Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 2A, 2019, 587–612
DOI: 10.12775/TMNA.2019.078

QUASILINEAR ELLIPTIC PROBLEMS ON NON-REFLEXIVE ORLICZ–SOBOLEV SPACES

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Abstract. In the paper the existence, uniqueness and the multiplicity of solutions for a quasilinear elliptic problems driven by the Φ-Laplacian operator is established. Here we consider the non-reflexive case taking into account the Orlicz and Orlicz–Sobolev framework. The non-reflexive case occurs when the N-function $\widetilde{\Phi}$ does not verify the Δ_2-condition. In order to prove our main results we employ variational methods, regularity results and truncation arguments.

1. Introduction

In this work we consider the existence and uniqueness of solutions for quasilinear elliptic problems given by

$$\begin{cases} -\Delta_\Phi u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary: 35J20, 35J25; Secondary: 35J60, 35J92, 58E05.

Key words and phrases. Variational methods; quasilinear elliptic problems; positive solutions; reflexive and non-reflexive Banach spaces.

Research supported in part by INCTmat/MCT/Brazil, CNPq and CAPES/Brazil.

The authors was partially supported by Fapeg/CNPq grants 03/2015-PPP and CNPq grant 408604/2018-2.

Furthermore, we shall consider the existence and multiplicity of solutions for the following quasilinear elliptic problem

$$\begin{cases} -\Delta_\Phi u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary. The function Φ is an even N-function defined by

$$\Phi(t) = \int_0^t s\phi(s)\,ds, \quad t \in \mathbb{R}.$$

Later on, we shall consider some assumptions on ϕ, f and g. It is important to recall that Φ satisfies the so called Δ_2-condition whenever

$$\Phi(2t) \leq C\Phi(t), \quad t \geq 0, \tag{1.3}$$

holds true for some $C > 0$, in short, we write $\Phi \in \Delta_2$.

Quasilinear elliptic problems driven by the Φ-Laplacian operator have been widely considered in the last years. Here we refer the reader to [6], [8], [13]–[20], [23], [24], [28]. Most of them considered the Orlicz and Orlicz–Sobolev framework taking into account that Φ and $\widetilde{\Phi}$ verify the so called Δ_2-condition. Under these conditions the Orlicz and Orlicz–Sobolev spaces are separable and reflexive Banach spaces, see [27]. Hence we can use the weak convergence in order to guarantee that problems (1.1) and (1.2) admits at least one weak solution by applying variational methods.

The main novel in this work is to consider quasilinear elliptic problem driven by the Φ-Laplacian operator where the Δ_2-condition is not satisfied for $\widetilde{\Phi}$, that is, the conjugate function defined by

$$\widetilde{\Phi}(t) = \max_{s \geq 0}\{ts - \Phi(s)\}, \quad t \geq 0,$$

does not verifies the Δ_2-condition. The main difficulty in this work arises from the fact that $W_0^{1,\Phi}(\Omega)$ is not reflexive anymore. In order to overcome this difficulty we consider a sequence of quasilinear elliptic problems modeled in an appropriate reflexive Orlicz–Sobolev space obtaining a sequence of solutions u_ε for each $\varepsilon > 0$. Then we take the limit as $\varepsilon \to 0$ getting a weak solution u for the quasilinear elliptic problems (1.1) and (1.2).

The approach here is purely variational using an energy functional associated to the elliptic problems (1.1) or (1.2). In our setting we consider an auxiliary problem in order to recover some compactness for our energy functional which is crucial in variational methods.

Now we shall give the hypotheses for the functions ϕ, f and g. We assume that $\phi\colon \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies the following hypotheses:

(ϕ_1) (i) $t\phi(t) \to 0$ as $t \to 0$, (ii) $t\phi(t) \to \infty$ as $t \to \infty$;

(ϕ_2) $t\phi(t)$ is strictly increasing in $(0,\infty)$;

(ϕ_3) there exist $\ell, m \in [1,N)$ such that
$$\ell - 1 = \inf_{t>0} \frac{(t\phi(t))'}{\phi(t)} \le \frac{(t\phi(t))'}{\phi(t)} \le m-1, \quad t>0;$$

(ϕ_4) $a := \inf_{t>0} \dfrac{t^m}{\Phi(t)} > 0$.

Remark 1.1. Taking into account hypothesis (ϕ_3) we have that $t \mapsto t^m/\Phi(t)$ is a strictly increasing function. As a consequence we mention that
$$\inf_{t>0} \frac{t^m}{\Phi(t)} = \lim_{t\to 0_+} \frac{t^m}{\Phi(t)}.$$

For the non-homogeneous function $f\colon \Omega \to \mathbb{R}$ we assume that
$$f \in L^N(\Omega). \tag{1.4}$$

For the function $g\colon \Omega\times\mathbb{R} \to \mathbb{R}$ we suppose that $g \in C^0$ and $g(x,0)=0$ for each $x\in\Omega$. Furthermore, we suppose also the following assumptions:

(g_1) there exist a constant $C>0$ and a N-function
$$\Psi(t) = \int_0^t \psi(s)\,ds$$
with $\psi\colon [0,\infty)\to\mathbb{R}$ continuous and satisfying

(ψ_1) $1 < m < \ell_\Psi := \inf_{t>0} \dfrac{t\psi(t)}{\Psi(t)} \le \sup_{t>0} \dfrac{t\psi(t)}{\Psi(t)} =: m_\Psi < 1^* := \dfrac{N}{N-1}$;

such that $|g(x,t)| \le C(1+\psi(t))$,

(g_2) there is an N-function
$$\Gamma(t) = \int_0^t \gamma(s)\,ds$$
with $\gamma\colon [0,\infty)\to\mathbb{R}$ continuous and satisfying

(γ_1) $N < \ell_\Gamma := \inf_{t>0} \dfrac{t\gamma(t)}{\Gamma(t)} \le \sup_{t>0} \dfrac{t\gamma(t)}{\Gamma(t)} =: m_\Gamma < \infty$,

such that
$$\Gamma\bigg(\frac{G(x,t)}{|t|^\ell}\bigg) \le C\overline{G}(x,t), \quad x\in\Omega,\ |t|\ge R,$$
where C, R are positive constants,
$$G(x,t) := \int_0^t g(x,s)\,ds \quad \text{and} \quad \overline{G}(x,t) := tg(x,t) - mG(x,t),$$
for $x\in\Omega$, $t\in\mathbb{R}$.

In order to state our main result we consider the number $\lambda_1 > 0$ associated with Δ_Φ given by

$$\lambda_1 = \inf_{u \in W_0^{1,\Phi}(\Omega)} \bigg\{ \int_\Omega \Phi(|\nabla u|)\, dx \bigg/ \int_\Omega \Phi(|u|)\, dx,\ u \neq 0 \bigg\}.$$

It is important to emphasize that λ_1 is positive which can be proved, taking into account hypothesis (ϕ_3), thanks to the Poincaré inequality, (see e.g. [8], [18]). We consider some additional hypotheses:

(g$_3$) $\displaystyle\lim_{t\to\infty} \frac{g(x,t)}{|t|^{m-1}} = \infty$,

(g$_4$) $\displaystyle\limsup_{t\to 0} \frac{g(x,t)}{|t|\phi(t)} = \lambda < \lambda_1$,

Due to the nature of the operator $\Delta_\Phi u = \mathrm{div}(\phi(|\nabla u|)\nabla u)$ we need to consider the Orlicz–Sobolev framework. It is important to remember that the Φ-Laplacian operator is not homogeneous. This is a serious difficulty in order to use variational methods. In order to overcome this difficulty we shall consider some specific estimates in Orlicz and Orlicz–Sobolev spaces.

Definition 1.2. Let Φ, Ψ be two N-functions. We say that Φ and Ψ are equivalent, in short $\Phi \cong \Psi$, when there exist $c_1, c_2 > 0$ in such way that

$$c_1\Psi(t) \leq \Phi(t) \leq c_2\Psi(t) \quad \text{for each } t \geq t_0 \text{ and for some } t_0 \geq 0.$$

Our first main result can be stated in the following form

Theorem 1.3. *Assume* (ϕ_1)–(ϕ_4) *and* (1.4). *Then there exists an unique solution for the elliptic problem* (1.1)*, that is, there exists an unique function* $u \in W_0^{1,\Phi}(\Omega)$ *in such way that*

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla v\, dx = \int_\Omega f v\, dx, \quad v \in W_0^{1,\Phi}(\Omega). \tag{1.5}$$

Moreover, assuming that $\ell > 1$ *the solution belongs to* $L^\infty(\Omega)$ *whenever the function* $\Phi \cong |t|^r$ *for some* $r > 1$.

We point out that the function

$$\phi(t) = \frac{\log(1+|t|)}{|t|}, \quad t \in \mathbb{R}\backslash\{0\}$$

satisfies the hypotheses (ϕ_1)–(ϕ_4). In this case the operator in problem (1.1) has logarithmic growth with respect to the gradient which can be written in the following form

$$\begin{cases} -\mathrm{div}\bigg(\dfrac{\log(1+|\nabla u|)}{|\nabla u|}\,\nabla u\bigg) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Similarly, we also consider the following quasilinear elliptic problem

$$(1.7)\qquad \begin{cases} -\mathrm{div}\Big(\dfrac{\log(1+|\nabla u|)}{|\nabla u|}\,\nabla u+\dfrac{1}{1+|\nabla u|}\,\nabla u\Big)=f(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Here we observe that

$$\phi(t)=\frac{\log(1+|t|)}{|t|}+\frac{1}{1+|t|}\quad \text{and}\quad \Phi(t):=t\log(1+t).$$

It is important to emphasize that problem (1.7) was not treated in the literature. The main difficult occurs due the fact that $\ell=1$, $m=2$, $a>0$. More generally, fixing $\alpha>0$, $\beta\geq -1$, we define

$$\Phi(t)=(\beta-1)t-\beta\log(1+t)+(1+t)[\log(1+t^{\alpha})]^{1/\alpha},\quad t>0.$$

For this N-function $\ell=1$, $m=2$ which provide a new example for our setting. For the case $\beta=0$ and $\alpha=1$ we recover the quasilinear elliptic problem (1.6), while if $\beta=1$ and $\alpha=1$ we obtain the problem (1.7). Here we point out the examples listed just above give us concrete cases where the N-function Φ is in such way that $\widetilde{\Phi}$ does not verity the well known Δ_2 condition due the fact that $\ell=1$, $m=2$, see [27]. As a consequence the N-function $\Phi\colon\mathbb{R}\to\mathbb{R}$ given by

$$\Phi(t)=\int_0^t\log(1+s)\,ds,\quad s>0$$

is in such way that $W_0^{1,\Phi}(\Omega)$ is not reflexive. The problem (1.6) have been studied by many authors during the last years, see Boccardo et al. [3], Esposito et al. [12], Passarelli [10], Fuchs [13], [14], Zhang et al. [29] and references therein. For further results on Orlicz and Orlicz–Sobolev framework we refer the reader to [1], [15], [16], [18], [19], [23]. The main feature in this work is to find a weak solution for the problem (1.1) in the non-reflexive case using a sequence of approximating problems where in each term of this sequence the associated Orlicz–Sobolev space is reflexive. As a consequence, taking the limit we obtain at least one solution for the non-reflexive which is obtained by a careful analysis on continuous and compact embeddings for Orlicz–Sobolev spaces. Thanks to this approach we shall prove that the elliptic problem (1.2) admits existence and multiplicity of solutions for the non-reflexive case.

