

**THE EFFECT OF TOPOLOGY
ON THE NUMBER OF POSITIVE SOLUTIONS
OF ELLIPTIC EQUATION
INVOLVING HARDY–LITTLEWOOD–SOBOLEV
CRITICAL EXPONENT**

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ABSTRACT. In this article we are concerned with the following Choquard equation

$$\begin{aligned} -\Delta u &= \lambda |u|^{q-2} u + \left(\int_{\Omega} \frac{|u(y)|^{2^*_{\mu}}}{|x-y|^{\mu}} dy \right) |u|^{2^*_{\mu}-2} u, & u > 0, & \text{ in } \Omega, \\ u &= 0 & & \text{ on } \partial\Omega, \end{aligned}$$

where Ω is an open bounded set with continuous boundary in \mathbb{R}^N ($N \geq 3$), $2^*_{\mu} = (2N - \mu)/(N - 2)$ and $q \in [2, 2^*)$ where $2^* = 2N/(N - 2)$. Using Lusternik–Schnirelman theory, we associate the number of positive solutions of the above problem with the topology of Ω . Indeed, we prove that if $\lambda < \lambda_1$, then problem has $\text{cat}_{\Omega}(\Omega)$ positive solutions whenever $q \in [2, 2^*)$ and $N > 3$ or $4 < q < 6$ and $N = 3$.

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1. Introduction

The purpose of this article is to study the existence and multiplicity of solution of the following Choquard equation

$$(P_\lambda) \quad \begin{cases} -\Delta u = \lambda|u|^{q-2}u + \left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy \right) |u|^{2_\mu^*-2}u, & u > 0, \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded set with continuous boundary in \mathbb{R}^N ($N \geq 3$), $2_\mu^* = (2N - \mu)/(N - 2)$ and $q \in [2, 2^*)$ where $2^* = 2N/(N - 2)$.

It is not unfamiliar that nonlinear analysis fascinates many researchers. In particular, the study of elliptic equations is attractive both for theoretical pde's and real-world applications. There is an ample amount of literature regarding the existence and multiplicity of solutions of the following equation:

$$(1.1) \quad -\Delta u = \lambda|u|^{q-2}u + |u|^{2^*-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

In the pioneering work of Brezis and Nirenberg [7], authors studied the problem (1.1) with $q = 2$ for the existence of a nontrivial solution. Then many researchers studied the elliptic equations involving Sobolev critical exponent in bounded and unbounded domains. In [4], Bahri and Coron studied the problem (1.1) in case of $\lambda = 0$ and proved the existence of a positive solution when Ω is not a contractible domain using homology theory. Subsequently, Rey [27] studied critical elliptic problem (1.1) for $q = 2$ and proved that there exist at least $\text{cat}_\Omega(\Omega)$ solutions in $H_0^1(\Omega)$ whenever λ is sufficiently small. We cite e.g. [5], [6], [11], [2], [31] for issues on the existence and multiplicity of solutions of elliptic problems using variational methods, with no attempt to provide the complete list. In the framework of the fractional Laplacian, the effect of topology on the number of solutions of problems was discussed in [13], [14] and references therein.

Currently, nonlocal equations appealed a substantial number of researchers, especially the Choquard equations. The work on Choquard equations was started with the quantum theory of a polaron model given by S. Pekar [26] in 1954. After that in 1976, in the modeling of a one component plasma, P. Choquard [22] used the following equation with $\mu = 1$, $p = 2$ and $N = 3$:

$$(1.2) \quad -\Delta u + u = \left(\frac{1}{|x|^\mu} * |u|^p \right) |u|^{p-2}u \text{ in } \mathbb{R}^N.$$

For $\mu = 1$, $p = 2$ and $N = 3$, Lieb [22] proved existence, uniqueness of the ground state solution of (1.2) by using symmetric decreasing rearrangement inequalities. With the help of variational methods, Moroz and Schaftingen [24] established the existence of least energy solutions of (1.2) and prove properties about the symmetry, regularity, and asymptotic behavior at infinity of the least energy

solutions. For interested readers, we refer [3], [9], [10], [25] and references therein for the work on Choquard equations.

The Hardy–Littlewood–Sobolev inequality (2.1) plays a significant role in the variational formulation of Choquard equations. Observe that the integral

$$\int_{\Omega} \int_{\Omega} \frac{|u(x)|^q |u(y)|^q}{|x - y|^{\mu}} dy dx$$

is well defined if $(2N - \mu)/N \leq q \leq (2N - \mu)/(N - 2) = 2_{\mu}^*$. Choquard equations involving Hardy-Littlewood-Sobolev critical exponent (that is, $q = 2_{\mu}^*$) provoke the interest of the mathematical community due to the lack of compactness in the embedding

$$H_0^1(\Omega) \ni u \mapsto \frac{|u|^{2_{\mu}^*} |u|^{2_{\mu}^*}}{|x - y|^{\mu}} \in L^1(\Omega \times \Omega).$$

In [15], authors used variational methods to prove the existence and multiplicity of positive solutions for the critical Choquard problem involving convex and convex-concave type nonlinearities.

In this spirit, recently in [21] Goel, Rădulescu and Sreenadh, studied the Coron problem for Choquard equation and proved the existence of a positive high energy solution of the following problem

$$-\Delta u = \left(\int_{\Omega} \frac{|u(y)|^{2_{\mu}^*}}{|x - y|^{\mu}} dy \right) |u|^{2_{\mu}^* - 2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in \mathbb{R}^N ($N \geq 3$), $2_{\mu}^* = (2N - \mu)/(N - 2)$, $0 < \mu < N$ and satisfies the following conditions: There exists constants $0 < R_1 < R_2 < \infty$ such that

$$\{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \subset \Omega, \quad \{x \in \mathbb{R}^N : |x| < R_1\} \not\subset \bar{\Omega}.$$

In [18] Ghimenti and Pagliardini studied the following slightly subcritical Choquard problem

$$(1.3) \quad -\Delta u - \lambda u = \left(\int_{\Omega} \frac{|u(y)|^{p_{\varepsilon}}}{|x - y|^{\mu}} dy \right) |u|^{p_{\varepsilon} - 2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\varepsilon > 0$, Ω is a regular bounded domain of \mathbb{R}^N , $\lambda \geq 0$ and $p_{\varepsilon} = 2_{\mu}^* - \varepsilon$. Here authors proved that there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon}]$, Problem (1.3) has at least $cat_{\Omega}(\Omega)$ low energy solutions. Moreover, if Ω is not contractible, there exists another solution with higher energy.

Motivated by all these, in this paper, we study the existence of multiple solutions of the problem (P_{λ}) . Since the geometry of the domain plays an essential role, here we proved that the topology of the domain yields a lower bound on the number of positive solutions. More precisely, we show that the problem (P_{λ}) has at least $cat_{\Omega}(\Omega)$ solutions. Here $cat_{\Omega}(\Omega)$ is the Lusternik–Schnirelman category defined as follows

DEFINITION 1.1. Let X be a topological space and Y be a closed set in X . Then

$$\text{Cat}_X(Y) = \min \left\{ k \in \mathbb{N} : \text{there exist closed subsets } Y_1, \dots, Y_k \subset X \right. \\ \left. \text{such that } Y_j \text{ is contractible to a point in } X \text{ for all } j \text{ and } \bigcup_{j=1}^k Y_j = X \right\}.$$

In order to achieve our aim, we used the fact that Lusternik–Schnirelman category is invariant under Nehari manifold. Then using the blowup analysis involving the minimizers and the mountain pass lemma, we show the infimum of the functional associated with (P_λ) over the the Nehari manifold is achieved. Moreover, we define the barycenter mapping associated to Choquard nonlinear term and apply the machinery of barycenter mapping to prove our desired conclusion. With this introduction we will state our main result:

THEOREM 1.2. *Let Ω is an open bounded set with continuous boundary in \mathbb{R}^N ($N \geq 3$) and $q \in [2, 2^*)$ then there exists $0 < \Lambda^* < \lambda_1$ such that, for all $\lambda \in (0, \Lambda^*)$, there exists at least $\text{cat}_\Omega(\Omega)$ positive solutions of (P_λ) under the following conditions*

- (a) $q \in [2, 2^*)$ and $N > 3$, or
- (b) $4 < q < 6$ and $N = 3$.

