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REALISM AND PARADOX

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Abstract This essay addresses the question of the effect of Russell's paradox on Frege's distinctive brand of arithmetical realism. It is argued that the effect is not just to undermine Frege's specific account of numbers as extensions (courses of value) but more importantly to undermine his general means of explaining the object-directedness of arithmetical discourse. It is argued that contemporary neo-Fregean attempts to revive that explanation do not successfully avoid the central problem brought to light by the paradox. Along the way, it is argued that the need to fend off an eliminative construal of arithmetic can help explain the so-called Caesar problem in the *Grundlagen*, and that the "syntactic priority thesis" is insufficient to establish the claim that numbers are objects.

1. Introduction

The purpose of what follows is to discuss Frege's realism about mathematical objects, with the aim of trying to get somewhat clearer about two things: first, the location of Frege's distinctive brand of realism on the map of current debates about mathematical realism and secondly, the effect of the paradox on Frege's realism.

By a "realist" in what follows I shall mean a realist about mathematical objects, which is to say someone who takes it that the truths and falsehoods of mathematics are true and false in virtue of the properties of and relations between distinctively mathematical objects, objects of the kind a naïve reading of these truths and falsehoods would lead one to take them to be about. '$9 > 7$' is, on this view, true in virtue of the relation of *greaterness* holding between the two objects 9 and 7, or to put it better, in virtue of the fact that 9 is greater than 7.

Perhaps the most important objection these days to realist theories is the one made popular by Benacerraf [2], the problem of our access to mathematical objects. If mathematics were really about abstract objects, the objection has it, then since these objects would be causally disconnected from us, it follows via the best accounts of knowledge we have—namely, causal ones—that we would be unable to know

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any truths of mathematics. And since we do know some mathematical truths, the conclusion to be drawn is that they cannot, after all, be about abstract objects.

Recently, Burgess and Rosen have pointed out that our "best theories of knowledge" only went causal in response to Gettier problems, and that in this tradition, the causal connection between knower and object of knowledge only plays the role of elevating justified true beliefs to knowledge and is not a necessary element of justification (see Burgess and Rosen [5], pp. 35–37).

Having noticed this, it might seem that the appropriate realist response to the access problem is simply to note that justified true belief is really what ought to be at issue. The failure of justified and true mathematical beliefs to make the rarefied grade of the contemporary epistemologist's version of "knowledge" poses no problem, according to this response, for a mathematical ontology.

Though there's something right about this kind of reply, it does not entirely address the access problem. The central issue here is that in the absence of an acceptable account of some epistemically relevant kind of contact with abstract objects, whether that contact is causal or not, the realist has a large, unexplained gap at the heart of her theory. If in fact the realist's ontology rules out all access to the subject matter of mathematics, then it seems prima facie to rule out the acquisition even of justified, true mathematical beliefs. Without some account of how such belief acquisition is possible, then, there is a strong prima facie reason for rejecting a realist ontology.

Perhaps the most straightforward realist response to the access problem is what we shall call the *platonist* response, typically associated with the work of Plato and of Gödel. The distinctive idea of this tradition is that we do, after all, have some kind of direct contact with individual mathematical objects, contact that is different from but similar in principle to the sensory contact we have with physical objects. As Gödel famously puts it, we have "something like a perception . . . of the objects of set theory" (see Gödel [10], p. 271).

Along the same general lines is the response favored by Maddy [16] in 1990, according to which we have sensory contact with paradigmatic mathematical objects (sets) in virtue of having causal interactions with their members in much the same way that we have sensory contact with physical objects in virtue of having causal interactions with (time-slices of) their surfaces.

In both of these cases, the access problem is tackled head on by attempting to describe a kind of knowledge-conferring relationship that we do bear, contra the premise of the access challenge, to mathematical objects.

On the other side of the issue, the antirealist about mathematical objects has two options with respect to sentences such as

1. $2 + 2 = 4$;
2. There's a prime number between 11 and 16.

She can claim either that the sentences are false or that they don't wear their ontological commitments on their sleeve. We'll ignore the first of these in what follows, focusing on what we'll call the 'standard antirealist', one who takes the usually accepted statements of mathematics to be (probably, mostly) true, but who holds that they are shorthand for statements that do not call for the existence of such things as numbers. The articulation of this view standardly includes a recipe for eliminative paraphrase, that is, for rewriting ordinary arithmetical statements in such a way that they appear, so rewritten, not to require the existence of mathematical objects.

It's easy then to view the two available options regarding mathematical truth as follows:

1. Realism about mathematical objects, with a solution to the access problem;
2. Antirealism, motivated by the access problem and defended by a strategy for giving reductionist paraphrases of the truths of mathematics.