For the next result we shall consider the nonlinear elliptic problem (1.2) under some superlinear conditions at infinity. The main feature here is to consider non-reflexive problems without the well known Ambrosetti–Rabinowitz condition at infinity. Namely, the Ambrosetti–Rabinowitz condition, for the function g, in short (AR) condition, says that

(AR) $0<\theta G(x,t)\leq tg(x,t)$, for $x\in\Omega$ and $|t|\geq R$

holds true for some $\theta > m$ and $R > 0$. As a consequence the (AR) condition implies that

$$G(x,t) \geq c_1|t|^m - c_2, \quad x \in \Omega,\ t \in \mathbb{R} \tag{1.8}$$

holds for some $c_1, c_2 > 0$. Nevertheless, there are superlinear functions in such way that (1.8) is not satisfied. For example, we mention that

$$g(x,t) = |t|^{m-2}t\ln(1+|t|)$$

does not verity the superlinear condition given in (1.8) for each $m \in (1,N)$, which implies also that it does not verify the (AR) condition. It is worthwhile to mention that the (AR) condition implies some compactness properties such as the Palais–Smale condition which is crucial in variational methods. Since (AR) condition is not available in our setting we need to consider some compactness condition such as Cerami condition. This famous condition was first introduced by Cerami [4]. Latter on, we shall give a precise definition for the Cerami condition.

For our next result we shall consider hypotheses (g_1)–(g_4) proving that the associated functional for the problem (1.2) satisfies the well known Cerami condition which is sufficient in variational procedures.

Our main second result is the following:

Theorem 1.4. *Assume* (ϕ_1), (ϕ_2), (ϕ_4),

$(\phi_3)'$ $1 \leq \ell \leq \dfrac{\phi(t)t^2}{\Phi(t)} \leq m,\ t > 0,$

and (g_1)–(g_4). *Then problem* (1.2) *at least one solution* $u \in W_0^{1,\Phi}(\Omega)$, *that is, there exists a function* $u \in W_0^{1,\Phi}(\Omega)$ *in such way that*

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla v\,dx = \int_\Omega g(x,u)v\,dx, \quad v \in W_0^{1,\Phi}(\Omega). \tag{1.9}$$

Assuming (ϕ_3) *instead of* $(\phi_3)'$, *problem* (1.2) *admits at least two weak solutions* $u_1, u_2 \in W_0^{1,\Phi}(\Omega)/\{0\}$ *satisfying* $u_1 \geq 0$ *and* $u_2 \leq 0$ *in* Ω. *Furthermore, assuming also that* $\ell > 1$, *then solutions* u_1, u_2 *described just above, are strictly positive which belongs to* $C^{1,\alpha}$ *for some* $\alpha \in (0,1)$.

The paper is organized as follows: Section 2 is devoted to an overview on Orlicz and Orlicz–Sobolev framework considering the elliptic problem (1.1) for the reflexive case. In Section 3 we give some existence results for the problem (1.1) in the non-reflexive case. Section 4 is devoted to regularity results to the elliptic problem (1.1) and (1.2). In Section 5 we give the proofs of our main results.

2. Orlicz and Orlicz–Sobolev spaces

The reader is referred to [1], [11], [25], [27] regarding Orlicz–Sobolev spaces. The usual norm on $L_\Phi(\Omega)$ is (Luxemburg norm)

$$\|u\|_\Phi = \inf\bigg\{\lambda > 0 \ \bigg| \ \int_\Omega \Phi\bigg(\frac{u(x)}{\lambda}\bigg)\,dx \le 1\bigg\},$$

the Orlicz–Sobolev norm of $W^{1,\Phi}(\Omega)$ is

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \sum_{i=1}^N \bigg\|\frac{\partial u}{\partial x_i}\bigg\|_\Phi.$$

Since our N-function Φ verifies the Δ_2-condition we observe that $W_0^{1,\Phi}(\Omega)$ is given by the closure of $C_0^\infty(\Omega)$ with respect to the usual norm of $W^{1,\Phi}(\Omega)$, see [18], [19], [25].

Recall that $\widetilde{\Phi}(t) = \max_{s\ge0}\{ts-\Phi(s)\}$, $t \ge 0$. It turns out that when Φ and $\widetilde{\Phi}$ are N-functions satisfying the Δ_2-condition then $L_\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable, reflexive, Banach spaces, see [27]. However we shall consider the case when the function $\widetilde{\Phi}$ does not verify the Δ_2-condition.

REMARK 2.1. It is well known that (ϕ_3) implies $(\phi_3)'$. Furthermore, assuming $1 < \ell \le m < N$, we obtain $\Phi, \widetilde{\Phi} \in \Delta_2$. Conversely, assuming that $\Phi, \widetilde{\Phi} \in \Delta_2$ then $1 < \ell \le m < \infty$.

By the Poincaré inequality, (see e.g. [18]),

$$\int_\Omega \Phi(u)\,dx \le \int_\Omega \Phi(2d|\nabla u|)\,dx$$

where $d = \operatorname{diam}(\Omega)$. It follows also that $\|u\|_\Phi \le 2d\|\nabla u\|_\Phi$ for $u \in W_0^{1,\Phi}(\Omega)$. As a consequence, $\|u\| := \|\nabla u\|_\Phi$ defines a norm in $W_0^{1,\Phi}(\Omega)$, equivalent to $\|\cdot\|_{1,\Phi}$. Here was used the fact that Φ satisfies the Δ_2-condition. Let Φ_* be the inverse of the function

$$t \in (0,\infty) \mapsto \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}\,ds$$

which extends to $\mathbb{R}$ by $\Phi_*(t) = \Phi_*(-t)$ for $t \le 0$. We say that a N-function Ψ grows essentially more slowly than Φ_* and we write $\Psi \ll \Phi_*$, if

$$\lim_{t\to\infty} \frac{\Psi(\lambda t)}{\Phi_*(t)} = 0, \quad \text{for all } \lambda > 0.$$

The embedding $W_0^{1,\Phi}(\Omega) \overset{\mathrm{cpt}}{\hookrightarrow} L_\Psi(\Omega)$, if $\Psi \ll \Phi_*$, will be used in this paper (cf. [1]). In particular, as $\Phi \ll \Phi_*$ (cf. [19, Lemma 4.14]), we obtain $W_0^{1,\Phi}(\Omega) \overset{cpt}{\hookrightarrow} L_\Phi(\Omega)$. Furthermore, we mention also that $W_0^{1,\Phi}(\Omega) \overset{\mathrm{cont}}{\hookrightarrow} L_{\Phi_*}(\Omega)$. It is worthwhile to mention that under hypotheses (ϕ_1)–(ϕ_2) and (ϕ_3) (cf. [7, Lemma D.2]) the continuous embedding $L^m(\Omega) \overset{\mathrm{cont}}{\hookrightarrow} L_\Phi(\Omega) \overset{\mathrm{cont}}{\hookrightarrow} L^\ell(\Omega)$ holds.

Now we refer the reader to [15], [16] for some elementary results on Orlicz and Orlicz–Sobolev spaces.

Proposition 2.2. *Assume that* ϕ *satisfies* (ϕ_1)–(ϕ_3). *Set*

$$\zeta_0(t) = \min\{t^{\ell}, t^{m}\}, \quad \zeta_1(t) = \max\{t^{\ell}, t^{m}\}, \quad t \geq 0.$$

Then Φ *satisfies*:

$$\begin{aligned} \zeta_0(t)\Phi(\rho) \leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho), &\qquad \rho, t > 0, \\ \zeta_0(\|u\|_\Phi) \leq \int_\Omega \Phi(u)\,dx \leq \zeta_1(\|u\|_\Phi), &\quad u \in L_\Phi(\Omega). \end{aligned}$$

Proposition 2.3. *Assume that* ϕ *satisfies* (ϕ_1)–(ϕ_3). *Set*

$$\zeta_2(t) = \min\{t^{\ell^*}, t^{m^*}\}, \quad \zeta_3(t) = \max\{t^{\ell^*}, t^{m^*}\}, \quad t \geq 0,$$

where $1 < \ell, m < N$ *and* $m^* = mN/(N-m)$, $\ell^* = \ell N/(N-\ell)$. *Then*

$$\begin{aligned} \ell^* \leq \frac{t\Phi_*'(t)}{\Phi_*(t)} \leq m^*, &\qquad t > 0, \\ \zeta_2(t)\Phi_*(\rho) \leq \Phi_*(\rho t) \leq \zeta_3(t)\Phi_*(\rho), &\qquad \rho, t > 0, \\ \zeta_2(\|u\|_{\Phi_*}) \leq \int_\Omega \Phi_*(u)\,dx \leq \zeta_3(\|u\|_{\Phi_*}), &\quad u \in L_{\Phi_*}(\Omega). \end{aligned}$$

In order to conclude the present section we prove the following:

Theorem 2.4 (The reflexive case). *Suppose* (ϕ_1)–(ϕ_2), $(\phi_3)'$ *and* $\ell > 1$. *Then problem* (1.1) *admits exactly one solution* $u \in W_0^{1,\Phi}(\Omega)$.

