

NEW OPERATIONS ON ORTHOMODULAR LATTICES: “DISJUNCTION” AND “CONJUNCTION” INDUCED BY MACKEY DECOMPOSITIONS

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Abstract New conjunctionlike and disjunctionlike operations on orthomodular lattices are defined with the aid of formal Mackey decompositions of not necessarily compatible elements. Various properties of these operations are studied. It is shown that the new operations coincide with the lattice operations of join and meet on compatible elements of a lattice but they necessarily differ from the latter on all elements that are not compatible. Nevertheless, they define on an underlying set the partial order relation that coincides with the original one. The new operations are in general nonassociative: if they are associative, a lattice is necessarily Boolean. However, they satisfy the Foulis-Holland-type theorem concerning associativity instead of distributivity.

1. Operations “sharp” and “flat”—Motivation and Definition

1.1 Introduction In the Lindenbaum-Tarski algebra of any theory governed by the laws of the classical logic, disjunctions and conjunctions of propositions are represented, respectively, by the lattice-theoretic operations of join (least upper bound, supremum) and meet (greatest lower bound, infimum). Birkhoff and von Neumann, the founding fathers of the quantum logic theory, assumed in their historic 1936 paper [3] that the same holds also in the case of quantum theory. However, it is a characteristic feature of quantum mechanics, by which it differs a lot from any “classical” theory, that here there do exist experimentally testable propositions about quantum physical systems that cannot be tested simultaneously. Such propositions are represented in orthomodular lattices, which are believed to be algebraic representations of families of experimentally testable propositions about quantum systems, by elements that are not compatible (see, e.g., Beltrametti and Cassinelli [1], Kalmbach [12], Beran [2],

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Pták and Pulmannová [16]). On the other hand, compatible elements of an orthomodular lattice (OML) necessarily belong to a Boolean subalgebra of an OML, that is, to a “locally classical” fragment of an OML. Therefore, Birkhoff and von Neumann’s assumption that conjunctions and disjunctions should be represented by meets and joins is natural and obvious as far as compatible propositions are concerned, but it can be neither confirmed nor falsified experimentally on propositions that are not compatible since such propositions cannot be tested simultaneously. Thus, we see that, in principle, conjunctions and disjunctions of propositions about quantum systems could be represented by other operations on OMLs that should coincide with meets and joins on compatible elements of OMLs but could be possibly different from them on noncompatible elements of a lattice.

The problem of finding the proper algebraic representation of conjunctions and disjunctions in more general “quantum structures”—orthoalgebras and effect algebras—is still more difficult since in these structures even for compatible elements meets and joins do not have to exist. Foulis [7] proposed to use for this purpose elements that appear in a Mackey decomposition of a pair of compatible elements a, b in an orthoalgebra L , provided that: (a) this decomposition is unique and (b) the meet of a and b exists in L . Later on it was argued by the present author [17] that neither of the assumptions (a), (b) is necessary in order to obtain reasonable models of conjunction and disjunction in orthoalgebras.

That paper [17] is the starting point for the present one, but now the direction of investigation is, in a sense, opposite: Instead of considering Mackey decompositions of compatible elements in an orthoalgebra, a *formal Mackey decomposition* of arbitrary elements in an orthomodular lattice is used to define new operations \sharp (sharp) and \flat (flat) on an OML. There are many properties of \sharp and \flat that are the same as properties of join and meet, moreover, $a \sharp b$ and $a \flat b$ coincide, respectively, with the join $a \vee b$ and the meet $a \wedge b$ whenever elements a and b are compatible. They also define on the underlying set a partial order relation that coincides with the original one. Therefore, it is possible to treat $a \sharp b$ and $a \flat b$ as new algebraic models of disjunction and conjunction of propositions a and b . We give some arguments that such interpretation of these operations is plausible in spite of the fact that some properties of \sharp and \flat are rather counterintuitive. However, we mostly confine this paper to investigations of the purely formal properties of the new operations. We commented on their possible physical and logical interpretation in another paper [5] which, however, does not contain proofs and only quotes several theorems that are proved in this paper and in the paper “New operations on orthomodular lattices II: ‘Disjunctions’ and ‘conjunctions’ generated by Kotas conditionals,” (an unpublished paper by D’Hooghe) which is, in a sense, a continuation of the present paper.

When the first version of this paper was completed we were informed that the operation flat was considered by Länger [13] as one of two possible ring multiplications (the other was ordinary meet) in ringlike structures that correspond to OMLs in the same way as Boolean rings correspond to Boolean algebras. However, Länger did not consider any operation “dual” to \flat by de Morgan laws that could be interpreted as disjunctionlike operation as our operation sharp (two ring additions considered by Länger were generalizations of Boolean symmetric difference, not of the ordinary join). Nevertheless, some of the properties of the operation flat studied in the present paper appeared to be already obtained by Länger and some other facts concerning

operations sharp and flat were conjectured, proved, or can be inferred from the results of other researchers. We decided to include them in this paper for the sake of completeness, but all such (known to us) instances are indicated in remarks following relevant theorems. We apologize for all the possible omissions that, due to the extensiveness of the literature on OMLs, are still likely to appear.