Turning to the layout of the article: In Section 2, we give the variational framework and preliminary results. In Section 3, we give the Palais–Smale analysis and existence of a solution of (P_λ) . In Section 4, we prove some technical lemmas and proof Theorem 1.2. Finally, in the appendix we study the behavior of optimizing sequence of the best constant $S_{H,L}$ defined in (2.2).

2. Variational framework and the preliminary results

To study the problem (P_λ) by variational approach we will start by stating the celebrated Hardy–Littlewood–Sobolev inequality.

PROPOSITION 2.1 (Hardy–Littlewood–Sobolev Inequality, [23]). *Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, r, \mu, N)$ independent of f, h , such that*

$$(2.1) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x - y|^\mu} dy dx \leq C(t, r, \mu, N) \|f\|_{L^t} \|h\|_{L^r}.$$

If $t = r = 2N/(2N - \mu)$, then

$$C(t, r, \mu, N) = C(N, \mu) = \pi^{\mu/2} \frac{\Gamma(N/2 - \mu/2)}{\Gamma(N - \mu/2)} \left\{ \frac{\Gamma(N/2)}{\Gamma(\mu/2)} \right\}^{-1 + \mu/N}.$$

Equality holds in (2.1) if and only if $f \equiv (\text{constant})h$ and

$$h(x) = A(\gamma^2 + |x - a|^2)^{(2N-\mu)/2},$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

The Sobolev space $D^{1,2}(\mathbb{R}^N)$ is defined as

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N, \mathbb{R}^N)\},$$

endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{1/2}.$$

The best constant for the embedding $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ (where $2^* = 2N/(N-2)$) is defined as

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : \int_{\mathbb{R}^N} |u|^{2^*} dx = 1 \right\}.$$

Consequently, we define

$$(2.2) \quad S_{H,L} = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} dx dy = 1 \right\}.$$

LEMMA 2.2 ([16]). *The constant $S_{H,L}$ defined in (2.2) is achieved if and only if*

$$u = C \left(\frac{b}{b^2 + |x - a|^2} \right)^{(N-2)/2}$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0, \infty)$ are parameters. Moreover,

$$S = S_{H,L}(C(N, \mu))^{(N-2)/(2N-\mu)}.$$

LEMMA 2.3 ([16]). *For $N \geq 3$ and $0 < \mu < N$. Then*

$$\|\cdot\|_{NL} := \left(\int_{\Omega} \int_{\Omega} \frac{|\cdot|^{2^*} |\cdot|^{2^*}}{|x-y|^\mu} dy dx \right)^{1/(2 \cdot 2^*)}$$

defines a norm on $L^{2^*}(\Omega)$, where Ω is an open bounded set with continuous boundary in \mathbb{R}^N .

The energy functional $J_\lambda: H_0^1(\Omega) \rightarrow \mathbb{R}$ associated with (P_λ) , is defined by

$$J_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \frac{1}{2 \cdot 2^*} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} dy dx.$$

Employing the Hardy–Littlewood–Sobolev inequality (2.1), we have

$$\left(\int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} dy dx \right)^{1/2^*} \leq C(N, \mu)^{(2N-\mu)/(N-2)} \|u\|_{L^{2^*}}^2.$$

It implies the functional $J_\lambda \in C^1(H_0^1(\Omega), \mathbb{R})$. We know that there exists a one to one correspondence between the critical points of J_λ and solution of (P_λ) .

Notation. We denote λ_1 be the first eigenvalue of $-\Delta$ with zero Dirichlet boundary data, which is given by

$$\lambda_1 = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \left\{ \int_\Omega |\nabla u|^2 dx : \int_\Omega |u|^2 dx = 1 \right\}.$$

We also denote (Q) as the following condition:

- (Q) Assume $0 < \lambda < \lambda_1$. Moreover, $q \in [2, 2^*)$ and $N > 3$ or $4 < q < 6$ and $N = 3$.

LEMMA 2.4. Assume $N \geq 3$ and $\lambda \in (0, \lambda_1)$. Then J_λ satisfies the following conditions:

- (a) There exists $\alpha, \rho > 0$ such that $J_\lambda(u) \geq \alpha$ for $\|u\| = \rho$.
- (b) There exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $J_\lambda(e) < 0$.

PROOF. (a) Using Hölder’s inequality, Sobolev’s inequality and Hardy–Littlewood inequality, we have

$$J_\lambda(u) \geq \begin{cases} \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^2 - \frac{S_{H,L}^{-1}}{2 \cdot 2_\mu^*} \|u\|^{2 \cdot 2_\mu^*} & \text{if } q = 2, \\ \frac{1}{2} \|u\|^2 - \frac{\lambda S^{-q/2} |\Omega|^{(2^*-q)/2^*}}{q} \|u\|^q - \frac{S_{H,L}^{-1}}{2 \cdot 2_\mu^*} \|u\|^{2 \cdot 2_\mu^*} & \text{if } q \in (2, 2^*). \end{cases}$$

Using the given assumption on λ and the fact that $2 < 2 \cdot 2_\mu^*$, we can choose $\alpha, \rho > 0$ such that $J_\lambda(u) \geq \alpha$ whenever $\|u\| = \rho$.

- (b) Let $u \in H_0^1(\Omega)$ then

$$J_\lambda(tu) = \frac{t^2}{2} \|u\|^2 - \frac{t^q}{q} \int_\Omega |u|^q dx - \frac{t^{2 \cdot 2_\mu^*}}{2 \cdot 2_\mu^*} \int_\Omega \int_\Omega \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy \rightarrow -\infty$$

as $t \rightarrow \infty$. Hence we can choose $t_0 > 0$ such that $e := t_0 u$ such that (b) follows. \square

The Nehari manifold associated to J_λ defined as

$$N_\lambda^\Omega := \{u \in H_0^1(\Omega) \setminus \{0\} : \langle J'_\lambda(u), u \rangle = 0\}.$$

LEMMA 2.5. Let u be a critical point on N_λ^Ω . Then u is a critical point of J_λ on $H_0^1(\Omega)$.

PROOF. The proof follows from [12]. \square

LEMMA 2.6. Assume $\lambda \in (0, \lambda_1)$. Then $N_\lambda^\Omega \neq \emptyset$ and J_λ is bounded below on N_λ^Ω .

PROOF. Let $u \in H_0^1(\Omega) \setminus \{0\}$. Consider the function

$$\phi_u(t) = J_\lambda(tu) = \frac{t^2}{2} \|u\|^2 - \frac{\lambda t^q}{q} \int_\Omega |u|^q dx - \frac{t^{2 \cdot 2_\mu^*}}{2 \cdot 2_\mu^*} \|u\|_{NL}^{2 \cdot 2_\mu^*}.$$

Then $\phi_u(t) = 0$, $\phi_u(t) \rightarrow -\infty$ as $t \rightarrow \infty$. We now show that there exists unique $t_0 > 0$ such that $\phi'_u(t_0) = 0$. Since

$$\phi'_u(t) = t \|u\|^2 - \lambda t^{q-1} \int_\Omega |u|^q dx - t^{2 \cdot 2_\mu^* - 1} \|u\|_{NL}^{2 \cdot 2_\mu^*} = t m_u(t),$$

where $m_u(t) = \|u\|^2 - b_u(t)$ and

$$b_u(t) = \lambda t^{q-2} \int_\Omega |u|^q dx + t^{2 \cdot 2_\mu^* - 2} \|u\|_{NL}^{2 \cdot 2_\mu^*}.$$

Observe that b_u is a continuous function, $\lim_{t \rightarrow \infty} b_u(t) = \infty$ and $b'_u(t) > 0$ for all $t > 0$. Therefore, there exists unique $t_0 > 0$ such that $b_u(t_0) = \|u\|^2$. That is, $\phi'_u(t_0) = 0$. It implies $t_0 \phi'_u(t_0) = 0$ and $t_0 u \in N_\lambda^\Omega$. It implies $N_\lambda^\Omega \neq \emptyset$. Now, if $u \in N_\lambda^\Omega$, then $J_\lambda(u)$ reduced to

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega |u|^q dx + \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) \|u\|_{NL}^{2 \cdot 2_\mu^*} > 0.$$

Therefore, $\inf_{u \in N_\lambda^\Omega} J_\lambda(u) > 0$. That is, J_λ is bounded below on N_λ^Ω . □

Now we set

$$(2.3) \quad \theta_\lambda := \inf_{u \in N_\lambda^\Omega} J_\lambda(u) \quad \text{and} \quad \widehat{\theta}_\lambda := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \sup_{t \geq 0} J_\lambda(tu),$$

where $\widehat{\theta}_\lambda$ denote the Mountain Pass (MP, in short) level.