While it is clear that both the realist and the antirealist must give an explanation of mathematical knowledge (or at least enough of a sketch of an explanation that it is clear that their position is compatible with a reasonable epistemology), it seems often to be supposed that the realist must do this by giving, in the first instance, an account of access to mathematical *objects*. That is, the assumption is that the realist's account of mathematical knowledge must explain this knowledge in terms of some knowledge-conferring relationship we bear to numbers and the like. For it is only under this assumption that the "access problem" gets off the ground. And it is only under this assumption that the two options (realism together with an explanation of mathematical knowledge in terms of access to mathematical objects and reductionist antirealism) exhaust the space of possibility for anyone who accepts the truth of standardly accepted mathematical statements.

2. Frege, Access, and Propositional versus Objectual Priority

Frege does not, of course, fit neatly into either of these camps. Frege believes that the numbers are self-subsistent objects, but his account of mathematical knowledge does not proceed via a solution to the access problem. Rather the reverse: for Frege, the explanation of knowledge "about" mathematical objects is given in terms of knowledge of mathematical propositions. And this propositional knowledge, in turn, is explained in terms of our capacity to reason in purely general ways in accordance with principles of logic. The fact that knowledge of purely general logical propositions is at the same time knowledge about mathematical objects is guaranteed, on this account, by the fact that the logical propositions (about, say, the coextensiveness of concepts) can be restated as facts about objects (specifically, about the extensions of those concepts) some of which are numbers.

Consider, for example, Frege's account of our knowledge that

(F) The extension of G = the extension of H.

As Frege sees it, (F) is not grammatically misleading; it is an identity statement, one which is true if and only if the object referred to by the first singular term is identical with the object referred to by the second singular term. But Frege's explanation of our knowledge that (F) will not typically proceed by appeal to some prior contact with these objects. To know that (F) is, as Frege sees it, to know that

(F′) $\forall x(Gx \longleftrightarrow Hx)$.

If 'G' and 'H' are short for the predicate phrases '. . . is an apple on my desk' and '. . . is a piece of fruit on my desk', then we know (F′) by whatever means suffice for knowing that all and only the pieces of fruit on my desk are apples on my desk—namely, by ordinary empirical means. When G and H are nonempirical, the route to knowledge of (F′) will be different, sometimes appealing to geometrical knowledge, sometimes to logical reasoning, and so on. The important point is that, in the usual cases, no appeal is made to prior "contact with" the extensions in question despite the fact that, as Frege sees it, knowledge of (F′) is at the same time knowledge of (F).

Similarly, via the account of numbers in terms of extensions, knowledge about numbers is explained in terms of our knowledge of purely general propositions. As Wright has put it,

> Frege's platonism . . . escapes the need to be bound up with the usual kind of platonist epistemological metaphors, the sort of idea which Gödel voiced when he spoke in terms of our having a kind of 'perception' of mathematical objects. The key epistemological issue will be not: how can we get into cognitive relations with the objects which constitute the subject matter of number-theory, but: how can we get into cognitive relations with the states of affairs that make for the truth of number-theoretic statements. (Wright [20], p. 25)

Frege's "reduction" of arithmetical statements does not imply, as he sees it, that the statements are not, after all, about numbers. Rather, it shows that the statements about numbers are equivalent to purely general statements, and hence that statements about numbers can be known without any kind of prior access to those numbers.

The Fregean account of our knowledge of numbers, and of abstract objects generally, turns on his context principle. "How . . . are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a sentence that words have any meaning, our problem becomes this: To define the sense of a sentence in which a number word occurs" (Frege [8], §73). Numbers are given to us via the understanding of propositions in whose expression number-words occur.

The two models of object-directed knowledge we have been looking at are these (where '$P(a)$' is short for any proposition about the object a):

(OF) [*Objects-First*]:	One comes to know that $P(a)$ by means of some contact with the object a;
(PF) [*Propositions-First*]:	One comes to know about an object a by means of coming to know some proposition(s) $P(a)$.

The objects-first picture is the natural one when the objects we are thinking about are physical or otherwise empirically accessible. It looks plausible to say that knowledge about ordinary physical objects must in some sense "stem from" physical contact with those objects—not necessarily on the part of the knower, but on the part of someone from whom the knowledge was passed on. If this is right, then a sure way of arguing against the possibility of knowledge that $P(a)$ is to demonstrate that nobody can have had the right kind of "contact" with a.