1.2 Mackey Decomposition of Compatible Elements in Orthomodular Posets

Let L be a σ -orthomodular poset, that is, a partially ordered set with the least element 0 and the greatest element 1, endowed with the unary operation of orthocomplementation $' : L \rightarrow L$ that satisfies, for any $a, b \in L$,

- (i) $(a')' = a$,
- (ii) if $a \leq b$, then $b' \leq a'$,
- (iii) $a \wedge a' = 0, a \vee a' = 1$,

and such that the following σ -orthocompleteness condition,

- (iv) if $a_i \leq a'_j$ for $i \neq j$, then $\bigvee_i a_i \in L$,

and the orthomodular law,

- (v) if $a \leq b$, then $b = a \vee (a' \wedge b)$,

hold. Compatibility of two elements a, b of L is usually defined in the following way: a and b are called *compatible* (abbr. aCb) if and only if there exist in L mutually orthogonal elements a_1, b_1, c : $a_1 \perp b_1, a_1 \perp c, b_1 \perp c$ (i.e., $a_1 \leq b'_1, a_1 \leq c', b_1 \leq c'$) such that $a = a_1 \vee c$ and $b = b_1 \vee c$. Since this way of defining compatibility of elements in an orthomodular poset goes back to the fundamental book by Mackey [14], the triple $\{a_1, b_1, c\}$ is often referred to as a *Mackey decomposition* of a and b (see, e.g., Younce [18] or [7]).

Actually, there are many equivalent ways of defining compatibility in orthomodular posets. Some of them are listed among the following facts about compatibility and will be used in the sequel. Elementary proofs of these facts can be found in the standard textbooks (see, e.g., [1], [12], [2], [16]) on quantum logic theory.

- (C1) If aCb , then the join and the meet of any two among a, a', b, b' exist in L .
- (C2) If aCb , then the elements of the Mackey decomposition of a and b are uniquely determined as $a_1 = a \wedge b', b_1 = b \wedge a', c = a \wedge b$.
- (C3) aCb iff bCa .
- (C4) aCb iff aCb' iff $a'Cb$ iff $a'Cb'$.
- (C5) aCb iff $a \wedge b, a \wedge b' \in L$ and $a = (a \wedge b) \vee (a \wedge b')$; iff $a \wedge b, a' \wedge b \in L$ and $b = (a \wedge b) \vee (a' \wedge b)$.
- (C6) If $a \leq b$, then aCb .
- (C7) If $a \perp b$, then aCb .
- (C8) aCb iff $a \vee b, a \vee b', a' \vee b, a' \vee b' \in L$ and $(a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b') = 0$.
- (C9) In an orthomodular lattice aCb iff $a \wedge b = a \wedge (a' \vee b) = b \wedge (b' \vee a)$.

The left-hand side of the equality (C8), usually considered only in orthomodular lattices, is called the *commutator* of a and b (see, e.g., Bruns and Greechie [4], Greechie and Herman [9]) and will be denoted $a * b$ throughout this paper.

1.3 Definition of Operations Sharp and Flat Let us now assume that the underlying set L is not merely an orthomodular poset but an orthomodular lattice. This change does not only make (C1) void of importance and assumptions about the existence of meets and joins in (C5) and (C8) unnecessary but also allows us to consider a *formal Mackey decomposition*: $a_1 = a \wedge b'$, $b_1 = b \wedge a'$, $c = a \wedge b$ of any, not necessarily compatible, pair of elements $a, b \in L$, and allows us to define the following operations on an orthomodular lattice L .

Definition 1.1 Let L be an OML. For any $a, b \in L$ we define

$$\text{“}a \text{ sharp } b\text{”} = a \sharp b =_{\text{df}} a_1 \vee b_1 \vee c = (a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \quad (1)$$

$$\text{“}a \text{ flat } b\text{”} = a \flat b =_{\text{df}} (a' \sharp b')' = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b). \quad (2)$$

Of course, the operation flat is defined by “de Morgan formula” in order to make it dual to the operation sharp, which secures the validity of both de Morgan laws:

$$(a \sharp b)' = a' \flat b' \quad (3)$$

$$(a \flat b)' = a' \sharp b'. \quad (4)$$

The explicit form of $a \flat b$ follows from straightforward calculations and since right-hand sides of both (1) and (2) are invariant with respect to interchanging a with b , the operations \sharp and \flat are commutative:

$$a \sharp b = b \sharp a \quad (5)$$

$$a \flat b = b \flat a. \quad (6)$$

2. Sharp, Flat, and Compatibility

We begin with the fact of the utmost importance: new operations coincide with join and meet on all compatible elements of a lattice but are necessarily different from them whenever applied to elements that are not compatible.

Theorem 2.1 Let L be an orthomodular lattice. For any $a, b \in L$ the following conditions are equivalent:

$$(i) \quad aCb \quad (7)$$

$$(ii) \quad a \sharp b = a \vee b \quad (8)$$

$$(iii) \quad a \flat b = a \wedge b \quad (9)$$

Proof: If aCb , then by (C5), idempotency and commutativity of the traditional lattice operations

$$\begin{aligned} a \sharp b &= (a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \\ &= ((a \wedge b) \vee (a \wedge b')) \vee ((a \wedge b) \vee (a' \wedge b)) = a \vee b, \end{aligned} \quad (10)$$

so (8) holds. De Morgan laws applied to the lattice operations imply that

$$a' \flat b' =_{\text{df}} (a \sharp b)' = (a \vee b)' = a' \wedge b'. \quad (11)$$

Therefore, if (8) holds, then

$$\begin{aligned} 0 &= (a \vee b) \wedge (a \vee b)' = (a \vee b) \wedge (a' \wedge b') = (a \vee b) \wedge (a' \flat b') \\ &= (a \vee b) \wedge (a' \vee b') \wedge (a' \vee b) \wedge (a \vee b') = a * b, \end{aligned} \quad (12)$$

which, because of (C8), implies that aCb .

To finish the proof it is enough to notice that by (C4) aCb is equivalent to $a'Cb'$ and that by exchanging elements a, b with their orthocomplements we get from the previously obtained implications the other chain of implications:

$$a'Cb' \Rightarrow (a' \sharp b' = a' \vee b') \Rightarrow (9) \Rightarrow a'Cb' \quad (13)$$

which forces the theorem to hold true. \square

Remark 2.2 The equivalence of (7) and (9) was also noticed by Langer ([13], Theorem 2.1 (viii)).