3. The Palais–Smale condition and estimates of the functional

In this section we will give the Palais–Smale analysis and prove the existence of a minimizer of the functional J_λ over the Nehari manifold.

LEMMA 3.1. *Let $N \geq 3$, $\lambda \in (0, \lambda_1)$ and $q \in [2, 2^*)$. Then the functional J_λ satisfies the $(PS)_c$ condition for all*

$$c < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

PROOF. Let $\{u_n\}$ be a sequence in $H_0^1(\Omega)$ such that

$$(3.1) \quad J_\lambda(u_n) \rightarrow c \quad \text{and} \quad \left\langle J'_\lambda(u_n), \frac{u_n}{\|u_n\|} \right\rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

CLAIM 1. u_n is a bounded sequence in $H_0^1(\Omega)$.

On the contrary assume that $\|u_n\| \rightarrow \infty$. Let $\widetilde{u}_n = u_n/\|u_n\|$ be a sequence in $H_0^1(\Omega)$ then $\|\widetilde{u}_n\| = 1$ for all n . Therefore we can assume there exists \widetilde{u} , up to subsequences, such that

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \text{ weakly in } H_0^1(\Omega), \quad \widetilde{u}_n \rightarrow \widetilde{u} \text{ strongly in } L^r(\Omega)$$

for all $r \in [1, 2^*)$. Using (3.1) we have

$$\begin{aligned} \frac{1}{2}\|\widetilde{u}_n\|^2 - \frac{\lambda}{q}\|u_n\|^{q-2} \int_{\Omega} |\widetilde{u}_n|^q dx - \frac{1}{2 \cdot 2_\mu^*} \|u_n\|^{2 \cdot 2_\mu^* - 2} \|\widetilde{u}_n\|_{NL}^{2 \cdot 2_\mu^*} &= o_n(1), \\ \|\widetilde{u}_n\|^2 - \lambda\|u_n\|^{q-2} \int_{\Omega} |\widetilde{u}_n|^q dx - \|u_n\|^{2 \cdot 2_\mu^* - 2} \|\widetilde{u}_n\|_{NL}^{2 \cdot 2_\mu^*} &= o_n(1). \end{aligned}$$

It implies that

$$\left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right)\|\widetilde{u}_n\|^2 = \left(\frac{1}{q} - \frac{1}{2 \cdot 2_\mu^*}\right)\lambda\|u_n\|^{q-2} \int_{\Omega} |\widetilde{u}_n|^q dx + o_n(1).$$

Now, if $q > 2$ and $\lambda > 0$ then by the assumption $\|u_n\| \rightarrow \infty$, we get $\|\widetilde{u}_n\| \rightarrow \infty$, which is not possible. If $q = 2$ and $\lambda \in (0, \lambda_1)$, then $0 < (1 - \lambda/\lambda_1)\|u_n\|^2 \leq o_n(1)$, which is again not possible, this concludes the proof of Claim.

Hence we can assume, there exists a $u_0 \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, $u_n \rightarrow u_0$ strongly in $L^r(\Omega)$ for all $r \in [1, 2^*)$ and $u_n \rightarrow u_0$ almost every on Ω . Using all this and proceeding with the same assertions as in [16, Lemma 2.4], we get $J'_\lambda(u_0) = 0$. Now the Brezis–Leib Lemma (see [8], [16]) leads to

$$J_\lambda(u_n) = J_\lambda(u_0) + \frac{1}{2}\|u_n - u_0\|^2 - \frac{1}{2 \cdot 2_\mu^*} \|u_n - u_0\|^{2 \cdot 2_\mu^*} + o_n(1)$$

and

$$\begin{aligned} (3.2) \quad o_n(1) &= \langle J'_\lambda(u_n) - J'_\lambda(u_0), u_n - u_0 \rangle \\ &= \|u_n\|^2 - \|u_0\|^2 - \|u_n\|_{NL}^{2 \cdot 2_\mu^*} + \|u_0\|_{NL}^{2 \cdot 2_\mu^*} \\ &= \|u_n - u_0\|^2 - \|u_n - u_0\|_{NL}^{2 \cdot 2_\mu^*}. \end{aligned}$$

It implies

$$J_\lambda(u_0) + \frac{N - \mu + 2}{2(2N - \mu)} \|u_n - u_0\|^2 = c + o_n(1)$$

and, if $\|u_n - u_0\|^2 \rightarrow M$ as $n \rightarrow \infty$, then by (3.2), $\|u_n - u_0\|_{NL}^{2 \cdot 2_\mu^*} \rightarrow M$ as $n \rightarrow \infty$. If $M = 0$ then we are done otherwise, if $M > 0$ then using the definition of $S_{H,L}$, we have $M^{1/2_\mu^*} S_{H,L} \leq M$ that is, $S_{H,L}^{(2N-\mu)/(N-\mu+2)} \leq M$. Since $\langle J'_\lambda(u_0), u_0 \rangle = 0$, it gives

$$J_\lambda(u_0) = \left(\frac{1}{2} - \frac{1}{q}\right)\|u_0\|^2 + \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right)\|u_0\|_{NL}^{2 \cdot 2_\mu^*} \geq 0.$$

Resuming the information collected so far, what we have gained is that,

$$o_n(1) + c = J_\lambda(u_0) + \frac{N - \mu + 2}{2(2N - \mu)} M \geq \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)},$$

which yields a contradiction to the range of c . Hence compactness of the sequence follows. \square

LEMMA 3.2. *Let $N \geq 3$ and $\lambda \in (0, \lambda_1)$ then J_λ constraint to N_λ^Ω satisfies the $(PS)_c$ condition for all*

$$c < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

PROOF. Let $u_n \in N_\lambda^\Omega$ be such that $J_\lambda(u_n) \rightarrow c$ and there exists a sequence $\{\alpha_n\}$ in \mathbb{R} with

$$(3.3) \quad \sup\{|\langle J'_\lambda(u_n) - \alpha_n T'_\lambda(u_n), \phi \rangle| : \phi \in H_0^1(\Omega), \|\phi\| = 1\} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where the functional T_λ is defined as

$$T_\lambda(u) = \|u\|^2 - \lambda \int_\Omega |u|^q dx - \|u\|_{NL}^{2 \cdot 2_\mu^*}.$$

First of all, we will show that u_n is a bounded sequence in $H_0^1(\Omega)$. From the fact that $J_\lambda(u_n) \rightarrow c$, it is easy to see that there exists a positive constant C_1 such that $|J_\lambda(u_n)| < C_1$. If $q \in (2, 2^*)$ then, using the fact that $u_n \in N_\lambda^\Omega$, we deduce that

$$\begin{aligned} C_1 &> J_\lambda(u_n) - \frac{1}{q} \langle J'_\lambda(u_n), u_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) \|u_n\|^2 + \left(\frac{1}{q} - \frac{1}{2 \cdot 2_\mu^*}\right) \|u_n\|_{NL}^{2 \cdot 2_\mu^*} \geq \left(\frac{1}{2} - \frac{1}{q}\right) \|u_n\|^2. \end{aligned}$$