In order to prove Theorem 2.4 we need some preliminaries results:

Proposition 2.5. *Let* $\phi\colon (0,\infty) \to (0,\infty)$ *be a fixed* N*-function satisfying hypotheses* (ϕ_1) *and* (ϕ_2). *Then*

$$\begin{aligned} &(\phi(|x|)x - \phi(|y|)y, x - y) \geq 0, \quad \text{for all } x, y \in \mathbb{R}^n, \\ &(\phi(|x|)x - \phi(|y|)y, x - y) > 0, \quad \text{for all } x, y \in \mathbb{R}^n,\ x \neq y. \end{aligned}$$

Proof. We will split the proof into three parts. In the first part we choose $x, y \in \mathbb{R}^N$ in such way that $|x| = |y|$. In this case we easily see that

$$(\phi(|x|)x - \phi(|y|)y, x - y) = \phi(|x|)|x - y|^2 \geq 0, \quad x, y \in \mathbb{R}^N,\ |x| = |y|.$$

This estimate proves the proposition in the first part. In the second part we shall consider $x, y \in \mathbb{R}^N$ in such that $|x| < |y|$. Thanks to hypothesis (ϕ_2) we mention that

$$\begin{aligned} (\phi(|x|)x - \phi(|y|)y, x - y) &\geq \phi(|x|)|x|(|x| - |y|) + \phi(|y|)|y|(|y| - |x|) \\ &= (\phi(|x|)|x| - \phi(|y|)|y|)(|x| - |y|) > 0. \end{aligned} \tag{2.1}$$

This ends the proof in the second case. In the last part we shall consider $x, y \in \mathbb{R}^N$ in such way that $|x| > |y|$. Using the same ideas discussed just above we conclude one more time that

$$\begin{aligned} (2.2) \qquad (\phi(|x|)x - \phi(|y|)y, x - y) &\geq \phi(|x|)|x|(|x| - |y|) + \phi(|y|)|y|(|y| - |x|) \\ &= (\phi(|x|)|x| - \phi(|y|)|y|)(|x| - |y|) > 0. \qquad \square \end{aligned}$$

The next result shows that the Φ-Laplacian is also strictly monotonic.

PROPOSITION 2.6. *Suppose* (ϕ_1)–(ϕ_2). *Then we have that*

$$\int_\Omega (\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v)(\nabla u - \nabla v)\, dx > 0, \quad u, v \in W_0^{1,\Phi}(\Omega), \ u \neq v.$$

PROOF. Let $u, v \in W_0^{1,\Phi}(\Omega)$ be fixed functions in such way that $u \neq v$. Using Proposition 2.5 we deduce that

$$(\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v)(\nabla u - \nabla v) \geq 0 \quad \text{a.e. in } \Omega.$$

Using the fact that $u \neq v$ there exits $\Omega_0 \subset \Omega$ with positive Lebesgue measure such that

$$(\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v)(\nabla u - \nabla v) > 0 \quad \text{a.e. in } \Omega_0.$$

As a consequence we obtain

$$\int_{\Omega_0} (\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v)(\nabla u - \nabla v)\, dx > 0.$$

The last estimate implies that

$$\int_\Omega (\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v)(\nabla u - \nabla v)\, dx > 0. \qquad \square$$

Now we apply Proposition 2.6 to prove that problem (1.1) has at most one solution.

PROPOSITION 2.7. *Suppose* (ϕ_1)–(ϕ_2). *Then the problem* (1.1) *admits at most one solution in* $W_0^{1,\Phi}(\Omega)$.

As a consequence, the uniqueness stated in Theorem 2.4 is proved. For the existence we refer the reader to Fukagai et al. [15], [16] for $\ell > 1$ and hence the proof of Theorem 2.4 is completed.

3. Problem (1.1) for the non-reflexive case

In this section we prove some useful results in order to ensure existence of solutions for problem (1.1) in the non-reflexive case. The first result in this direction is to consider a sequence of approximating quasilinear elliptic problems driven by the Φ-Laplacian operator given by

$$\begin{cases} -\Delta_{\Phi\varepsilon} u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $f \in L^N(\Omega)$ and $\Phi_\varepsilon(t) := \Phi(t) + \varepsilon t^m/m$, $t \in \mathbb{R}$, $\varepsilon > 0$. It is important to remember that $v \in W_0^{1,\Phi_\varepsilon}(\Omega)$ is a weak solution for the problem (3.1) whenever

$$\varepsilon \int_\Omega |\nabla v|^{m-2} \nabla v \nabla w \, dx + \int_\Omega \phi(|\nabla v|) \nabla v \nabla w \, dx = \int_\Omega f w \, dx \tag{3.2}$$

holds true for each $w \in W_0^{1,\Phi_\varepsilon}(\Omega)$. We list now some useful properties of the functions Φ_ε:

Lemma 3.1. *Suppose* (ϕ_1)–(ϕ_4). *Then the function* Φ_ε *satisfies the following properties*:

(a) $\Phi_\varepsilon \to \Phi$ *as* $\varepsilon \to 0$;

(b) $1 < \ell_\varepsilon \leq \dfrac{\Phi'_\varepsilon(t)t}{\Phi_\varepsilon(t)} \leq m$, $t > 0$, $\varepsilon > 0$;

(c) $\ell_\varepsilon \to 1$ *as* $\varepsilon \to 0$;

(d) Φ_ε *is equivalent to the* N*-function* t^m *for each* $\varepsilon > 0$.

Proof. First of all, we mention that Φ and $t \mapsto \varepsilon t^m/m$ are N-functions. Hence Φ_ε is also a N-function. The proof for the limit in (a) is obvious.

Now we prove the item (b). Taking into account (ϕ_3) we have that

$$1 \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m, \quad t > 0. \tag{3.3}$$

As a consequence, we infer that

$$\frac{\Phi'_\varepsilon(t)t}{\Phi_\varepsilon(t)} = \frac{\varepsilon t^m + \phi(t)t^2}{\Phi(t) + \dfrac{\varepsilon}{m} t^m} = \frac{\varepsilon \dfrac{t^m}{\Phi(t)} + \dfrac{\phi(t)t^2}{\Phi(t)}}{1 + \dfrac{\varepsilon}{m} \dfrac{t^m}{\Phi(t)}} \leq \frac{\varepsilon \dfrac{t^m}{\Phi(t)} + m}{1 + \dfrac{\varepsilon}{m} \dfrac{t^m}{\Phi(t)}} = m.$$

On the other hand, using one more time (3.3), we also have that

$$\frac{\Phi'_\varepsilon(t)t}{\Phi_\varepsilon(t)} = \frac{\varepsilon \dfrac{t^m}{\Phi(t)} + \dfrac{\phi(t)t^2}{\Phi(t)}}{1 + \dfrac{\varepsilon}{m} \dfrac{t^m}{\Phi(t)}} \geq \frac{\varepsilon \dfrac{t^m}{\Phi(t)} + 1}{1 + \dfrac{\varepsilon}{m} \dfrac{t^m}{\Phi(t)}} = h\left(\frac{t^m}{\Phi(t)}\right), \tag{3.4}$$

where we define $h(s) := (\varepsilon s + 1)/(\varepsilon s/m + 1)$. It is easy to see that h is increasing. Furthermore, we observe that

$$\frac{d}{dt}\left(\frac{t^m}{\Phi(t)}\right) = \frac{t^{m-1}}{\Phi(t)}\left(m - \frac{\phi(t)t^2}{\Phi(t)}\right) \geq 0.$$

As a consequence we obtain $t \mapsto t^m/\Phi(t)$ is nondecreasing. Hence the function $t \mapsto h(t^m/\Phi(t))$ is also nondecreasing. As a consequence, using the estimate (3.4), we observe that

$$\frac{\Phi_\varepsilon'(t)t}{\Phi_\varepsilon(t)} \geq \lim_{t\to 0} h\left(\frac{t^m}{\Phi(t)}\right) = \frac{\varepsilon a+1}{\dfrac{\varepsilon}{m}\, a+1} = 1+\frac{(m-1)\varepsilon a}{\varepsilon a+m} =: \ell_\varepsilon > 1.$$

Here we have used the fact that Φ is a N-function satisfying $m > 1$. According to the last estimate we see that $\lim\limits_{\varepsilon\to 0+} \ell_\varepsilon = 1$. This ends the proof for the item (c). Moreover, using Proposition 2.2, we infer that

$$\frac{\varepsilon}{m}\, t^m \leq \Phi_\varepsilon(t) \leq \left(\Phi(1)+\frac{\varepsilon}{m}\right)t^m, \quad t \geq 1.$$

So that the proof of item (d) is now achieved. These facts finish the proof of this proposition. $\square$

In what follows we shall consider the approximating elliptic problem (3.1) that admits exactly one solution u_ε for each $\varepsilon > 0$. In fact, due to the inequality $\ell_\varepsilon > 1$, we can apply the Theorem 2.4. Now we provide some a priori estimates to the family u_ε.

Proposition 3.2. *Suppose (ϕ_1)–(ϕ_3) where $\ell = 1$. Let $\varepsilon > 0$ be fixed. Then the sequence (u_ε) belongs to $L^\infty(\Omega)$, i.e. there exists $C_\varepsilon > 0$ such that $\|u_\varepsilon\| \leq C_\varepsilon$. Furthermore (u_ε) is bounded in $W_0^{1,\Phi}(\Omega)$ and $W_0^{1,1}(\Omega)$.*

Proof. Thanks to Lemma 3.1 (d) we infer that $\Phi_\varepsilon \approx t^m$. Taking $q = N > N/m$ the Theorem 4.1 ensures that any solution for the problem (3.1) is bounded, that is, the unique solution for the quasilinear elliptic problem (3.1) (u_ε) is in $L^\infty(\Omega)$ for each $\varepsilon > 0$. In other words, we know that $u_\varepsilon \in W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ for each $\varepsilon > 0$. Now, using the results discussed in Section 2 for Orlicz and Orlicz–Sobolev spaces, we mention that the following embedding are continuous $W_0^{1,m}(\Omega) \hookrightarrow W_0^{1,\Phi_\varepsilon}(\Omega) \hookrightarrow W_0^{1,\ell_\varepsilon}(\Omega)$ and $W_0^{1,\Phi_\varepsilon}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$, see [1], [7].

On the other hand, we observe that $\Phi(t), \varepsilon t^m/m \leq \Phi_\varepsilon(t)$, $t \geq 0$. As a consequence $L^{\Phi_\varepsilon}(\Omega) \hookrightarrow L^\Phi(\Omega)$ and $L^{\Phi_\varepsilon}(\Omega) \hookrightarrow L^m(\Omega)$. Furthermore, we infer that $W_0^{1,\Phi_\varepsilon}(\Omega) \hookrightarrow W_0^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi_\varepsilon}(\Omega) \hookrightarrow W_0^{1,m}(\Omega)$. As a consequence the last embedding says also that $W^{1,\Phi_\varepsilon}(\Omega) = W_0^{1,m}(\Omega)$. In particular, we obtain that $u_\varepsilon \in W_0^{1,m}(\Omega)$ for each $\varepsilon > 0$.

