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EFFECT OF EXTERNAL POTENTIALS IN A COUPLED SYSTEM OF MULTI-COMPONENT INCONGRUENT DIFFUSION

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Abstract. This work is devoted to investigations of some interesting aspects of a multi-component Reaction-Diffusion system of the form

$$\partial_t z = \mathbf{D}\Delta_x z + M(x)z + W(x)|z|^{p-2}\beta z, \quad z\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^{2K},\ N\geq 2$$

where M, W are external potential functions, $\mathbf{D}$ and β are matrices of diffusion coefficients and coupling constants respectively. When the diffusion rate is small, we show that the geometric shapes of the external potential functions will influence the multiplicity of solutions to the system. It is also of interest to know that, for $z=(u,v)$, we shall deal with standard diffusion coefficients $D_u>0$ and the incongruent diffusion coefficients $D_v<0$ which has generally been overlooked in the study of Reaction–Diffusion systems.

1. Introduction

1.1. Some backgrounds and previous results. A system of Reaction–Diffusion (RD) equations comprises of reaction terms and diffusion terms, i.e. the typical form is as follows:

$$\partial_t z = \mathrm{div}_x\big(\mathbf{D}\nabla_x z\big) + M(x)z + f(x,z) \tag{1.1}$$

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where $z(t,x) \in \mathbb{R}^m$, $m > 1$, is a state variable and describes density or concentration of multi-component substances, populations at position $x \in \mathbb{R}^N$ at time t. The first term on the right hand side describes the "diffusion", including $\mathbf{D}$ as a matrix of diffusion coefficients D_{ij} (the diagonal elements of $\mathbf{D}$ describe the main-term diffusion rate and the off-diagonal elements express the the cross-terms diffusion which was suggested firstly in 1932 see [21]). Creation and killing in the reaction process (birth, death, etc) are described by the scalar field $M(x)$ which could be a matrix-valued function, and the nonlinear part, $f(x,z)$ is smooth function $f\colon \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^m$ called reaction kinetics and describes processes that really "change" the present z, i.e. coupling actions, chemical reactions, not just diffusion in the space.

Being one of the major transport processes in liquids RD processes, especially in multi-component systems, have attracted increasing attention from the scientific community in recent years as investigators have begun to seek insights into the fascinating patterns that occur in living organisms, in ecological systems, in geochemistry and in physicochemical systems. The rapid growth of the field of systems biology has further contributed to interest in RD systems (1.1). Unfortunately, multi-component diffusion is more complicated than is often realized. In general, systems of RD equations allow for much more complex behavior than a scalar RD equation does. For example, a ternary system (two solutes in a solvent) may has four coefficients in the diffusion matrix, not just two. And the diffusion coefficients can be large or small and be positive or negative, thus having a substantial effect on flows of matter. Meanwhile, the interacting reaction terms are of interest and lead to interesting behavior. For instance, oscillating phenomena can evolve – as these oscillations can spread in space via diffusion and instabilities may develop spatial phenomena like pattern formation can be observed. Here for a detailed survey, we would refer the readers to [18].

On the macroscopic level, fluxes of chemical components (species) are due to convection and molecular fluxes, where the latter essentially refers to diffusive transport. The almost exclusively employed constitutive "law" to model diffusive fluxes within continuum mechanical models is Fick's law (which requires positive diffusion coefficients), stating that the flux of a chemical component is proportional to the gradient of the concentration of this species, directed opposite to the gradient. This leads to positive coefficients D_{ii} in $\mathbf{D}$, however, thermodynamic conditions do not require the diagonal of $\mathbf{D}$ to be positive. Contrary to popular belief, there are systems with negative D_{ii} for a particular choice of the solvent component. Typical example is acetic acid + chloroform with water being chosen as solvent, a totally unexpected result is the negative chloroform main term diffusion coefficient, that is a negative D_{22} value (see [17], [25] for the chemical details). This behavior has been interpreted as a thermodynamic effect caused by the diffusion of salted-out chloroform down the water concentration gradient produced by the chloroform gradient under conditions of constant acetic acid concentration.