The important question, with respect to realism in mathematics, is whether the principle which looks plausible with respect to knowledge about empirical objects is also plausible when applied to knowledge about abstract objects. Is it the case that knowledge of a proposition which is "about" a mathematical object can only be explained in terms of some conceptually prior contact with that object? From the Fregean point of view, extensions (and numbers) are not the kinds of things with which we have such contact on the way to coming to know propositions about them. Or, to put the point slightly differently: to have the relevant kind of contact with extensions (or numbers) is just to come to know propositions about them. That knowledge of propositions of the form (F) can be explained in terms of the unproblematic knowledge of those of the form (F′), together with the fact that the former are about extensions, gives the Fregean realist a means of accounting, nonmysteriously, for knowledge of extensions.

The importance of noting alternatives to the simple (OF) account of knowledge-acquisition is that in the absence of an independent argument for (OF), the access problem does not get off the ground. The disconnectedness of abstract objects from our basic knowledge-gathering activities is, barring such an independent argument, no reason to doubt the fitness of these entities as objects of knowledge nonmysteriously acquired.

Frege's central example of (PF)-style knowledge-acquisition, and his most important counterexample to (OF), is, of course, unsuccessful: as revealed by the paradox, there are no Fregean extensions. The question for the realist of a Fregean stripe is whether a case can be made for understanding object-directed knowledge in some such propositions-first way despite the failure of Frege's own central example.

3. The "Broadly Fregean" Strategy

Take the "broadly Fregean" strategy of explaining our knowledge of abstract objects to be that of:

1. explaining knowledge of abstract objects via an explanation of knowledge of propositions that are explicitly about those objects; and
2. explaining knowledge of these "apparently committed" propositions in terms of knowledge of other propositions that are not so apparently committed and from which the apparently committed ones are obtained by what Frege calls the "recarving" of their content.

The strategy is illustrated not just by the case of extensions, but also, as Frege sees it, by the examples of directions, shapes, and numbers. To be "given directions" is to understand propositions involving directions, and the central of these, namely, propositions of the form 'The direction of $a =$ the direction of b' are understood by understanding their equivalence with 'a is parallel to b'.

> The judgement "line a is parallel to line b," or, using symbols, $a//b$, can be taken as an identity. If we do this, we obtain the concept of direction, and say: "the direction of line a is identical with the direction of line b. Thus we replace the symbol // by the more generic symbol =, through removing what is specific in the content of the former and dividing it between a and b. We carve up the content in a way different from the original way, and this yields us a new concept. ([8], §74)

This way of "obtaining" new concepts goes hand in hand with the recognition of previously unrecognized objects: when we arrive at the concept of direction via the recarving just described, we recognize the "new" object(s), *the direction of a* and *the direction of b*.

Similarly with numbers. While the recognition of directions turns on understanding equivalences of the form

(D) The direction of $a =$ the direction of b if and only if $a//b$,

the recognition of cardinal numbers turns on understanding equivalences of the form

(HP) $NxFx = NxGx$ if and only if $F \sim G$, now known collectively (following Boolos) as Hume's Principle.[1] Frege defines directions and numbers not directly via (D) and (HP), but rather (for reasons discussed below) as the extensions of appropriate concepts. (D) and (HP) are themselves derivable from the official definitions, and Frege's arithmetical results are all derivable just from (HP).[2] Because the definitions are given in terms of extensions, they fall prey to Russell's paradox and must be replaced if anything like the Fregean program is to succeed.

The replacement that most closely reflects Frege's intentions is that championed by Wright [20]. The heart of this neo-Fregean approach is to follow up on the point just noted, that (HP) together with the extension-free portion of Frege's logic suffices for the derivation of all the arithmetic Frege actually derived and, in particular, suffices for the derivation of the Peano/Dedekind axioms. The strategy, then, is to take (HP) as a fundamental principle of the system, justified by its status as an explication of cardinal number, and proceed from there in essentially Frege's own way.

A much discussed question about the neo-Fregean program concerns the analyticity of (HP). If (HP) is not analytic, then presumably the success of the program does not establish what Frege hoped to establish, namely, that the truths of arithmetic are demonstrably analytic.[3] We will leave this question aside here to focus on another: does the appeal to (HP), and the abandonment of extensions, leave Frege's *realism* intact?

