [ZF] there is no cardinal number that is the number of all the sets there are. The worry is that the theory of number [based on Hume's Principle] is incompatible with Zermelo-Fraenkel set theory plus standard definitions. ([3], p. 260)

Wright [32] accepts the force of this objection, and grants "the plausible principle . . . that there is a determinate number of F's just provided that the F's compose a *set*." Since "Zermelo-Fraenkel set theory implies that there is no set of all sets . . . it would follow that there is no number of sets." Wright's proposed response is to restrict the second-order variables in Hume's Principle, so that some properties do not have numbers—those which are what Dummett calls "indefinitely extensible." According to Dummett, an "*indefinitely extensible* concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it" (Dummett [8], p. 441, emphasis added). Ordinal numbers and cardinal numbers are paradigm cases of indefinitely extensible notions. Wright writes:

> I do not know how best to sharpen [the notion of indefinite extensibility] . . . . But Dummett could . . . be emphasizing an important insight concerning certain very large totalities—ordinal number, cardinal number, set, and indeed "absolutely everything." If there is anything at all in the notion of an indefinitely extensible totality . . . one principled restriction on Hume's Principle will surely be that [cardinal numbers] *not* be associated with such totalities. ([32], p. 13–14)

Thus, Wright suggests that the second-order variables in Hume's Principle be restricted to definite properties—those not indefinitely extensible. Hale [18] follows suit.

If this is sound, it suggests a general thesis that the second-order variables in acceptable abstractions be restricted to definite properties. This would rule out $(\Pi, \prec) - (\mathrm{CP})$, since the sets of ordinals constitute an indefinitely extensible totality, if anything does. Moreover, let (GCP$-$) be the result of restricting the initial second-order variables of (GCP) to definite properties (i.e., to sets). The resulting principle is still universally inflationary, thanks to Theorem 6.1 above: given any *set* b with $|b| = \kappa$, there is a linear order on a subset of b which has more than κ cuts. As Cook notes, Theorem 6.1 shows that the notion of being a cut of a linear order is itself indefinitely extensible.

Nevertheless, it is a theorem of Zermelo-Fraenkel set theory that if '$\prec$' is a linear order on a set h, then there is a function defined on the powerset of h that satisfies the consequent of (GCP). So the restricted (GCP$-$) does not go beyond Zermelo Fraenkel set theory (with choice). It is satisfiable on the iterative hierarchy.

The problem now is to formulate the restriction to abstraction principles like Hume's Principle and (GCP) more rigorously. What is it for a property to be indefinitely extensible? Wright [32] conceded that he does not have a more rigorous, internal articulation of the notion of indefinite extensibility. The details of the proposal go beyond the scope of this already lengthy section, and I do not have much to add in any case. I agree with Cook that the proposal to restrict abstractions to definite properties is "[p]erhaps the most promising . . . restriction on the applicability of cut abstraction." The present state is programmatic. Articulating the Dummettian notion of indefinite extensibility is a central item on the agenda of neologicism.[13]

7. Brief Philosophical Epilogue

Heck [20] makes an important distinction between interpreting a theory (like arithmetic or analysis) in an analytically true theory (or a theory based on abstraction principles) and showing that the theory *itself* can be derived from abstraction principles. Frege himself surely knew that Euclidean geometry could be interpreted in real analysis, and yet he did not hold that Euclidean geometry was analytic. Heck's thesis is that Frege's Theorem does not, by itself, provide an epistemic foundation for *arithmetic*. We need to make sure that the relevant abstracts are indeed the natural numbers that we all know and love.

What can be said about the present case? Can one claim that the cuts on bounded, instantiated properties of rational numbers are the real numbers that we all know and love? I do not know how to even begin a definitive resolution of this issue, but one or two points can be made. It seems to me that continuity is essential to the real numbers. So their neologicist characterization should have continuity built in. And both the present account and Hale's [18] rival account do so. This is probably the most important place where we are indebted to Dedekind, who showed us what continuity is and used only logical resources in the process.

Another sticky philosophical matter, related to Heck's requirement, is Frege's insistence that an account of the applications of a mathematical structure must be built into its characterization. Hume's Principle, for example, recapitulates an important application of the natural numbers—to measure cardinalities of sortal properties. The contrast is with Dedekind's [7] account of the natural numbers, in his second great foundational work. Dedekind provides a direct description of the natural number structure, and after that provides an account of the application of this structure—to both cardinal numbers and ordinal numbers. Frege would complain that this account of application comes too late. It should be built into the very constitution of the natural numbers.

Frege's Constraint is completely ignored in the present account. By now, we know exactly which *structure* we were looking for, and the present account zeros in on this very structure. To be sure, it would be easy to tack on an account of applications—the measurement of quantities—to this structure. We know how to do this for any complete ordered field. But for Frege, this account of application comes too late. It should be built into the application.

Hale's [18] rival account of the real numbers is more in line with Frege's Constraint, since he develops the real number structure from one of its applications, the measurement of ratios of complete quantitative domains which have no negative or zero quantities and which are not "cyclical" (like angles).

As a structuralist [24], I do not know what to make of Frege's Constraint. Hale's account and the present one deliver the same structure (eventually). With Dedekind, I'd say they are both accounts of *the* real number structure. If we take Frege's Constraint concerning applications seriously, however, then at least one of us—me presumably—has delivered an isomorphic imposter, perhaps in the same sense that $\mathbb{R}^3$ is an isomorphic imposter to Euclidean space. This metaphysical issue is pursued in Wright's contribution [33] to this issue.

Notes

1. I am indebted to John Burgess for suggesting that the abstraction be put this way.

2. If we think of Π and the various P as sets, then Q is the union of Π : $Q = \{r : \exists P(r \in P \;\&\; P \in \Pi)\}$.

3. See, for example, van Heijenoort [28], Goldfarb [17], and Shapiro [24], Chapter 5.

4. That is, we restrict the first-order quantifiers to $\neg S$ and restrict the second-order quantifiers to the properties and relations on $\neg S$.

5. This version of the conservativeness requirement is close to that formulated in Field ([9], p. 12) in a different context. Fine ([10], p. 626–27) suggests something in the neighborhood of this requirement, on behalf of neologicism. See also [25].

6. The same goes for the restricted version of Basic Law V that Hale [18] employs to construct a complete quantitative domain. As far as I know, it is open whether Hume's Principle is deductively conservative over relatively weak, consistent theories.

7. This result uses the axiom of choice. In particular, we assume that the domain of M can be well ordered. This use of choice is necessary since there are models of Zermelo-Fraenkel set theory in which Hume's Principle cannot be satisfied on the real numbers. I do not know whether one can show that Hume's Principle is uniformly compatible with (or model-theoretically conservative over) every theory, without using the axiom of choice.

8. Cook restricts κ to infinite cardinals. Since I do not follow that here, my definitions are slightly different from his.

9. The usefulness of this notion turns on the axiom of choice since in that case, for any distinct cardinals κ, κ', either $\kappa < \kappa'$ or $\kappa' < \kappa$.

10. Actually, the construction can be carried out on any binary relation, but I will stick to linear orders for convenience.

11. The neologicist herself cannot formulate this restriction internally unless she has a coherent formulation of definability.

12. The abstraction principle New V entails that the universe is well ordered (see [25]). So, as Cook notes, New V is inconsistent with $(\Pi, \prec) - (\mathrm{CP})$.

13. See Clark [4] and Shapiro [27]. The basic theme of the latter is restrict Basic Law V similarly and resurrect set theory along neologicist lines. That paper contains a more detailed discussion of the notion of indefinite extensibility.

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