Our current work is motivated exactly by the unusual phenomenon of negative main term diffusion. Let us consider the multi-component nonlinear RD system (1.1) in the situation $z = (u, v) \in \mathbb{R}^{2K}$, some $K \geq 1$, and

$$\mathbf{D} = \begin{pmatrix} D_u & * \\ * & D_v \end{pmatrix} = \varepsilon^2 \mathcal{J}, \quad \text{where} \quad \mathcal{J} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix} \tag{1.2}$$

and $\varepsilon > 0$ being a parameter characterizing the pace of the diffusion process (for simplicity we have ignored the cross-term diffusions). To the best of our knowledge negative main term diffusion, the phenomenon in which one species to be driven from lower to higher concentrations, has generally been neglected in the study of RD systems, only a few results are available so far. An early work of Brézis and Nirenberg [6] considered the 2-component coupled system

$$\begin{cases} \partial_t u = \Delta_x u - v^5 + f(x), \\ \partial_t v = -\Delta_x v - u^3 + g(x), \end{cases} \quad \text{in } (0, T) \times \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f, g \in L^\infty(\Omega)$. Subject to the boundary conditions $u(t, x) = v(t, x) = 0$ on $(0, T) \times \partial\Omega$ and $u(0, x) = v(T, x) = 0$ on Ω, the authors obtained a solution (u, v) with $u \in L^4$ and $v \in L^6$ of (1.3) by using Schauders fixed point theorem. And in [8], Clément, Felmer and Mitidieri considered the problem (with a Fujita-type nonlinearity, see [11], [12])

$$\begin{cases} \partial_t u = \Delta_x u + |v|^{q-2}, \\ \partial_t v = -\Delta_x v - |u|^{p-2} u, \end{cases} \quad \text{in } (-T, T) \times \Omega, \tag{1.4}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^N$, and $N/(N + 2) < 1/p + 1/q < 1$. By variational arguments, they proved that there exists $T_0 > 0$ such that for each $T > T_0$, (1.4) has at least one positive solution satisfying the 0-boundary condition: $u(t, \cdot)|_{\partial\Omega} = 0 = v(t, \cdot)|_{\partial\Omega}$ for all $t \in (-T, T)$, and the periodicity condition: $u(-T, \cdot) = u(T, \cdot)$, $v(-T, \cdot) = v(T, \cdot)$. Moreover, by passing to the limit as $T \to \infty$, they showed that (1.4) has at least one positive solution defined on $\mathbb{R} \times \Omega$ satisfying the 0-boundary condition and

$$\lim_{t\to\infty} u(t, x) = \lim_{t\to\infty} v(t, x) = 0 \quad \text{uniformly in } x \in \Omega.$$

For later developments, we would mention that Bartsch and Ding [3] investigated the following $2K$-component system

$$\begin{cases} \partial_t u = \Delta_x u - V(x)u + \partial_v H(x, u, v), \\ \partial_t v = -\Delta_x v + V(x)v - \partial_u H(x, u, v), \end{cases} \quad \text{in } \mathbb{R} \times \mathbb{R}^N. \tag{1.5}$$

In [3], the authors established a proper variational framework and proved the existence and multiplicity of solutions of homoclinic type to (1.5) under appropriate conditions on the nonlinearities (see also [9]). All the above mentioned systems have constant diffusion coefficients, and if denoted by $z = (u, v)$, they all have the form

$$\partial_t z = \mathcal{J}\Delta_x z + M(x)z + f(x, z)$$

with different types of creation and killing field M and reaction kinetics f.

For the case where the diffusion is parameterized by ε, a very recent paper [10] considered exactly the system (1.1) with $\mathbf{D}$ being defined in (1.2) and the creation and killing field being specified as

$$M(x) = \begin{pmatrix} -\mathrm{id} & -V(x) \\ V(x) & \mathrm{id} \end{pmatrix}$$

for some bounded function $V\colon \mathbb{R}^N \to \mathbb{R}$. In [10], the authors prove that there must be a solution concentrating around the local minimums of the scalar potential $V(x)$ for small diffusion coefficients. This provides a natural and intrinsic characterization of the pattern generalizing dependence on the varying parameters and the spatial distributions of chemical potentials.

In this paper, we are interested in the following aspects which have not been dealt with before and are new in the case of RD systems. Namely we intend

(1) to apply concentration and rescaling techniques to non-autonomous nonlinearities, and in particular to characterize the concentration phenomenon in terms of the different potential functions;
(2) to show that the presence of a field M and a non-autonomous nonlinearity f give better information concerning the existence of a solution, i.e. they can provide multiplicity result to problem (1.1).

1.2. Specific models and main results. In this paper we study the evolution of patterns in solutions of a singularly perturbed RD system, and the simplest model is the following:

$$\partial_t z = \mathbf{D}\Delta_x z + M(x)z + W(x)|z|^{p-2}\beta z, \quad z\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^{2K}, \tag{1.6}$$

for $N \geq 2$, $K \geq 1$, where $\mathbf{D}$ is defined in (1.2) with $\varepsilon > 0$ being a small parameter, $2 < p < 2(N+2)/N$, the external potential M is defined by one of the following form

$$M(x) = \begin{pmatrix} -\mathrm{id} - V(x) & 0 \\ 0 & \mathrm{id} + V(x) \end{pmatrix} \quad \text{or} \quad M(x) = \begin{pmatrix} -\mathrm{id} & -V(x) \\ V(x) & \mathrm{id} \end{pmatrix} \tag{1.7}$$