Now we shall prove that u_ε is bounded in $W_0^{1,\Phi}(\Omega)$. Putting u_ε as testing function in (3.2) we easily see that

$$\varepsilon\int_\Omega |\nabla u_\varepsilon|^m\,dx + \int_\Omega \phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|^2\,dx = \int_\Omega f u_\varepsilon\,dx.$$

Using the Hölder inequality we also see that

$$\varepsilon\int_\Omega |\nabla u_\varepsilon|^m\,dx + \int_\Omega \phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|^2\,dx \leq \|f\|_N \|u_\varepsilon\|_{1^\star}.$$

Taking into account the embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{1^\star}(\Omega)$ there exists $S = S(N,\Omega) > 0$ in such way that

$$\|v\|_{1^\star} \le S\|v\|_{W_0^{1,1}(\Omega)}, \quad v \in W_0^{1,1}(\Omega).$$

As a consequence the last embedding and hypothesis (ϕ_3) imply that

$$\begin{aligned}\int_\Omega \Phi(|\nabla u_\varepsilon|)\,dx &\le \int_\Omega \phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|^2\,dx \\ &\le m\varepsilon \int_\Omega |\nabla u_\varepsilon|^m\,dx + \int_\Omega \phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|^2\,dx \\ &\le \|f\|_N \|u_\varepsilon\|_{1^\star} \le S\|f\|_N \|u_\varepsilon\|_{W_0^{1,1}(\Omega)}.\end{aligned} \tag{3.5}$$

Let $K > 0$ be fixed. Using the last estimate and hypothesis (ϕ_2) it follows that

$$\begin{aligned}\|u_\varepsilon\|_{W_0^{1,1}(\Omega)} &= \int_{|\nabla u_\varepsilon| \le K} |\nabla u_\varepsilon|\,dx + \int_{|\nabla u_\varepsilon| > K} |\nabla u_\varepsilon|\,dx \\ &\le K|\Omega| + \frac{1}{K\phi(K)} \int_{|\nabla u_\varepsilon| > K} \phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|^2\,dx.\end{aligned}$$

Combining these estimates we obtain

$$\|u_\varepsilon\|_{W_0^{1,1}(\Omega)} \le K|\Omega| + \frac{S\|f\|_N}{K\phi(K)} \|u_\varepsilon\|_{W_0^{1,1}(\Omega)}. \tag{3.6}$$

Now due to the fact that $\lim\limits_{K\to\infty} K\phi(K) = \infty$ there exists $K_0 > 0$ such that $S\|f\|_N/(K\phi(K)) < 1$ for any $K \ge K_0$. In particular, using inequality (3.6), we infer that

$$\|u_\varepsilon\|_{W_0^{1,1}(\Omega)} \le \frac{K|\Omega|}{1 - \dfrac{S\|f\|_N}{K\phi(K)}}.$$

Furthermore, taking into account (3.5) and the estimate just above, we conclude that

$$\int_\Omega \Phi(|\nabla u_\varepsilon|)\,dx \le \frac{S\|f\|_N K|\Omega|}{1 - \dfrac{S\|f\|_N}{K\phi(K)}}.$$

Now we define

$$R = \max\bigg\{\frac{S\|f\|_N K|\Omega|}{1 - S\|f\|_N/(K\phi(K))}, \frac{K|\Omega|}{1 - S\|f\|_N/(K\phi(K))}\bigg\}.$$

As a consequence we have that

$$\int_\Omega \Phi(|\nabla u_\varepsilon|)\,dx \le R, \quad \int_\Omega \phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|^2\,dx \le R, \quad \int_\Omega |\nabla u_\varepsilon|\,dx \le R. \tag{3.7}$$

Accordingly to Lemma 2.2 it follows that

$$\min\Big\{\|u_\varepsilon\|_{W_0^{1,\Phi}(\Omega)}, \|u_\varepsilon\|^m_{W_0^{1,\Phi}(\Omega)}\Big\} \le \int_\Omega \Phi(|\nabla u_\varepsilon|)\,dx \le R.$$

Hence the sequence (u_ε) is now bounded in $W_0^{1,\Phi}(\Omega)$ and $W_0^{1,1}(\Omega)$. $\square$

Now we prove that the Φ-Laplacian operator is of $(S)^+$ type, see [5]. This is a powerful tool in order to restore some compactness required in variational methods.

PROPOSITION 3.3. *Suppose* (ϕ_1), (ϕ_2), $(\phi_3)'$. *Let* $(u_n) \in W_0^{1,\Phi}(\Omega)$ *be a sequence satisfying*

(a) $u_n \overset{*}{\rightharpoonup} u$ *in* $W_0^{1,\Phi}(\Omega)$;
(b) $\limsup\limits_{n\to\infty} \langle -\Delta_\Phi u_n, u_n - u\rangle \le 0$.

Then we obtain that $u_n \to u$ *in* $W_0^{1,\Phi}(\Omega)$.

PROOF. The proof is similar to the proof of [9, Proposition 3.5] replacing the weak convergence $u_n \rightharpoonup u$ by the weak star convergence $u_n \overset{*}{\rightharpoonup} u$. For the reader convenience we give here a sketch of the proof. Here we emphasize one more time that $W_0^{1,\widetilde{\Phi}}(\Omega)$ is not reflexive anymore. However, the Orlicz–Sobolev space $W_0^{1,\widetilde{\Phi}}(\Omega)$ is isomorphic to a closed set in the weak star topology. More precisely,

$$W_0^{1,\widetilde{\Phi}}(\Omega) \subseteq \prod_{j=1}^{N+1} L^{\widetilde{\Phi}}(\Omega) \simeq \bigg(\prod_{j=1}^{N+1} E_{\widetilde{\Phi}}\bigg)^{\star}$$

where $E_{\widetilde{\Phi}}$ is a separable space. Under these conditions the proof following the same ideas discussed in [9, Proposition 3.5]. $\square$

For the next result we borrow some ideas discussed by Boccardo et al. [3]. The main idea is to find at least one solution u for the problem (1.1) using some fine estimates on the gradient of u.

PROPOSITION 3.4. *Suppose* (ϕ_1)–(ϕ_4) *where* $\ell = 1$. *Then the problem* (1.1) *admits at least one solution* $u \in W_0^{1,\Phi}(\Omega)$.

PROOF. Let $u_\varepsilon \in W_0^{1,\Phi_\varepsilon}(\Omega)$ be the unique solution for the auxiliary elliptic problem (3.1). Accordingly to Proposition 3.2 we know that u_ε is bounded in $W_0^{1,\Phi}(\Omega)$ and $W_0^{1,1}(\Omega)$. As a consequence $u_\varepsilon \overset{*}{\rightharpoonup} u$ in the weak star topology in $W_0^{1,\Phi}(\Omega)$. Indeed, the Orlicz–Sobolev space $W_0^{1,\Phi}(\Omega)$ is isomorphic to a closed set in the weak star topology. More precisely, as was mentioned before we observe that

$$W_0^{1,\Phi}(\Omega) \subseteq \prod_{j=1}^{N+1} L^{\Phi}(\Omega) \simeq \bigg(\prod_{j=1}^{N+1} E_{\Phi}\bigg)^{\star}$$

where E_Φ is a separable space. For further results on weak star topologies we refer the reader to Gossez [18], [19].

Now, using the weak star convergence of u_ε, we observe that

$$\int_\Omega |\nabla u||\nabla u_\varepsilon|\phi(|\nabla u_\varepsilon|)\,dx \le C$$

holds true for some $C > 0$. In fact, using Young's inequality and the Δ_2 condition for Φ, we have that

$$\begin{aligned}|\nabla u||\nabla u_\varepsilon|\phi(|\nabla u_\varepsilon|) &\leq \Phi(|\nabla u|) + \widetilde{\Phi}(|\nabla u_\varepsilon|\phi(|\nabla u_\varepsilon|)) \\ &\leq \Phi(|\nabla u|) + \Phi(2|\nabla u_\varepsilon|) \leq \Phi(|\nabla u|) + 2^m\Phi(|\nabla u_\varepsilon|).\end{aligned}$$

Hence the last estimate together with (3.7) imply that

$$\int_\Omega |\nabla u||\nabla u_\varepsilon|\phi(|\nabla u_\varepsilon|)\,dx \leq \int_\Omega [\Phi(|\nabla u|) + 2^m\Phi(|\nabla u_\varepsilon|)]\,dx \leq R + 2^m R.$$

Now we claim that u is a weak solution to the elliptic problem (1.1). Note also that u is not in general a testing function for the auxiliary elliptic problem (3.1). In this way, we shall consider a density argument in order to prove the claim just above. More specifically, we know that $C_0^\infty(\Omega)$ is dense in $W_0^{1,1}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$. As a result there exists a sequence (U_k) in $C_0^\infty(\Omega)$ in such way that

$$\|u - U_k\|_{W_0^{1,1}(\Omega)}, \|u - U_k\|_{W_0^{1,\Phi}(\Omega)} \leq \frac{1}{k}. \tag{3.8}$$

Using $u_\varepsilon - U_k$ as testing function in the problem (3.1) we mention that

$$\varepsilon\langle -\Delta_m u_\varepsilon, u_\varepsilon - U_k\rangle + \langle -\Delta_{\Phi_\varepsilon} u_\varepsilon, u_\varepsilon - U_k\rangle = \int_\Omega f(u_\varepsilon - U_k)\,dx. \tag{3.9}$$

The last identity is equivalently to

$$-\varepsilon\int_\Omega |\nabla u_\varepsilon|^{m-2}\nabla u_\varepsilon\nabla U_k\,dx + \int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla(u_\varepsilon - U_k)\,dx \leq \int_\Omega f(u_\varepsilon - U_k)\,dx.$$

The last inequality can be written in the following form

$$\begin{aligned}&-\varepsilon\int_\Omega |\nabla u_\varepsilon|^{m-2}\nabla u_\varepsilon\nabla U_k dx + \int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla(u_\varepsilon - u)\,dx \\ &\qquad + \int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla(u - U_k) \leq \int_\Omega f(u_\varepsilon - u)\,dx + \int_\Omega f(u - U_k)\,dx.\end{aligned}$$

Moreover, we mention that $\phi(|\nabla u|)|\nabla u||\nabla(u_\varepsilon - u)| \in L^1(\Omega)$.

