Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 2A, 2019, 807–815
DOI: 10.12775/TMNA.2019.071

REMARKS ON SOME LIMITS APPEARING IN THE THEORY OF ALMOST PERIODIC FUNCTIONS

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Abstract. In this note we are going to present new short proofs concerning either the existence or the non-existence of some limits appearing in the theory of almost periodic functions. Our proofs are completely different from those presented in the papers [1] and [3].

1. Introduction

In the rich theory of almost periodic functions (see e.g. [6]) problems concerning the evaluation of the limit

$$\lim_{x\to+\infty}\frac{f(x)}{2+\cos x+\cos(x\sqrt{2})}, \tag{1.1}$$

where $f\colon \mathbb{R}\to\mathbb{R}$ is an exponential function or a polynomial (cf. [1] or [3]), quite frequently appear. This is connected to the fact that the function

$$x\mapsto \frac{1}{2+\cos x+\cos(x\sqrt{2})}\quad \text{for } x\in\mathbb{R}$$

constitutes a classical example of a function which is either almost periodic in the sense of Levitan (briefly: LAP) or almost periodic with respect to the Lebesgue measure (briefly: μ.a.p.) (see e.g. [4]). In particular, in [1] the authors used the theory of continued fractions to prove that the limit (1.1) is equal to zero if

2010 *Mathematics Subject Classification.* Primary: 42A75; Secondary: 41A10.
Key words and phrases. Almost periodic functions; asymptotic behavior of functions; algebraic numbers; transcendental numbers; Pell's equation, quinary system.

$f(x) = e^{\lambda x}$ for $x \in \mathbb{R}$ ($\lambda < 0$). Let us notice that the proof of that fact given in [1] is quite long.

On the other hand, it was proven in [3] that the limit (1.1) remains equal to zero if one replaces the exponential function by a polynomial $x \mapsto x^{-2-\varepsilon}$, $x \in \mathbb{R}^+$ and $\varepsilon > 0$. The proof of that result given in [3] is based on the well-known Liouville's Theorem (see e.g. [2]). Further, as it was proven in [3], the limit (1.1) does not exist if $f(x) = x^{-2}$ for $x \in \mathbb{R}^+$.

In this note we are going to present two short proofs of the fact, that the limit (1.1) is equal to zero if $f(x) = e^{-x}$ for $x \in \mathbb{R}$. Next, we consider a more general case, namely we investigate the limit

$$\lim_{x\to+\infty} \frac{x^{-2n+2-\varepsilon}}{2+\cos x+\cos(x\alpha)}, \tag{1.2}$$

where α is an algebraic number of degree n. We also examine the upper limit of the quotient appearing in (1.1), in the case when $f(x) = x^{-2}$ for $x \in \mathbb{R}^+$. In that investigation we use the classical Pell equation. At the end of this note we present another short proof of Theorem 7 from [1]. In our proof we extensively use the quinary system.

2. Main results

By $\{x\}$ and $[x]$ we will denote the fractional part and the entier of x, respectively. Now, let us define relations $\gg$, $\ll$, $\approx$ on the set of real functions admitting positive values.

Definition 2.1. Let $f, g\colon \mathbb{R} \to \mathbb{R}^+$. We say that $f \gg g$ if and only if the following condition holds:

$$\exists\, C > 0 \quad \exists\, x_1 \in \mathbb{R} \quad \forall x > x_1 \quad f(x) \geq Cg(x).$$

Obviously, we admit that

$$f \ll g \quad \text{if and only if} \quad g \gg f.$$

Finally,

$$f \approx g \quad \text{if and only if} \quad f \ll g \quad \text{and} \quad f \gg g.$$

The above definition obviously implies that $\approx$ is an equivalence relation. Moreover, if the limit, lower limit or upper limit of f at infinity is equal to 0 or $+\infty$, then the limit, lower limit or upper limit of g is equal to 0 or $+\infty$, respectively.

In [1, Theorem 6.13] the following result was proven.

Theorem 2.2. *It holds*:

$$\lim_{x\to+\infty} \frac{e^{-x}}{2+\cos x+\cos(x\sqrt{2})} = 0.$$

Now we will prove the above result using two various techniques.