with $V\colon \mathbb{R}^N \to \mathbb{R}$ bounded (possibly, sign-changing) and $W\colon \mathbb{R}^N \to \mathbb{R}$ is positive, and β is a coupling matrix defined by

$$\beta = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}.$$

Let us introduce, for $r \geq 1$, the Banach space

$$\mathscr{B}^r\big(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2K}\big) := W^{1,r}\big(\mathbb{R}, L^r\big(\mathbb{R}^N, \mathbb{R}^{2K}\big)\big) \cap L^r\big(\mathbb{R}, W^{2,r}\big(\mathbb{R}^N, \mathbb{R}^{2K}\big)\big)$$

equipped with the norm

$$\|z\|_{\mathscr{B}^r} := \bigg(\iint_{\mathbb{R}\times\mathbb{R}^N} \big(|z|^r + |\partial_t z|^r + |\Delta_x z|^r\big)\, dx\, dt \bigg)^{1/r}, \tag{1.8}$$

and in the sequel when no confusion can arise, we will use $\mathscr{B}^r$ for short. By variational frameworks as developed by [3] and [10], we consider (1.6) in the function space $E := [\mathscr{B}^2, L^2]_{1/2}$ which is an interpolation space between $\mathscr{B}^2$ and $L^2\big(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2K}\big)$. We point out here that this space embeds into the corresponding L^q-spaces for $2 \leq q \leq 2(N+2)/N$. It is easy to see that (1.6) is subcritical in the sense that p is smaller than the critical embedding exponent.

In order to gain further insight into the effect of potential functions on the concentrating process and the multiplicity of solutions, we will deal with the following more general class of RD system with critical growth, namely

$$\partial_t z = \mathbf{D}\Delta_x z + M(x)z + \big(W(x)|z|^{p-2} + Q(x)|z|^{4/N}\big)\beta z, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{1.9}$$

Of course, (1.9) reduces to (1.6) when Q is identically zero. Associated to (1.9), let us introduce the following problem

$$\partial_t z = \mathcal{J}\Delta_x z - (\mathcal{J} + \beta\omega_\nu)z + \big(\lambda|z|^{p-2} + \kappa|z|^{4/N}\big)\beta z \tag{1.10}$$

where $\nu \in (-1,1)$, $\lambda > 0$, $\kappa \geq 0$ and ω is in one of the following form (accordingly to $M(x)$)

$$\omega_\nu = \nu \quad \text{or} \quad \omega_\nu = \begin{pmatrix} 0 & \nu \\ \nu & 0 \end{pmatrix}.$$

This equation appears as the limit equation for (1.9).

Let us remark that systems (1.9) and (1.10) can be viewed as infinite-dimensional Hamiltonian systems of the form

$$\beta \frac{dz}{dt} = \nabla\mathcal{H}(z),$$

where $\mathcal{H}$ is some energy functional on a real Hilbert space $\mathscr{H} \subset L^2(\mathbb{R}^N, \mathbb{R}^{2K})$, and the matrix β can be regarded as a skew-symmetric operator. Such an interesting feature of these problems makes variational method applicable, especially, the techniques developed for variational problems with strongly indefinite structure can be employed.

Now, let us state our main results. To be more precisely, we shall apply the global variational arguments, described in Section 3, to define the minimal energy (or ground state energy) associated to (1.10) as $\gamma(\omega_\nu,\lambda,\kappa)$. Roughly speaking, $\gamma(\omega_\nu,\lambda,\kappa)$ is a positive function, and is increasing in the factor ν and decreasing in the factors λ and κ. To relate (1.9) with $\gamma(\omega_\nu,\lambda,\kappa)$, let us introduce the function $c_0\colon \mathbb{R}^N\to\mathbb{R}$ as

$$c_0(y):=\gamma(\omega_{V(y)},W(y),Q(y)).$$

By letting

$$\nu_\infty:=\liminf_{|x|\to\infty}V(x),\qquad \lambda_\infty:=\limsup_{|x|\to\infty}W(x),\qquad \kappa_\infty:=\limsup_{|x|\to\infty}Q(x)$$

and

$$c_*=\min_{y\in\mathbb{R}^N}c_0(y),\qquad \mathscr{C}=\{y\in\mathbb{R}^N:c_0(y)=c_*\},$$

our main theorem can be formulated as

Theorem 1.1. *Assume the matrix M is of the form given in* (1.7), $p\in(2,2(N+2)/N)$ *and V, W, Q are bounded, Hölder continuous functions on $\mathbb{R}^N$ satisfying*

(A1) $\|V\|_{L^\infty}<1$, $\inf_{x\in\mathbb{R}^N}W(x)>0$ *and* $Q(x)\ge 0$ *for all* x.

