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CARDINALITY, COUNTING, AND EQUINUMEROSITY

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Abstract Frege, famously, held that there is a close connection between our concept of cardinal number and the notion of one-one correspondence, a connection enshrined in Hume's Principle. Husserl, and later Parsons, objected that there is no such close connection, that our most primitive conception of cardinality arises from our grasp of the practice of counting. Some empirical work on children's development of a concept of number has sometimes been thought to point in the same direction. I argue, however, that Frege was close to right, that our concept of cardinal number is closely connected with a notion like that of one-one correspondence, a more primitive notion we might call just as many.

1. Opening

It is by now widely recognized that Frege's *Foundations of Arithmetic* [8] pursues its philosophical goals, at least in part, by pursuing mathematical ones. Indeed, the centerpiece of the book is what we now know as Frege's Theorem: Given appropriate definitions of arithmetical notions, the basic laws of arithmetic can be derived from a single (nonlogical) axiom, HP, which says that the number of Fs is the same as the number of Gs if and only if the Fs are in one-one correspondence with the Gs. This beautiful result has been the object of a good deal of study and its inner workings are well understood. It is far less clear, however, what philosophical significance it might have. Wright, who was the first clearly to recognize the Theorem's philosophical potential,[1] has suggested that it supports a reformed logicism, according to which the basic laws of arithmetic are, though not *logical* truths, still *analytic* or *conceptual* truths, insofar as they are logical consequences of an analytic or conceptual truth, namely, HP.

Almost every aspect of Wright's proposal has been the subject of controversy.[2] When it has not just been met with the Incredulous Stare Objection, the claim that HP

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is a conceptual truth has been vehemently disputed. There have also been doubts about the epistemological neutrality of the "logic" needed to derive arithmetic from HP: The principles one needs include impredicative instances of second-order comprehension; one might wonder whether the class of conceptual truths is closed under deduction of this kind. But suppose we accept that HP is a conceptual truth and that analyticity is preserved by second-order deduction. Must we also agree that the basic laws of arithmetic are analytic? Counterexamples to the inference one is tempted to make abound. To use one I have used before, Frege held that analysis, as well as arithmetic, was analytic. He did not, however, regard all of mathematics as analytic, since he agreed with Kant that Euclidean geometry is synthetic a priori. But the truths of Euclidean geometry can be proven in analysis (given suitable definitions). Were Frege's views inconsistent then? And since he knew of analytic geometry himself, were his views simply incoherent? I take it they were not.

Similarly, we know from Gödel that one can "do syntax" in arithmetic: Given appropriate definitions, one can prove the basic principles of syntax, for any of a wide class of languages, in arithmetic. Is it then an *arithmetic truth*, say, that there are infinitely many sentences of the language of arithmetic or that '$0 = S$' is not one of them? I would suppose not. The truths of arithmetic, it seems to me, are quite independent of the existence of even one linguistic expression.

What Gödel showed is that syntax is *interpretable* in arithmetic: One can map sentences of the language of syntax onto those of the language of arithmetic in such a way that axioms of syntax go over into theorems of arithmetic and derivations in syntax go over into (typically more complicated) derivations in arithmetic; for certain purposes, therefore, one can think of arithmetic as if it contained syntax as a subtheory. (Similarly, Euclidean geometry is interpretable in analysis.) But nothing follows about the epistemological status of the laws of syntax. What one proves when showing that arithmetic interprets syntax are not the laws of syntax but their "translations" into arithmetic—and the notion of translation that is in play here is a purely formal one.

What Frege's Theorem shows, most fundamentally, is that arithmetic—by which I mean second-order arithmetic as standardly axiomatized, following Dedekind and Peano, that is, PA—is interpretable in Frege Arithmetic (FA), second-order logic plus HP. Even if HP is analytic and second-order logic is rightly so called, it does not follow that arithmetic is analytic. Even granting these assumptions, it follows only that arithmetic is interpretable in an analytically true theory; it does not follow that its truths are provable in such a theory. To draw that conclusion, we need to know that, when we "translate" sentences of FA into sentences of arithmetic via the "definitions" of arithmetical notions Frege bequeathed to us, we are actually *translating*. We need to know, that is, that sentences of FA express the very propositions about numbers we prephilosophically employ: We need to know, for example, that the sentence of FA that translates, say, 'Every number has a successor', *says* that every number has a successor.[3]

I am not going to discuss that problem directly, however. I mention it only to fix our attention upon the relationship between HP and our pretheoretic concept of number, for it is that relationship that I want to discuss here. We can approach it by considering some remarks Boolos made in connection with Wright's suggestion that HP embodies an explanation of the concept of number:

> [HP is] a biconditional whose right limb is a formula defining an equivalence relation between concepts F and G and whose left limb is a formula stating when the cardinal numbers of F and G are the same. Since the sign for cardinal number does not occur in the right limb, can one not appropriately say that HP *explains* the concept of cardinal number by saying what it is for two cardinal numbers, both referred to by expressions of the form "the number of . . . ", to be identical? Certainly. HP states a necessary and sufficient condition for an identity [concerning numbers] to hold. . . . So if one wants merely to sum up this state of affairs by saying that HP explains the concept of a cardinal number, I would not object. However, it is hard to avoid the impression that more is meant.[4]

And more, of course, is meant. Wright intends the claim that HP embodies an explanation of the concept of number to imply that it is *analytic of* the concept of cardinal number—and so that it is an analytic or conceptual truth, in much the way a definition would be. Boolos wants to block this argument:

> It is certainly true that *one* of the ways in which HP can be used is to fix the character of a certain concept. Here's how: lay Hume down. . . . But Hume is no different in this regard from any other statement that we might choose to take as an axiom. The axiom of choice fixes the concept of set in a similar manner. . . . The principle of mathematical induction fixes the character of the natural numbers. The statement that bananas are yellow fixes the character of the concept of a banana. So nothing is said when it is said that one of the roles of HP is to fix the character of the concept of a cardinal number. (Boolos [1], p. 311)

Boolos's claim is that, in whatever sense HP fixes the concept of cardinal number, *all* truths help "to fix the character" of the concepts they concern. Note especially the remark about bananas: No one anymore would think it *analytic* that bananas are yellow. What is behind these remarks, obviously, is a Quinean skepticism about the analytic-synthetic distinction—one that can do any philosophical *work*, anyway. Boolos does misplay his hand a bit, since Wright's claim is not that HP is "a super-hard truth," as Boolos suggests: The issue is not certainty or unrevisability. But Wright *is* claiming that HP is a special sort of truth in *some* way: It is supposed to be the *fundamental* truth about cardinality, *the* truth that fixes the character of our concept of cardinality and so, in that sense, the truth on which all other truths about cardinality rest; in particular, HP is supposed to be more fundamental, in some sense, than the axioms isolated by Dedekind and popularized by Peano. So Boolos's Quinean skepticism is entirely appropriate: Quine's real point, I was taught, is that there is no distinction among truths *period* that will do any philosophical work. But one does not have to accept this extraordinarily strong claim to be worried, and Boolos was sometimes given to press his point in a different way.[5]

What reason have we to think that HP is the "fundamental" truth about cardinality, whatever exactly that may mean? In particular, what reason do we have to think HP is more fundamental than the Dedekind-Peano axioms? It is true, of course, that the axioms of PA can be proven from HP and Frege's "definitions"; but, as I shall explain below, it is also true that (something very like) HP can be proven from (something very like) the axioms of PA and those same "definitions." Those are the mathematical facts—and Boolos would never have suggested that they do not profoundly illuminate the concept of number. But Wright's view is that one of these mathematical facts is much *more*, or at least quite *differently*, illuminating than the other. He thinks that the interpretability of Peano arithmetic in Frege arithmetic is more significant, epistemologically, than the (approximate) converse. But why? Waive the Quinean skepticism. Allow that talk of foundations makes sense. Allow that the question what follows from HP *might* be of more epistemological significance than what follows from the Peano axioms. Why *should* it be thought more significant? We all know what the Fregean answer is: It is that the concept of number is fundamentally bound up with the notion of one-one correspondence. But the question Boolos wants answered is what reason we have to suppose that it is any *more* bound up with the notion of one-one correspondence than it is with the facts about the structure of the natural numbers that the axioms of PA codify.