A crucial question for the Fregean and neo-Fregean is as follows: What reason do we have to agree with him that the ontological commitments of sentences like the left-hand side of the equivalence (HP), namely, those of the form

(LHS) $\quad NxFx = NxGx$

are to be "read off" from the grammatical structure of these very sentences and not (just) from the structure of the admittedly equivalent instances of

(RHS) $\quad F \sim G,$

that is, of the right-hand sides of (HP)? Similar questions arise for directions, shapes, and so on. Why, that is, shouldn't we take the Fregean equivalences (which there must be if there is to be a Fregean strategy) as demonstrating the eliminability of direction-theoretic and number-theoretic discourse, and *hence* as showing that we are not after all committed to such things as directions and numbers when we assert the left-hand sides of the equivalences? Why not read the right-hand sides of, for example, (HP), (D), and so on as eliminative paraphrases?[4]

A parallel question arises for the eliminative antirealist. As Alston has pointed out, it is by no means obvious that the equivalence between an apparently ontologically-committed sentence (say, one about numbers) and an intended eliminative paraphrase shows that the original sentence's commitments are only apparent. The equivalence can, other things being equal, just as well be read as pointing out that the apparently uncommitted sentence is, despite appearances, about the suspect objects (see Alston [1]; also see [20], pp. 31–32).

While Frege, the neo-Fregean, and the typical eliminativist will all agree that each instance of (LHS) is strongly (logically, analytically, or definitionally) equivalent with the corresponding instance of (RHS), the equivalence licenses very different conclusions on each part. The eliminativist takes the equivalence to license the claim that the grammatical structure of (LHS) is misleading as to its logical structure. The Fregean and neo-Fregean, on the other hand, take the equivalence to show that the content expressed by both sides can be understood accurately either as an identity statement or as a claim about the equinumerosity of two concepts. This malleability of content is no ad hoc maneuver on Frege's part: it is essential to his overall account of abstract objects that contents are "recarvable" in precisely this way and that neither "carving" is to be preferred when assessing ontological commitment. The central difference between the eliminativist and the (neo-)Fregean is, in short, that the eliminativist takes only (RHS) to accurately reflect the underlying ontology of the claim expressed by each side, while the (neo-)Fregean takes *both* sides to accurately reflect that ontology.

The equivalences given by Hume's Principle do not by themselves, of course, provide a complete reductive paraphrase of numerical discourse. But they can be read as providing a crucial part of such a reduction. If talk apparently about numerical identity is "really" talk about 1-1 correlations, it is a short step to the view that all of number theory concerns simply claims about the cardinalities of arbitrary collections and not about peculiarly arithmetical objects. Consider, for example, the proposal of Hodes [15] on which apparent reference to natural numbers is construed as really a disguised means of invoking cardinality quantifiers. An axiom of infinity along the lines of Hodes's account, as asserting essentially the possibility of ever larger collections, would complete the antirealist construal of number theory suggested by the eliminative reading of the Fregean equivalences.

Against the background of an objects-first epistemology, the eliminativist's reading of equivalences like (HP) has a clear advantage. For under such an assumption, knowledge of, for example, (LHS), is problematic if and only if (LHS) is really about the objects $NxFx$ and $NxGx$. But without the presumption of the objects-first epistemology, this argument against the (neo-)Fregean reading of equivalences like (HP) is undermined. We are left with the question of whether there are any good arguments for or against the (neo-)Fregean claim that arithmetical discourse ought to be taken at face value.

I would like to turn here to one strand in Frege's writings which might seem to give the (neo-)Fregean realist an immediate answer to the eliminativist challenge, namely, the position known as the *syntactic priority thesis*.

4. The Syntactic Priority Thesis

The syntactic priority thesis, attributed to Frege first by Dummett, is the thesis that an object is, by definition, just whatever can be referred to by a singular term.[5] On this view, the existence of an object is evidenced immediately by a true (or false) sentence containing a singular term. It might seem that the holding of the syntactic priority thesis means that the eliminativist challenge never gets off the ground since, on Frege's conception of objects, the mere fact that the sentences in question *appear* to be committed to such things as directions and numbers suffices to show that they are. I think in the end that this is not a successful way either for Frege or for neo-Fregeans to respond to this challenge, and it is worth seeing why.

If the syntactic priority thesis is to be usable in determining when an object has been referred to, it is, of course, essential that there be an independent characterization of the notion of *singular term*, a characterization that does not turn on any prior grasp of the notion of *object*. Frege himself offers no such characterization, but Dummett [6] has proposed that singular terms be those terms *a* which, in addition to being associated with a criterion of identity, render the following inference patterns valid:

1. From $A(a)$, to infer that there is something such that $A(\text{it})$;
2. From $A(a)$ and $B(a)$, to infer that there is something such that $A(\text{it})$ and $B(\text{it})$;
3. From the claim that it's true of a that either $A(\text{it})$ or $B(\text{it})$, to infer the claim that $A(a)$ or $B(a)$.