If $q = 2$, for $\lambda \in (0, \lambda_1)$, we obtain, for any $n \in \mathbb{N}$,

$$\begin{aligned} C_1 &> J_\lambda(u_n) - \frac{1}{2 \cdot 2_\mu^*} \langle J'_\lambda(u_n), u_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) \|u_n\|^2 - \lambda \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) \int_\Omega |u_n|^2 dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) \left(1 - \frac{\lambda}{\lambda_1}\right) \|u_n\|^2. \end{aligned}$$

This proves that u_n is a bounded sequence in $H_0^1(\Omega)$. So $\{\langle T'_\lambda(u_n), u_n \rangle\}$ is a bounded sequence in \mathbb{R} and there exists $\kappa \in (-\infty, 0]$ such that, up to a subsequence, $\langle T'_\lambda(u_n), u_n \rangle \rightarrow \kappa$ as $n \rightarrow \infty$. Let if possible, $\kappa < 0$ then using the fact that $u_n \in N_\lambda^\Omega$ and (A.5), we have $\langle \alpha_n T'_\lambda(u_n), u_n \rangle \rightarrow 0$ as $n \rightarrow \infty$. This implies $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. That is,

$$\sup\{|\langle J'_\lambda(u_n), \phi \rangle| : \phi \in H_0^1(\Omega), \|\phi\| = 1\} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which, by employing Lemma 3.1, gives that u_n has a convergent subsequence.

At last suppose $\kappa = 0$. Since

$$\langle T'_\lambda(u_n), u_n \rangle = \lambda(2 - q) \int_\Omega |u_n|^q dx + (2 - 2 \cdot 2_\mu^*) \|u_n\|_{NL}^{2 \cdot 2_\mu^*} \rightarrow \kappa,$$

then

$$\int_\Omega |u_n|^q dx \rightarrow 0 \quad \text{and} \quad \|u_n\|_{NL}^{2 \cdot 2_\mu^*} \rightarrow 0.$$

Taking into account the fact $u_n \in N_\lambda^\Omega$ we have $\|u_n\| \rightarrow 0$. That is, $u_n \rightarrow 0$ strongly in $H_0^1(\Omega)$. \square

To proceed further we will use the minimizer of $S_{H,L}$. From Lemma 2.2 we know that

$$U_\varepsilon(x) = S^{(N-\mu)(2-N)/(4(N-\mu+2))} (C(N, \mu))^{(2-N)/(2(N-\mu+2))} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{(N-2)/2},$$

for $0 < \varepsilon < 1$, are the minimizers of $S_{H,L}$. Without loss of generality, let us assume that $0 \in \Omega$. This implies there exists a $\delta > 0$ such that $B_{4\delta}(0) \subset \Omega$. Now define $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$ in \mathbb{R}^N , $\eta \equiv 1$ in $B_\delta(0)$ and $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_{2\delta}(0)$ and $|\nabla \eta| < C$. Let $u_\varepsilon \in H_0^1(\Omega)$ be defined as $u_\varepsilon(x) = \eta(x)U_\varepsilon(x)$.

PROPOSITION 3.3. *Let $N \geq 3$, $0 < \mu < N$ and $q \in (2, 2^*)$ then the following holds:*

- (a) $\|u_\varepsilon\|^2 \leq S_{H,L}^{(2N-\mu)/(N-\mu+2)} + O(\varepsilon^{N-2})$.
- (b) $\|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*} \leq S_{H,L}^{(2N-\mu)/(N-\mu+2)} + O(\varepsilon^N)$ and $\|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*} \geq S_{H,L}^{(2N-\mu)/(N-\mu+2)} - O(\varepsilon^N)$.
- (c) $\int_\Omega |u_\varepsilon|^2 dx \geq C \begin{cases} \varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N > 4, \\ \varepsilon^2 |\log \varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ \varepsilon^{N-2} + O(\varepsilon^2) & \text{if } N < 4. \end{cases}$
- (d) $\int_\Omega |u_\varepsilon|^q dx \geq O(\varepsilon^{N-(N-2)q/2})$ whenever $q \in (2, 2^*)$ and $N > 3$ or $4 < q < 6$ and $N = 3$.

PROOF. For (a) and (c) see [30, Lemma 1.46]. For (b) See [20, Proposition 2.8]. For (d), first let $N > 3$ and $2 < q < 2^*$ then $0 < (N - 2)q - N < N$. Now let $N = 3$ and $4 < q < 6$ then $1 < q - 3 < 3$. Hence we have the following estimate

$$\begin{aligned} \int_\Omega |u_\varepsilon|^q dx &\geq C \int_{|x|<\delta} |U_\varepsilon|^q dx \geq C \varepsilon^{N-(N-2)q/2} \int_1^{\delta/\varepsilon} r^{N-1-(N-2)q} dx \\ &= \frac{C \varepsilon^{N-(N-2)q/2}}{(N-2)q - N} \left[1 - \left(\frac{\varepsilon}{\delta} \right)^{(N-2)q - N} \right] = O(\varepsilon^{N-(N-2)q/2}). \quad \square \end{aligned}$$

LEMMA 3.4. *Let $N \geq 3$ and $\lambda > 0$ and condition (\mathcal{Q}) holds. Then*

$$\widehat{\theta}_\lambda < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

PROOF. By the definition of $\widehat{\theta}_\lambda$, it is enough to show that, for $u_\varepsilon \in H_0^1(\Omega)$,

$$\sup_{t \geq 0} J_\lambda(tu_\varepsilon) < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

Let

$$\mathcal{G}(t) = J_\lambda(tu_\varepsilon) = \frac{t^2}{2} \|u_\varepsilon\|^2 - \frac{\lambda t^q}{q} \int_\Omega |u_\varepsilon|^q dx - \frac{t^{2 \cdot 2_\mu^*}}{2 \cdot 2_\mu^*} \|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*},$$

then using the same assertions as in Lemma 2.6 for the function \mathcal{G} , we deduce that there exists unique $t_\varepsilon > 0$ such that $\sup_{t \geq 0} \mathcal{G}(t) = \mathcal{G}(t_\varepsilon) = J_\lambda(t_\varepsilon u_\varepsilon)$ and $\mathcal{G}'(t_\varepsilon) = 0$, provided $\lambda \in (0, \lambda_1)$. As a result, we obtain

$$(3.4) \quad t_\varepsilon^2 \|u_\varepsilon\|^2 - \lambda t_\varepsilon^q \int_\Omega |u_\varepsilon|^q dx - t_\varepsilon^{2 \cdot 2_\mu^*} \|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*} = 0.$$

It implies

$$\|u_\varepsilon\|^2 = \lambda t_\varepsilon^{q-2} \int_\Omega |u_\varepsilon|^q dx + t_\varepsilon^{2 \cdot 2_\mu^* - 2} \|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*}.$$

Therefore, using Proposition 3.3, Sobolev embedding, definition of $S_{H,L}$ and the fact that $\lambda \in (0, \lambda_1)$, we deduce

$$1 \leq \lambda C_1 t_\varepsilon^{q-2} \|u_\varepsilon\|^{q-2} + C_2 t_\varepsilon^{2 \cdot 2_\mu^* - 2} \|u_\varepsilon\|^{2 \cdot 2_\mu^* - 2},$$

for some suitable constants $C_1, C_2 > 0$. It gives that there exists a $T_1 > 0$ such that $t_\varepsilon \geq T_1$. Also, from (3.4), $t_\varepsilon^{2 \cdot 2_\mu^*} \|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*} \leq t_\varepsilon^2 \|u_\varepsilon\|^2$. That is,

$$t_\varepsilon \leq \left(\frac{\|u_\varepsilon\|^2}{\|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*}} \right)^{1/(2 \cdot 2_\mu^* - 2)}.$$