PROOF 1. Fix $\varepsilon \in (0;2)$. Let it be equal to $2+\cos x+\cos(x\sqrt{2})$ for some $x>0$. Then

$$\cos x = \varepsilon - \cos(x\sqrt{2}) - 2 \le \varepsilon - 1.$$

Let $a = \pi - \arccos(\varepsilon - 1)$. Then $x \in [2\pi n + \pi - a, 2\pi n + \pi + a]$ for some $n \in \mathbb{N}_0$. Analogously $x\sqrt{2} \in [2\pi m + \pi - a, 2\pi m + \pi + a]$ for some $m \in \mathbb{N}_0$. The function $x \mapsto 2+\cos x+\cos(x\sqrt{2})$ takes positive values, but they can be arbitrarily small. Thus, for any constant $M>0$, for small enough $\varepsilon>0$ and for each x fulfilling the equality $2+\cos x+\cos(x\sqrt{2}) = \varepsilon$ we have $x \ge M$. Since $a<\pi$, the intervals $[2\pi n + \pi - a, 2\pi n + \pi + a]$ and $[2\pi m + \pi - a, 2\pi m + \pi + a]$ do not contain any negative numbers, so

$$\sqrt{2} = \frac{x\sqrt{2}}{x} \in \left[\frac{2\pi m + \pi - a}{2\pi n + \pi + a}, \frac{2\pi m + \pi + a}{2\pi n + \pi - a}\right].$$

Let us substitute $k = 2m+1$, $l = 2n+1$, $b = a/\pi$. We get:

$$\sqrt{2} \in \left[\frac{k-b}{l+b}, \frac{k+b}{l-b}\right]$$

and thus $k-b \le \sqrt{2}(l+b)$, so $k - l\sqrt{2} \le b(1+\sqrt{2})$. Analogously, we get $k - l\sqrt{2} \ge -b(1+\sqrt{2})$, so $|k-l\sqrt{2}| \le b(1+\sqrt{2})$. Therefore

$$|k^2 - 2l^2| \le b(1+\sqrt{2})(k+l\sqrt{2}).$$

The number k^2-2l^2 is an integer and $k^2-2l^2 \ne 0$, so $|k^2-2l^2| \ge 1$ and therefore

$$k + l\sqrt{2} \ge \frac{1}{b(1+\sqrt{2})}. \tag{2.1}$$

As ε approaches 0, numbers a and b (treated as functions of ε) approach 0, so for sufficiently small $\varepsilon>0$ we have $b < 1/(2(1+\sqrt{2}))$, which implies

$$k - l\sqrt{2} \ge -b\big(1+\sqrt{2}\big) \ge -\frac{1}{2} \ge -\frac{k}{2},$$

so $5k/2 \ge k + l\sqrt{2}$, and, by (2.1), we obtain $k \ge 2/\big(5(1+\sqrt{2})b\big)$. For $\varepsilon>0$ small enough the number $x\sqrt{2}-(2\pi m+\pi)$ can be arbitrarily small, so $x\sqrt{2}/(2\pi m+\pi)$ is arbitrarily close to 1. Therefore, we get

$$\begin{aligned} x &= \frac{x\sqrt{2}}{2\pi m+\pi}\cdot\frac{\pi}{\sqrt{2}}\cdot(2m+1) \ge \frac{x\sqrt{2}}{2\pi m+\pi}\cdot\frac{\pi}{\sqrt{2}}\cdot\frac{2}{5(1+\sqrt{2})b} \\ &= \frac{x\sqrt{2}}{2\pi m+\pi}\cdot\frac{\pi}{\sqrt{2}}\cdot\frac{2\pi}{5(1+\sqrt{2})a} = \frac{x\sqrt{2}}{2\pi m+\pi}\cdot\frac{2\pi^2}{5(2+\sqrt{2})a} \ge \frac{1}{a}. \end{aligned}$$

Thus, for a small enough $\varepsilon>0$, we have

$$\frac{e^{-x}}{2+\cos x+\cos(x\sqrt{2})} \le \frac{e^{-1/a}}{\varepsilon} = \frac{e^{-1/a}}{\cos(\pi-a)+1} = \frac{e^{-1/a}}{1-\cos a} =: h(\varepsilon). \tag{2.2}$$