Corollary 2.3 *In Boolean algebras, operations \sharp and \flat coincide, respectively, with \vee and \wedge .*

As was mentioned in the Introduction, in the quantum logic theory it is assumed that one can simultaneously test only such propositions about physical systems that are represented by compatible elements of a lattice. Therefore, Theorem 2.1 implies that operations \sharp and \flat , although in general different from the lattice operations of join and meet, are “experimentally indistinguishable” from the latter. Consequently, from the “experimental” point of view, they are as good algebraic models of disjunction and conjunction of experimentally verifiable propositions about physical systems as traditionally used join and meet.

Since the operations \sharp and \flat coincide, respectively, with \vee and \wedge on all pairs of compatible elements, we immediately get the following.

Corollary 2.4 *Operations \sharp and \flat satisfy the following laws for all elements a, b of an orthomodular lattice L .*

Laws of idempotency:

$$a \sharp a = a, \quad a \flat a = a. \quad (14)$$

Law of excluded middle:

$$a \sharp a' = 1. \quad (15)$$

Law of contradiction:

$$a \flat a' = 0. \quad (16)$$

0-1 laws:

$$a \sharp 1 = 1, \quad a \flat 1 = a, \quad a \sharp 0 = a, \quad a \flat 0 = 0. \quad (17)$$

Orthomodular law:

$$\text{if } a \leq b, \text{ then } b = a \sharp (a' \flat b). \quad (18)$$

Finally, let us notice that from the very definition of a commutator, $a * b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$ it follows that

$$a * b = (a \flat b) \wedge (a' \vee b') = (a \sharp b)' \wedge (a \vee b). \quad (19)$$

Let us now study the problem of compatibility of $a \sharp b$ and $a \flat b$ for arbitrary (not necessarily compatible) elements a, b . Let us notice that since any two comparable elements of an OML are compatible, $a \wedge bCa \vee b$ and $a \wedge bCa, bCa \vee b$. Therefore,

the natural question arises whether this holds also for operations sharp and flat. The following theorem answers this question in the positive.

Theorem 2.5 *Let L be an orthomodular lattice. For any $a, b \in L$ the elements $a \sharp b$ and $a \flat b$ are compatible and each of them is also compatible with a and with b .*

Proof: By multiple application of conditions (C3) and (C4), along with the laws $aCa \wedge b, aCa \vee b, aCbCc \implies bCa \wedge c, a \vee c$, we build up the expressions corresponding to each side of the relations $a \sharp bCa \flat b$ and $a \sharp bCa, bCa \flat b$. \square

Remark 2.6 Theorem 2.5 was presented as a conjecture at the IV Biannual Congress of the International Quantum Structures Association held in Liptovský Ján, Slovakia, in September 1998 and it was immediately shown to be true by Navara.

3. Definability of Lattice Operations by Sharp and Flat

The operations sharp and flat were defined in Section 1 by the lattice operations and the operation of orthocomplementation. The natural question arises whether it is possible to go in the opposite direction, that is, to express the lattice operations by the operations \sharp and \flat (and, possibly, orthocomplementation). The following theorem answers this question in the positive.

Theorem 3.1 *For any two elements a, b of an orthomodular lattice L*

$$a \vee b = (a \flat b) \sharp (a \sharp b) \quad (20)$$

$$= (a \flat b') \sharp b \quad (21)$$

$$= (a' \flat b) \sharp a, \quad (22)$$

$$a \wedge b = (a \sharp b) \flat (a \flat b) \quad (23)$$

$$= (a \sharp b') \flat b \quad (24)$$

$$= (a' \sharp b) \flat a. \quad (25)$$

Proof: Since $a \sharp bCa \flat b$, by Theorem 2.1,

$$(a \sharp b) \sharp (a \flat b) = (a \sharp b) \vee (a \flat b). \quad (26)$$

Since all meets and joins of a, a', b, b' are compatible, they generate a distributive sublattice of L (Greechie [10]; see also [12], Chapter 2, §7, Theorem 4) so we can calculate:

$$\begin{aligned} (a \sharp b) \vee (a \flat b) &= [(a \wedge b) \vee (a \wedge b') \vee (a' \wedge b)] \vee [(a \vee b) \wedge (a \vee b') \wedge (a' \vee b)] \\ &= (a \vee b) \wedge [(a' \wedge b) \vee (a \vee b')] \wedge [(a \wedge b') \vee (a' \vee b)] \\ &= (a \vee b) \wedge 1 \wedge 1 = a \vee b. \end{aligned} \quad (27)$$

The proof of formula (21) also follows from the fact that all joins of a, a', b, b' belong, together with a and b , to a distributive sublattice of L , from (C4), and from Theorem 2.5:

$$\begin{aligned} (a \flat b') \sharp b &= (a \flat b') \vee b = [(a \vee b') \wedge (a \vee b) \wedge (a' \vee b')] \vee b \\ &= (a \vee b' \vee b) \wedge (a \vee b \vee b) \wedge (a' \vee b' \vee b) \\ &= 1 \wedge (a \vee b) \wedge 1 = a \vee b. \end{aligned} \quad (28)$$

Formula (22) can be obtained from formula (21) by simple interchanging a with b , which is allowed by the commutativity of join.