Two sorts of responses to this question come immediately to mind. The first sort of reply we might call metaphysical. It would hold that among the truths themselves, as it were, some are more fundamental than others, and that HP is, as it happens, more fundamental than the axioms of PA. One does sometimes find intimations of such a conception in Frege, when, for example, he speaks of "the dependence of truths upon one another" [8], but I find such ideas obscure and suspect I'm not the only one who does.[6]

The other sort of reply is an epistemological one: On this view, HP somehow grounds our *knowledge* of arithmetic. That may seem more promising, but is it? After all, one's first skeptical thought might be, it is not at all clear that HP is even *known* by ordinary, unreflective masters of the concept of number, let alone that it grounds their knowledge, for example, that $7 + 5 = 12$. Of course, it will quickly be said, no one is claiming that our ordinary knowledge of arithmetic is really *derived from* some prior knowledge of HP. Something like the distinction between 'context of discovery' and 'context of justification' (which we do, of course, find in Frege) must be invoked if the epistemological conception of HP's centrality is to be defensible. But how should that distinction be put to use?

One way would be to deny that our ordinary knowledge of arithmetic is even at issue. Perhaps the question should be not how we *do* come by knowledge of arithmetic but how someone *could* do so. Even if no one ever does, the idea might be, still someone *could* acquire a concept of cardinal number by receiving HP as an explanation. If she did, HP would then be analytic of her concept of cardinal number and hence would be a conceptual truth for her. So, if she had powers of thought and reasoning similar to ours, she would then be in a position to define the basic arithmetical notions in Frege's way and to prove the basic laws of arithmetic, the axioms of PA, in particular, from HP. So (assuming questions set aside above to have been answered), she would have analytic, a priori knowledge of those laws. Hence . . . well, *what* exactly? It is tempting to want to say: So arithmetical truths are analytic. But what would that add to the claim that someone *could* have analytic, and so a priori, knowledge of the laws of arithmetic?

Indeed, of what interest is that claim anyway? Even if we could earn a right to it, we would have been given no reason to suppose that *we* have, or even *could* have, analytic, or even a priori, knowledge of any arithmetical truths at all. And, I take it, when Kant claimed that arithmetical knowledge is synthetic a priori, he meant to be making a claim about *our* knowledge of arithmetic, not God's: As far as I can see, that is, a Kantian could just respond that it is *impossible* for possessors of finite minds to acquire the concept of cardinal number in the way that Wright suggests. And even if that objection could be met, it would remain an open question what our *actual* knowledge of arithmetic has or might have to do with HP, what the *actual* relationship is between our capacity for reasoning (the faculty of understanding, in Kant's framework) and our arithmetical knowledge.

What follows will, I hope, contribute to the study of those (ultimately empirical) questions by clarifying the relationship between our concept of number—the one we employ in ordinary, unreflective applications of arithmetic—and the notion of one-one correspondence. I shall argue that the relation is close and that there is a sense in which any master of the concept of cardinal number must know something akin to HP: in that sense, something akin to HP is a conceptual truth. I shall, however, draw no conclusions about the relation between our *knowledge* of HP and our *knowledge* of the properties of the natural numbers enshrined in the axioms of PA: that issue I must leave for another time. Even if there is no significant such relationship, though, a nonepistemological account of the philosophical significance of Frege's Theorem may be available, one that still requires a close relationship between HP and our ordinary concept of number. Unfortunately, I shall only be able to gesture at a possible view, since my own thought on the matter remains in flux.

2. Technical Preliminaries

Before we begin that discussion, however, I need to make a few remarks about technical matters and to make one further clarification about the scope of the present inquiry.

Stated in a form that is useful for comparison with Frege arithmetic, the Peano axioms are:

1. $\mathbb{N}0$
2. $\mathbb{N}x \ \& \ xPy \rightarrow \mathbb{N}y$
3. $\forall x\forall y\forall z(\mathbb{N}x \ \& \ xPy \ \& \ xPz \rightarrow y = z)$
4. $\forall x\forall y\forall z(\mathbb{N}x \ \& \ \mathbb{N}y \ \& \ xPz \ \& \ yPz \rightarrow x = y)$
5. $\neg\exists x(\mathbb{N}x \ \& \ xP0)$
6. $\forall x(\mathbb{N}x \rightarrow \exists y xPy)$
7. $\forall F[F0 \ \& \ \forall x\forall y(Fx \ \& \ xPy \rightarrow Fy) \rightarrow \forall x(\mathbb{N}x \rightarrow Fx)]$.

Here, obviously, '0' is supposed to denote zero, and '$\mathbb{N}$' is to be true of exactly the natural numbers. 'P' is a relational predicate that does the work of the more familiar functional expression 'S':[7] 'xPy' means: x (immediately) precedes y, or: y is an (immediate) successor of x. **PA** is the (full) second-order theory with nonlogical axioms (1)–(7).

HP is:

$$Nx : Fx = Nx : Gx \text{ iff } F \sim G,$$

where, of course, '$F \sim G$' abbreviates one of the usual formulas that define: the Fs and Gs are in one-one correspondence, for example,

$$(\exists R)[\forall x\forall y\forall z\forall w(Rxy \ \& \ Rzw \rightarrow x = z \equiv y = w) \ \& \\ \forall x(Fx \rightarrow \exists z(Gz \ \& \ Rxz)) \ \& \ \forall z(Gz \rightarrow \exists x(Fx \ \& \ Rxz))].$$

HP, in this familiar form, is *impredicative* in the sense that terms of the form '$Nx : Fx$' are terms and are of the same logical type as the first-order variables that appear in the formula displayed and, indeed, are of the same type as those bound by 'N' itself. So, for example, '$Nx : (x = Ny : y \neq y)$' is well formed. **FA** is (full) second-order logic with HP the sole nonlogical axiom.[8] The definitions Frege uses to interpret **PA** in **FA** are

(DZ) $0 = Nx : x \neq x$,
(DP) mPn iff $\exists F \exists y[n = Nx : Fx \;\&\; Fy \;\&\; m = Nx : (Fx \;\&\; x \neq y)]$,
(DN) $\mathbb{N}x$ iff $0P^{*=}x$,

where, in general, $R^{*=}$ is the weak ancestral of R defined thus:

$$aR^{*=}b \text{ iff } \forall F(Fa \;\&\; \forall x \forall y(Fx \;\&\; xRy \rightarrow Fy) \rightarrow Fb).$$

Frege's Theorem (in one form) is that (1)–(7), the axioms of **PA**, are provable in **FA** via the definitions (DZ), (DP), and (DN) which we shall collectively call **FD**. No converse is possible, however: It can be shown that HP is *not* provable in **PA**+**FD**, though, of course, **FA** is relatively interpretable in **PA** and so the two are equiconsistent.