There exits a constant $\widehat{\kappa}>0$ such that if $\|Q\|_{L^\infty}<\widehat{\kappa}$ and

(A2) *there exists $x_0\in\mathbb{R}^N$ such that $Q(x_0)=\max_{x\in\mathbb{R}^N}Q(x)$ and*

$$\nu_\infty\ge\nu_0:=V(x_0),\qquad \lambda_\infty\le\lambda_0:=W(x_0),\qquad \kappa_\infty\le Q(x_0)$$

with one of the first two inequalities being strict,

then RD system (1.9) *possesses at least θ distinct solutions z_ε^k, $k=1,\dots,\theta$ for small $\varepsilon>0$, where θ is the largest integer such that*

$$\theta<\left(\frac{1+\nu_\infty}{1+\nu_0}\right)^{(4-N(p-2))/(2(p-2))}\left(\frac{\lambda_0}{\lambda_\infty}\right)^{2/(p-2)}. \tag{1.11}$$

Moreover, among these solutions, z_ε^1 lies in the ground state energy level and has the following properties:

(a) $|z_\varepsilon^1(t,\cdot\,)|$ *has exactly one global maximum point at some x_ε with*

$$\lim_{\varepsilon\to0}\operatorname{dist}(x_\varepsilon,\mathscr{C})=0,$$
$$\lim_{\substack{R\to\infty\\ \varepsilon\to0}}\|\widetilde{z}_\varepsilon(t,\cdot\,)\|_{L^\infty(\mathbb{R}^N\setminus B_{\varepsilon R}(x_\varepsilon))}=0\quad \textit{uniformly for } t\in\mathbb{R};$$

(b) *the rescaled function $w_\varepsilon(t,x)=z_\varepsilon(t,\varepsilon x+x_\varepsilon)$ converges as $\varepsilon\to0$ uniformly to a ground state solution $z\colon\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^{2K}$ of*

$$\partial_t z=\mathcal{J}\Delta_x z+M(y_0)z+\big(W(y_0)|z|^{p-2}+Q(y_0)|z|^{4/N}\big)\beta z.$$

Remark 1.2. (a) A possible explicit formula for the constant $\widehat{\kappa}$ could be defined in terms of the factors $\|V\|_{L^\infty}$ and $\inf W$; see (5.1). It satisfies $\widehat{\kappa} \to c(p) > 0$ as $\|V\|_{L^\infty} \to 0$. Thus we do allow critical growth in the nonlinear part but the function Q cannot be too large. It is an interesting open problem whether this restriction on Q can be removed.

(b) Assumptions $p \in \big(2, 2(N+2)/N\big)$ and (A2) imply that

$$L(\nu_0, \nu_\infty, \lambda_0, \lambda_\infty, p) := \left(\frac{1+\nu_\infty}{1+\nu_0}\right)^{(4-N(p-2))/(2(p-2))} \left(\frac{\lambda_0}{\lambda_\infty}\right)^{2/(p-2)} > 1.$$

This suggests that (1.9) has at least one solution. We can always find a function W to make the ratio λ_0/λ_∞ large and, hence, there are examples showing that (1.9) has a very large number of solutions. We would like to mention here that assumption $(A2)$ inherited a similar spirit of [23], and it makes the variational structure of (1.9) satisfy a variant of Palais–Smale condition.

(c) Concerning the number of solutions in Theorem 1.1, for the case that $L(\nu_0, \nu_\infty, \lambda_0, \lambda_\infty, p)$ is an integer, we have $\theta = L(\nu_0, \nu_\infty, \lambda_0, \lambda_\infty, p) - 1$. And hence the choice of θ is slightly different from the function of taking the integer part of $L(\nu_0, \nu_\infty, \lambda_0, \lambda_\infty, p)$. Moreover, we can see from the statement of Theorem 1.1 that the number θ is irrelevant to the function Q. And hence the number of solutions is dominated by the functions V and W.

As a consequence of Theorem 1.1, we obtain the following corollary for the subcritical equation (1.6).