Now, let us consider ε as a variable and, consequently, a as a variable depending on ε. Using de l'Hopital's rule we can easily calculate that

$$\lim_{a\to 0^+} \frac{e^{-1/a}}{1-\cos a} = 0.$$

Fix $\widetilde{\epsilon} > 0$. Since $\lim\limits_{\varepsilon\to 0} h(\varepsilon) = 0$, there exists an $\varepsilon_0 > 0$ such that $h(\varepsilon) < \widetilde{\epsilon}$ for all $0 < \varepsilon < \varepsilon_0$. Let ε_1 be such a positive number, that for $0 < \varepsilon < \varepsilon_1$ the inequality (2.2) is satisfied. Let $\varepsilon_2 = \min\{\varepsilon_0; \varepsilon_1\}$. Fix $x > 0$. If $2+\cos x+\cos x\sqrt{2} < \varepsilon_2$, then, by (2.2), we have

$$\frac{e^{-x}}{2+\cos x+\cos x\sqrt{2}} < \widetilde{\epsilon}.$$

If $2+\cos x+\cos x\sqrt{2} \geq \varepsilon_2$, then

$$\frac{e^{-x}}{2+\cos x+\cos x\sqrt{2}} \leq \frac{e^{-x}}{\varepsilon_2},$$

and since for large enough x the number e^{-x} is arbitrarily small, x for large enough, we have $e^{-x}/\varepsilon_2 \leq \widetilde{\epsilon}$. Finally, for any $\widetilde{\epsilon} > 0$ for large enough x we have

$$\frac{e^{-x}}{2+\cos x+\cos x\sqrt{2}} < \widetilde{\epsilon}$$

and thus

$$\lim_{x\to+\infty} \frac{e^{-x}}{2+\cos x+\cos x\sqrt{2}} = 0. \qquad \square$$

PROOF 2. First, let us notice that

$$\begin{aligned}\sqrt{2+\cos(2\pi x)+\cos(2\pi x\sqrt{2})} &= \sqrt{1+\cos(2\pi x)+1+\cos(2\pi x\sqrt{2})}\\ &= \sqrt{2\cos^2(\pi x)+2\cos^2(\pi x\sqrt{2})} \approx |\cos(\pi x)|+|\cos(\pi x\sqrt{2})|\\ &= |\cos(\pi\{x\})|+|\cos(\pi\{x\sqrt{2}\})| \approx \left|\{x\}-\frac{1}{2}\right| + \left|\{x\sqrt{2}\}-\frac{1}{2}\right| =: g(x).\end{aligned}$$

The graphs of the two components of the function g consist of line segments. The first component can have a slope of 1 and -1, respectively; the second one: $\sqrt{2}$ and $-\sqrt{2}$, respectively. Thus the graph of that function consists of line segments of slopes $1+\sqrt{2}$; $1-\sqrt{2}$; $-1+\sqrt{2}$; $-1-\sqrt{2}$. Consider the local minima of that function. They must occur in the points of non-differentiability, which are $n/2$ and $n\sqrt{2}/4$ for $n \in \mathbb{N}$, because for other points we can find a neighbourhood, which is a line segment of a non-zero slope. We can easily check that there are no minima of the first type, and that minima and maxima of the second type occur alternately. Let $x_n = (2n+1)\sqrt{2}/4$. Obviously, the function g is continuous and it achieves minima in points x_n, so $g(x)$ is greater or equal to the value in one of the two consecutive elements of (x_n), between which lies x. Let us assign to every $x > x_1$ the point $f(x)$, so that $f(x_n) = x_n$ for all $n \in \mathbb{N}$ and if $x_n < x < x_{n+1}$, the value of $f(x)$ is x_n if $g(x_n) < g(x_{n+1})$ and x_{n+1} otherwise. Since, as we established, consecutive minima differ by $\sqrt{2}/2$, so $|f(x)-x| \le \sqrt{2}/2$ and thus

$$\frac{e^{-\pi x}}{|\{x\}-1/2|+|\{x\sqrt{2}\}-1/2|} \le \frac{e^{-\pi f(x)} \cdot e^{\pi(f(x)-x)}}{|\{f(x)\}-1/2|} \approx \frac{e^{-\pi f(x)}}{|\{f(x)\}-1/2|}.$$