HP is therefore stronger than it needs to be for the proof of Frege's Theorem. But now let 'Finite$_x(Fx)$' abbreviate some second-order formula defining the notion of (simple, not Dedekind) finitude; we may take it to abbreviate, say, the formula Frege uses in his work.[9] Now consider

(FHP) $\text{Finite}_x(Fx) \bigvee \text{Finite}_x(Gx) \rightarrow Nx : Fx = Nx : Gx \text{ iff } F \sim G.$

FHP says that HP holds so long as at least one of F and G is finite: it leaves it open what the standard of equality is when both F and G are infinite. The axioms of **PA** are provable from FHP and **FD**. More importantly, for our purposes here, FHP itself is provable in **PA**+**FD** given the additional axiom that every predecessor of a natural number is a natural number:[10]

(PAF) $\forall x \forall y(\mathbb{N}x \;\&\; yPx \rightarrow \mathbb{N}y)$.

For the most part, however, I won't fuss about the distinction between HP and FHP or that between **PA** and **PA**+(PAF).

The study of Frege's Theorem has revealed a good deal about HP and how it "gets its strength" as Boolos puts it. Axioms (1), (2), and (7) are consequences of **FD** alone, in fact, of (DN) alone. Axioms (3), (4), and (5) are derivable from HP and **FD** using only *predicative* comprehension; moreover, Frege's definition of '$\mathbb{N}$', of natural number—which is the only *really* controversial definition—is not needed for these proofs. Nor need one make any impredicative application of 'N' in these derivations. To put this more precisely, suppose we formulate HP in a two-sorted language, so that terms of the form '$Nx : Fx$' are *not* of the same logical type as the first-order variables bound by 'N', in which case terms such as '$Nx : (x = Ny : y \neq y)$' are *not* well formed. Moreover, take the logic of this theory to be *predicative* second-order logic. All axioms of **PA**, other than axiom (6), are provable in this theory via **FD**.

To prove axiom (6), on the other hand, one needs both to appeal to instances of impredicative comprehension and to allow impredicative applications of 'N'. What is surprising, though, is that axiom (6) is, as it turns out, provable *from axiom* (3) *alone*, given **FD**. Moreover—and this point, unlike those I've been making thus far, has not been made before—one needs only $\prod_1^1$ (or equivalently, $\sum_1^1$) comprehension for this argument.[11]

The axioms of **PA** thus divide into three groups. Axioms (1), (2), and (7) follow from Frege's definitions of arithmetical notions, in fact, just from the definition of number. There are all kinds of issues here, but these are quite independent of the foundational status of HP, so I'll set them aside for now. Axiom (6) is often taken to be the crux of the matter: But it follows from axiom (3) and Frege's definitions. There are big issues here, too, involving particularly the validity of "impredicative" applications of the cardinality operator. But I want to set these aside, as well, since they too are independent of the foundational status of HP.

The issue thus concerns the status of the remaining axioms, (3), (4), and (5), which are derivable from a predicative version of HP using only predicative comprehension (and whose proofs do not, as it happens, involve the definition of natural number at all). One might think that, if we regard HP as a conceptual truth, we must regard these axioms too as conceptual truths, since they follow logically, by just about anyone's lights, from HP. But we saw above that matters aren't so straightforward. For one thing, the derivation of the axioms depends upon the definitions of zero and predecession: One might wonder whether the definitions really *define* those notions. More importantly, we might wonder whether HP is a more fundamental truth than those expressed by the axioms mentioned. That is the question I aim to discuss here.

3. Frege and Husserl

We may approach the problem by considering one of the many disagreements between Frege and Husserl. Husserl famously argued that one should not explain the concept of number in terms of that of equinumerosity (or one-one correspondence), but should explain equinumerosity in terms of sameness of number, which in turn should be characterized in terms of the practice of *counting*:[12] that is, Frege ought to have said, not that the number of Fs is the number of Gs just in case F and G can be put in one-to-one correspondence, but that the number of Fs is the number of Gs just in case you get the very same number when you count the Fs as you get when you count the Gs. Now, Frege does consider a similar objection concerning what he regards as the analogous case of directions: The objection in this case is that parallelism ought to be explained in terms of sameness of direction, not conversely. Frege says, in response, that parallelism is more fundamental because all geometrical notions must originally be given in intuition ([8] §64). But it is difficult, to say the least, to see how to apply such an observation to the case of numbers and equinumerosity.[13] Frege's response, in his review of *Philosophie der Arithmetik*, is that "counting rests itself on a one-one correlation, namely, of the numerals 1 to n and the objects" to be counted. That there are n such objects follows, Frege intimates, only from the fact that, since the objects in question are in one-one correspondence with the numerals between 1 and n, the number of such objects is the same as the number of numerals between 1 and n which number is n.[14] Note that this argument appeals to (an instance of) HP.

Husserl, so far as I know, never responded to Frege. But there is a response to be had, and it has been forcefully developed by Parsons [16]. Parsons concedes (as one obviously must) that counting does *in fact* establish a one-one correspondence between the objects to be counted and some numerals. But simply to rest with that observation is to miss Husserl's point which concerns *conceptual priority*. Frege's observation, right though it may be, concerns, as we might put it, what someone who is counting *accomplishes*, not what she is *doing*: Frege gives us no reason to suppose that one must *set out* to establish a one-one correspondence when one counts. Of course, to be able to count,[15] one must in some sense realize that each object is to be counted once, and only once. But one can grasp that rule without thinking of oneself as establishing a one-one correspondence: One can have, as it were, an *operational* grasp of the rule of counting. For consider: Suppose that, instead of spoken numerals, we had a stack of Post-Its with numerals on them and that counting involved *literally* tagging objects with Post-Its. It would be a rule that one was to tag each object once and only once. But it's obviously one thing to be able to check whether all the objects in a given group have Post-Its on them and whether any object has more than one Post-It on it. It's quite another thing to have a *general* conception of one-one correspondence and to conceive of oneself, in tagging objects with Post-Its, as establishing such a relation between them.

Parsons himself develops a treatment of counting that is, in relevant respects, similar. Counting, he suggests, can be conceived on a demonstrative model. Think of '1' as meaning something like: the first. So when one counts, one is saying, as it were, 'the first, the second, the third, the fourth', where it is a rule of the game that one is supposed to "tag" each object exactly once, and then, when done, report how many objects there are by converting the last ordinal used to a cardinal. It is clear, once again, that one can grasp the "once and only once" rule without thinking of oneself as establishing a one-one correspondence between the objects and the numerals. Indeed, there is a more powerful point in the vicinity, namely, that *one need not conceive of the numerals as objects in their own right* in order to count. To miss this point is simply to confuse use and mention: The numerals are not *mentioned* in counting—they are not treated as objects to be correlated with baseball players, for example—but are *used*. One no more correlates the numerals one-one with the members of the Boston Red Sox when one counts them than one correlates their *names* with them when one lists the members of the team.

A conversation I had with my daughter Isobel, when she was about three, illustrates this point. She was, by then, pretty good at counting. So, sometime neo-Fregean that I am, I decided to investigate how well she understood the connection between number and one-one correspondence and set up a little experiment. We had some Barbies and some hats and put them on the table. "How many Barbies are there?" I asked her. "One, two, three, four. Four Barbies!" she said proudly. And then we spent some time with the hats. We saw that we could put a hat on each Barbie—just one—there not being any left once each Barbie had a hat. "Just enough hats for the Barbies!" she said. So now the question: How many hats are there? No amount of prompting would elicit the inference: Four Barbies; one hat for each; so four hats.