The equalities (23), (24), and (25) follow from the equalities (20), (21), (22), de Morgan laws, and commutativity of \sharp and \flat :

$$\begin{aligned} (a \wedge b) &= (a' \vee b')' = [(a' \sharp b') \sharp (a' \flat b')] = (a' \sharp b')' \flat (a' \flat b')' = (a \sharp b) \flat (a \flat b) \\ &= [(a' \flat b) \sharp b']' = (a' \flat b)' \flat b = (a \sharp b)' \flat b \\ &= [(a \flat b') \sharp a']' = (a \flat b')' \flat a = (a' \sharp b) \flat a. \end{aligned} \quad (29)$$

□

Remark 3.2 Equalities (20) and (23) were conjectured by Harding in May 1998 when the first version of this paper was presented at the workshop on orthomodular lattices held at Vrije Universiteit Brussel. They are also implicitly contained in the condition UJ(5).1 of the paper by Pavičić and Megill [15]. The equality (23) can be also derived from Länger’s Theorem ([13], 2.1 (iii)) by application of de Morgan law (4). The possibility of expressing the join by (21) and (22) and the meet by (24) and (25) is implicitly contained in the formulas G2 and G3 of the paper by Georgacarakos [8]. However, it has to be mentioned that neither [15] nor [8] is devoted to the studies of disjunctionlike and conjunctionlike operations on OMLs. Both these papers are focused on the problems connected with implications in OMLs, but since one of the implications studied there, called the *relevance implication*, can be expressed in terms of our operation sharp as follows,

$$a \rightarrow b =_{\text{df}} (a \wedge b) \vee (a' \wedge b) \vee (a' \wedge b') = a' \sharp b, \quad (30)$$

Georgacarakos’s and Pavičić-Megill’s formulas for $a \vee b$ expressed in terms of the relevance implication can be immediately “translated” into the language that uses \sharp and \flat instead of \rightarrow . The thorough studies of other disjunctionlike and conjunctionlike operations on OMLs generated by other implicationlike operations is contained in [5].

We agree with an anonymous referee of this paper that since all expressions with two variables in an OML are reducible to one of ninety-six canonical forms [2] and since there exist even computer programs for such reductions, the proof of formulas (20)–(25), and also some other 2-variable formulas of this paper, can be “automatized.” However, we also agree with the referee that explicit proofs are more instructive to a reader who should not be forced to rely on computer programs.

4. Sharp, Flat, and Partial Order

Order relations concerning $a \sharp b$, $a \flat b$, and $a \vee b$, $a \wedge b$, $a * b$ follow from the very definition of the operations \sharp and \flat , and formula (19).

$$a \wedge b \leq a \sharp b \leq a \vee b, \quad (31)$$

$$a \wedge b \leq a \flat b \leq a \vee b, \quad (32)$$

$$a * b \leq a \flat b, \quad a \sharp b \leq (a * b)'. \quad (33)$$

Remark 4.1 The relation $a \wedge b \leq a \flat b$ was also noticed by Länger ([13], Theorem 2.1 (vii)).

However, contrary to the lattice operations that always satisfy $a \wedge b \leq a \vee b$, $a \sharp b$ and $a \flat b$ may be noncomparable if a and b are noncompatible.

Example 4.2 Let $L = G_{12}$.

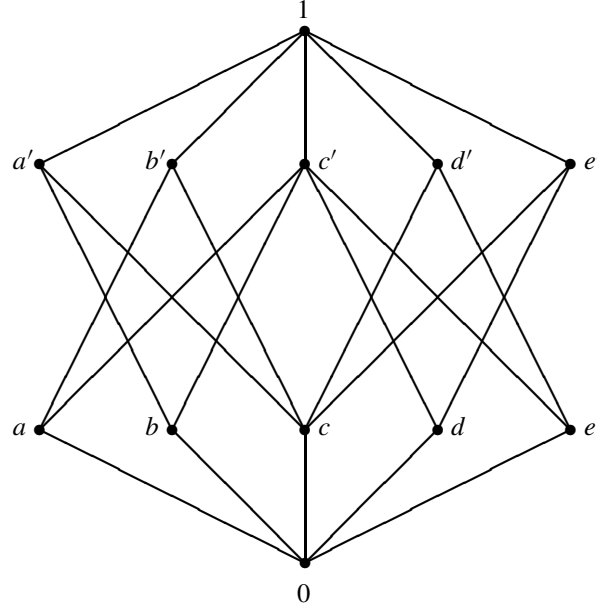


Fig. 1

Since this lattice also will be extensively used in the other examples of this paper we write for future use tables for $x \sharp y$ and $x \flat y$ when x and y are noncompatible elements of G_{12} .

\sharp	d	d'	e	e'
a	0	c	0	c
a'	c	c	c	c
b	0	c	0	c
b'	c	c	c	c

Table 1. Values of $x \sharp y$ for noncompatible elements $x, y \in G_{12}$.

\flat	d	d'	e	e'
a	c'	c'	c'	c'
a'	c'	1	c'	1
b	c'	c'	c'	c'
b'	c'	1	c'	1

Table 2. Values of $x \flat y$ for noncompatible elements $x, y \in G_{12}$.

Now we see that, for instance, $a \sharp e' = c$ while $a \flat e' = c'$, that is, $a \sharp e'$ and $a \flat e'$ are noncomparable.

Moreover, even when $a \sharp b$ and $a \flat b$ are comparable, the order between them can be (of course, only for a noncompatible with b) surprisingly “reversed” with respect

to what could have been expected if \sharp and \flat were to act as join and meet on the whole OML.

Example 4.3 Let $L = G_{12}$ as in Example 4.2. We see that $a \sharp d = 0 < a \flat d = c'$.

Examples 4.2 and 4.3 suggest that the “natural” order between $a \sharp b$ and $a \flat b$ (i.e., $a \flat b \leq a \sharp b$), and between them and a, b (i.e., $a \flat b \leq a, b$; $a, b \leq a \sharp b$) is preserved if and only if a and b are compatible, in which case, of course, “new” operations coincide with the traditional ones. Indeed, the following theorem holds.