Corollary 1.3. *Under the hypotheses of Theorem* 1.1 *and assuming Q vanishes identically the RD system* (1.6) *possesses at least θ distinct solutions z_ε^k, $k = 1, \dots, \theta$ for small $\varepsilon > 0$, where θ is the integer in* (1.11)*. Moreover, among these solutions, z_ε^1 lies in the ground state energy level and has the following properties*:

(a) *$|z_\varepsilon^1(t, \cdot)|$ has exactly one global maximum point at some x_ε with*

$$\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathscr{C}) = 0,$$
$$\lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \|\widetilde{z}_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^N \setminus B_{\varepsilon R}(x_\varepsilon))} = 0 \quad \textit{uniformly for } t \in \mathbb{R};$$

(b) *the rescaled function $w_\varepsilon(t, x) = z_\varepsilon(t, \varepsilon x + x_\varepsilon)$ converges as $\varepsilon \to 0$ uniformly to a ground state solution $z \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^{2K}$ of*

$$\partial_t z = \mathcal{J} \Delta_x z + M(y_0) z + W(y_0) |z|^{p-2} \beta z.$$

Remark 1.4. In terms of V, W and Q, the assumptions (A2) is sufficient conditions to guarantee $c_0(x_0) < \limsup\limits_{|y|\to\infty} c_0(y)$, and hence, $\mathscr{C}$ is a compact set in $\mathbb{R}^N$. Without loss of generality, throughout this paper, we may always make the assumption that $x_0 = 0 \in \mathbb{R}^N$. We emphasize that $(A2)$ can be replaced by other sufficient conditions, for example,

(a) $\nu_\infty = \sup_{\mathbb{R}^N} V$, $\lambda_\infty = \inf_{\mathbb{R}^N} W$, $Q \equiv$ constant.

(b) Denoted by

$$g(x) = \frac{Q(x)}{W(x)^{4/(N(p-2))}} \quad \text{and} \quad \mu_\infty := \limsup_{|x|\to\infty} g(x),$$

there exists $x_0 \in \mathbb{R}^N$ such that $g(x_0) = \max_{x\in\mathbb{R}^N} g(x)$ and

$$\nu_\infty \geq \nu_0 := V(x_0), \qquad \lambda_\infty \leq \lambda_0 := W(x_0), \qquad \mu_\infty \leq g(x_0)$$

with one of the first two inequalities being strict.

If V and W are not constants, then each of the above conditions (a), (b) is a sufficient condition to guarantee $c_0(x_0) < \limsup_{|y|\to\infty} c_0(y)$.

Remark 1.5. The nonlinearity of power functions was considered firstly by H. Fujita in his classical papers [11], [12] on Cauchy problems for a single RD equation; and such nonlinearity also plays an important role in the analysis of steady-state solutions to RD equations (see a series of remarkable papers of Ni and Takagi [19], [20], and with Lin [15]). Here in this work, we focus on (1.6) and (1.9) as multi-component versions of nonlinear problems in Fujita's type. It is quite natural that, in the case $Q \equiv 0$, the set $\mathscr{C}$ is defined as the "middle ground" between minima of V and maxima of W. Indeed, as one will see in Section 3, minimum points of $c_0(\,\cdot\,)$ coincides with minimum points of the function $V(x)^{(N+4)/2+2/(p-2)}/W(x)^{2/(p-2)}$ provided that $M(x)$ is of the first form in (1.7). As a consequence, if V is constant, then $\mathscr{C}$ coincides with the global maxima set of the potential W. And conversely, if W is constant, then $\mathscr{C}$ is the set of global minima of the function V. However, in the general case ($M(x)$ is of the second form in (1.7), and $Q \not\equiv 0$), it is impossible to have an explicit formula for $c_0(y)$, and hence we cannot explicitly characterize the location of concentration of minimal energy solutions in term of the functions V, W and Q.

The proof of Theorem 1.1 will use variational techniques. Since we are working on the unbounded domain $\mathbb{R} \times \mathbb{R}^N$, we will employ the concentration-compactness argument explored in [16]. It consists in finding a suitable energy threshold for the energy functional of (1.9) such that the Palais–Smale condition holds below this threshold, constructing different minimax levels and then showing these minimax levels are indeed below the energy threshold. We emphasize here that, in the usual concept, the energy threshold tricks are well adapted for the study of variational problems in geometry and physics where lack of compactness occurs. The most notorious example is Yamabe's problem. Here, due the Hamiltonian structure of the RD system, we note that it is not easy to obtain compactness in view of the critical growth of the nonlinearity even in finding the energy threshold. To overcome this, we will need a delicate analysis for the limit problem (1.10) on the ground state energy level and use the concentration-compactness principle to control the factor Q in the critical growth.

To obtain multiple solutions of the problem, the main ingredient is to make use of the invariance of (1.9) under some group actions, for instance the multiplication by ± 1 and the translation in time-coordinate will not change a solution to (1.9). This kind of invariance will lead us to build a pseudo-index theory for the associated functional. More precisely, the number of solutions is related to the frequency that the pseudo-index changes, see an abstract setting in Theorem 2.11.