As it happens, this sort of phenomenon is well known to psychologists. In their important early study, Gelman and Galistel write:

> The preschooler's normal principle for determining whether two sets are numerically equal is "Count them and see." . . . [T]he child's procedure actually presupposes the establishment of a one-to-one correspondence. In counting, the child establishes a one-to-one correspondence between the elements in his count sequence and the elements in the set being counted. From a logical point of view, the child's procedure for deciding numerical equivalence depends upon the fact that the numerosities of both sets can be placed in a relation of one-to-one correspondence with the same set of counting tags. But the child does not ordinarily take cognizance of the transitivity of one-to-one correspondence. He ignores or is indifferent to the fact that the cardinal numerons representing two equally numerous sets are identical precisely because both sets have been placed in one-to-one correspondence with that cardinal numeron. (Gelman and Galistel [10])

I take Gelman and Galistel's observation to show that Parsons is right, and not just about the abstract conceptual possibilities: The conceptual structures of very young children are just as Parsons describes. They know how to count—and, to that extent, can answer questions of the form 'How many?' and 'Are there just as many?'. But such children have no, or only a very minimal, understanding of one-one correspondence and its bearing upon questions of cardinality.[16]

If we look at the situation formally, we might come to a similar conclusion on different grounds: It turns out, when one looks closely, that HP plays a very limited role in the proofs of axioms (3)–(5). Consider, for example, the case of axiom (5) which says that zero has no predecessor. HP allows us to prove that a concept is 'nonempty'—that it is true of some object—just in case it has the number zero. That is, HP allows us to prove that Zero-Concepts are Empty:

(ZCE) $\neg(\exists x)Fx \equiv Nz : Fz = 0.$

From (ZCE), axiom (5) then follows almost immediately. Applying Frege's definitions to axiom (5), we get

$$\neg\exists y[\mathbb{N}y \;\&\; \exists F\exists x(0 = Nz : Fz \;\&\; Fx \;\&\; y = Nz : (Fy \;\&\; x \neq z))]$$

which (ZCE) immediately implies. Similarly, in the cases of axioms (3) and (4), HP allows us to prove that Adjunction Preserves Cardinality and that Removal Preserves Cardinality:

(APC) $Nx : Fx = Nx : Gx \;\&\; \neg Fa \;\&\; \neg Gb \rightarrow Nx : (Fx \bigvee x = a) = Nx : (Gx \bigvee x = b);$

(RPC) $Nx : Fx = Nx : Gx \;\&\; Fa \;\&\; Gb \rightarrow Nx : (Fx \;\&\; x \neq a) = Nx : (Gx \;\&\; x \neq b).$

Again, axioms (3) and (4) follow nearly immediately, in the presence of **FD**.

There is a case to be made that these Three Principles, as I shall call them, are absolutely fundamental to the notion of cardinal number. Consider what (ZCE) adds to the definition of '0': If there is an F, the number of Fs is not zero; or conversely, if the number of Fs is zero, there aren't any Fs. That's pretty basic. If you don't realize that there being zero books on the table means that there *aren't any* books on the table, then you simply don't understand what 'zero' means—or, to put the point without mention of language, you don't have the concept *zero*. Or consider (APC): What it says is that, if you have the same number of Fs as Gs, and you adjoin an object to the Fs and an object to the Gs, then you will still have the same number. That too is pretty basic. If you have the same number of dolls as hats, and if you go get another doll and another hat, and you don't realize that you *still* have the same number of dolls as hats— then, or so it seems to me, you really don't understand what it means to say that you've got the same number of dolls as hats. You don't have the concept *sameness of number*. And similarly for (RPC).[17]

So the Three Principles are pretty basic: Arguably, one could not have even the most *primitive* conception of cardinality without at least tacitly grasping them. Is HP so basic? It seems easy to imagine someone's grasping the Three Principles and applying them in their dealings with numbers while not grasping HP. Husserl, Parsons, and Isobel at age three give us models for just this possibility. Further, since sameness of number is now being conceived in terms of the results of counting, and counting itself is done with a collection of numerals, features of the numerals with which one counts can't help but be *reflected*, as it were, in properties of the numbers. And that is what we have here: The Three Principles simply reflect features of the numerals. That (APC) holds, for example, is a reflection of the fact that there is always a definite "next" numeral. For assume the antecedent of (APC), that the number of Fs is the same as the number of Gs, that is, that in counting the Fs and the Gs I end with the same numeral. Now suppose I add an object to each pile. To count what I have now, I need only continue the previous counts: Hence, the only way I could end up with different numerals now is if the *next* numeral might be different in the two cases. But the structure of the numerals makes that impossible.[18]

Furthermore, one might argue, the Three Principles implicitly characterize sameness of number.[19] They do not *quite* fix the extension of '$Nx : Fx = Nx : Gx$': They will not, in particular, tell us whether the number of evens is the number of odds. But it is easy to see that the Three Principles *do* fix its extension on finite concepts; it is only a little harder to see that they fix its extension so long as *at least one* of F and G is finite.[20] So, on this view, that sameness of number is (at least as concerns finite concepts) one-one correspondence is, far from being the fundamental fact about cardinality, a *theorem*, a consequence of the *more* fundamental Three Principles.

I think that one or another version of the view I've just outlined underlies much of the opposition to Fregean accounts of the source of our knowledge of (certain of) the axioms of arithmetic. Consider, for example, structuralist views. The analogue of the idea that the numbers are given via the numerals is that the numbers are given, in the first instance, as elements of a certain sort of structure, one *exhibited* by the numerals. The analogue of the idea that the finite cardinals are characterized in terms of counting is the idea that they are characterized in terms of one-one functions onto initial segments of such structures (as in Dedekind). The structures in question are, of course, completely characterized (that is to say, characterized up to isomorphism) by the axioms of PA. The question whether our knowledge of those axioms depends upon our grasp of the notion of one-one correspondence thus becomes the question whether our grasp of such structures and our ability to employ them in thought requires grasp of the concept of one-one correspondence. And what we have just seen is that there is at least some reason to doubt it does: It appears that a grasp of the rules that govern the practice of counting—which presupposes some sort of grasp of the structure of at least *one* ω-sequence, that of the numerals—does not require a grasp of the notion of one-one correspondence. If not, then it would seem that one can have a perfectly serviceable concept of cardinality without so much as *having* the concept of one-one correspondence.

4. Counting and Cardinality

It can seem like so much common sense that a child who has learned to count and so has learned to answer questions like "How many cookies are on the table?" has acquired a concept of cardinality. But that is wrong: A child who can, *in that sense*, answer questions of the form 'How many?' need have no idea what the answers to such questions *mean*—or, indeed, what the questions mean. Such a child need have no concept of cardinality.[21]

Something like this point is, again, well known to psychologists. Here are a couple illustrations of it. Recall the story I told above about my daughter Isobel. Asked how many hats we had, her impulse was to count them, despite already knowing that we had four dolls and one hat for each doll. I didn't say before, though, that if I'd asked her again how many *dolls* there were, *she would not just have told me*: What children of that age do when asked how many Fs there are is to count, even if they've just finished counting those same objects. In fact, children of that age will count a group of objects as many times as one asks them "How many?" (until they get bored, anyway). It's almost as if they understand "How many dolls?" as meaning: Would you please count the dolls?

Such children do not appear to understand ascriptions of number, as Frege called them, either. If I had asked Isobel to give me three hats, her response would have been to grab some hats and hand them over. Whether she gave me *three* hats would have been a matter of chance. She wouldn't have given me *one* hat: Children of that age do understand the difference between one and not one. But it seems not to occur to them to *count out* three hats. Or again, if I'd shown Isobel two cards and asked her which had three balloons on it, she would have responded at chance—unless one of the cards had just one balloon on it, since children of that age do know one from not one. Perhaps one of the most amusing observations here is that children at this level of development will occasionally count with sequences other than the conventional numerals: For example, 'a, b, c', or even 'Monday, Tuesday, Wednesday'.[22]

Such children may well understand the numerals as mere tags, having no independent significance. For them, 'There are four hats on the table' *really does* mean something like: I ended with 'four' when I counted the hats. But they seem to have no grasp at all of the *point* of such "ascriptions of number".