Theorem 4.4 Let L be an orthomodular lattice. For any $a, b \in L$ the following conditions are equivalent:

- (i) $a \flat b \leq a \sharp b$,
- (ii) $a, b \leq a \sharp b$,
- (iii) $a \flat b \leq a, b$,
- (iv) aCb .

Proof: Since (i) means that $a \flat b \vee a \sharp b = a \sharp b$ and since comparable elements are compatible, we obtain from formula (20) with the aid of Theorem 2.1 that

$$a \vee b = (a \flat b) \sharp (a \sharp b) = a \flat b \vee a \sharp b = a \sharp b \quad (34)$$

which, by the further use of Theorem 2.1, implies (iv). In order to show the implication (ii) \Rightarrow (iv) it is enough to notice that $a, b \leq a \sharp b$ implies that $a \vee b \leq a \sharp b$. Since the opposite inequality (31) always holds, (iv) follows. The proof of the implication (iii) \Rightarrow (iv) is analogous. Since due to Theorem 2.1 implications (iv) \Rightarrow (i), (ii), (iii) are obvious, the theorem is proved. \square

The next theorem shows that operations sharp and flat can be used to *define* partial order relation \leq on an underlying set L exactly in the same way as the lattice operations can be used to define the original partial order relation \leq on L . In view of the different behavior of \sharp and \flat , and the lattice operations with respect to the original partial order relation \leq shown in Examples 4.2 and 4.3 it is a surprising fact that the partial order relation \leq defined on L by \sharp or \flat do coincide with the original one!

Theorem 4.5 *The relation \leq defined on an orthomodular lattice L by*

$$a \leq b \quad \text{iff} \quad a \flat b = a \quad \text{iff} \quad a \sharp b = b \quad (35)$$

is a partial order relation and it coincides with the original partial order on L , that is,

$$a \leq b \quad \text{iff} \quad a \leq b \quad \text{for any } a, b \in L. \quad (36)$$

Proof: (i) Let us assume that $a = a \flat b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b)$. By taking the meet of both sides of this equality with the element b we obtain

$$a \wedge b = b \wedge (a \vee b) \wedge (a \vee b') \wedge (a' \vee b), \quad (37)$$

and by the absorption law applied two times to the suitable terms of the right-hand side of the last equality we get

$$a \wedge b = b \wedge (a' \vee b) \wedge (a \vee b') = b \wedge (a \vee b'). \quad (38)$$

The equality (38), because of (C9), implies that aCb which, because of Theorem 2.1, means that $a \flat b = a \wedge b$. Therefore, we get an implication

$$\text{if } a = a \flat b, \text{ then } a = a \wedge b, \quad (39)$$

which in turn means that $a \leq b$, that is,

$$\text{if } a \leq b, \text{ then } a \leq b. \quad (40)$$

Since all elements comparable with respect to \leq are compatible, the reverse implication is obvious.

(ii) Let us assume that $a \sharp b = b$. Then it is enough to notice that by taking the join of both sides of the equality

$$b = a \sharp b = (a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \quad (41)$$

with the element a and by applying two times the absorption law to the right-hand side of the equality obtained in this way we get

$$a \vee b = a \vee (a' \wedge b) \quad (42)$$

which, because of (C4) and (C9), also implies that aCb . The rest of the proof follows as in (i). \square

Theorem 4.5 is less surprising if we take into account the basic fact that enables one to use OMLs as algebraic models of families of experimentally testable propositions about physical systems: making tests on pairs of compatible propositions (and only such pairs can be tested simultaneously!) suffices to establish the partial order relation on the whole set of propositions. Simultaneous testing of noncompatible propositions is neither possible from the experimental, nor necessary from the theoretical point of view. Therefore, since by Theorem 2.1 new and old operations coincide on and only on pairs of compatible propositions, new operations are bound to establish the same partial order as traditional ones.

5. Distributivity and Associativity

Since the operations sharp and flat resemble in many aspects lattice operations on an OML it is not surprising that they are in general nondistributive, which can be demonstrated, of course, only when considered elements do not belong to the same Boolean subalgebra of an OML.

Example 5.1 Let $L = G_{12}$ as in the previous examples. One can easily check that

$$a \sharp (c \flat e) = a \sharp (c \wedge e) = a \sharp 0 = a \vee 0 = a \quad (43)$$

while

$$(a \sharp c) \flat (a \sharp e) = (a \vee c) \flat 0 = b' \flat 0 = b' \wedge 0 = 0, \quad (44)$$

and that

$$a \flat (c' \sharp e) = a \flat (c' \vee e) = a \flat c' = a \wedge c' = a \quad (45)$$

while

$$(a \flat c') \sharp (a \flat e) = (a \wedge c') \sharp c' = a \sharp c' = a \vee c' = c'. \quad (46)$$

Since in this example aCc , cCe , and aCc' , $c'Ce$, that is, we were *focusing*, respectively, on the elements c and c' we see that no counterpart of the Foulis-Holland theorem holds for \sharp and \flat (Foulis [6], Holland [11]; see also [1], p. 128; [12], p. 25).

The next example shows that \sharp and \flat are, in general, nonassociative.