The remainder part of the paper is organized as follows. Section 2 is devoted to introduce some notations and to briefly recall some preliminary results such as the linking geometry and a Lyapunov–Schmidt type reduction. An abstract theorem regarding the multiplicity of critical points for strongly indefinite functionals is also introduced, and the proof will be postponed to Appendix B. In Section 3, we investigate the associated autonomous problem (1.10). This study allows us to show the role played by the critical factor κ at the ground state energy level. The Palais–Smale condition, which does not hold in general case since we allow critical growth, will then be studied in Section 4. Here we perform the crucial criterion for compactness in terms of a energy threshold. Next, in Section 5, we provide the main components of our proof. The first part is the analysis on the concentration behavior of the ground state energy solution for (1.9). In the second part we build a finite-dimensional linking argument to construct specific minimax schemes which can be applied to (1.9) such that these minimax levels stay below the energy threshold. The proof will be then completed by applying our abstract theorem in Section 2. Finally, for the sake of completeness, in the Appendix A we collect some regularity results which are used in the paper.

2. Notation, known results and main ingredients

In this section we establish some preliminary results which are needed for the proof of our main theorems. Given V, W and Q as in the previous section, we consider the RD system

$$\partial_t z = \mathbf{D}\Delta_x z + M(x)z + \big(W(x)|z|^{p-2} + Q(x)|z|^{4/N}\big)\beta z \tag{2.1}$$

with coefficients $\mathbf{D}$ defined in (1.2) and $M(x)$ being of the form (1.7).

For a clearer expression let us set

$$\mathcal{J}_0 = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix},$$

and by noting that $\mathbf{D} = \varepsilon^2 \mathcal{J}$, we shall consider the scaling $x \to \varepsilon x$ so that (2.1) equivalently transforms as

$$\partial_t z = \mathcal{J}\Delta_x z + M_\varepsilon(x) z + \big(W_\varepsilon(x)|z|^{p-2} + Q_\varepsilon(x)|z|^{4/N}\big)\beta z \tag{2.2}$$

with $M_\varepsilon(x) = M(\varepsilon x)$, $W_\varepsilon(x) = W(\varepsilon x)$ and $Q_\varepsilon(x) = Q(\varepsilon x)$. Remark that $\beta^{-1} = -\beta$, hence if denoted by $\mathscr{L} = -\beta\partial_t + \mathcal{J}_0(-\Delta_x + 1)$ and $\mathcal{V}(x) =$ either $V(x)$ or $V(x)\mathcal{J}_0$, we have (2.2) to be rewritten as

$$\mathscr{L} z + \mathcal{V}_\varepsilon(x) z = W_\varepsilon(x)|z|^{p-2} z + Q_\varepsilon(x)|z|^{4/N} z \quad \text{for } z(t,x) \in \mathbb{R}^{2M} \tag{2.3}$$

where obviously we have used $\mathcal{V}_\varepsilon(x) = \mathcal{V}(\varepsilon x)$.

In what follows, we will focus on the equivalent system (2.3), and, throughout the paper we make use of the following notations: for $1 \le q \le \infty$ we set $L^q := L^q(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2K})$, and by $|\cdot|_q$ we denote the usual L^q-norm, and particularly denoted by $(\,\cdot\,,\,\cdot\,)_2$ the usual L^2-inner product.

Considering the differential operator $\mathscr{L}$ acts on the Hilbert space L^2, it is quite standard to see that $\mathscr{L}$ is a self-adjoint operator with domain

$$\mathcal{D}(\mathscr{L}) = \mathscr{B}^2 := W^{1,2}\big(\mathbb{R}, L^2(\mathbb{R}^N, \mathbb{R}^{2K})\big) \cap L^2\big(\mathbb{R}, W^{2,2}(\mathbb{R}^N, \mathbb{R}^{2K})\big).$$

Let $\sigma(\mathscr{L})$ and $\sigma_e(\mathscr{L})$ be respectively the spectrum and essential spectrum of $\mathscr{L}$, we have $\sigma(\mathscr{L}) = \sigma_e(\mathscr{L}) = \mathbb{R} \setminus (-1,1)$ (cf. [9, Lemma 8.7]). And as a direct consequence, L^2 possesses the orthogonal decomposition:

$$L^2 = L^+ \oplus L^-, \qquad z = z^+ + z^-, \tag{2.4}$$

so that L is positive definite (resp. negative definite) in L^+ (resp. L^-).