Think of the matter from the child's point of view. I have been taught how to "count": I have, that is to say, been taught to point at objects and say certain words. I have to say the words in a certain order. I'm supposed to count each of the objects once and only once. And most importantly, when I get to the end, I'm supposed to say the last of the words loudly and proudly. But what all of this has to do with anything, who knows? The grownups seem to like it and they're forever fawning over me when I do it right. That's more than enough reason for me to play.

The moral of the story is that mastery of the practice of (transitive) counting is compatible with one's having no concept of cardinality at all. One can be able to answer questions such as "Are there as many Fs as Gs?" in the terms allowed by one's mastery of the practice of counting while yet having no idea what the significance of an answer to the question might be or even what the question means. And that is precisely the situation in which one will find oneself if one understands the question "How many Fs are there?" as meaning "With which numeral does one end when one counts the Fs?" and the question whether there are as many Fs as Gs as meaning "Does one end with the same numeral when one counts the Fs as when one counts the Gs?" Very young children simply illustrate this conceptual point.

So what *is* involved in understanding "How many?" questions as how many questions? What *is* the point of ascriptions of number? Well, suppose one knows that there are five cookies. By itself, there's not a lot one can do with that knowledge. But suppose one also knows there are four children. Now one has knowledge one can use, for one can infer that there are *more* cookies than children. But what does that mean? Or suppose one knew that there were five children. Then one could infer that there were *just as many* cookies as children. But what does *that* mean?

There is a familiar answer to this question, namely: To know that there are just as many cookies as children is to know that there is a one-one correspondence between the cookies and the children. That is a *bad* answer. It is far from obvious that knowing what 'just as many' means requires knowing what a one-one correspondence is, in general: The notion of a one-one correspondence is very sophisticated; it is far from clear that five-year-olds, who do seem to grasp the concept *just as many*, have any general grasp of one-one correspondence.

Another option would be to take the concept *just as many* to be characterized by analogues of the Three Principles. Let '$JAM_x(Fx, Gx)$' be read: There are just as many Fs as Gs, and consider the following principles:

(ZCE*) $\neg(\exists x)Fx \rightarrow [JAM_x(Fx, Gx) \equiv \neg\exists x(Gx)]$;
(APC*) $JAM_x(Fx, Gx) \,\&\, \neg Fa \,\&\, \neg Gb \rightarrow JAM_x(Fx \bigvee x = a, Gx \bigvee x = b)$;
(RPC*) $JAM_x(Fx, Gx) \,\&\, Fa \,\&\, Gb \rightarrow JAM_x(Fx \,\&\, x \neq a, Gx \,\&\, x \neq b)$.

If it is hard to see how one could have even the most primitive grasp of the concept *just as many* without at least tacitly grasping the Three Principles, much the same can be said about these "Starred Principles." And the remarks made above, about how the Three Principles characterize sameness of number, apply here, *mutatis mutandis*: Though the Starred Principles do not fix the extension of '$JAM_x(Fx, Gx)$' since they do not tell us whether there are just as many evens as odds, they do fix its extension so long as at least one of the concepts involved is finite. So the Starred Principles implicitly characterize equinumerosity, that is, just as many.

(This account of equinumerosity need not prevent us from characterizing sameness of number, and so the concept of cardinality, in terms of equinumerosity:

(HPJ) $Nx : Fx = Nx : Gx$ iff $JAM_x(Fx, Gx)$.

It's obvious that the original Three Principles follow immediately from HPJ and the Starred Principles. So axioms (3)–(5) follow from HPJ, the Starred Principles, and Frege's definitions of zero and predecession.)

The position just described has some attractive elements. Its answer to the question what 'just as many' means does have the desirable and necessary feature of being *un*sophisticated. But it suffers from problems not unlike those that afflict the Husserl-inspired view that gave a central place to counting. It seems obvious that someone could accept the Starred Principles and still have no idea what to make of the claim that there are just as many children as cookies. What we need, then, is some other unsophisticated answer to the question what 'just as many' and 'more' mean. Suppose that there are more cookies than children. What follows? It follows that there are more than enough cookies for each child to have one. From a *logical* point of view, that means that there is a one-one function from the children to the cookies which omits some cookie from its range. But from a *child's* point of view, it need mean no more than that, if you start giving cookies to children, and you're careful not to give another cookie to anyone to whom you've already given a cookie, then you won't run out and will even have some left. Or suppose there are just as many children as cookies. Then it follows that there are *just* enough cookies for each child to have one. From a logical point of view, that means that there is a one-one correspondence between the cookies and the kids; but from the child's point of view, it need mean no more than that, if you start giving cookies to children, and you're careful not to give another cookie to anyone to whom you've already given one, then you won't run out, though you won't have any left. There are enough, that is to say, for everyone to have one, but not for anyone to have two.[23]

The suggestion, then, is that grasping the concepts *more, just as many*, and so on, involves connecting them, in the right way, with what we might call their practical correlates: *enough, just enough*, and so on. Of the practical notions, the child has, as it were, an operational understanding:[24] Her understanding of what it is for there to be just enough is bound up with the idea of giving (just) one to everyone, or putting (just) one hat on each doll's head, or what have you; to say that there are just enough is to say that one *could* give (just) one cookie to each child. Of course, to give just one cookie to each child is, in fact, to establish a one-one correspondence between the cookies and the children, and to say that one could do so is to say that there exists such a correspondence. But this sort of contrast is now familiar from the case of counting. And from the case of renates and cordates, too: The concept *just as many* has the same extension as the concept *is in one-one correspondence with*, but that simply does not imply that the concepts themselves are the same. They are not.

On this view then, grasp of the concept *just as many* is in principle independent of the ability to count. It is, therefore, a mistake to attempt to characterize equinumerosity as Husserl did, by reference to the practice of counting. Moreover, grasp of the concept *just as many* is clearly independent even of the concept of number, that is, of any grasp of the significance of ascriptions of number: One can understand what it is for there to be just enough cookies to go around without understanding what it means to say *how many* cookies there are. If so, then it is wrong to attempt to explain equinumerosity in terms of sameness of number, whether characterized in terms of counting or not. On the other hand, it would seem that a grasp of the concept of cardinal number *does* require a grasp of the concept of equinumerosity: One will understand answers to how many questions *as* answers to how many questions—*as* ascriptions of number, rather than statements about the results of countings—only if one grasps the concept *just as many* and its relation to ascriptions of number. That relation, of course, is just the relation HPJ reports.

5. Counting and Ascriptions of Number

I have been arguing, to this point, that our concept of equinumerosity is independent of any connection with the practice of counting. As I just mentioned, however, the argument also shows that our concept of equinumerosity is independent of any grasp of the significance of ascriptions of number, that is, of the meanings of such sentences as 'There are four dolls on the table'. That is, according to me, our grasp of the concept of equinumerosity is independent of any grasp of the concept of cardinal number. It does *not* follow, however, that our grasp of the concept of cardinal number—our grasp of what *specific* assignments of number mean—is similarly independent of counting. Grant that when we say 'There are four dolls on the table' we don't just mean that you end with 'four' if you count the dolls: If that were all we meant, then we wouldn't be ascribing number at all. But there is an obvious alternative, namely, that 'There are four dolls on the table' means: There are as many dolls on the table as there are numerals between 'one' and 'four'. If that is what we mean, then it would seem that our understanding of ascriptions of number *is* bound up with our ability to count: Establishing that there is such a relation between the dolls and the numerals between 'one' and 'four' would, on this conception, be just what counting the dolls did—as Frege suggested, more or less.