Hence

$$\begin{aligned} \sup_{t \geq 0} \mathcal{G}(t) &= \frac{t_\varepsilon^2}{2} \|u_\varepsilon\|^2 - \frac{\lambda t_\varepsilon^q}{q} \int_\Omega |u_\varepsilon|^q dx - \frac{t_\varepsilon^{2 \cdot 2_\mu^*}}{2 \cdot 2_\mu^*} \|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*} \\ &\leq \sup_{t \geq 0} \mathcal{V}(t) - \frac{\lambda T_1^q}{q} \int_\Omega |u_\varepsilon|^q dx, \end{aligned}$$

where

$$\mathcal{V}(t) = \frac{t^2}{2} \|u_\varepsilon\|^2 - \frac{t^{2 \cdot 2_\mu^*}}{2 \cdot 2_\mu^*} \|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*}.$$

Now, using Proposition 3.3 and the fact that $\mathcal{V}(t)$ has maximum at

$$t^* = \left(\frac{\|u_\varepsilon\|^2}{\|u_\varepsilon\|_{NL}^{2 \cdot 2_\mu^*}} \right)^{1/(2 \cdot 2_\mu^* - 2)},$$

we get

$$(3.5) \quad \sup_{t \geq 0} \mathcal{G}(t) \leq \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)} + C_1 \varepsilon^{N-2} - \frac{\lambda T_1^q}{q} \int_{\Omega} |u_{\varepsilon}|^q dx.$$

Case 1. $N > 3$ and $q \in (2, 2^*)$ or $N = 3$ and $4 < q < 6$.

As a consequence of Proposition 3.3 and (3.5), we have

$$\begin{aligned} \sup_{t \geq 0} \mathcal{G}(t) &\leq \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)} + C_1 \varepsilon^{N-2} - \frac{\lambda T_1^q}{q} \int_{\Omega} |u_{\varepsilon}|^q dx \\ &\leq \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)} + C_1 \varepsilon^{N-2} - \frac{\lambda T_1^q}{q} C_2 \varepsilon^{N-(N-2)q/2}. \end{aligned}$$

Now, using the condition of N and q , we have $N - (N - 2)/2q < N - 2$ then, for ε sufficiently small,

$$C_1 \varepsilon^{N-2} - \frac{\lambda T_1^q}{q} C_2 \varepsilon^{N-(N-2)q/2} < 0.$$

Therefore,

$$\sup_{t \geq 0} J_{\lambda}(tu_{\varepsilon}) = \sup_{t \geq 0} \mathcal{G}(t) < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

Case 2. If $q = 2$ and $N > 3$.

When $N > 4$ then by Proposition 3.3 and (3.5),

$$\sup_{t \geq 0} \mathcal{G}(t) \leq \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)} + C_1 \varepsilon^{N-2} - \frac{\lambda T_1^2}{2} C_2 \varepsilon^2.$$

Therefore, for ε sufficiently small,

$$C_1 \varepsilon^{N-2} - \frac{\lambda T_1^2}{2} C_2 \varepsilon^2 < 0,$$

we obtain

$$\sup_{t \geq 0} J_{\lambda}(tu_{\varepsilon}) < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

When $N = 4$ then, again by Proposition 3.3 and (3.5), for an appropriate constant $C_3 > 0$, we have

$$\begin{aligned} \sup_{t \geq 0} \mathcal{G}(t) &\leq \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)} + C_1 \varepsilon^2 - \frac{\lambda T_1^2}{2} C_2 (\varepsilon^2 |\log \varepsilon| + \varepsilon^2) \\ &\leq \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)} + C_3 \varepsilon^2 - \frac{\lambda T_1^2}{2} C_2 \varepsilon^2 |\log \varepsilon|. \end{aligned}$$

Since $|\log \varepsilon| \rightarrow \infty$ as $\varepsilon \rightarrow 0$, for ε sufficiently small,

$$C_3 \varepsilon^2 - \frac{\lambda T_1^2}{2} C_2 \varepsilon^2 |\log \varepsilon| < 0.$$

Thus

$$\sup_{t \geq 0} J_{\lambda}(tu_{\varepsilon}) < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}. \quad \square$$

LEMMA 3.5. *If condition (Q) holds then the following holds:*

- (a) $\widehat{\theta}_\lambda = \theta_\lambda$.
- (b) $0 < \theta_\lambda < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}$.
- (c) *There exists $u_\lambda^\Omega \in N_\lambda^\Omega$ such that $J_\lambda(u_\lambda^\Omega) = \inf_{u \in N_\lambda^\Omega} J_\lambda(u) = \theta_\lambda$ and $u_\lambda^\Omega \geq 0$.*

PROOF. (a) By Lemmas 3.1, 3.4, 2.4 and the Mountain Pass Lemma, there exists a $u_\lambda^\Omega \in H_0^1(\Omega)$ such that $J_\lambda(u_\lambda^\Omega) = \widehat{\theta}_\lambda$ and $J'_\lambda(u_\lambda^\Omega) = 0$. It implies $u_\lambda^\Omega \in N_\lambda^\Omega$. Hence, $\theta_\lambda \leq J_\lambda(u_\lambda^\Omega) = \widehat{\theta}_\lambda$. Also from Lemma 2.6, for each $v \in N_\lambda^\Omega$, there exists a unique $t_0 > 0$ such that $\sup_{t \geq 0} J_\lambda(tv) = J_\lambda(t_0v)$. Since $u_\lambda^\Omega \in N_\lambda^\Omega$, it implies

$$\widehat{\theta}_\lambda \leq \sup_{t \geq 0} J_\lambda(tu) = J_\lambda(u). \text{ Therefore, } \widehat{\theta}_\lambda \leq \theta_\lambda.$$

(b) By Lemma 2.6, $\theta_\lambda > 0$ and, by Lemma 3.4,

$$\theta_\lambda = \widehat{\theta}_\lambda < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

(c) By part (a), there exists a $u_\lambda^\Omega \in N_\lambda^\Omega$ such that $J_\lambda(u_\lambda^\Omega) = \widehat{\theta}_\lambda = \theta_\lambda = \inf_{u \in N_\lambda^\Omega} J_\lambda(u)$. Since $J_\lambda(u_\lambda^\Omega) = J_\lambda(|u_\lambda^\Omega|)$, we can assume $u_\lambda^\Omega \geq 0$. □

4. Proof of Theorem 1.2

In this section, first we gather some information which is needed to estimate the $\text{cat}_\Omega(\Omega)$. Before that, we prove some lemmas necessary for the proof of Theorem 1.2.

LEMMA 4.1. *Let $N \geq 3$ and $\{u_n\}$ be a sequence in $H_0^1(\Omega)$ such that*

$$\|u_n\|_{NL}^{2 \cdot 2_\mu^*} = \|u_n\|^2 \leq S_{H,L}^{(2N-\mu)/(N-\mu+2)} + o_n(1) \text{ as } n \rightarrow \infty.$$

Then, there exist sequences $z_n \in \mathbb{R}^N$ and $\alpha_n \in \mathbb{R}^+$ such that the sequence

$$v_n(x) = \alpha_n^{(N-2)/2} u_n(\alpha_n x + z_n)$$

have a convergent subsequence, still denoted by v_n . Moreover, $v_n \rightarrow v \not\equiv 0$ in $D^{1,2}(\mathbb{R}^N)$, $z_n \rightarrow z \in \overline{\Omega}$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.

PROOF. Let $\{w_n\}$ be a sequence such that $w_n = u_n/\|u_n\|_{NL}$ then $\|w_n\|_{NL} = 1$, $\|w_n\|^2 = \|u_n\|^2/\|u_n\|_{NL}^2 = \|u_n\|^{2((N-\mu+2)/(2N-\mu))} \leq S_{H,L} + o_n(1)$. By definition of $S_{H,L}$, $\|w_n\|^2 \geq S_{H,L}$, it implies $\|w_n\|^2 \rightarrow S_{H,L}$ as $n \rightarrow \infty$. Now, using Proposition A.1 for the sequence $\{w_n\}$, we have the desired result. □

Since Ω is a smooth bounded domain of \mathbb{R}^N , we can pick $\delta > 0$ small enough so that $\Omega_\delta^+ = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \delta\}$ and $\Omega_\delta^- = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) > \delta\}$ are homotopically equivalent to Ω .