Example 5.2 Let us consider again $L = G_{12}$. We read from Figure 1 and Tables 1 and 2 that

$$a \sharp (b \sharp e) = a \sharp 0 = a \vee 0 = a \quad (47)$$

while

$$(a \sharp b) \sharp e = (a \vee b) \sharp e = c' \sharp e = c' \vee e = c', \quad (48)$$

and

$$a \flat (b \flat e) = a \flat c' = a \wedge c' = a \quad (49)$$

while

$$(a \flat b) \flat e = (a \wedge b) \flat e = 0 \flat e = 0 \wedge e = 0. \quad (50)$$

However, it should be noticed that although in “classical” theories conjunction and disjunction of propositions are generally assumed to be associative, it is possible to consider situations admitting nonassociativity of these operations. In the realm of quantum physics such situations could be considered in the so-called contextual hidden variables theories where truth-value of the proposition b could depend on whether it is tested in the context defined by the simultaneous testing of the proposition a or by the simultaneous testing of the proposition c , so one might expect that $(a \text{ AND } b) \text{ AND } c \neq a \text{ AND } (b \text{ AND } c)$. Another such situation can be found in psychology where answers given to questions a and c can bias the state of mind of an interrogated person (i.e., they also define specific “contexts”) and can influence his/her answer given to the question b .

It is a surprising fact that although in general the operations \sharp and \flat are nonassociative, the Foulis-Holland-type theorem concerning associativity instead of distributivity holds for \sharp and \flat .

Theorem 5.3 *If, in an orthomodular lattice, one of the elements a, b, c is compatible with the other two, then $\{a, b, c\}$ is an associative triple with respect to both operations \sharp and \flat .*

Proof: Let us assume that $aCbCc$. Since new operations coincide with the lattice operations on compatible elements, we get

$$(a \sharp b) \sharp c = (a \vee b) \sharp c = [(a \vee b) \wedge c] \vee [(a \vee b) \wedge c'] \vee [(a \vee b)' \wedge c]. \quad (51)$$

Let us consider the first two square brackets of the right-hand side of this equality. Since $aCbCc$, (C4) and (C7) imply that also $aCbCc'$ and $bCcCc'$. Therefore, by the Foulis-Holland theorem the triples $\{a, b, c\}$, $\{a, b, c'\}$, and $\{b, c, c'\}$ are distributive, and we get

$$\begin{aligned} [(a \vee b) \wedge c] \vee [(a \vee b) \wedge c'] &= (a \wedge c) \vee (b \wedge c) \vee (a \wedge c') \vee (b \wedge c') \\ &= (a \wedge c) \vee (a \wedge c') \vee [(b \wedge c) \vee (b \wedge c')] \\ &= (a \wedge c) \vee (a \wedge c') \vee [b \wedge (c \vee c')] \\ &= (a \wedge c) \vee (a \wedge c') \vee b. \end{aligned} \quad (52)$$

After inserting this result into (51) and applying de Morgan law to its last term we obtain

$$\begin{aligned} (a \sharp b) \sharp c &= (a \wedge c) \vee (a \wedge c') \vee b \vee (a' \wedge b' \wedge c) \\ &= (a \wedge c) \vee (a \wedge c') \vee b \vee [b' \wedge (a' \wedge c)]. \end{aligned} \quad (53)$$

Since if $aCbCc$, then b and b' are compatible to all formulas in a and c , by the Foulis-Holland theorem the triple $\{b, a' \wedge c, b'\}$ is distributive and we get

$$b \vee [b' \wedge (a' \wedge c)] = (b \vee b') \wedge [b \vee (a' \wedge c)] = b \vee (a' \wedge c), \quad (54)$$

which inserted into (53) yields

$$(a \sharp b) \sharp c = (a \wedge c) \vee (a' \wedge c) \vee (a \wedge c') \vee b = (a \sharp c) \vee b. \quad (55)$$

In a fully analogous way we can show also that $a \sharp (b \sharp c)$ equals $(a \sharp c) \vee b$:

$$a \sharp (b \sharp c) = (a \sharp c) \vee b. \quad (56)$$

Therefore, the equality

$$(a \sharp b) \sharp c = a \sharp (b \sharp c) \quad (57)$$

is proved.

Commutativity of the operation \sharp implies that

$$(a \sharp b) \sharp c = (b \sharp a) \sharp c = c \sharp (b \sharp a) = c \sharp (a \sharp b), \quad (58)$$

$$a \sharp (b \sharp c) = a \sharp (c \sharp b) = (c \sharp b) \sharp a = (b \sharp c) \sharp a, \quad (59)$$

and

$$(a \sharp c) \sharp b = (c \sharp a) \sharp b = b \sharp (c \sharp a) = b \sharp (a \sharp c). \quad (60)$$

Due to the already mentioned fact that under our assumptions b and b' are compatible to all formulas in a and c , Theorem 2.1 implies that

$$(a \sharp c) \sharp b = (a \sharp c) \vee b. \quad (61)$$

Therefore, by (55) and (57) all elements of equalities (58), (59), and (60), which exhaust all possible permutations of $\{a, b, c\}$ and all possible positions of brackets, are equal.

Finally, let us notice that since $aCbCc$ implies $a'Cb'Cc'$, associativity of \flat follows from associativity of \sharp and de Morgan laws, for example,

$$(a \flat b) \flat c = [(a' \sharp b') \sharp c']' = [a' \sharp (b' \sharp c')] = a \flat (b \flat c), \quad (62)$$

which finishes the proof. \square

Remark 5.4 The fact that $aCbCc$ implies $(a \flat b) \flat c = a \flat (b \flat c)$ was also proved by Länger ([13], Theorem 4.2). However, in view of our Theorem 5.3, Länger's remark "one can easily see that the associative and distributive laws which we have formulated in Theorem 4.4.2 do not hold if we assume that another of the three elements a, b, c commutes [in our terminology: is compatible] with the remaining two elements" is clearly unjustified as far as it concerns operation \flat considered in part (ii) of his Theorem 4.4.2 (although it remains true with respect to parts (i) and (iii) of his Theorem 4.4.2 where the considered operations are different from ours).