I am shortly going to argue that this conception of the relation between counting and ascriptions of cardinality is mistaken. I should first like to emphasize, however, that the arguments I shall be giving do not focus upon the reference to *numerals*. They apply as well, for example, to Frege's proposal that 'four' means 'the number of numbers between zero and three'. And to Dedekind's, and Cantor's, closely related view that the cardinals may be defined in terms of the ordinals: The cardinal number of a set S is the least ordinal onto whose predecessors the members of S can be mapped one-one. My specific remarks are directed against the view that ties cardinality to counting, but their extension to these similar views should be fairly obvious.

I have three objections to bring against such conceptions of cardinality. None of them, I am prepared to admit, is independently conclusive, but their combined force is substantial. Sufficient, I say, when combined with the alternative conception I shall offer.

First of all, children, and adults, seem to be able to understand some attributions of cardinality quite independently of any connection with counting. For example, when I think that there are two blocks on the table, I do not seem to be thinking that there are as many blocks on the table as there are numerals from 'one' to 'two'. The significance of this point is unclear, however. One might still want to hold that our conception of specific cardinalities is, in general, given by the connection with counting, even if there are specific cases in which it need not be so given.

Second, there seems to be an important distinction between the sort of knowledge one has when one knows that there are as many Fs as numerals from 'one' to 'nine' and when one knows that there are nine Fs. In the latter case, one actually knows how many Fs there are; or, as Kripke (to whom the observation is due) puts it, one has *de re* knowledge in the latter case but only *de dicto* knowledge in the former.[25] Of course, one can easily calculate how many Fs there are in the former case. But consider a different case. If I am told that there are as many Fs as there are numerals between '1' and '21' base-16, then I, anyway, do *not* thereby come to know how many Fs there are. Presumably, there are people who *do* know how many 21_{16} is, but I don't, not immediately, not without calculating how many it is. What calculating means here, for me, is finding the decimal numeral, because knowing how many Fs there are, for me, is knowing such things as that there are $33F$s. It's not that the rule linking numerals to numbers is any more complicated in the hexadecimal case than in the decimal case: Just as there are 33 decimal numerals between '1' and '33', so there are 21_{16} hexadecimal numerals between '1_{16}' and '21_{16}'. The difference, rather, is that, once this simple rule has been applied, there is more work to be done in the latter case—but not in the former. That is because, in knowing that there are 33 numerals, I *thereby* know how many numerals there are; whereas in knowing that there are 21_{16} numerals, I do *not* thereby know how many.

Knowing how many is *knowing what number*. So, to reformulate the foregoing: I do not, without calculating, know what number the base-16 numeral '21' denotes, even though I *do*, of course, know that it denotes 21_{16} and know, moreover, that it denotes the number of hexadecimal numerals between '1' and '21_{16}'. My knowing what number '33' denotes, therefore, cannot consist in my knowing that it denotes the number of decimal numerals between '1' and '33': I have *that* kind of knowledge in the hexadecimal case, too.

Of course, one would like more to be said here, and I shall say none of it. But let me address one response I have frequently heard. It's easy to want to dismiss the foregoing, saying that the difference is simply that I am *more familiar* with decimal numerals than I am with hexadecimal ones. In some sense, that must, of course, be right: It's not as if there is anything intrinsically special about decimal numerals; the difference obviously has to be a matter of my *relation* to the decimal numerals, in the end. But it's not just my *familiarity* with decimal numerals that matters; it's not just a matter of my "comfort level," as one objector once put it. It's my *facility* with decimal numerals that matters. The important difference is that I am able to use decimal numerals immediately in thought, whereas I almost always have to translate hexadecimal numerals before I can do much of anything with them. That, however, doesn't undermine the point I'm making: It illustrates and establishes it.

Third, a point made above, in a somewhat different context, is also relevant here. The knowledge that there are five numerals between 'one' and 'five' is actually quite sophisticated. Isobel, who clearly has to put up with a lot, was not able to acknowledge this fact until some time after she had mastered counting and the concept *just as many* and had developed a thorough understanding of such claims as that there are five Beanie Babies on the sofa. For a long time, she had essentially no understanding of the question how many numerals (or numbers) there are between 'one' and 'five'. What she lacked, I think, was the concept of a numeral (or number) as an object: She had no concept of the numerals *as objects which can themselves be counted*. Numerals, as emphasized above, are *used* in counting; they are not *mentioned*. There is really no good reason that one who is able to count *with* numerals need be able to conceive of them as objects in their own right. If so, then the idea that 'There are five Beanie Babies on the sofa' means that there are as many Beanie Babies as numerals between 'one' and 'five' confuses use and mention. In doing so, it presupposes a conception of numerals as objects that the ability to count, and even to make judgments of cardinality, does not require.

How, though, *should* we understand attributions of number? What *does* the statement that there are nine Fs mean, if not that there are as many Fs as numerals between 'one' and 'nine'? A related question is how counting establishes how many there are, if not by establishing that there are as many Fs as numerals between 'one' and whatever.

The short answer to the first question is that attributions of number answer how many questions. What that means is, admittedly, less than obvious. To some extent, I think, the development of this famous Fregean doctrine awaits progress in semantics: It is not at all clear what the logical form of, say, 'Two men went for a walk' should be taken to be. But Frege's central idea is really quite simple: It is that one grasps the concept *two* just in case one knows how many Fs there are when there are two Fs. In this simple case, perhaps that's just knowing that there are two Fs if there are this F, and the other F, and no more Fs: Perhaps, that is, it's just grasping the usual first-order definition of 'there are two Fs'. Of course, larger numbers will pose more of a challenge: Understanding '145', say, does not consist in knowing that there are 145 Fs if and only if—forgive me if I omit the first-order equivalent. Rather, understanding '145' has something to do with one's understanding of the decimal system. It is a nice question (an empirical one, of course) what that involves—but it is not a question that bears fundamentally upon what is at stake here. For the reasons already given above, knowing how many '145' is isn't, in any event, a matter of knowing that there are 145 Fs if and only if there are just as many Fs as numerals between '1' and '145'.

More important is the question how we should understand counting. What alternative is there to the idea that counting establishes how many Fs there are by establishing that there are as many Fs as numerals from '1' to whatever? Some time ago, I was counting out tulip bulbs, and I did so this way: 2, 3, 5, 7, 11. That wasn't because I was using a nonstandard count sequence consisting only of prime numbers. It was, rather, because I was counting the bulbs not, as one says, by twos, but by groups. We count that way all the time. Some children learn to do so at a very young age, and without ever learning to count by twos intransitively (that is, just saying the words), let alone by groups (which would just be randomly, though increasing). What are we doing when we count like that? We are, I suggest, making a series of judgments of cardinality. One is saying, as it were, "That's two bulbs so far; now three; two more, so five"; and so forth. We can think of ordinary counting in the same way: That's one; now two; now one more, so three; one more, so four; four cookies in all.