Let's consider the application of this criterion in the context of a debate about whether there are such abstract objects as *books*, that is, those things of which you and I can be said to have read the same one in virtue of having read different tokens. The realist who accepts Dummett's criterion for singular termhood will take the validity of, for example,

1. I have read *Jane Austen*;
2. You have read *Jane Austen*;
3. Therefore, there's something that both you and I have read;

as evidencing the claim that '*Jane Austen*', in this context, is a singular term. This should be enough, on the syntactic priority thesis, to demonstrate that there are books.

The difficulty is that the antirealist about books will be inclined either (a) to read the conclusion (3) at face value but deny the validity of the argument, or (b) to accept the validity of the argument while offering an eliminativist account of what's "really meant" by (3), for example, that you and I have read appropriately similar tokens.

The availability of the first response highlights the fact that agreement with Dummett's Frege about the syntactic priority thesis and even, additionally, about the criteria for singular termhood, will not bring in its wake agreement about which objects there are. For starting from such agreement, the validity of those arguments on whose status the singularity of the terms in question depends cannot be determined without a prior decision about whether those terms in fact pick out determinate objects. We can't tell, despite acceptance of the syntactic priority thesis, whether (3), read literally, is true unless we already know whether there's an abstract object to which '*Jane Austen*' refers.

The second response is more to the point in the mathematical case. If the apparently existential conclusions in question are misleadingly phrased and do not literally assert the existence of an object (indeed, are consistent with the *denial* of the existence of any object referred to by the terms in question), then the terms qualified via these arguments' validity as "singular" cannot be said to refer to objects in any ordinary sense. The syntactic priority thesis, and its associated criterion of singular termhood, together give us a reasonable criterion of objecthood only if restricted to arguments whose apparently existential conclusions are not shorthand for something else altogether. And since it is precisely this issue—whether or not to prefer an eliminativist paraphrase of apparently existentially committed mathematical discourse—that is at issue between the (neo-)Fregean and the eliminativist, appeals to the syntactic priority thesis can do nothing to move the debate forward.[6]

5. The Caesar Problem

Frege's own definitions of the numbers would, if successful, have fended off the kind of eliminative reconstrual of arithmetical discourse discussed above, which is to say that they would have done what the syntactic priority thesis alone cannot do.[7] It is worth retracing Frege's own steps toward these definitions in order to see the difficulties left for the neo-Fregean. We begin with the case of directions.

Frege's basic idea, again, is that we arrive at the concept of direction via "a process of intellectual activity which takes its start from the intuition" of straight lines ([8], p. 75). What we know, via this process, is that the direction of a line a is what a shares with all and only the lines parallel to it. We might try to put this idea into a definition, Frege notes, by stipulating that 'the direction of $a =$ the direction of b' is henceforth to mean what 'a is parallel to b' already means. But though this definition reflects our means of acquiring knowledge of directions, it ultimately fails as a definition. It is unacceptable, says Frege, because it fails to settle the truth conditions of a wide variety of statements involving the term 'the direction of a'. It fails, in fact, to establish truth conditions for any such statements aside from those of the form 'the direction of $a =$ the direction of b'; and it fails, most importantly for Frege, with respect to identity statements 'the direction of $a = q$', for q not of the form 'the direction of b' (see [8], pp. 76–79).

In response to this problem, Frege proposes the definition in terms of extensions:

The direction of $a =$ the extension of the concept *line parallel to a*.

Given the assumption that, as Frege puts it, "it is known what the extension of a concept is" ([8], p. 80), this definition does provide truth conditions for all statements involving the phrase 'the direction of a'. It also continues to reflect the all-important connection between sentences of the form 'a is parallel to b' and 'the direction of $a =$ the direction of b' since the biconditional (D) linking these sentences is derivable from the definition.

Precisely the same issue arises with the attempt to introduce numerical terminology via (HP). The central Fregean idea, namely, that we arrive at a knowledge of cardinal numbers by recognizing that the cardinality of a concept is just what it shares with all and only equinumerous concepts, might lead one to define 'the number of Fs' contextually by stipulating that 'the number of Fs $=$ the number of Gs' is to mean what is already meant by 'F is mappable 1-1 onto G'—in short, to take (HP) as a definition. But, of course, such a stipulation will fail to give truth-conditions for a wide variety of statements involving the terms in question, and in particular will fail to give truth conditions for statements of the form 'the number of Fs $= c$' for c not of the form 'the number of Gs'. This problem, the so-called Julius Caesar problem, is apparently the primary reason Frege opts for the definition in terms of extensions.