Define $2l+1 = 2\sqrt{2}f(x)$ and $k = [f(x)]$, where k and l are treated as functions of x. We can easily see that k and l are integers, and that $k \approx l \approx f$. Thus

$$\left|\{f(x)\}-\frac{1}{2}\right| = \left|\frac{l\sqrt{2}}{2}+\frac{\sqrt{2}}{4}-\frac{1}{2}-k\right| = \frac{|(2l+1)\sqrt{2}-2(2k+1)|}{4}$$
$$= \frac{1}{4}\left|\frac{2(2l+1)^2-4(2k+1)^2}{(2l+1)\sqrt{2}+2(2k+1)}\right| \ge \frac{1}{4}\left|\frac{1}{(2l+1)\sqrt{2}+2(2k+1)}\right| \gg \frac{1}{l} \approx \frac{1}{f(x)}.$$

Hence, finally we get

$$\frac{e^{-\pi x}}{\sqrt{2+\cos(2\pi x)+\cos(2\pi x\sqrt{2})}} \approx \frac{e^{-\pi x}}{|\{x\}-1/2|+|\{x\sqrt{2}\}-1/2|}$$
$$\le \frac{e^{-\pi f(x)} \cdot e^{\pi(f(x)-x)}}{|\{f(x)\}-1/2|} \approx \frac{e^{-\pi f(x)}}{|\{f(x)\}-1/2|} \ll f(x)e^{-\pi f(x)}.$$

As x approaches plus infinity, $f(x)$ approaches plus infinity, so

$$0 \le \lim_{x\to+\infty} \frac{e^{-x}}{2+\cos x+\cos(x\sqrt{2})}$$
$$= \left(\lim_{x\to+\infty} \frac{e^{-\pi x}}{\sqrt{2+\cos(2\pi x)+\cos(2\pi x\sqrt{2})}}\right)^2 \le \left(\lim_{x\to+\infty} xe^{-\pi x}\right)^2 = 0,$$

which completes the proof. □

The following corollary, connected with the limit (1.2), actually is an extension of Corollary 1 from [3] (see also [5]).

COROLLARY 2.3. *If α is an irrational algebraic number of degree n, then*

$$\lim_{x\to+\infty} \frac{x^{-2n+2-\varepsilon}}{2+\cos x+\cos(x\alpha)} = 0 \quad \textit{for any } \varepsilon > 0.$$

PROOF. We can easily assume that $\alpha > 1$ by showing that

$$\lim_{x\to+\infty} \frac{x^{-2n+2-\varepsilon}}{2+\cos x+\cos(x\alpha)} = 0 \quad \Leftrightarrow \quad \lim_{u\to+\infty} \frac{u^{-2n+2-\varepsilon}}{2+\cos u+\cos(u/\alpha)} = 0.$$

Analogously to the above proof we can consider the function

$$x \mapsto \left|\{x\}-\frac{1}{2}\right| + \left|\{x\alpha\}-\frac{1}{2}\right|$$

and conclude that its local minima occur at points $x_m = (2m+1)/(2\alpha)$. Defining the function f as above and letting $k = [f(x)]$ and $2l+1 = 2\alpha f(x)$ we have $k \approx l \approx f$ and we can apply Liouville's Theorem to get

$$\left|\{f(x)\} - \frac{1}{2}\right| = \frac{|(2l+1)-(2k+1)\alpha|}{2\alpha} \approx (2k+1)\left|\frac{2l+1}{2k+1} - \alpha\right| \gg (2k+1)^{-n+1} \approx f(x)^{-n+1}.$$

Then carrying out similar calculations to the ones in the above proof yields the desired result. □

In the following theorem we examine the asymptotic behavior of the function

$$x \mapsto \frac{x^{-2}}{2 + \cos x + \cos(x\sqrt{2})}.$$

Actually, we are going to prove a generalization of Theorem 4 from [3].