At this moment we claim that

$$\left|\int_\Omega |\nabla u_\varepsilon|^{m-2}\nabla u_\varepsilon\nabla U_k\,dx\right| \leq C$$

holds for some $C > 0$ independent on $\varepsilon > 0$. Indeed, the continuous embedding $W^{1,\Phi_\varepsilon}(\Omega) \hookrightarrow W_0^{1,m}(\Omega)$ provide a positive number $C > 0$ in such way that

$$\|v\|_{W_0^{1,m}(\Omega)} \leq C\|v\|_{W_0^{1,\Phi_\varepsilon}(\Omega)}, \quad v \in W_0^{1,\Phi_\varepsilon}(\Omega).$$

Taking $v = u_\varepsilon$ in the previous estimate we obtain

$$\|u_\varepsilon\|_{W_0^{1,m}(\Omega)} \leq C\|u_\varepsilon\|_{W_0^{1,\Phi}(\Omega)} \leq C.$$

In other words, we have shown that (u_ε) is bounded in $W_0^{1,m}(\Omega)$ for any $\varepsilon > 0$. Hence, using the Hölder inequality and the estimate just above, we deduce

$$\begin{aligned}\left|\int_\Omega |\nabla u_\varepsilon|^{m-2}\nabla u_\varepsilon \nabla U_k\,dx\right| &\le \||\nabla u_\varepsilon|^{m-1}\|_{m/m-1}\|U_k\|_{W_0^{1,m}(\Omega)} \\ &\le \|u_\varepsilon\|^{m-1}_{W_0^{1,m}(\Omega)}\|U_k\|_{W_0^{1,m}(\Omega)} \le C\|U_k\|_{W_0^{1,m}(\Omega)}.\end{aligned}$$

It follows tha,t if we take the limit in the last inequality, we see that

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega |\nabla u_\varepsilon|^{m-2}\nabla u_\varepsilon\nabla U_k\,dx = 0. \tag{3.10}$$

On the other hand, due to the weak star convergence, we also see that

$$\lim_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon - u)\,dx = 0. \tag{3.11}$$

Now, using one more time the Hölder inequality, we observe that

$$\lim_{k\to\infty}\int_\Omega f(u - U_k)\,dx = 0. \tag{3.12}$$

In fact, using Orlicz–Sobolev embedding and (3.8), we easily see that

$$\left|\int_\Omega f(u-U_k)\,dx\right| \le \|f\|_N\|u-U_k\|_{1^\star} \le C\|f\|_N\|u-U_k\| \le \frac{C\|f\|_N}{k}.$$

as $k\to\infty$. Furthermore, we claim also that

$$\lim_{k\to\infty}\int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla(u-U_k)\,dx = 0.$$

Indeed, from the Hölder inequality we have that

$$\begin{aligned}&\left|\int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla(u-U_k)\,dx\right| \\ &\qquad\le 2\|\nabla(u-U_k)\|_\Phi\|\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|\|_{\widetilde\Phi} \le \frac{2}{k}\,\|\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|\|_{\widetilde\Phi}.\end{aligned} \tag{3.13}$$

On the other hand, due to the Δ_2 condition for Φ and estimation (3.7), we get

$$\int_\Omega \widetilde\Phi(\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|) \le \int_\Omega \Phi(2|\nabla u_\varepsilon|)\,dx \le 2^m\int_\Omega \Phi(|\nabla u_\varepsilon|) \le 2^m R.$$

Now, using one more time that Φ is convex, we deduce that

$$\|\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|\|_{\widetilde\Phi}\int_\Omega \widetilde\Phi\left(\frac{\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|}{\|\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|\|_{\widetilde\Phi}}\right) \le \int_\Omega \widetilde\Phi(\phi(|\nabla u_\varepsilon||\nabla u_\varepsilon|)\,dx \le 2^m R$$

holds true whenever $\|\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|\|_{\widetilde\Phi} \ge 1$. Hence the last estimate shows that

$$\|\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|\|_{\widetilde\Phi} \le \max(1, 2^m R).$$

Now, taking into account (3.13), we obtain that

$$\lim_{k\to\infty}\int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla(u-U_k)\,dx = 0.$$

At this moment, using (3.10)–(3.12) and taking the limits as $\varepsilon \to 0$ and $k \to \infty$ in the inequality (3.9), we get

$$\limsup_{\varepsilon\to 0} \int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon \nabla(u_\varepsilon - u) dx = 0$$

Summing up, due to the (S^+) condition, see Proposition 3.3, for the Φ-Laplacian operator, we have that $u_\varepsilon \to u$ as $\varepsilon \to 0$ in $W_0^{1,\Phi}(\Omega)$. It follows from Boccardo and Murat [2] that $\nabla u_\varepsilon \to \nabla u$ almost everywhere in Ω. Since $u_\varepsilon \to u$ we infer that

$$\lim_{\varepsilon\to 0} \int_\Omega \Phi(|\nabla(u_\varepsilon - u)|)\, dx \to 0.$$

Thanks to the Dominated Convergence Theorem $\Phi(|\nabla(u_\varepsilon - u)|) \leq h$ almost everywhere in Ω for some function $h \in L^1(\Omega)$. The last estimate says that

$$|\nabla(u_\varepsilon - u)| \leq \Phi^{-1}(h) \quad \text{a.e. in } \Omega.$$

As a consequence

$$|\nabla u_\varepsilon| \leq |\nabla(u_\varepsilon - u)| + |\nabla u| \leq \Phi^{-1}(h) + |\nabla u|.$$

In particular, using one more time Young's inequality and Δ_2 condition for Φ, we have that

$$\begin{aligned}\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon||\nabla v| &\leq \Phi(|\nabla v|) + \widetilde{\Phi}(\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|) \\ &\leq \Phi(|\nabla v|) + \Phi(2|\nabla u_\varepsilon|) \leq \Phi(|\nabla v|) + 2^m\Phi(|\nabla u_\varepsilon|).\end{aligned}$$

Now, using the last estimate and due to the convexity of Φ, we obtain

$$\begin{aligned}\phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon||\nabla v| &\leq \Phi(|\nabla v|) + 2^m\Phi(|\nabla u| + \Phi^{-1}(h)) \\ &\leq \Phi(|\nabla v|) + 2^{2m}[\Phi(|\nabla u|) + h].\end{aligned}$$

As a consequence the Lebesgue convergence theorem implies that

$$\lim_{\varepsilon\to 0} \int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon \nabla v\, dx = \int_\Omega \phi(|\nabla u|)\nabla u \nabla v\, dx, \quad v \in W_0^{1,\Phi}(\Omega).$$

Putting all estimates together and taking the limit as $\varepsilon \to 0$ in the equation

$$\varepsilon\langle -\Delta_m u_\varepsilon, v\rangle + \int_\Omega \phi((|\nabla u_\varepsilon|)\nabla u_\varepsilon \nabla v\, dx = \int_\Omega f v\, dx, \quad v \in W_0^{1,\Phi}(\Omega),$$

we conclude that

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla v\, dx = \int_\Omega f v\, dx, \quad v \in W_0^{1,\Phi}(\Omega)$$

In conclusion, $u \in W_0^{1,\Phi}(\Omega)$ is a weak solution for the problem (1.1). □

Proposition 3.5. *Suppose* (ϕ_1)–(ϕ_4) *where* $\ell = 1$. *Then the problem* (1.1) *admits exactly one solution* $u \in W_0^{1,\Phi}(\Omega)$.

PROOF. The proof follows using the same ideas discussed in the proof of Proposition 2.7. The main point here is to ensure that the Φ-Laplacian operator is strictly monotonic. As a consequence problem (1.1) admits at most one solution $u \in W_0^{1,\Phi}(\Omega)$. Besides that, using Proposition 3.4, there exists at least one solution $u \in W_0^{1,\Phi}(\Omega)$ for the problem (1.1). These facts imply that problem (1.1) admits exactly one weak solution for the non-reflexive case. We omit the details. $\square$

In what follows we shall consider the elliptic problem (1.2) assuming that g is superlinear. One more time we study the auxiliary elliptic problem

$$\begin{cases} -\Delta_{\Phi_\varepsilon} u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

where $\varepsilon > 0$ and $\Phi_\varepsilon(t) = \varepsilon t^m/m + \Phi(t)$, $t \geq 0$. Here is important to recover the definition for weak solution $u \in W_0^{1,\Phi_\varepsilon}(\Omega)$ to the problem (3.14) which is given by

$$\int_\Omega \phi_\varepsilon(|\nabla u|)\nabla u \nabla w\, dx = \int_\Omega g(x,u) w\, dx, \quad w \in W_0^{1,\Phi_\varepsilon}(\Omega).$$

Recall that weak solutions for (3.14) are precisely the critical point for the functional $J\colon W_0^{1,\Phi_\varepsilon}(\Omega) \to \mathbb{R}$ given by

$$J(u) = \int_\Omega \Phi_\varepsilon(|\nabla u|)\, dx - \int_\Omega G(x,u)\, dx$$

where $G(x,t) = \int_0^t g(x,s)\, ds$, $t \in \mathbb{R}, x \in \Omega$. As a consequence finding weak solutions to the problem (1.2) is equivalent to find critical points for J. Using the approximating problem (3.14) we observe that J satifies the Cerami condition for each $\varepsilon > 0$, see Carvalho et al. [5]. In addition, using hypotheses (ϕ_1)–(ϕ_3) and (g_1)–(g_4), the functional J possesses the mountain pass geometry, see Carvalho et al. [5]. In this way, we shall consider the following existence result

PROPOSITION 3.6. *Suppose* (ϕ_1)–(ϕ_3) *where* $\ell = 1$. *Assume also that* (g_1)–(g_4) *holds. Then the problem* (3.14) *admits at least one weak solution in* $u_\varepsilon \in W_0^{1,\Phi_\varepsilon}(\Omega)$ *for each* $\varepsilon > 0$. *Furthermore, using regularity results, we also mention that* u_ε *is in* $C^{1,\alpha_\varepsilon}(\overline{\Omega})$, *for some* $\alpha_\varepsilon > 0$.