The first question to raise is why Frege cares so much about the unusual identity contexts, those of the form 'the direction of $a = q$' and 'the number of $Fs = c$' for q and c not of the form, respectively, 'the direction of b' and 'the number of Gs'. They are not, after all, statements whose truth conditions we shall ever care very much about: it presumably matters not a bit to the mathematical usefulness of an account of the number seven whether it follows from that account that seven is not Caesar, as long as enough mathematics does follow from it.

The answer, at least in part, is that for Frege the ability of a linguistic item to appear on either side of the identity sign in true or false statements is the hallmark of object denotation (see [8], p. 116). Objects are exactly the arguments of the identity function, with the result that if a sentence of the form $\alpha = \beta$ lacks a truth value, then either α or β fails to refer to an object. If 'the number of Fs $=$ Caesar' lacks a truth value, then 'the number of Fs' does not refer to an object. And it is essential to Frege's program that his numerical terms refer to objects.

We can now see the close connection between the Caesar problem and the eliminativist challenge. If we try to give the meaning of the functor 'the number of . . . ' by stipulating that each instance of (LHS) is to have the same meaning as the corresponding instance of (RHS), then the eliminativist is, by Frege's own criteria of objecthood, right to claim that (LHS) is a mere shorthand for (RHS) and is not an identity statement strictly speaking. Its apparent singular terms fail to refer to objects, and the Fregean reduction is not a realist one.

Frege's own response to the problem was to define (LHS)'s singular terms directly as object-denoting terms, thereby eliminating any question about the object-directedness of (LHS). Once we take '$NxFx$' and '$NxGx$' to stand for determinate extensions, it is clear (given the Fregean assumption that extension-talk is clearly about determinate objects) that (LHS) is indeed about objects and that its equivalence with (RHS) can give credence only to Frege's account of the epistemology of abstract objects and not to the eliminativist's story about (LHS). In short, if the numerals stand for extensions, then they clearly stand for objects.

The neo-Fregean, however, has no such recourse. What's required for the persuasiveness of the neo-Fregean program is some reason to take terms of the form '$NxFx$' as functioning to pick out objects and not merely as syntactically misleading parts of statements better phrased as (RHS).

Looked at in this light, we can see that there are real problems with some of the more popular strategies for meeting the Caesar problem. The strategy suggested by Wright is to hold that the introduction of the numerical terms via (HP) does, despite Frege's worries, suffice to settle the truth conditions of the nonstandard identity statements. As Wright sees it, the recognition that (HP) is explicative of the concept of number suffices to establish that the open sentence '$NxFx = q$' is not satisfied by any objects whose identity conditions are given in terms other than that of the equinumerosity of associated concepts. Since people are not individuated in this way, for example, it follows that '$NxFx =$ Caesar' is false, as desired (see [20], pp. 113–17).

But this, as a general principle, supposes that knowing that the As are individuated by R and the Bs by the different R' suffices for knowing that no A is a B. And this cannot be right. Consider, for example, the open questions of the identities of mental states and brain states, of people and their bodies, and so on. If any questions of this kind, ones concerning entities that appear prima facie to be of kinds with different criteria of individuation, have affirmative answers, then Wright's principle fails straightforwardly. And indeed the mere conceptual possibility of such cross-type identities shows that the principle cannot be used to settle any such open questions.

We cannot settle the nonstandard identity statements by stipulating that '$NxFx = c$' is false whenever c is an object of such a kind that its identity criteria cannot be given in terms of the equinumerosity of associated concepts. For, to begin with, identity criteria only make sense with respect to a *kind* of object, not a particular object, and as long as a given object can in principle be of multiple kinds (mental event and physical event, number and set), there is no sense to the idea of the identity criteria of the object c. Secondly and more importantly, to know of an object c that it is not of a kind whose identity criteria are tied to equinumerosity requires that one know already that c is not a number.

If we knew that the (HP) identity criteria reflected in some sense the "essences" of cardinal numbers and that objects of the target kind K (say, people) failed to have essences of this kind, then we would know immediately via the indiscernibility of identicals that no numbers were objects of kind K. This, I take it, is something like the intuition behind the view that people are just the wrong *kinds of things* to be identified with numbers and that this is to be seen by noting their different identity criteria. But this compelling intuition about people will not help to settle identities of the form 'the number of Fs $= c$' in cases in which we do not take ourselves to be privy to the "essences" of the items referred to on the right-hand side—which, of course, is the general case. And, perhaps more to Frege's point, the appeal to such extra knowledge (about something like "essences") in settling the identities in question shows just what Frege claims: that (HP) is insufficient to settle them. That it is inadequate in this way is, again, the central problem, since this inadequacy makes (HP) unfit as a guarantor of the all-important claim that arithmetical discourse is really about objects.