Theorem 2.4. *It holds*:

$$0 < \limsup_{x\to+\infty} \frac{x^{-2}}{2 + \cos x + \cos(x\sqrt{2})} < +\infty.$$

Proof. Notice that by Definition 2.1 and the second proof of Theorem 2.2 we just need to prove that

$$0 < \limsup_{x\to+\infty} \frac{x^{-1}}{|\{x\} - 1/2| + |\{x\sqrt{2}\} - 1/2|} < +\infty.$$

We know that

$$\frac{x^{-1}}{|\{x\} - 1/2| + |\{x\sqrt{2}\} - 1/2|} \ll \frac{f(x)^{-1}}{|\{f(x)\} - 1/2|} \ll f(x)f(x)^{-1} = 1,$$

where f is defined as in the second proof of Theorem 2.2. To complete the proof we will indicate a sequence (l_n) for which this function does not approach 0. Consider the Pell equation $k^2 - 2l^2 = -1$. It is well known that it is fulfilled by infinitely many pairs of integers, and considering this equation modulo 4 we easily see that k and l must be odd. Let (l_n) and (k_n) be increasing sequences of positive integers l_n and k_n appearing in those pairs. Then

$$\left|\left\{\frac{l_n}{2}\right\} - \frac{1}{2}\right| + \left|\left\{\frac{l_n\sqrt{2}}{2}\right\} - \frac{1}{2}\right| = \left|\left\{\frac{l_n\sqrt{2}}{2} - \frac{k_n}{2} + \frac{1}{2}\right\} - \frac{1}{2}\right| = \left|\left\{\frac{1}{2k_n + 2l_n\sqrt{2}} + \frac{1}{2}\right\} - \frac{1}{2}\right| = \frac{1}{2k_n + 2l_n\sqrt{2}}.$$

Thus

$$\frac{2l_n^{-1}}{|\{l_n/2\} - 1/2| + |\{l_n\sqrt{2}/2\} - 1/2|} = 4\,\frac{k_n}{l_n} + 4\sqrt{2} > 1$$

and

$$1 < \limsup_{x\to+\infty} \frac{x^{-1}}{|\{x\} - 1/2| + |\{x\sqrt{2}\} - 1/2|} < +\infty.$$ □

At the end of this note we are going to present another short proof of Theorem 7 from [1] connected with the limit (1.1).

THEOREM 2.5. *For every function* $f\colon \mathbb{R} \to \mathbb{R}^+$, *every* $a \in \mathbb{R}$ *and every* $\varepsilon > 0$, *there exists* $\alpha \in \mathbb{R}$ *such that*

$$|a-\alpha| < \varepsilon \quad \text{and} \quad \limsup_{x\to+\infty} \frac{f(x)}{2+\cos x+\cos(x\alpha)} = +\infty.$$

PROOF. Fix $a \in \mathbb{R}$ and $\varepsilon > 0$. We will construct a number α and a sequence (πl_n) such that

$$\lim_{n\to+\infty} \frac{f(\pi l_n)}{2+\cos(\pi l_n)+\cos(\pi l_n \alpha)} = +\infty.$$

Let $\beta = \sum_{i=1}^{\infty} 5^{-a_i}$, where (a_n) is defined recursively as follows: $a_1 = 1$ and a_{i+1} is the smallest integer greater than a_i for which

$$\frac{f(\pi 5^{a_i})}{2\pi \cdot 5^{a_i - a_{i+1}}} > i.$$

Such a_{i+1} obviously exists and, since (a_n) is increasing, the series defining β is convergent. Consider the interval $(a-\beta-\varepsilon, a-\beta+\varepsilon)$. It must contain a number of the form $m/5^N$ for some integers m and N (expressions $[5^n(a-\beta+\varepsilon)]/5^n$, while smaller than $a-\beta+\varepsilon$, can be arbitrarily close to it, so for large enough $n \in \mathbb{N}$ they have to be larger than $a-\beta-\varepsilon$). Now, let $\alpha = \beta + m/5^N$. Let us define the sequences (l_n) and (k_n) as $l_n = 5^{a_n}$ and $k_n = [5^{a_n} \cdot \alpha]$. Notice that