PROOF. First of all, we recall that $W_0^{1,\Phi_\varepsilon}(\Omega)$ is a reflexive Banach space due the fact that $\ell_\varepsilon > 1$ for each $\varepsilon > 0$. Here the Lemma 3.1 was used. As a consequence, using the Mountain Pass Theorem, we know that the problem (1.2) admits at least one solution $u_\varepsilon \in W_0^{1,\Phi_\varepsilon}(\Omega) \cap C^{1,\alpha_\varepsilon}(\overline{\Omega})$ for each $\varepsilon > 0$, see Carvalho et al. [5]. We omit the details. $\square$

PROPOSITION 3.7. *Suppose* (ϕ_1)–(ϕ_4) *where* $\ell = 1$. *Assume also that* (g_1)–(g_4) *holds. Then the problem* (1.2) *admits at least one weak solution* $u \in W_0^{1,\Phi}(\Omega)$.

The proof is similar to that discussed in the proof of Proposition 3.4. We omit it here.

Let X be a Banach space endowed with the norm $\|\cdot\|$. Consider a functional $J\colon X \to \mathbb{R}$ of C^1 class. Recall that a sequence $(u_n) \in X$ is said to be a Cerami sequence at the level $c \in \mathbb{R}$, in short $(Ce)_c$ sequence, whenever $J(u_n) \to c$ and $(1 + \|u_n\|)\|J'(u_n)\| \to 0$ as $n \to \infty$. The functional J satisfies the Cerami condition at the level $c \in \mathbb{R}$, in short $(\mathrm{Ce})_c$ condition, whenever any Cerami condition at the level c possesses a convergent subsequence. When J satisfies the Cerami condition at any level $c \in \mathbb{R}$ we say purely that J satisfies the Cerami condition, in short, we write (Ce) condition.

At this moment we shall truncate the function g in the following ways

$$g^+(x,t) = \begin{cases} g(x,t) & \text{for } t \geq 0,\ x \in \Omega, \\ 0 & \text{for } t < 0,\ x \in \Omega, \end{cases}$$

and

$$g^-(x,t) = \begin{cases} g(x,t) & \text{for } t \leq 0,\ x \in \Omega, \\ 0 & \text{for } t > 0,\ x \in \Omega. \end{cases}$$

At the same time we define the functionals $J^\pm\colon W_0^{1,\Phi_\varepsilon}(\Omega) \to \mathbb{R}$ given by

$$J^\pm(u) = \int_\Omega \Phi_\varepsilon(|\nabla u|)\,dx - \int_\Omega G^\pm(x,u)\,dx$$

where $G^\pm(x,t) = \int_0^t g^\pm(x,s)\,ds$, $t \in \mathbb{R}$, $x \in \Omega$. It is not hard to verify that $J^\pm$ admits the mountain pass geometry.

Proposition 3.8. *Suppose* (ϕ_1)–(ϕ_4) *and* $\ell = 1$ *holds true. Assume also that* (g_1)–(g_4) *holds. Then the problem* (1.2) *admits at least two nontrivial weak solutions* $u_1, u_2 \in W_0^{1,\Phi}(\Omega)$.

Proof. The proof follows from the Mountain Pass Theorems for the functionals $J^\pm$. Once again we observe that $J^\pm$ satisfies the Cerami condition for each $\varepsilon > 0$, see Carvalho et al. [5]. Here was used the fact that $\ell_\varepsilon > 1$. In this way we obtain two sequences $u_\varepsilon^+, u_\varepsilon^- \in W_0^{1,\Phi_\varepsilon}(\Omega)$ of critical points for J^+ and J^-, respectively.

At this stage we claim that there exists $r_0 > 0$ in such way that $J^\pm(u_\varepsilon^\pm) \geq r_0$ where r_0 does not depend on $\varepsilon > 0$. In fact, using (ψ_1) and (g_4), given $0 < \eta < \lambda_1$ there exist $C, \delta > 0$ such that

$$G^\pm(x,t) < (\lambda_1 - \eta)\Phi(t) + C\Psi(t), \quad t \in \mathbb{R}.$$

Hence, taking into account the Poincaré inequality and using the estimate $\Phi_\varepsilon(t) \geq \Phi(t)$, $t \in \mathbb{R}$, $W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Psi(\Omega)$, it follows that

$$J^\pm(u) \geq \int_\Omega \Phi_\varepsilon(|\nabla u|)\,dx - (\lambda_1 - \eta)\int_\Omega \Phi(u)\,dx - C\int_\Omega \Psi(u)\,dx$$

$$\begin{aligned}
&\geq \frac{\eta}{\lambda_1}\int_\Omega \Phi(|\nabla u|)\,dx - C\int_\Omega \Psi(u)\,dx\\
&\geq \frac{\eta}{\lambda_1}\min\{\|u\|,\|u\|^m\} - C\max\{\|u\|^{\ell_\Psi},\|u\|^{m_\Psi}\}\\
&= \|u\|^m\bigg(\frac{\eta}{\lambda_1} - C\|u\|^{\ell_\Psi - m}\bigg)
\end{aligned}$$

holds true for each $\|u\| \leq 1$.

Now using the same ideas discussed in the proof of Proposition 3.4 we point out that $u_\varepsilon^+ \overset{*}{\rightharpoonup} u_1$ and $u_\varepsilon^- \overset{*}{\rightharpoonup} u_2$ in the weak star topology. Furthermore, the functional $J^\pm$ is weak star lower semicontinuous. Now, applying Proposition 3.3, we deduce that $u_n^+ \to u_1$ and $u_n^- \to u_2$ in $W_0^{1,\Phi}(\Omega)$. Hence, taking the negative part of u_1 as testing function, we obtain that $u_1 \geq 0$ in Ω. Similarly, we also obtain $u_2 \leq 0$ in Ω. As a consequence u_1, u_2 are nontrivial critical points to the functional J which give us weak nontrivial solutions to the elliptic problem (1.2). $\square$

4. Regularity results for some quasilinear elliptic problems

In this section we prove a regularity result for the problem (1.1). More precisely, we shall prove that each weak solution for the quasilinear elliptic problem (1.1) remains bounded whenever $f \in L^q(\Omega)$ for some $q > N/m$. This result is well known for the Laplacian and p-Laplacian operator. To the best of our knowledge this regularity result is new in the Orlicz and Orlicz–Sobolev framework. This can be done by using the Moser iteration. More precisely, using the Moser's method we shall prove the following regularity result:

THEOREM 4.1. *Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a N-function satisfying the Δ_2-condition. Assume also that there exist positive constants C_1, C_2 such that*

$$C_1 \leq \frac{\Phi(t)}{t^m} \leq C_2 \quad \text{for all } t > 0. \tag{4.1}$$

Suppose that $f \in L^q(\Omega)$ where $q > N/m$ and u is a weak solution for the problem (1.1). Then we obtain that $u \in L^\infty(\Omega)$.

PROOF. For $R > 0$, define $\Omega_R = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > R\}$. For $0 < R_2 < R_1$, let $\varphi = \eta^m(\overline{u}_s^{\,m\alpha}\,\overline{u} - k^{\,m\alpha+1})$ where α is a parameter to be chosen conveniently later. Define also $\overline{u} = \max\{u, k\}$ for $k > 0$, $\overline{u}_s = \min\{\overline{u}, s\}$ and $\eta \in C_0^1(\Omega)$ which satisfies $\eta = 1$ in Ω_{R_1}, $\eta = 0$ in $\Omega \setminus \Omega_{R_2}$, $\eta \geq 0$ and $|\nabla\eta| \leq C/(R_1 - R_2)$ for some constant $C > 0$. Note that $\varphi \in W_0^{1,\Phi}(\Omega)$ and

$$\nabla\varphi = \eta^m\big[m\alpha\overline{u}_s^{\,m\alpha-1}\overline{u}\,\nabla\overline{u}_s + \overline{u}_s^{\,m\alpha}\,\nabla\overline{u}\big] + m\eta^{m-1}\big(\overline{u}_s^{\,m\alpha}\overline{u} - k^{m\alpha+1}\big)\nabla\eta.$$

At this stage, we substitute the function φ in the equation (1.1) proving that

$$m\alpha\int \eta^m\overline{u}_s^{\,m\alpha-1}\overline{u}\,\phi(|\nabla u|)\nabla u\nabla\overline{u}_s + \int \eta^m\overline{u}_s^{\,m\alpha}\phi(|\nabla u|)\nabla u\nabla\overline{u} \tag{4.2}$$

$$+ m\int \eta^{m-1}\big(\overline{u}_s^{m\alpha}\overline{u} - k^{m\alpha+1}\big)\phi(|\nabla u|)\nabla u\nabla\eta = \int f\eta^m\big(\overline{u}_s^{m\alpha}\,\overline{u} - k^{\alpha m+1}\big).$$

Notice that for $u \le k$ we obtain that $\nabla\overline{u}, \nabla\overline{u}_s = 0$ and $(\overline{u}_s^{m\alpha}\,\overline{u} - k^{m\alpha+1}) = 0$. It follows from (4.2) that

$$\begin{aligned}(4.3)\quad m\alpha\int \eta^m\,\overline{u}_s^{m\alpha-1}\overline{u}\phi(|\nabla\overline{u}_s|)|\nabla\overline{u}_s|^2 + \int \eta^m\overline{u}_s^{m\alpha}\phi(|\nabla\overline{u}|)|\nabla\overline{u}|^2&\\ + m\int \eta^{m-1}\big(\overline{u}_s^{m\alpha}\overline{u} - k^{m\alpha+1}\big)\phi(|\nabla\overline{u}|)\nabla\overline{u}\nabla\eta = \int f\eta^m\big(\overline{u}_s^{m\alpha}\overline{u} - k^{\alpha m+1}\big).&\end{aligned}$$

Note also that $\big(\overline{u}_s^{m\alpha}\overline{u} - k^{m\alpha+1}\big) \le \overline{u}_s^{m\alpha}\overline{u}$. The last assertion implies that

$$m\eta^{m-1}\big(\overline{u}_s^{m\alpha}\overline{u} - k^{m\alpha+1}\big)\phi(|\nabla\overline{u}|)\nabla\overline{u}\nabla\eta \le m\eta^{m-1}\,\overline{u}_s^{m\alpha}\overline{u}\,\phi(|\nabla\overline{u}|)|\nabla\overline{u}||\nabla\eta|.$$

It follows from Young's inequality that

$$\begin{aligned}(4.4)\quad m\eta^{m-1}\big(\overline{u}_s^{m\alpha}\overline{u} - k^{m\alpha+1}\big)\phi(|\nabla\overline{u}|)\nabla\overline{u}\,\nabla\eta&\\ \le m(\widetilde{\Phi}(\varepsilon\eta^{m-1}\phi(|\nabla\overline{u}||\nabla\overline{u}|)) + \Phi(\overline{u}|\nabla\eta|/\varepsilon))\overline{u}_s^{m\alpha}.&\end{aligned}$$