As a straightforward corollary from the proof of Theorem 5.3 we get the following.

Corollary 5.5 *If, in an orthomodular lattice, $aCbCc$, then*

$$a \sharp b \sharp c = (a \sharp c) \vee b, \quad (63)$$

and

$$a \flat b \flat c = (a \flat c) \wedge b. \quad (64)$$

If we interpret the operations sharp and flat as representing, respectively, disjunction and conjunction, Theorem 5.3 secures the unique meaning of propositions “ a AND b AND c ” and “ a OR b OR c ” for all possible permutations of $\{a, b, c\}$ when one of these propositions is simultaneously (but possibly, separately) verifiable with the remaining two, even if all three propositions a, b, c do not belong to the same Boolean subalgebra of an OML so they cannot be tested simultaneously.

Theorem 5.3 cannot be reversed, that is, from the fact that $a \sharp (b \sharp c) = (a \sharp b) \sharp c$ and $a \flat (b \flat c) = (a \flat b) \flat c$ one cannot infer that $aCbCc$. Indeed, if it were so, then by Theorem 5.3 one would get also $a \sharp (c \sharp b) = (a \sharp c) \sharp b$ which in turn would imply that aCc , and since $aC0Cc$ and $aC1Cc$ for all elements $a, c \in L$ this would lead to the compatibility of all elements of a lattice L .

The reasoning changes when we pass from the “local” situation described above, that is, associativity for a single triple of elements to the “global” one, that is, to considering associativity of all possible triples of elements of an OML.

Theorem 5.6 *If the operation \sharp or \flat on an orthomodular lattice L is associative, then all elements of L are compatible, that is, L is a Boolean algebra.*

Proof: Let us notice that because of de Morgan laws, associativity of \sharp implies the associativity of \flat and vice versa. Let us assume now associativity of \sharp : $(a \sharp b) \sharp c = a \sharp (b \sharp c)$. By putting here $c = b$ and by idempotency of \sharp we obtain

$$(a \sharp b) \sharp b = a \sharp (b \sharp b) = a \sharp b \quad (65)$$

which, because of Theorem 4.5, means that $b \leq a \sharp b$. By interchanging a and b one gets also $a \leq a \sharp b$. Therefore, Theorem 4.4 implies that aCb . Since this reasoning can be repeated for an arbitrary pair of elements of the OML, the theorem is proved. \square

Remark 5.7 This theorem was also proved for the operation \flat by Langer ([13] Theorem 4.3(ii)).

6. Sharp, Flat, and States

Let p be a *state* (another name, *probability measure*) on an orthomodular lattice L , that is, a mapping $p : L \rightarrow [0, 1]$ such that

- (i) $p(1) = 1$,
- (ii) if $a_i \perp a_j$ (i.e., $a_i \leq a'_j$) for $i \neq j$, then $p(\bigvee_i a_i) = \sum_i p(a_i)$,

(one allows in (ii) countable joins and sums in the case of σ -lattices). In particular, it follows from this very definition that for any $a \in L$ and any state p on L ,

$$p(a) + p(a') = p(a \vee a') = p(1) = 1. \quad (66)$$

The next theorem shows that it is possible to express the values $p(a \sharp b)$ and $p(a \flat b)$ by the values that p takes on meets and joins of some pairs chosen from the set $\{a, a', b, b'\}$.

Theorem 6.1 *If L is an orthomodular lattice, then for any $a, b \in L$ and any state p on L ,*

$$p(a \sharp b) = p(a \wedge b) + p(a \wedge b') + p(a' \wedge b) \quad (67)$$

$$\begin{aligned} p(a \flat b) &= p(a \vee b) - p(a \wedge b') - p(a' \wedge b) \\ &= p(a \vee b) + p(a \vee b') + p(a' \vee b) - 2. \end{aligned} \quad (68)$$

Proof: Since elements of a formal Mackey decomposition of a pair $\{a, b\}$ used to define $a \sharp b$ are pairwise orthogonal, (67) follows immediately. In order to show (68), it is enough to calculate

$$\begin{aligned}
p(a \flat b) &= 1 - p(a' \sharp b') = 1 - [p(a' \wedge b') + p(a' \wedge b) + p(a \wedge b')] \\
&= 1 - [1 - p(a \vee b) + p(a' \wedge b) + p(a \wedge b')] \\
&= p(a \vee b) - p(a' \wedge b) - p(a \wedge b') \\
&= p(a \vee b) - p[(a \vee b)'] - p[(a' \vee b)'] \\
&= p(a \vee b) + p(a \vee b') + p(a' \vee b) - 2.
\end{aligned} \tag{69}$$

□

Let us assume now that, on the contrary, we know values that a state p takes on $a \sharp b$, $a \flat b$, $a \flat b'$, and $a' \flat b$. It occurs that this suffices to calculate $p(a \vee b)$ and $p(a \wedge b)$.

Theorem 6.2 *If L is an orthomodular lattice, then for any $a, b \in L$ and any state p on L*

$$p(a \vee b) = \frac{2}{3}p(a \sharp b) + \frac{1}{3}p(a \flat b) + \frac{1}{3}p(a \flat b') + \frac{1}{3}p(a' \flat b), \tag{70}$$

$$p(a \wedge b) = \frac{1}{3}p(a \sharp b) + \frac{2}{3}p(a \flat b) - \frac{1}{3}p(a \flat b') - \frac{1}{3}p(a' \flat b). \tag{71}$$

Proof: From Theorem 6.1 we obtain the following equations:

$$p(a \sharp b) = p(a \wedge b) + p(a \wedge b') + p(a' \wedge b), \tag{72}$$

$$p(a \flat b) = p(a \vee b) - p(a \wedge b') - p(a' \wedge b), \tag{73}$$

$$p(a \flat b') = p(a \vee b) - p(a \wedge b) - p(a' \wedge b), \tag{74}$$

$$p(a' \flat b) = p(a \vee b) - p(a \wedge b) - p(a \wedge b'). \tag{75}$$

This is a system of linear equations and by solving it we get (70) and (71). □

By adding equations (67) and (68) we obtain the following corollary.