Without loss of generality, we can assume that $B_\delta = B_\delta(0) \subset \Omega$. Consequently, we consider the functional $J_\lambda^{B_\delta} : H_{0,\text{rad}}^1(B_\delta) \rightarrow \mathbb{R}^N$ defined as

$$J_\lambda^{B_\delta}(u) = \frac{1}{2} \int_{B_\delta} |\nabla u|^2 dx - \frac{\lambda}{q} \int_{B_\delta} |u|^q - \frac{1}{2 \cdot 2_\mu^*} \int_{B_\delta} \int_{B_\delta} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dy dx,$$

where $H_{0,\text{rad}}^1(B_\delta) = \{u \in H_0^1(B_\delta) : u \text{ is radial}\}$. And let $N_\lambda^{B_\delta}$ be the Nehari manifold associated to functional $J_\lambda^{B_\delta}$. Then all the results obtained in Section 3 are valid for the functional $J_\lambda^{B_\delta}$. In particular, by Lemma 3.5, we know that there exists $u_\lambda^{B_\delta} \in N_\lambda^{B_\delta}$ such that $u_\lambda^{B_\delta} \geq 0$ in B_δ . Moreover,

$$(4.1) \quad J_\lambda^{B_\delta}(u_\lambda^{B_\delta}) = \inf_{u \in N_\lambda^{B_\delta}} J_\lambda^{B_\delta}(u) < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

Now, with the help of $u_\lambda^{B_\delta}$, we will define the set $\mathcal{A}_\lambda = \{u \in N_\lambda^\Omega : J_\lambda(u) \leq J_\lambda^{B_\delta}(u_\lambda^{B_\delta})\}$, and the function $\phi_\lambda : \Omega_\delta^- \rightarrow \mathcal{A}_\lambda$ given by

$$(4.2) \quad \begin{cases} u_\lambda^{B_\delta}(x-z) & \text{if } x \in B_\delta(z), \\ 0 & \text{elsewhere.} \end{cases}$$

Next we define the barycenter mapping $\beta : N_\lambda^\Omega \rightarrow \mathbb{R}^N$ by setting

$$(4.3) \quad \beta(u) = \frac{1}{\|u\|_{NL}^{2 \cdot 2_\mu^*}} \int_\Omega \int_\Omega \frac{x|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dy dx$$

Using the fact that $u_\lambda^{B_\delta}$ is radial, $\beta(\phi_\lambda(z)) = z$ for all $z \in \Omega_\delta^-$.

LEMMA 4.2. *Let $N \geq 3$ and $q \in [2, 2^*]$. Then there exists $\Upsilon^* > 0$ such that if $u \in \mathcal{A}_\lambda$ and $\lambda \in (0, \Upsilon^*)$ then $\beta(u) \in \Omega_\delta^+$.*

PROOF. Assume to the contrary, that there exist sequences $\{\lambda_n\} \in \mathbb{R}^+$ and $u_n \in \mathcal{A}_{\lambda_n}$ such that $\lambda_n \rightarrow 0$ and $\beta(u_n) \notin \Omega_\delta^+$. Using the definition of \mathcal{A}_{λ_n} , we have $u_n \in N_{\lambda_n}^\Omega$ and $J_{\lambda_n}(u_n) \leq J_{\lambda_n}^{B_\delta}(u_{\lambda_n}^{B_\delta})$. Define

$$M(t) = J_{\lambda_n}(tu_n) = \frac{t^2}{2} \|u_n\|^2 - \frac{\lambda_n t^q}{q} \int_\Omega |u_n|^q dx - \frac{t^{2 \cdot 2_\mu^*}}{2 \cdot 2_\mu^*} \|u_n\|_{NL}^{2 \cdot 2_\mu^*},$$

using the same assertions and arguments as in Lemma 2.6, there exists a unique $t_0 > 0$ such that $M'(t_0) = 0$ and $t_0 u_n \in N_{\lambda_n}^\Omega$. Since $u_n \in N_{\lambda_n}^\Omega$, it implies that $M'(1) = 0$ and M is increasing for $t < 1$ and decreasing for $t > 1$. Therefore,

$$(4.4) \quad J_{\lambda_n}(u_n) = \sup_{t \geq 0} J_{\lambda_n}(tu_n).$$

As

$$\|u_n\|^2 - \lambda_n \int_\Omega |u_n|^q dx - \|u_n\|_{NL}^{2 \cdot 2_\mu^*} = 0,$$

employing this with definition of $S_{H,L}$ and Sobolev embedding, we have

$$1 = \frac{\lambda_n}{\|u_n\|^2} \int_\Omega |u_n|^q dx + \frac{\|u_n\|_{NL}^{2 \cdot 2_\mu^*}}{\|u_n\|^2} \leq \lambda_n c_1 \|u_n\|^{q-2} + S_{H,L}^{-2_\mu^*} \|u_n\|^{2 \cdot 2_\mu^* - 2},$$

where $c_1 > 0$ is a appropriate constant. It implies that for large n , there exists a constant $C > 0$ such that

$$(4.5) \quad \|u_n\| > C.$$

CLAIM 1. There exists an $l > 0$ such that up to a subsequence $\|u_n\|_{NL}^{2 \cdot 2_\mu^*} \rightarrow l$ as $n \rightarrow \infty$.

Since

$$J_{\lambda_n}(u_n) \leq J_{\lambda_n}^{B_\delta}(u_{\lambda_n}^{B_\delta}) < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N - \mu)/(N - \mu + 2)},$$

$J_{\lambda_n}(u_n)$ is bounded in \mathbb{R} , subsequently $\|u_n\|_{NL}$ is a bounded sequence. Moreover, from the fact that $u_n \in N_{\lambda_n}^\Omega$, it follows that

$$J_{\lambda_n}(u_n) = \lambda_n \left(\frac{1}{2} - \frac{1}{q} \right) \int_\Omega |u_n|^q dx + \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*} \right) \|u_n\|_{NL}^{2 \cdot 2_\mu^*}.$$

It implies that $\lambda_n \int_\Omega |u_n|^q dx$ is a bounded sequence. As a consequence, $\|u_n\|$ is bounded in \mathbb{R} . Therefore, there exists a $l \geq 0$ such that $\|u_n\|_{NL} \rightarrow l$ as $n \rightarrow \infty$. To prove the Claim 1, it is enough to show that $l \neq 0$. Using (4.5), we deduce

$$\|u_n\|_{NL}^{2 \cdot 2_\mu^*} = \|u_n\|^2 - \lambda_n \int_\Omega |u_n|^q dx \geq \|u_n\|^2 - \lambda_n c_1 \|u_n\|^q \geq C^2 - \lambda_n c_2,$$

where $c_2 > 0$ is a suitable constant. Since $\lambda_n \rightarrow 0$, so we have $l > 0$. This proves Claim 1.

CLAIM 2. For all $n \in \mathbb{N}$, there exists $t_n > 0$ such that $\|t_n u_n\|^2 = \|t_n u_n\|_{NL}^{2 \cdot 2_\mu^*}$. Furthermore, t_n is a bounded sequence in \mathbb{R} .

Assume $t_n = [\|u_n\|^2 / \|u_n\|_{NL}^{2 \cdot 2_\mu^*}]^{1/(2 \cdot 2_\mu^* - 2)}$ then $\|t_n u_n\|^2 = \|t_n u_n\|_{NL}^{2 \cdot 2_\mu^*}$ for all $n \in \mathbb{N}$. Using the fact that $\|u_n\|$ is bounded and by Claim 1, we deduce that t_n is a bounded sequence in \mathbb{R} , concludes the proof of Claim 2.