Furthermore, taking into account the Δ_2-condition for the function Φ, we see that

$$\Phi(\overline{u}|\nabla\eta|/\varepsilon) \le \max\{(|\nabla\eta|/\varepsilon)^\ell, (|\nabla\eta|/\varepsilon)^m\}\Phi(\overline{u}) = g_1(x,\varepsilon)\Phi(\overline{u})$$

where $g_1(x,\varepsilon) = \max\big\{(|\nabla\eta|/\varepsilon)^\ell, (|\nabla\eta|/\varepsilon)^m\big\}$. Again, applying the Δ_2-condition for the function Φ, we see also that

$$\begin{aligned}\widetilde{\Phi}(\varepsilon\eta^{m-1}\phi(|\nabla\overline{u}||\nabla\overline{u}|)) &\le \max\big\{(\varepsilon\eta^{m-1})^{\ell'}, (\varepsilon\eta^{m-1})^{m'}\big\}\widetilde{\Phi}(|\nabla\overline{u}||\nabla\overline{u}|)\\ &= g_2(x,\varepsilon)\widetilde{\Phi}(\phi(|\nabla\overline{u}|)|\nabla\overline{u}|),\end{aligned}$$

where $g_2(x,\varepsilon) = \max\big\{(\varepsilon\eta^{m-1})^{\ell'}, (\varepsilon\eta^{m-1})^{m'}\big\}$. Now, we observe that $\widetilde{\Phi}(\phi(t)t) \le C\Phi(t) \le C\phi(t)t^2$, $t \ge 0$ holds true for some constant $C > 0$.

It follows from the last inequalities and (4.4) that

$$\begin{aligned}(4.5)\qquad m\int \eta^{m-1}\big(\overline{u}_s^{m\alpha}\,\overline{u} - k^{m\alpha+1}\big)\phi(|\nabla\overline{u}|)\nabla\overline{u}\,\nabla\eta&\\ \le \int m\widetilde{\Phi}\big(\varepsilon\eta^{m-1}\phi(|\nabla\overline{u}|)|\nabla\overline{u}|\big) + \int m\Phi(\overline{u}|\nabla\eta|/\varepsilon)\,\overline{u}_s^{m\alpha}&\\ \le \int Cm\big[g_2(x,\varepsilon)\phi(|\nabla\overline{u}|)|\nabla\overline{u}|^2 + g_1(x,\varepsilon)\phi(\overline{u})\,\overline{u}^2\big]\overline{u}_s^{m\alpha}.&\end{aligned}$$

Now, we use (4.3) together with the last estimate showing that

$$\begin{aligned}(4.6)\quad m\alpha\int \eta^m\,\overline{u}_s^{m\alpha-1}\overline{u}\,\phi(|\nabla\overline{u}_s|)|\nabla\overline{u}_s|^2&\\ + \int \big(\eta^m - Cmg_2(x,\varepsilon)\big)\overline{u}_s^{m\alpha}\phi(|\nabla\overline{u}|)|\nabla\overline{u}|^2&\\ \le \int Cmg_1(x,\varepsilon)\phi(\overline{u})\,\overline{u}^2\,\overline{u}_s^{m\alpha} + \int |f|\eta^m\overline{u}_s^{m\alpha}\,\overline{u}.&\end{aligned}$$

Note that, for each $\varepsilon > 0$ small enough, $g_2(x,\varepsilon) = \varepsilon^{m'}\eta^{(m-1)m'} = \varepsilon^{m'}\eta^m$ implies that $\eta^m - Cmg_2(x,\varepsilon) = \eta^m - Cm\varepsilon^{m'}\eta^m = \eta^m\big(1 - Cm\varepsilon^{m'}\big)$. Thus we can choose a small ε in such a way that $\eta^m - Cmg_2(x,\varepsilon) > 0$. Hence, by fixing such ε and using the fact that $\overline{u}_s \le \overline{u}$, $|\nabla\overline{u}_s| \le |\nabla\overline{u}|$, it follows from (4.6) that

$$\begin{aligned}(4.7)\quad \big(m\alpha + 1 - Cm\varepsilon^{m'}\big) &\int \eta^m\, \overline{u}_s^{\,m\alpha}\, \phi(|\nabla\overline{u}_s|)|\nabla\overline{u}_s|^2 \\ &\le \int Cmg_1(x,\varepsilon)\,\phi(\overline{u})\,\overline{u}^2\,\overline{u}_s^{\,m\alpha} + \int |f|\eta^m\,\overline{u}_s^{\,m\alpha}\,\overline{u}.\end{aligned}$$

Now, we shall study the term with the f function in (4.7). We begin by noting that $\overline{u} \ge k$ implies that $\overline{u}^{\,m-1} \ge k^{m-1}$ whence $\overline{u} \le \overline{u}^{\,m}/k^{m-1}$. Therefore, by using the Hölder inequality, Young inequality and interpolation inequalities we obtain that

$$\begin{aligned}(4.8)\qquad \int |f|\eta^m\,\overline{u}_s^{\,m\alpha}\,\overline{u} &\le \|f\|_q\|\eta^m\overline{u}_s^{\,m\alpha}\overline{u}\|_{q'} \le \frac{\|f\|_q}{k^{m-1}}\big\|\eta^m\,\overline{u}_s^{\,m\alpha}\,\overline{u}\big\|_{q'} \\ &\le \frac{\|f\|_q}{k^{m-1}}\|\eta\overline{u}_s^{\,\alpha}\overline{u}\|_m^{m(1-N/mq)}\|\eta\overline{u}_s^{\,\alpha}\overline{u}\|_{m^\star}^{mN/mq} \\ &\le \frac{\|f\|_q}{k^{m-1}\delta^{mq/(mq-N)}}\int \eta^r\overline{u}_s^{\,m\alpha}\,\overline{u}^{\,m} + \frac{\|f\|_q\delta^{mq/N}}{k^{m-1}}\|\eta\,\overline{u}_s^{\,\alpha}\,\overline{u}\|_{m^\star}^m,\end{aligned}$$

where $\delta > 0$ will be chosen later and $m^\star$ denotes the Sobolev critical exponent $mN/(N-m)$.

It follows from (4.7) and (4.8) that

$$\begin{aligned}(4.9)\quad \big(m\alpha + 1 - Cm\varepsilon^{m'}\big) &\int \eta^m\,\overline{u}_s^{\,m\alpha}\,\phi(|\nabla\overline{u}_s|)|\nabla\overline{u}_s|^2 \\ \le \int Cmg_1(x,\varepsilon)\,\phi(\overline{u})\,\overline{u}^2\,\overline{u}_s^{\,m\alpha} &+ \frac{\|f\|_q}{k^{m-1}\delta^{mq/(mq-N)}}\int \eta^m\overline{u}_s^{\,m\alpha}\overline{u}^{\,m} \\ &+ \frac{\|f\|_q\delta^{mq/N}}{k^{m-1}}\|\eta\overline{u}_s^{\,\alpha}\overline{u}\|_{m^\star}^m.\end{aligned}$$

In order to apply our argument, noticing that

$$\big|\nabla\big(\eta\overline{u}_s^{\,\alpha+1}\big)\big|^m \le C\big(\overline{u}_s^{\,m(\alpha+1)}|\nabla\eta|^m + \eta^m\,\overline{u}_s^{\,m\alpha}|\nabla\overline{u}_s|^m\big)$$

and by using (4.1) and the fact that Φ satisfies the Δ_2-condition, we infer that

$$(4.10)\qquad \int \big|\nabla\big(\eta\overline{u}_s^{\,\alpha+1}\big)\big|^m \le C\int \big(\overline{u}_s^{\,m(\alpha+1)}|\nabla\eta|^m + \eta^m\,\overline{u}_s^{\,m\alpha}\,\phi(|\nabla\overline{u}_s|)|\nabla\overline{u}_s|^2\big).$$

Now, using the Sobolev embedding, (4.9) and (4.10), we ensure the following inequality

$$\begin{aligned}(4.11)\quad \big\|\eta\overline{u}_s^{\,\alpha+1}\big\|_{m^\star}^m \le C&\int \overline{u}_s^{\,m(\alpha+1)}|\nabla\eta|^m \\ &+ \frac{C}{m\alpha + 1 - Cm\varepsilon^{m'}}\int mg_1(x,\varepsilon)\,\phi(\overline{u})\,\overline{u}^2\,\overline{u}_s^{\,m\alpha}\end{aligned}$$

$$+ \frac{C}{m\alpha + 1 - Cm\varepsilon^{m'}} \bigg(\frac{\|f\|_q}{k^{m-1}\delta^{mq/(mq-N)}} \int \eta^m \,\overline{u}_s^{m\alpha}\, \overline{u}^{\,m} + \frac{\|f\|_q \delta^{mq/N}}{k^{m-1}} \big\|\eta\, \overline{u}_s^{\,\alpha}\, \overline{u}\big\|_{m^\star}^m \bigg).$$

Furthermore, by letting $s \to \infty$ and using the monotone convergence theorem, we conclude from (4.11) that

$$\|\eta\overline{u}^{\alpha+1}\|_{m^\star}^m - \frac{C}{m\alpha + 1 - Cm\varepsilon^{m'}} \frac{\|f\|_q\delta^{mq/N}}{k^{m-1}} \|\eta\,\overline{u}^{\,\alpha+1}\|_{m^\star}^m \le C \int \overline{u}^{\,m(\alpha+1)} |\nabla\eta|^m + \frac{C}{m\alpha + 1 - Cm\varepsilon^{m'}} \bigg(\int m g_1(x,\varepsilon) \overline{u}^{\,m(\alpha+1)} + \frac{\|f\|_q}{k^{m-1}\delta^{mq/(mq-N)}} \int \eta^m \,\overline{u}^{\,m(\alpha+1)} \bigg). \tag{4.12}$$

In this way, we choose

$$\delta = \bigg(\frac{(m\alpha + 1 - Cm\varepsilon^{m'})k^{m-1}}{2C\|f\|_q} \bigg)^{N/mq}.$$