Corollary 6.3 *For any two elements a, b of an orthomodular lattice L and any state p on L ,*

$$p(a \sharp b) + p(a \flat b) = p(a \vee b) + p(a \wedge b). \tag{76}$$

Let us calculate now the value that a state p takes on the commutator $a * b$ of any two elements $a, b \in L$. By taking orthocomplements of all terms that appear in formula (19) we get

$$(a * b)' = (a \sharp b) \vee (a \vee b)' = (a \wedge b) \vee (a \flat b)'. \tag{77}$$

From formulas (31) and (32) we infer that

$$a \sharp b \leq a \vee b = [(a \vee b)']' \tag{78}$$

and

$$a \wedge b \leq a \flat b = [(a \flat b)']', \tag{79}$$

which means that $a \sharp b$ is orthogonal to $(a \vee b)'$ and $a \wedge b$ is orthogonal to $(a \flat b)'$. Therefore, for any state p on L we get

$$1 - p(a * b) = p[(a * b)'] = p(a \sharp b) + 1 - p(a \vee b) \tag{80}$$

which yields

$$p(a * b) = p(a \vee b) - p(a \sharp b) \tag{81}$$

and

$$1 - p(a * b) = p[(a * b)'] = p(a \wedge b) + 1 - p(a \flat b) \quad (82)$$

which yields

$$p(a * b) = p(a \flat b) - p(a \wedge b). \quad (83)$$

Therefore, we proved the following the following theorem.

Theorem 6.4 *If L is an orthomodular lattice, then for any $a, b \in L$ and any state p on L ,*

$$p(a * b) = p(a \vee b) - p(a \sharp b) = p(a \flat b) - p(a \wedge b). \quad (84)$$

We infer from this theorem that the value $p(a * b)$, which, of course, equals 0 when elements a and b are compatible and, therefore, is in a sense a “measure of noncompatibility of a and b ,” can be treated also as a measure of a “distance” between the new and the traditional operations on a lattice.

7. Open Problems

7.1 Definability of an OML by \sharp , \flat , and $'$ What axioms should be imposed on \sharp , \flat , and $'$, treated as abstract binary and unary operations on some set L , in order to force L to be an orthomodular lattice? In view of Theorem 3.1 this problem seems to be solved since it is sufficient to take well-known axioms that define an OML in terms of \vee , \wedge , and $'$, and “translate” them into the language of \sharp , \flat , and $'$ with the aid of Theorem 3.1. However, such solution could not be treated as satisfactory since the axioms become very complicated: for example, associativity of join expressed in terms of \sharp and \flat with the aid of the formula (20) yields the following, rather lengthy, equation.

$$\{a \sharp [(b \sharp c) \sharp (b \flat c)]\} \sharp \{a \flat [(b \sharp c) \sharp (b \flat c)]\} = \\ \{[(a \sharp b) \sharp (a \flat b)] \sharp c\} \sharp \{[(a \sharp b) \sharp (a \flat b)] \flat c\}. \quad (85)$$

Therefore, the problem should be posed as follows: find a system of *simple* (and, preferably, logically “plausible” or “natural”) axioms for operations \sharp , \flat , and $'$ that would force $(L, \sharp, \flat, ')$ to be an orthomodular lattice.

7.2 Generalization of Theorem 5.3 and Corollary 5.5 What can be proved in the situation when one element is compatible with more than two (not necessarily compatible) elements?

7.3 Partition in the family of pairs of noncompatible propositions Theorem 4.4 implies that if a and b are not compatible, then either $a \sharp b < a \flat b$ or the elements $a \sharp b$ and $a \flat b$ are noncomparable. Is this partition of any (mathematical, logical, or physical) significance? In particular, is the “reverse” order— $a \sharp b < a \flat b$ —of any special meaning? Similar questions arise also in cases not covered by $a, b \leq a \sharp b$ or $a \flat b \leq a, b$.

7.4 Further links between various operations on OMLs Are there any other (interesting) links between \sharp , \flat , and other operations on orthomodular lattices except those listed in this paper? Some results of such investigations are contained in [5] and in the already mentioned unpublished paper by D’Hooghe, but they concern, however, other “conjunctionlike” and “disjunctionlike” operations that also were not studied before. Also in the often-mentioned paper by Länger [13] several relations between \flat treated as ring multiplication and two OML generalizations of Boolean symmetric difference (logical *exclusive disjunction*) are studied, but its “de Morgan dual” operation \sharp is not considered.

7.5 Generalizations to ortholattices What features would have operations \sharp and \flat defined by the formulas (1) and (2) if L were assumed to be orthocomplemented but not necessarily orthomodular lattice?

7.6 Which operations are better for physical purposes According to the recent results of [5] there are five pairs of conjunctionlike and disjunctionlike operations on orthomodular lattices that coincide with the lattice operations on compatible elements but are necessarily different from them and from each other on noncompatible elements of a lattice. Since one can test simultaneously only compatible propositions about quantum systems, it seems that we cannot distinguish between all these operations by making experiments, that is, from the experimental point of view all these operations seem to be equivalent. This yields the last problem: looking for the other than experimental, maybe indirect arguments, which ones among these operations are better suited to be used as algebraic models of conjunctions and disjunctions of propositions about quantum systems.

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