$$\sum_{n=i+1}^{\infty} 5^{a_i - a_n} \le \frac{1}{5-1} < 1,$$

so for $a_i > N$ we have

$$\begin{aligned} k_i &= [m \cdot 5^{a_i - N} + 5^{a_i} \cdot \beta] = m \cdot 5^{a_i - N} + [5^{a_i} \cdot \beta] \\ &= m \cdot 5^{a_i - N} + 5^{a_i - a_1} + 5^{a_i - a_2} + \cdots + 5^{a_i - a_i} \end{aligned}$$

and since $5^{a_i}\beta - [5^{\alpha_i}\beta] > 0$, we have

$$0 < l_i \alpha - k_i = \sum_{j=i+1}^{\infty} 5^{a_i - a_j} < 2 \cdot 5^{a_i - a_{i+1}} < 1.$$

Furthermore, if $a_i > N$, the numbers k_i and k_{i+1} have different parity, so one of the sequences (k_{2n}) and (k_{2n+1}) contains only odd numbers for large enough $n \in \mathbb{N}$. Thus

$$\begin{aligned} \frac{f(\pi l_i)}{2+\cos(\pi l_i)+\cos(\pi l_i \alpha)} &= \frac{f(\pi l_i)}{1-\cos(\pi l_i \alpha - \pi k_i)} \\ &> \frac{f(\pi l_i)}{\pi l_i \alpha - \pi k_i} > \frac{f(\pi 5^{a_i})}{2\pi \cdot 5^{a_i - a_{i+1}}} > i \end{aligned}$$

for the subsequences (k_{2n}) and (l_{2n}) or (k_{2n+1}) and (l_{2n+1}) for large enough n. Thus α and one of the sequences (πl_{2n}), (πl_{2n+1}) fulfills desired conditions. □

Remark 2.6. In the above proof one could substitute 5 in the definition of β by any odd integer greater than 1 (and consequently use it in the rest of the proof). If we tried substituting it by an even number, e.g. 10, our l_n would be even and $\cos(\pi l_n)$ would be equal to 1 instead of -1.

Remark 2.7. Let us notice that the substitution $f(x) = e^{-x}$ in Theorem 2.5 and the application of Corollary 2.3 leads to a commonly known fact concerning the existence of transcendental numbers. Indeed, since $e^x \gg x^a$ for any $a \in \mathbb{R}$, any number α satisfying

$$\limsup_{x\to+\infty} \frac{e^{-x}}{2+\cos x+\cos(x\alpha)} = +\infty$$

also satisfies

$$\limsup_{x\to+\infty} \frac{x^{-n+1-\varepsilon}}{2+\cos x+\cos(x\alpha)} = +\infty$$

and so, from Corollary 2.3, we obtain that α is not an algebraic number of degree n for any $n \in \mathbb{N}$, which means that α is a transcendental number. Then Theorem 2.5 implies that the set of all transcendental numbers is dense in real numbers.

Moreover, let us notice that if the sequence $(a_{n+1}/a_n)_{n\in\mathbb{N}}$ is unbounded for some increasing sequence $(a_n)_{n\in\mathbb{N}}$ of positive integers, then using Liouville's Theorem we can easily prove that $\sum_{i=1}^{\infty} 5^{-a_i}$ is a transcendental number. Let $(e_n)_{n\in\mathbb{N}}$ be any sequence such that $e_n \in \{0,1\}$ for all $n \in \mathbb{N}$. It is well-known that the set of all such sequences is uncountable. If we then define $a_n = (n+1)! + e_n$ then it can be easily checked that $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence of integers, $a_{n+1}/a_n > n$ and that the numbers $\sum_{i=1}^{\infty} 5^{-a_i}$ are distinct for different sequences $(e_n)_{n\in\mathbb{N}}$. This implies that the set of all transcendental numbers is uncountable.

Acknowledgements. I would like to thank my teacher prof. D. Bugajewski for the introduction to the topic discussed in the paper and for comments, which essentially improved the paper.

I would also like to thank the referees for all their comments and suggestions, which allowed me to improve the quality of this note.

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Manuscript received November 11, 2018

accepted April 3, 2019

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