Heck suggests that Frege's point in requiring truth conditions for all sentences of the form '$NxFx = c$' was to ensure the clear individuation of numbers from all other objects in order to be able to treat numbers as things to be counted, which is essential to Frege's way of providing an infinity of objects (see Heck [11]). As Heck points out, if this is Frege's idea, then it's misguided: we can take the numbers to be sui generis and still manage to count them: we deny that Caesar-type sentences make sense, but still use the numbers themselves to count both collections of numbers and collections of (other) objects and even mixed collections.

This point, as important as it is, does not resolve the whole of the Caesar problem. If the suggestion made above, that an essential part of such a solution must be to establish that the apparent singular terms are indeed genuine singular terms, that is, to fend off a particular eliminativist challenge, then Heck's suggestion does not do all Frege needs it to do. The introduction of numerical terminology via Hume's Principle leaves open the possibility that statements of the form '$NxFx = NxGx$' are simply misleading ways of claiming that there is a 1-1 correspondence between the Fs and the Gs, and that there are no numbers at all.

6. Conclusion

So far, it has looked like something of a standoff between the neo-Fregean realist and our sample eliminative reductionist. The most compelling argument in favor of an eliminative reading of arithmetical discourse (including (HP)) has been found wanting in its requirement of an as yet unsupported objects-first epistemology. But so far, we have seen no compelling reason to favor the (neo-)Fregean realist account of this discourse either.

There is, however, a good deal more for the realist to say. For both the Fregean and the neo-Fregean realist, our recognition of mathematical objects is the result of an extremely general kind of intellectual activity, one that is not limited to the mathematical realm. As Frege puts it, we "carve up" the contents of our judgments into objects and functions, that is, into those entities of which we predicate things and those whose invocation constitutes predication. Frege himself notes the recognition of directions, shapes, sets, and judgment-contents (propositions). If something like the Fregean propositions-first account of our knowledge of (abstract) objects is right, then presumably the "recarving" story is the beginning of an explanation of this kind of knowledge-acquisition. And something like this story must be invoked if we are to explain our knowledge of such things as languages, sentences, books, universities, nations, families, contracts, symphonies, and so on virtually ad infinitum. These are not entities with which we come into contact on the way to learning things about them; instead we come into contact with other, related entities (their members, or performances, or inscriptions, and so on) on the way to forming judgments, the more sophisticated of which can (or must) be understood as being about the abstract objects themselves. And though the Fregean account of "recarving" as a route to knowledge of mathematical objects is sorely lacking in detail, its strength is that it takes this means of acquiring knowledge to be of the same type, broadly speaking, as it seems prima facie must be invoked in any case to account for a large amount of ordinary nonmathematical knowledge.

The challenge to the eliminativist is now sharpened. Prima facie, it seems evident that we have a good deal of knowledge about a virtually unlimited number of ineffable entities, entities like political organizations and musical compositions. If the eliminativist is to claim that there is something incoherent about the general object-disclosing activity involved in both the discovery of numbers via the recarving of claims about concept-equinumerosity and the discovery, say, of books via the recarving of claims about token-similarity, then we are owed an explanation of what "really" is going on when the talk seems to be about such things as books. What is required in this case, from the eliminativist, is a *general* eliminativist strategy, not a piecemeal recipe for the replacement of number-talk by something else, but rather a global strategy for the replacement of the whole vast realm of abstract-object talk by something more streamlined. For, again, the antirealist's epistemological complaint about realism is a complaint that concerns all (purported) abstract objects, not just numbers. The complaint issues in an impossibility claim: that knowledge about such ineffable entities would be impossible (or, at least, problematically mysterious). In order to make such a claim plausible, the antirealist must say something about what appears to be a wealth of counterexamples. Because unreconstructed discourse and knowledge about such things as magazines, societies, songs, and so on stands directly in conflict with the antirealist's epistemological strictures, the plausibility of those strictures themselves requires a general reconstruction of this kind of discourse and knowledge. Without such a general reconstructive strategy, the antirealist's epistemological scruples are on shaky ground. And not only is it difficult to see how such a reconstructive project might proceed, but the failure of every early twentieth-century attempted reduction gives us good reason to suspect that no such strategy is workable.

The Fregean approach to abstract objects is, I think we've learned, in its detail much too simple. We are not "given" objects just by being given a straightforward template for equivalences of the (HP) kind. We are given much more, and what we are given is essentially a whole practice, a way of reasoning and speaking and acting. This is part of the lesson of the Caesar problem. But despite perhaps an oversimplification, the broadly Fregean program has a great deal to be said for it if anything like the general epistemological story just sketched is on the right track. If we do acquire knowledge of nations, sentences, years, and so on via an understanding of propositions about these things (and not vice versa), then the broadly Fregean account of mathematical knowledge is the one that takes mathematical knowledge to form a relatively nonexceptional, seamless part of ordinary human knowledge.