In order to construct the energy functionals whose critical points are the solutions of (2.3), we introduce $E := \mathcal{D}(|\mathscr{L}|^{1/2})$ (which is the form domain of L) be equipped with the inner product

$$\langle z_1, z_2 \rangle = \big(|\mathscr{L}|^{1/2} z_1, |\mathscr{L}|^{1/2} z_2\big)_2$$

and the induced norm $\|z\| = \langle z, z\rangle^{1/2}$, where $|\mathscr{L}|$ and $|\mathscr{L}|^{1/2}$ denote respectively the absolute value of $\mathscr{L}$ and the square root of $|\mathscr{L}|$. Since $\sigma(\mathscr{L}) = \mathbb{R} \setminus (-1,1)$, one has

$$|z|_2^2 \le \|z\|^2 \quad \text{for all } z \in E. \tag{2.5}$$

As an interpolation space between $\mathscr{B}^2$ and L^2, E (being a Hilbert space) has the decomposition

$$E = E^+ \oplus E^-, \quad \text{where } E^\pm = E \cap L^\pm$$

which is orthogonal with respect to both $(\,\cdot\,,\,\cdot\,)_2$ and $\langle\,\cdot\,,\,\cdot\,\rangle$. We write $z = z^+ + z^-$ for $z \in E$ according to this decomposition. Remarkably, this decomposition of E induces also a natural decomposition of L^q for every $q \in (1, +\infty)$:

PROPOSITION 2.1. *Let $E^+ \oplus E^-$ be the decomposition of E according to the positive and negative part of $\sigma(\mathscr{L})$. Then, set $E_q^\pm := E^\pm \cap L^q$ for $q \in (1,\infty)$, there holds*

$$L^q = \mathrm{cl}_q\, E_q^+ \oplus \mathrm{cl}_q\, E_q^-$$

with cl_q denoting the closure in L^q. More precisely, there exists $d_q > 0$ for every $q \in (1,\infty)$ such that

$$d_q |z^\pm|_q \le |z|_q \quad \text{for all } z \in E \cap L^q.$$

In L^q's (for $q \ne 2$), by $\oplus$ we mean the topologically direct sum. Before proving Proposition 2.1 we would like to introduce the following definition for *Multipliers* (see [24, Chapter 4]) which plays an important role in our arguments.

DEFINITION 2.2. Let m be a bounded measurable function on $\mathbb{R}^n$, we associate a linear operator T_m on $L^2 \cap L^q$ by $(T_m u)\widehat{\ }(\xi) = m(\xi)\,\widehat{u}(\xi)$ where $\widehat{u}$ denotes the Fourier transform of u. We say that m is a multiplier for L^q $(1 \le q \le \infty)$ if whenever $u \in L^2 \cap L^q$ then $T_m u \in L^q$ (notice it is automatically in L^2), and T_m is bounded, that is,

$$|T_m u|_q \le C \cdot |u|_q, \quad u \in L^2 \cap L^q \text{ (with } C \text{ independent of } u). \tag{2.6}$$

Observe that if (2.6) is satisfied, and $p < \infty$, then T_m has a unique bounded extension to L^q, which again satisfies the same inequality.

PROOF OF PROPOSITION 2.1. First we remark that in this context, the spatial domain is $\mathbb{R} \times \mathbb{R}^N$. Now recall the definitions for the matrices β and $\mathcal{J}_0$, let us study $\mathscr{L} := -\beta\partial_t + \mathcal{J}_0(-\Delta_x + 1)$. It is a differential operator with real constant coefficients. In the Fourier domain $\xi = (\xi_0, \dots, \xi_N)$, it becomes the operator of multiplication by the matrix:

$$\widehat{\mathscr{L}}(\xi) = \begin{pmatrix} 0 & \overline{A(\xi)} \\ A(\xi) & 0 \end{pmatrix} \quad \text{with } A(\xi) = \left(i\xi_0 + 1 + \sum_{k=1}^N \xi_k^2 \right) \cdot \mathrm{id}.$$

Here "id" denotes the $K \times K$ identity matrix.

Denoted by $\lambda(\xi) = \sqrt{\xi_0^2 + \left(1 + \sum_{k=1}^N \xi_k^2\right)^2}$. By classical calculus, we have that $\widehat{\mathscr{L}}(\xi)$ has two eigenvalues: $\pm\lambda(\xi)$. Now, denote $P^\pm$ the projections on E with kernel $E^\mp$. We see that in the Fourier domain, $P^\pm$ are multiplication operators by bounded smooth matrix-valued functions of ξ:

$$(P^+ z)\widehat{\ }(\xi) = \frac{1}{2\lambda(\xi)} \begin{pmatrix} \lambda(\xi) & \overline{A(\xi)} \\ A(\xi) & \lambda(\xi) \end{pmatrix} \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix},$$

$$(P^- z)^\frown(\xi) = \frac{1}{2\lambda(\xi)} \begin{pmatrix} \lambda(\xi) & -\overline{A(\xi)} \\ -A(\xi) & \lambda(\xi) \end{pmatrix} \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix}.$$

Here we have used the notation $z = (u, v)$ and $\widehat{z} = (\widehat{u}, \widehat{v})$.