Of course, I am not denying that counting can be done mindlessly, without making judgments of cardinality along the way. It is certainly done that way by very young children, and it can even be done that way by adults. But the question here is whether counting is, as it were, *fundamentally* a mindless exercise, and I mean to be denying that it is: I mean to be denying that we best understand what counting is, and how it functions, by conceiving it simply as a matter of tagging objects with symbols which do not, qua symbols used in counting, function to assign cardinal number. Conceived as my opponent wants to conceive it, counting can issue in judgments of cardinality only in one of two ways: Either 'There are four Fs' must mean: You end with 'four' when you count, or it must mean: There are as many Fs as there are numerals from 'one' to 'four'. I have been arguing against both of these construals, but let me not simply repeat myself. What is, in the end, most worrisome, or so it seems to me, is that, if counting were a mere tagging, then it wouldn't matter whether we counted with numerals or with days of the week or with letters of the alphabet. If counting were mere tagging, we should be just as happy to say 'There are 'i' dolls', meaning: There are as many dolls as there are letters from 'a' to 'i', as we are to say that there are nine. But we are not, and the reason is because counting is *not* mere tagging: It is the successive assignment of cardinal number to increasingly large collections of objects. And the symbols we use in counting are not mere tags: Even qua symbols used in counting, they function to assign cardinal number.

Note that counting, conceived as I am suggesting it should be, has nothing essential to do with equinumerosity (let alone one-one correspondence): Husserl was, we see again, absolutely right about this point. Counting 'by ones' does, of course, establish a one-one correspondence between the objects counted and an initial segment of the numerals. But that fact isn't at all fundamental to what counting is.

6. Closing

I have argued that something close to HP, namely, HPJ, is a conceptual truth: An appreciation of the connection between sameness of number and equinumerosity that it reports is essential to even the most primitive grasp of the concept of cardinal number. If so, then it is arguably a conceptual truth also that the (finite) cardinal numbers form (at least) an initial segment of an ω-sequence. That they do is the content of the axioms of **PA**, other than axiom (6), existence of successor (whose inclusion would license the removal of the phrase 'an initial segment of' and whose status I am setting aside). Of these, axioms (1), (2), and (7) recall, follow immediately from Frege's definitions: Acceptance of these definitions as capturing the content of our ordinary notions of *zero*, *one more*, and *natural number*—not uncontroversial, but hardly unreasonable—therefore gives us reason to regard these axioms as logical consequences of conceptual truths and so, I take it, themselves conceptual truths. The other axioms, (3)–(5), follow, as we saw earlier, from the Three Principles and so from HPJ and the Starred Principles. But the Starred Principles themselves can arguably be established purely conceptually. Consider, for example, (APC*):

$$JAM_x(Fx, Gx) \mathbin{\&} \neg Fa \mathbin{\&} \neg Gb \rightarrow JAM_x(Fx \bigvee x = a, Gx \bigvee x = b)$$

or in a less formal version:

> If, for example, there are just as many children as cookies, and if one more child shows up, and we find one more cookie, then there will still be just as many children as cookies.

Suppose there are just as many children as cookies, and one more child shows up, and we find one more cookie. So there were just enough of the "original" cookies for each of the "original" children to have one. So, if we start giving the original cookies to the original children, and are careful not to give one to anyone to whom we've already given one, we will be able to give a cookie to each, but won't have any left. But then we can just give the new cookie to the new child, and there will have been just enough cookies for each child to have one. So there are just as many cookies as children.

Of course, one might question whether the sort of reasoning employed in this informal argument yields a conceptual truth—and that, of course, is a question to which Frege himself was acutely sensitive. To show that it did, we would have to formalize the argument and "follow it up right back to the primitive truths" ([8], §3) from which it proceeds. Doing so, however, raises difficult issues. Obviously, the informal argument relies heavily upon our informal understanding of the notion 'just as many': No formalization of the argument will be possible unless we either uncover some deductive principles—introduction and eliminations rules, say—specific to this notion or analyze it in other terms, so that other, more familiar sorts of deductive principles become applicable to it. For present purposes, we can ignore the difference between these two approaches and focus on the latter.[26]

Frege famously offers a version of this latter approach: In effect, he suggests that 'just as many', equinumerosity, should be analyzed as one-one correspondence. But even granting that Frege is *right* that equinumerosity is one-one correspondence, extensionally speaking, in what sense can we regard this as an 'analysis' of the notion of equinumerosity? I have argued already that equinumerosity is not the same concept as being in one-one correspondence with. Still, though, the connection is very tight: To say that there are just as many Fs as Gs is to imply that there are just enough Gs for each F to have one, that is, that, if we start giving Gs to Fs, and are careful not to give another F to any G to which we've already given one, then we won't run out, but won't have any left. Something like the familiar Fregean analysis of equinumerosity seems buried in there somewhere.

It is, I think, a difficult question in what sense it might be buried and what exactly one is doing when one digs it out. There is some good recent work by Demopoulos that seeks to articulate a notion of 'analysis' that would apply not just here but to the similar case of continuity.[27] But let me not attempt to pursue this matter here. I find this territory terribly confusing, and my views are unstable, to say the least. If, however, we assume that some such conception of analysis is available and that it is, in the relevant sense, an analytic truth that[28]

(Eq) There are just as many Fs as Gs if and only if the Fs are in one-one correspondence with the Gs,

then HP, in its predicative form, ought to be an analytic truth, in the same sense, since it follows from (Eq) and HPJ. It ought similarly to be an analytic truth that the (finite) cardinals form an initial segment of an ω-sequence, since that follows from HP, in its predicative form.

Take that as a conjecture. Before we try to develop and defend it, however, it is worth asking what interest would such a conclusion have, even if it could be established. It would give us no reason to believe that we come by our knowledge of the relevant properties of the cardinals by deriving them from HPJ, and I see no reason to believe that we do. The suggestion that someone *could* come by knowledge of the axioms of PA by deriving them from HPJ suffers from problems I discussed some time ago: It is unclear what it adds to the claim that the axioms of PA are derivable from HPJ; if so, then someone who had sufficient powers of reasoning could, of course, derive them from it, but what of it? It is unclear to me, then, that there is any way of giving *epistemological* content to Frege's Theorem.[29]

What then *does* Frege's Theorem offer us? Something, I should like to suggest, like an answer to the question *why* the (finite) cardinal numbers satisfy the Dedekind-Peano axioms. I expect that question will strike some as bizarre: Certain varieties of structuralism, for example, would encourage the thought that it is ill-conceived. The finite cardinals are the objects of arithmetic, and arithmetic, the claim would be, is just the study of the properties of ω-sequences—or something like that. So, one simply can't ask *why* the finite cardinals form an ω-sequence: If the axioms of PA just characterize the natural numbers as forming such a structure, the question why they satisfy those axioms is just silly, right? But the question seems to me to be a perfectly sensible one—all the more so since it seems to have a perfectly sensible answer. The finite cardinals satisfy the axioms of PA because they satisfy HP, that is, because of the connection between cardinality and one-one correspondence it reports. So the interest of the Theorem lies there: It offers us an *explanation* of the fact that the numbers satisfy the Dedekind-Peano axioms.

One might ask, though, why it should matter, if that is what Frege's Theorem offers us, whether HP or HPJ is a conceptual truth.[30] And, in fact, I am happy to concede that, in a sense, it doesn't matter: HPJ does not need to be a conceptual truth for the Theorem to provide such an explanation; the notion of a conceptual truth doesn't even have to be coherent. But HP does need to be *more fundamental*, in some significant way, than the axioms of PA if Frege's Theorem is to have any explanatory force, for not every derivation of a conclusion from premises counts as *explaining*, in terms of the premises' holding, why the conclusion holds: And if HP itself were *less* fundamental than the axioms of PA, the explanatory value of Frege's Theorem would be nil. What I have argued here is that recognition of the truth of something very much like HP is required if one is even to have a concept of cardinality. If so, then what Frege's Theorem shows is that the fact that the finite cardinals form an initial segment of an ω-sequence is implicit in our very concept of cardinal number—'implicit' in the sense that their forming such a sequence is logically required by the character of our concept of cardinal number.