Hence, using this $\delta > 0$ in (4.12) we get

$$\|\eta\,\overline{u}^{\alpha+1}\|_{m^\star}^m \le C \int \overline{u}^{\,m(\alpha+1)} |\nabla\eta|^m + \frac{C}{m\alpha + 1 - Cm\varepsilon^{m'}} \int m g_1(x,\varepsilon)\, \overline{u}^{\,m(\alpha+1)} + C\bigg(\frac{\|f\|_q}{(m\alpha + 1 - Cm\varepsilon^{m'})k^{m-1}} \bigg)^{N/(mq-N)} \int \eta^m \,\overline{u}^{\,m(\alpha+1)}. \tag{4.13}$$

Now, we define

$$G(R_1, R_2) = \bigg(\frac{1}{R_1 - R_2}\bigg)^{\ell} + \bigg(\frac{1}{R_1 - R_2}\bigg)^{m}.$$

Hence we obtain from (4.13) and the definition of η and g_1 that

$$\|\eta\overline{u}^{\,\alpha+1}\|_{m^\star}^m \le \frac{C}{(R_1 - R_2)^m} \int_{\Omega_{R_2}} \overline{u}^{m(\alpha+1)} + \frac{CG(R_1,R_2)}{m\alpha + 1 - Cm\varepsilon^{m'}} \int_{\Omega_{R_2}} \overline{u}^{\,m(\alpha+1)} + C\bigg(\frac{\|f\|_q}{(m\alpha + 1 - Cm\varepsilon^{m'})k^{m-1}} \bigg)^{N/(mq-N)} \int_{\Omega_{R_2}} \overline{u}^{\,m(\alpha+1)}. \tag{4.14}$$

Since $\eta = 1$ in Ω_{R_1} and (4.14) we easily see that

$$\|\overline{u}^{\,\alpha+1}\|_{L^{m^\star}(\Omega_{R_1})} \le C\bigg[\frac{1}{R_1 - R_2} + \bigg(\frac{G(R_1,R_2)}{m\alpha + 1 - Cm\varepsilon^{m'}} \bigg)^{1/m} \bigg] \|\overline{u}^{\,\alpha+1}\|_{L^m(\Omega_{R_2})} \tag{4.15}$$

$$+ C\left(\frac{\|f\|_q}{(m\alpha + 1 - Cm\varepsilon^{m'})k^{m-1}}\right)^{N/m(mq-N)} \|\overline{u}^{\alpha+1}\|_{L^m(\Omega_{R_2})}.$$

Let $\chi = N/(N-m)$ be a fixed function. It follows from (4.15) that

$$\text{(4.16)}\quad \|\overline{u}\|_{L^{m\chi(\alpha+1)}(\Omega_{R_1})} \leq C^{1/(\alpha+1)}\Bigg[\frac{1}{(R_1-R_2)} + \left(\frac{G(R_1,R_2)}{m\alpha+1-Cm\varepsilon^{m'}}\right)^{1/m} + \left(\frac{\|f\|_q}{(m\alpha+1-Cm\varepsilon^{m'})k^{m-1}}\right)^{N/m(mq-N)}\Bigg]^{1/(\alpha+1)} \|\overline{u}\|_{L^{m(\alpha+1)}(\Omega_{R_2})}.$$

In this way, for $n \in \{0,1,\ldots\}$, define $\alpha + 1 = \chi^n$ and $R_n = R_2 + (R_1 - R_2)/2^n$. It is not hard to verify that (4.16) implies

$$\text{(4.17)}\quad \|\overline{u}\|_{L^{m\chi^{n+1}}(\Omega_{R_{n+1}})} \leq C^{1/\chi^n}\Bigg[\frac{1}{R_n - R_{n+1}} + \left(\frac{G(R_n,R_{n+1})}{m(\chi^n-1)+1-Cm\varepsilon^{m'}}\right)^{1/m} + \left(\frac{\|f\|_q}{(m(\chi^n-1)+1-Cm\varepsilon^{m'})k^{m-1}}\right)^{N/m(mq-N)}\Bigg]^{1/\chi^n} \|\overline{u}\|_{L^{m\chi^n}(\Omega_{R_n})}.$$

Here we emphasize that $R_n - R_{n+1} = (R_1 - R_2)/2^{n+1}$ goes to zero as n goes to infinity. Therefore there exists a positive constant $C > 0$ (independent of n) in such way that

$$G(R_n, R_{n+1})^{1/m} \leq \frac{C}{(R_n - R_{n+1})}.$$

As a consequence, applying (4.17) and the last inequality, we obtain

$$\text{(4.18)}\quad \|\overline{u}\|_{L^{m\chi^{n+1}}(\Omega_{R_{n+1}})} \leq C^{1/\chi^n}\Bigg[\frac{2^{n+1}}{R_1-R_2} + \left(\frac{2^{n+1}}{R_1-R_2}\frac{1}{m(\chi^n-1)+1-Cm\varepsilon^{m'}}\right)^{1/m} + \left(\frac{\|f\|_q}{(m(\chi^n-1)+1-Cm\varepsilon^{m'})k^{m-1}}\right)^{N/m(mq-N)}\Bigg]^{1/\chi^n} \|\overline{u}\|_{L^{m\chi^n}(\Omega_{R_n})}.$$

From now on, by choosing $n = 0, 1, \ldots$, we obtain that $\|\overline{u}\|_{L^{m\chi^n}(\Omega_{R_n})}$ is finite for every n. Moreover, using the fact that $\chi > 1$, there exists n_0, which depend upon χ, such that

$$\left(\frac{1}{m(\chi^n-1)+1-Cm\varepsilon^{m'}}\right)^{1/m\chi^n} \leq 1$$

and

$$\left(\frac{1}{m(\chi^n-1)+1-Cm\varepsilon^{m'}}\right)^{N/m(mq-N)\chi^n} \leq 1, \quad \text{for all } n \geq n_0.$$

As a consequence, we mention that

$$\text{(4.19)}\quad \|\overline{u}\|_{L^{m\chi^{n+1}}(\Omega_{R_{n+1}})} \leq C^{1/\chi^n}\sigma\|\overline{u}\|_{L^{m\chi^n}(\Omega_{R_n})},$$

holds for any $n \geq n_0$ where

$$\sigma = \left[\frac{2^{n+1}}{R_1 - R_2} + \left(\frac{2^{n+1}}{R_1 - R_2}\right)^{1/m} + \left(\frac{\|f\|_q}{k^{m-1}}\right)^{\beta}\right]^{1/\chi^n}$$

and $\beta = N/m(mq - N)$. Now, we can also assume that for $n \geq n_0$ proving that

$$\left(\frac{2^{n+1}}{R_1 - R_2}\right)^{1/m} \leq \frac{2^{n+1}}{R_1 - R_2} \quad \text{and} \quad 1 \leq \frac{2^{n+1}}{R_1 - R_2}.$$

Therefore, using (4.19), we observe that

$$\|\overline{u}\|_{L^{m\chi^{n+1}}(\Omega_{R_{n+1}})} \leq \left[\frac{C}{R_1 - R_2}\left(1 + \frac{\|f\|_q^\beta}{k^{\beta(m-1)}}\right)\right]^{1/\chi^n} 2^{(n+1)/\chi^n} \|\overline{u}\|_{L^{m\chi^n}(\Omega_{R_n})}$$

holds for all $n \geq n_0$. Hence the last assertion implies that (after an argument of iteration)

$$\|\overline{u}\|_{L^{m\chi^{n+1}}(\Omega_{R_{n+1}})} \leq \prod_{i=n_0}^{n} \left[\frac{C}{R_1 - R_2}\left(1+\frac{\|f\|_q^\beta}{k^{\beta(m-1)}}\right)\right]^{1/\chi^i} 2^{(i+1)/\chi^i} \|\overline{u}\|_{L^{m\chi^{n_0}}(\Omega_{R_{n_0}})}.$$

holds true for $n \geq n_0$. Now, doing $n \to \infty$ we infer that

$$\|\overline{u}\|_{L^\infty(\Omega_{R_1})} \leq \left[\frac{C}{R_1 - R_2}\left(1 + \frac{\|f\|_q^\beta}{k^{\beta(m-1)}}\right)\right]^{1/(1-\chi)} \|\overline{u}\|_{L^{m\chi^{n_0}}(\Omega_{R_{n_0}})}. \tag{4.20}$$

As a consequence, the estimate (4.20) implies that $u^+ \in L^\infty(\Omega_{R_1})$. Using the same ideas discussed just above we also obtain that $u^- \in L^\infty(\Omega_{R_1})$. In order to extend the result the estimate for the boundary, for small $s > 0$, we define $U_s = \{x \in \mathbb{R}^N \setminus \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) < s\}$. Here we define also $\Omega_s = \overline{\Omega} \cup U_s$. Let $\widetilde{u}, \widetilde{f} : \Omega_s \to \mathbb{R}$ be extensions by zero of u, f, respectively. Note that $\widetilde{u}$ is a solution of the problem (1.1) with $\widetilde{f}$ in the place of f and Ω_s in the place of Ω. Now we apply the same argument as before in order to conclude that $u \in L^\infty(\Omega)$. □

5. The proof our main theorems

In this section we prove our main theorems using the Orlicz–Sobolev framework discussed in previous sections.

Proof of Theorem 1.3. First of all, using Proposition 3.5 we know that problem (1.1) admits exactly one solution $u \in W_0^{1,\Phi}(\Omega)$. According to Theorem 4.1 we mention that u is in $L^\infty(\Omega)$ whenever Φ is equivalent to the function $t \to |t|^r$, $r \in (1, \infty)$ and $\ell > 1$. □

Proof of Theorem 1.4. Initially, using hypothesis (ϕ_3) and Proposition 3.7, we obtain at least one weak solution $u \in W_0^{1,\Phi}(\Omega)$. Furthermore, using (ϕ_3) instead of $(\phi_3)'$, we obtain two weak solutions $u_1, u_2 \in W_0^{1,\Phi}(\Omega)$ in such way $J(u_1), J(u_2) > 0$, see Proposition 3.8. Hence, taking the negative part of u_1 as testing function, we deduce that $u_1 \geq 0$. At the same time, using the positive part of u_2, we observe that $u_2 \leq 0$. Now, we know also that $u_1, u_2 \in L^\infty(\Omega)$

whenever $\ell > 1$, see [28]. So that regularity results on quasilinear elliptic problems imply that $u_1, u_2 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, see Lieberman [21], [22] whenever $\ell > 1$. Furthermore, the solutions $u_1 > 0$ and $u_2 < 0$ in Ω thanks to the Maximum Principle for quasilinear elliptic equations, see [26]. $\square$

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Manuscript received July 24, 2018

accepted April 30, 2019

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