By the definition of J_{λ_n} and taking into account (4.1), (4.4), Claim 2, $u_n \in \mathcal{A}_{\lambda_n}$, $\lambda_n \rightarrow 0$, and $\int_\Omega |u_n|^q dx$ is bounded, we obtain

$$\begin{aligned} \frac{N - \mu + 2}{2(2N - \mu)} \|t_n u_n\|^2 &= J_{\lambda_n}(t_n u_n) + \lambda_n t_n^q \int_\Omega |u_n|^q dx \leq J_{\lambda_n}(u_n) + o_n(1) \\ &\leq J_{\lambda_n}^{B_\delta}(u_{\lambda_n}^{B_\delta}) + o_n(1) < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N - \mu)/(N - \mu + 2)} + o_n(1). \end{aligned}$$

From Claim 2 and Lemma 4.1, there exists a sequences $z_n \in \mathbb{R}^N$ and $\alpha_n \in \mathbb{R}^+$ such that the sequence $v_n(x) = \alpha_n^{(N-2)/2} t_n u_n(\alpha_n x + z_n)$ have a convergent subsequence, still denoted by v_n . Moreover, $v_n \rightarrow v \neq 0$ in $D^{1,2}(\mathbb{R}^N)$, $z_n \rightarrow z \in \overline{\Omega}$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. Let $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $\psi(x) = x$ for all $x \in \overline{\Omega}$.

Consider

$$\begin{aligned} \beta(u_n) = \beta(t_n u_n) &= \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\psi(x) |u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dy dx}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dy dx} \\ &= \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\psi(\alpha_n x + z_n) |v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dy dx}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dy dx} \rightarrow z \in \bar{\Omega}, \end{aligned}$$

where the last one follows from regularity of ψ and Lebesgue dominated theorem. This contradicts the assumption $\beta(u_n) \notin \Omega_\delta^+$. It concludes the proof. \square

LEMMA 4.3. Assume $N \geq 3$, $q \in [2, 2^*)$ and $\lambda \in (0, \Upsilon^*)$ (defined in Lemma 4.2). Then $\text{cat}_{\mathcal{A}_\lambda}(\mathcal{A}_\lambda) \geq \text{cat}_\Omega(\Omega)$.

PROOF. The proof can be done by using the same assertions as in [2, Lemma 4.3]. \square

Next we need following lemma in order to proof Theorem 1.2.

LEMMA 4.4. [1] Suppose that X is a Hilbert manifold and $F \in C^1(X, \mathbb{R})$. Assume that there are $c_1 \in \mathbb{R}$ and $k \in \mathbb{N}$, such that

- (a) F satisfies the Palais–Smale condition for energy level $c \leq c_1$;
- (b) $\text{Cat}(\{x \in X : F(x) \leq c_1\}) \geq k$.

Then F has at least k critical points in $\{x \in X : F(x) \leq c_1\}$.

PROOF OF THEOREM 1.2. By Lemma 3.2, J_λ satisfies $(\text{PS})_c$ condition on N_λ^Ω for any

$$c < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)},$$

provided $\lambda \in (0, \lambda_1)$. If condition (\mathcal{Q}) holds then from Lemma 3.5,

$$0 < \theta_\lambda < \frac{N - \mu + 2}{2(2N - \mu)} S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

Hence if condition (\mathcal{Q}) holds then Lemmas 4.3 and 4.4, we have at least $\text{cat}_\Omega(\Omega)$ critical points of J_λ restricted to N_λ for any $\lambda \in (0, \Lambda^*)$, where $\Lambda^* = \min\{\lambda_1, \Upsilon^*\}$. Thus using Lemma 2.5, we obtain J_λ has at least $\text{cat}_\Omega(\Omega)$ critical points on $H_0^1(\Omega)$. From [15, Lemma 4.4] and [19, Theorem 2.2], we have at least $\text{cat}_\Omega(\Omega)$ positive solutions of problem (P_λ) . \square

Appendix A

Here we will discuss the behavior of the optimizing sequence of $S_{H,L}$. For the local case, Proposition A.1 has been proved in [29] and [30]. Combining the ideas of [17] and [30], one expects the Proposition A.1 to hold for critical Choquard case, but as best of our knowledge this type of result has not been proved exclusively anywhere. For $N = 3$, Proposition A.1 has been proved in [28].

PROPOSITION A.1. *Let $\{u_n\}$ be a sequence in $H_0^1(\Omega)$ such that*

$$\int_{\Omega} \int_{\Omega} \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^*}}{|x-y|^{\mu}} dy dx = 1 \quad \text{and} \quad \|u_n\|^2 \rightarrow S_{H,L} \quad \text{as } n \rightarrow \infty.$$

Then, there exists a sequences $z_n \in \mathbb{R}^N$ and $\alpha_n \in \mathbb{R}^+$ such that the sequence

$$v_n(x) = \alpha_n^{(N-2)/2} u_n(\alpha_n x + z_n)$$

have a convergent subsequence, still denoted by v_n , such that $v_n \rightarrow v \not\equiv 0$ in $D^{1,2}(\mathbb{R}^N)$, $z_n \rightarrow z \in \bar{\Omega}$, and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. In particular, v is a minimizer of $S_{H,L}$.

PROOF. Define the Lévy concentration function

$$Q_n(\lambda) := \sup_{z \in \mathbb{R}^N} \int_{B(z,\lambda)} (|x|^{-\mu} * |u_n|^{2^*}) |u_n|^{2^*} dx.$$

It is easy to see that, for each n , $\lim_{\lambda \rightarrow 0^+} Q_n(\lambda) = 0$ and $\lim_{\lambda \rightarrow \infty} Q_n(\lambda) = 1$, there exists $\alpha_n > 0$ such that $Q_n(\alpha_n) = 1/2$. Also, there exist $z_n \in \mathbb{R}^N$ such that

$$\int_{B(z_n, \alpha_n)} (|x|^{-\mu} * |u_n|^{2^*}) |u_n|^{2^*} dx = Q_n(\alpha_n) = \frac{1}{2}.$$

Now define the function $v_n(x) = \alpha_n^{(N-2)/2} u_n(\alpha_n x + z_n)$ then

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*} |v_n(y)|^{2^*}}{|x-y|^{\mu}} dx dy = 1, \\ & \|\nabla v_n\|_{L^2}^2 \rightarrow S_{H,L} \quad \text{as } n \rightarrow \infty, \\ \text{(A.1)} \quad & \frac{1}{2} = \sup_{z \in \mathbb{R}^N} \int_{B(z,1)} (|x|^{-\mu} * |v_n|^{2^*}) |v_n|^{2^*} dx \\ & = \int_{B(0,1)} (|x|^{-\mu} * |v_n|^{2^*}) |v_n|^{2^*} dx. \end{aligned}$$

It implies $\{v_n\}$ is a bounded sequence in $D^{1,2}(\mathbb{R}^N)$. Therefore, there exist a subsequence, still denoted by $\{v_n\}$ such that $v_n \rightharpoonup v$ weakly in $D^{1,2}(\mathbb{R}^N)$, for some $v \in D^{1,2}(\mathbb{R}^N)$. Then we can assume that there exist ω, τ, ν such that

$$\begin{aligned} v_n \rightarrow v \quad \text{a.e. on } \mathbb{R}^N, \quad & |\nabla v_n|^2 \rightharpoonup \omega, \quad |v_n|^{2^*} \rightharpoonup \tau, \\ & (|x|^{-\mu} * |v_n|^{2^*}) |v_n|^{2^*} \rightharpoonup \nu \quad \text{in the sense of measure.} \end{aligned}$$

Now, using the Brezis–Leib lemma in sense of measure, we have

$$|\nabla(v_n - v)|^2 \rightharpoonup \varpi := \omega - |\nabla v|^2, \quad |v_n - v|^{2^*} \rightharpoonup \chi := \tau - |v|^{2^*},$$

$$(|x|^{-\mu} * |v_n - v|^{2^*})|v_n - v|^{2^*} \rightharpoonup \kappa := \nu - (|x|^{-\mu} * |v|^{2^*})|v|^{2^*}.$$