Notes

1. I think this is the correct way to state one of Wright's important contributions: The other was the conjecture that HP is consistent. For both, see Wright [18]. Wright was not the first to realize that Frege's Theorem could be proven, that is, that Frege's proofs could proceed from HP, rather than from Basic Law V: Parsons makes this observation in [17]; I have argued that Frege also knew of the fact. But Parsons, he tells me, did not suspect that HP might be consistent and Frege (even before he got Russell's letter) did not recognize the philosophical significance of Frege's Theorem—as opposed to that of the fateful derivation beginning with definition of number in terms of extensions. Only once both of those pieces are in place do we get the explosion of interest in Frege's Theorem we have seen over the last twenty years.

2. Even terminology gets people bent out of shape. The term 'Hume's Principle', introduced by Boolos, I believe, has itself been the object of intense controversy. Here, I shall follow another suggestion of Boolos's and use the term 'HP' whose relation to 'Hume's Principle' is intended to be similar to that between 'LF' and 'logical form'.

3. For a similar route to this same conclusion, see "Frege and the paradox of analysis" in Dummett [6], pp. 17–52. My own interest in this topic was inspired many years ago by hearing Dummett make similar remarks during lectures on Frege he delivered in Oxford.

4. "Is Hume's Principle Analytic?" in [1], pp. 301–14 at p. 310. I've omitted a footnote. Boolos is responding to Wright [19]. The two papers were originally published together in Heck [13].

5. One can find indications of what follows in many of Boolos's writings on this topic, all of which are now collected in [1]. But the formulation is mine and is based loosely on memories of conversations.

6. The quote is from Frege [8], §2. Burge has struggled mightily to make sense of this aspect of Frege's position. See Burge [2] and [3]. Much of the dissatisfaction Burge expresses with Frege's conception of apriority in the latter paper has its source, or so it seems to me, in its metaphysical, or Leibnizian, aspects.

7. Using which allows us to state the assertions that every number has a successor and that no number has more than one successor as nonlogical axioms, namely, as (6) and (3): In the more familiar formulation, using a function symbol 'S', these are consequences of conventions that govern the use of all function symbols.

8. It is convenient to take second-order logic here to include a principle guaranteeing the extensionality of operators of the type of 'N'. See "On the proof of Frege's Theorem" in [1], pp. 275–91 at pp. 278 ff.

9. For discussion of which, see Heck [14].

10. See Heck [12] for discussion of this result and some suggestions about its philosophical implications. I do not know how much induction is needed for this argument, but I believe at least $\prod_2^1$ comprehension is required because the formula defining finitude is $\prod_2^1$. If

so, then we need more comprehension for this direction, PAF $\rightarrow$ HPF, than we do for the other, HPF $\rightarrow$ PAF, since, as will be noted below, only $\prod_1^1$ comprehension is needed for the latter.

11. Of course, one will only get as much induction as one puts into the comprehension axioms.

12. Frege quotes Husserl as having written: "The simplest criterion of equality of number is just that the *same* number results in counting the sets to be compared." See Frege [9], p. 199, original p. 319. The Husserl of whom I speak here is Husserl as seen through Frege's eyes. I hereby confess that I have not actually read *Philosophie der Arithmetik*.

13. For extended discussion of this matter, see Hale [11].

14. Frege [9], p. 199, original pp. 318–19. This remark dovetails nicely with certain of the results Frege proves in the later parts of the first volume of Frege [7] which was published in 1893 just a year before the review appeared. See [14] for discussion of the results in question.

15. To count 'transitively', as it is sometimes put—that is, to count a group of objects—as opposed to counting 'intransitively', which is just to recite the numerals in order.

16. It is, of course, possible that children *do* grasp the connection between cardinality and one-one correspondence, even from this young age, but that, for various reasons, that grasp is not revealed in the sorts of experiments so far devised. If so, then wonderful. But it is important to understand that the issue being discussed here concerns children's *conscious* knowledge of the relation between number and cardinality. Though it may be true, it would not advance the argument I am making to claim, say, that the connection is *tacitly* known. See notes 21 and 24 for more on this matter.

17. Whose converse is the natural direction: If you have *different* numbers of dolls and hats, getting another doll and another hat isn't going to solve your problem.

18. (RPC) is a somewhat more complicated case, but that it holds reflects the fact that, if one says a numeral just before **n**, there is a definite numeral one always says just before **n**. For (RPC) to fail, it would have to be that, though in counting the Fs and the Gs, I ended up at the same numeral, still, if I'd stopped *just before* counting the last object, I would have ended up at different numerals.

19. I use the term 'implicitly' in a nontechnical sense here. In the mathematical sense, they do not do so, since they do not completely determine the extension of '$Nx : Fx = Nx : Gx$'.

20. The proof is, in essentials, just the proof of FHP in **PAF**+ **FD**, which I discuss in detail in [12]. The technique of that proof can be applied here, since the Three Principles are, as noted above, the result of applying Frege's definitions to axioms (3) – (5) and doing some simple manipulations. The Three Principles can therefore play the role these axioms play in the original proof, for which no appeal to axiom (6) is needed.

21. Actually, what I will be arguing is simply that such young children are not *deploying* a concept of cardinality in answering "How many?" questions. They may, of course, *have* such a concept, but not yet have connected it with counting and such.

22. The results mentioned in the last two paragraphs are summarized in Carey [4] at pp. 105 ff. Wynn and her colleagues did much of this work.

23. The notion of having should be understood abstractly and is by children. They understand that they can have a mother, a bicycle, a cookie, a home, a teacher, a friend, and a God. A doll can have a hat, and a hat can have a doll—and in a nonmetaphorical sense, too. (I am half-tempted to suggest that 'have' is a free variable over relations.)

24. My use of the term 'operational' should not be taken too seriously: I do not mean to be suggesting a return to behaviorism. For all I have said here, grasp of the concept *just as many* might involve *tacitly* knowing that there are just as many Fs as Gs if and only if they are in one-one correspondence. Here the point is simply that the child's grasp of the concept involves no conscious, theoretical identification of it in other terms.

25. The observation was made in Kripke [15]. Kripke made a somewhat similar application of his point, but I do not recall his connecting it with this particular question.

26. My own preference, as it happens, is for the former approach, but I will not discuss the matter here.

27. See Demopoulos [5]. Another important such case, not often mentioned in this connection, but of great interest, is that of computability: Is Church's Thesis an analytic truth, in some reasonable sense?

28. My own view is that, in fact, only something weaker may be analytic, namely, the analogue of FHP: So long as at least one of F and G is finite, the Fs are equinumerous with the Gs if and only if they are in one-one correspondence. If so, then only FHP is analytic, in the relevant sense.

29. One idea that might be worth investigating is that our knowledge of the relevant axioms of PA is *based upon* our knowledge of HPJ, in the sense in which that notion is used in epistemology: Perhaps our knowledge that no natural number precedes zero is ultimately responsible to, and causally sustained by, our knowledge of HPJ.

30. The account of the significance of Frege's Theorem offered here thus does not rest upon or require the elaborate framework Wright sets up and deploys in [18]: It does not, in particular, need to make use of the idea that, in general, we understand a sortal concept by understanding a criterion of identity for it. Of course, for that very reason, the view I take here contributes nothing to our understanding of *ontological* problems involving numbers.

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