The primary effects of the paradox on the Fregean account are twofold. As noted above, the paradox undermines Frege's straightforward way of resolving the Caesar problem, and hence opens up his general strategy to the eliminativist challenge. The (neo-)Fregean realist is left with the extremely important problem of saying something about the "alternative contexts," that is, about statements involving number-words that are not of or reducible to the form '$NxFx = NxGx$'. Perhaps, as hinted at above, the appropriate realist response to this challenge is to recognize the role of propositions in addition to (HP) the understanding of which gives rise to knowledge about numbers.

Secondly and perhaps more importantly, the paradox affects Frege's account of propositional content. The broadly Fregean strategy turns, again, on the possibility of "recarving" the content of a sentence in such a way that propositions not obviously about particular objects (and knowable without any prior contact with those objects) are seen to be equivalent to propositions that are in fact about those objects. Because Axiom V is an instance of recarving, we learn from the paradox that recarvings are not as easy as they might appear. We cannot take ourselves unproblematically to be able to "discover" objects willy-nilly whenever we have, say, an equivalence relation between objects or functions. As Frege puts it, recarving involves "the transformation of a concept into an object," and this is what we have seen is not as easy as he took it to be. The upshot of this is that the Fregean realist needs a strategy for defending particular recarvings and for defending the claim that particular (apparent) singular terms are genuinely referring. Perhaps, again, the avenue to be pursued here will involve both the recognition of the close connections between mathematical singular terms and nonmathematical ones, and the recognition of the role of the broader use of those terms, in addition to their use in such central statements as (HP).

Despite these difficulties, things look worse for the antirealist. In order to pursue the strategy of the access challenge, the antirealist is faced with two rather difficult options. The first is to stick with the general objects-first epistemology, defending the access challenge to all purported knowledge of abstracta. This will require providing some reason to think that a global eliminativist project is workable. And the more we learn about both knowledge and the world, it seems, the less plausible this supposition becomes. Perhaps more plausible is the second option, which gives up on the attempt to defeat *all* knowledge of abstracta, but holds that there are specific problems with knowledge of mathematical abstracta. And perhaps there is indeed something more problematic about numbers than there is about the other various kinds of abstracta with which we have (or seem to have) commerce every day. But if so, then the burden of argument is squarely with the eliminativist. If there is no general problem about knowledge of abstract objects posed by their abstractness, then the claim that eliminativist paraphrase is to be preferred to realist original in the mathematical case needs a specific argument, one which has not yet made its way into this debate.

Notes

1. '$F \sim G$' is cashed out in terms of the 1-1 mapping of F onto G. '$NxFx$' is shorthand for 'the number of xs such that Fx', or more briefly 'the number of Fs' (see [8], pp. 73–74, pp. 81–85; and Boolos [3].)

2. That is, (HP) together with the system of higher-order logic obtained from Frege's *Grundgesetze* system by removal of Axioms V and VI suffices for Frege's results. This point has been noted by Parsons [17], p. 164. See also Wright [20], especially xix; also Heck [13].

3. See Boolos [3] and [4], Dummett [7], Shapiro and Weir [18] and [19], Wright [20] and [21], and Heck [12].

4. This question has been discussed by Heck in [14].

5. See [6], Chapter 4; also [20].

6. A side question is whether Frege himself actually held the syntactic priority thesis. Though Frege clearly held that objects are what singular terms refer to, the question is whether this statement on his part was intended as a true definition of 'object', or as a mere clarification for an audience assumed to be already in command of a rough concept/object distinction (along with the singular/predicative distinction). The latter would have been the case if, for example, Frege wanted to emphasize (as he frequently did) that by 'object' he did not mean anything so narrow as 'material object'. Two reasons to take Frege's statement not to have been intended as definitional (and hence to reject the idea that he held the syntactic priority thesis) are: (i) that his silence on the notion of *singular term* makes it unlikely that the all-important notion of *object* was to have been analyzed in terms of it; and (ii) that for Frege, the notions of concept and object are simple, unanalyzable, and indefinable, which would conflict directly with the syntactic priority thesis. See, for example, Frege [9].

7. In order to be "successful" here, not only would Frege's talk of extensions have to be consistent; additionally that talk would have to make it plain that terms of the form 'the extension of *F*' are genuinely object-denoting.

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