Moreover, if we define

$$\omega_\infty := \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla v_n|^2 dx,$$

$$\tau_\infty := \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |v_n|^{2^*} dx,$$

$$\nu_\infty := \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} (|x|^{-\mu} * |v_n|^{2^*})|v_n|^{2^*} dx$$

then, by using concentration-compactness principle [17, Lemma 2.5], we deduce that

$$\limsup_{n \rightarrow \infty} \|\nabla v_n\|_{L^2}^2 = \int_{\mathbb{R}^N} d\omega + \omega_\infty,$$

$$\limsup_{n \rightarrow \infty} \|v_n\|_{L^{2^*}}^{2^*} = \int_{\mathbb{R}^N} d\tau + \tau_\infty,$$

$$\limsup_{n \rightarrow \infty} \|v_n\|_{NL}^{2 \cdot 2^*} = \int_{\mathbb{R}^N} d\nu + \nu_\infty$$

and

$$C(N, \mu)^{-2N/(2N-\mu)} \nu_\infty^{2N/(2N-\mu)} \leq \tau_\infty \left(\int_{\mathbb{R}^N} d\tau + \tau_\infty \right),$$

$$S_{H,L}^2 \nu_\infty^{2/2^*} \leq \omega_\infty \left(\int_{\mathbb{R}^N} d\omega + \omega_\infty \right).$$

Also, if $v = 0$ and

$$\int_{\mathbb{R}^N} d\omega = S_{H,L} \left(\int_{\mathbb{R}^N} d\nu \right)^{1/2^*}$$

then ν is concentrated at a single point. By using [17, (2.11)], we have

$$(A.2) \quad S_{H,L} \left(\int_{\mathbb{R}^N} d\kappa \right)^{1/2^*} \leq \int_{\mathbb{R}^N} d\varpi.$$

It implies

$$(A.3) \quad S_{H,L} = \limsup_{n \rightarrow \infty} \|\nabla v_n\|_{L^2}^2 = \int_{\mathbb{R}^N} d\varpi + \|\nabla v\|_{L^2}^2 + \omega_\infty,$$

$$1 = \limsup_{n \rightarrow \infty} \|v_n\|_{NL}^{2 \cdot 2^*} = \int_{\mathbb{R}^N} d\kappa + \|v\|_{NL}^{2 \cdot 2^*} + \nu_\infty$$

$$S_{H,L} \nu_\infty^{2/2^*} \leq \omega_\infty.$$

Using the definition of $S_{H,L}$, (A.2) and (A.3), we obtain

$$S_{H,L} \geq S_{H,L} \left(\left(\|v\|_{NL}^{2 \cdot 2^*} \right)^{1/2^*} + \left(\int_{\mathbb{R}^N} d\kappa \right)^{1/2^*} + \nu_\infty^{2/2^*} \right),$$

that is

$$\int_{\mathbb{R}^N} d\kappa + \|v\|_{NL}^{2 \cdot 2^*_\mu} + \nu_\infty \geq \left(\|v\|_{NL}^{2 \cdot 2^*_\mu}\right)^{1/2^*_\mu} + \left(\int_{\mathbb{R}^N} d\kappa\right)^{1/2^*_\mu} + \nu_\infty^{2/2^*_\mu}.$$

Thanks to the fact that $\|v\|_{NL}$, $\int_{\mathbb{R}^N} d\kappa$, ν_∞ are non-negative, we get $\|v\|_{NL}$, $\int_{\mathbb{R}^N} d\kappa$, ν_∞ are equal to either 1 or 0. Using (A.1), we have $\nu_\infty \leq 1/2$. It implies $\nu_\infty = 0$. Now, if $\int_{\mathbb{R}^N} d\kappa = 1$ then $\|v\|_{NL} = 0$ that is, $v = 0$ almost everywhere on \mathbb{R}^N . Therefore

$$S_{H,L} = \int_{\mathbb{R}^N} d\varpi + \omega_\infty \geq \int_{\mathbb{R}^N} d\varpi.$$

Hence

$$(A.4) \quad S_{H,L} \left(\int_{\mathbb{R}^N} d\kappa\right)^{1/2^*_\mu} \geq \int_{\mathbb{R}^N} d\varpi.$$

Coupling (A.2), (A.4) with the fact that $v = 0$ almost everywhere on \mathbb{R}^N , we have ν is concentrated at a single point z_0 . From (A.1), we get

$$\begin{aligned} \frac{1}{2} &= \sup_{z \in \mathbb{R}^N} \int_{B(z,1)} (|x|^{-\mu} * |v_n|^{2^*_\mu}) |v_n|^{2^*_\mu} dx \\ &\geq \int_{B(z_0,1)} (|x|^{-\mu} * |v_n|^{2^*_\mu}) |v_n|^{2^*_\mu} dx \rightarrow \int_{\mathbb{R}^N} d\kappa = 1, \end{aligned}$$

which is not possible. Hence, $\|v\|_{NL}^{2 \cdot 2^*_\mu} = 1$. Also, $S_{H,L} = \lim_{n \rightarrow \infty} \|\nabla v_n\|_{L^2}^2 = \|\nabla v\|_{L^2}^2$. In particular, v is a minimizer of $S_{H,L}$. From [16, Lemma 1.2], we know $S_{H,L}$ is achieved if and only if

$$u = C \left(\frac{b}{b^2 + |x - a|^2}\right)^{(N-2)/2}$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0, \infty)$ are parameters. It implies

$$v = u = C \left(\frac{b}{b^2 + |x - a|^2}\right)^{(N-2)/2}.$$

In particular, $v \neq 0$. Now, we will prove that $\alpha_n \rightarrow 0$ and $z_n \rightarrow z_0 \in \bar{\Omega}$. Let if possible $\alpha_n \rightarrow \infty$. Since $\{u_n\}$ is a bounded sequence in $H_0^1(\Omega)$, $\{u_n\}$ is a bounded sequence in $L^2(\Omega)$. Thus, if we define $\Omega_n = (\Omega - z_n)/\alpha_n$, then

$$\int_{\Omega_n} |v_n|^2 dx = \frac{1}{\alpha_n^2} \int_{\Omega} |u_n|^2 dx \leq \frac{C}{\alpha_n^2} \rightarrow 0.$$

Contrary to this, by Fatou's Lemma we have

$$0 = \liminf_{n \rightarrow \infty} \int_{\Omega_n} |v_n|^2 dx \geq \int_{\Omega_n} |v|^2 dx.$$

This means $v \equiv 0$, which is not true. Hence $\{\alpha_n\}$ is bounded in \mathbb{R} that is, there exists $\alpha_0 \in \mathbb{R}$ such that $\alpha_n \rightarrow \alpha_0$ as $n \rightarrow \infty$. If $z_n \rightarrow \infty$ then for any $x \in \Omega$ and large n , $\alpha_n x + z_n \notin \bar{\Omega}$. Since $u_n \in H_0^1(\Omega)$ then $u_n(\alpha_n x + z_n) = 0$ for all

$x \in \Omega$, it yields a contradiction to the assumption $\|u_n\|_{NL}^{2 \cdot 2^*} = 1$. Therefore, z_n is bounded, it implies that $z_n \rightarrow z_0$. Now suppose $\alpha_n \rightarrow \alpha_0 > 0$ then $\Omega_n \rightarrow (\Omega - z_0)/\alpha_0 = \Omega_0 \neq \mathbb{R}^N$. Hence

$$\int_{\Omega_0} \int_{\Omega_0} \frac{|v_n(x)|^{2^*} |v_n(y)|^{2^*}}{|x-y|^\mu} dx dy = 1,$$

$$\int_{\Omega_0} |v_n|^2 dx \rightarrow \int_{\Omega_0} |v|^2 dx = S_{H,L} \quad \text{as } n \rightarrow \infty,$$

which is not true. Hence $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. Finally, arguing by contradiction, we assume that

$$(A.5) \quad z_0 \notin \bar{\Omega}.$$

In view of the fact that $\alpha_n x + z_n \rightarrow z_0$ for all $x \in \Omega$ as $n \rightarrow \infty$. Now, using (A.5), we have $\alpha_n x + z_n \notin \bar{\Omega}$ for all $x \in \Omega$ and n large enough. It implies that $u_n(\alpha_n x + z_n) = 0$ for n large enough. This yields a contradiction, therefore, $z_0 \in \bar{\Omega}$. \square

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