In order that $P^\pm$ are multipliers for L^q, we need to use the *Marcinkiewicz multiplier theorem* on $\mathbb{R} \times \mathbb{R}^N$ (see [24, Chapter 4, Theorem 6′]). A direct calculation shows that, for each $0 < k \le N + 1$, there holds

$$\bigg| \frac{\partial^k \big(A(\xi)/\lambda(\xi)\big)}{\partial \xi_{i_1} \ldots \partial \xi_{i_k}} \bigg| \le \frac{B}{\prod_{j=1}^{k} |\xi_{i_j}|} \quad \text{for some constant } B > 0.$$

And hence, as an immediate consequence, $P^\pm$ are *multipliers* for L^q for all $q \in (1, \infty)$. This implies that $P^\pm$ are continuous with respect to the L^q-norms. By noting that $P^\pm(E^\mp) = \{0\}$, one easily sees that $P^\pm$ extend to continuous projections on L^q (still denoted by $P^\pm$) with $P^\pm(\mathrm{cl}_q E_q^\mp) = \{0\}$. And this completes the proof. □

The embedding from E into L^q's can be concluded in the following lemma.

Lemma 2.3 (see e.g. [9]). *E is continuously embedded in L^q for $q \in [2, 2(N+2)/N]$ if $N \ge 2$, and compactly embedded in L^q_{loc} for $q \in [1, 2(N+2)/N)$ if $N \ge 2$.*

On E we consider the functionals Φ_ε and Φ_0 defined by

$$\begin{aligned} \Phi_\varepsilon(z) &= \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{1}{2} \iint \mathcal{V}_\varepsilon(x) z \cdot z \, dt \, dx - \Psi_\varepsilon(z), \\ \Phi_0(z) &= \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{1}{2} \iint \mathcal{V}(0) z \cdot z \, dt \, dx - \Psi_0(z), \end{aligned}$$

where the nonlinear part is defined by

$$\begin{aligned} \Psi_\varepsilon(z) &= \frac{1}{p} \iint W_\varepsilon(x) |z|^p \, dt \, dx + \frac{1}{2^*} \iint Q_\varepsilon(x) |z|^{2^*} \, dt \, dx, \\ \Psi_0(z) &= \frac{W(0)}{p} \iint |z|^p \, dt \, dx + \frac{Q(0)}{2^*} \iint |z|^{2^*} \, dt \, dx, \end{aligned}$$

(for simplicity, we set $2^* = 2(N+2)/N$ for the critical exponent and we shall omit the integration set $\mathbb{R} \times \mathbb{R}^N$ in the integrals). By virtue of Lemma 2.3, it is not difficult to check Φ_ε is 2-times Frechét differentiable on E and that its critical points correspond to the solutions of (2.3).

In order to study further the minimal energy level of Φ_ε, let us recall some known facts on a Lyapunov–Schmidt type reduction. Such reduction technique depends on the convexity of the nonlinearities, specifically, it requires that the second order derivative of Φ_ε is negative definite on E^-. And by the anti-coercion and concavity properties of $\Phi_\varepsilon|_{E^-}$, we can define $\ell_\varepsilon \colon E^+ \to E^-$ to be the bounded reduction map correspondingly such that, for any $u \in E^+$,

$$\Phi_\varepsilon\big(z + \ell_\varepsilon(z)\big) = \max_{w \in E^-} \Phi_\varepsilon(z + w).$$

And denote $I_\varepsilon(z) = \Phi_\varepsilon\big(z + \ell_\varepsilon(z)\big)$, we shall call $(\ell_\varepsilon, I_\varepsilon)\colon E^+ \times E^+ \to E^- \times \mathbb{R}$ the reduction couple associated to Φ_ε on E^+ (for details we refer to [1], [10]). Then, it is all clear that $I_\varepsilon \in C^2(E^+, \mathbb{R})$ and critical points of I_ε and Φ_ε are in one-to-one correspondence via the injective map $z \mapsto z + \ell_\varepsilon(z)$ from E^+ to E.

Now, on E^+, let us introduce

$$\mathscr{N}_\varepsilon = \big\{z \in E^+ \setminus \{0\} : I'_\varepsilon(z)[z] = 0\big\}. \tag{2.7}$$

The following lemmas collect the properties Φ_ε and I_ε have when the assumptions of our theorems hold.

Lemma 2.4. *Φ_ε possesses the linking structure:*

(a) *There are $r, \rho > 0$, both independent of ε, such that $\Phi_\varepsilon|_{B_r^+} \geq 0$ and $\Phi_\varepsilon|_{S_r^+} \geq \rho$, where*

$$B_r^+ = B_r \cap E^+ = \{z \in E^+ : \|z\| \leq r\},$$
$$S_r^+ = \partial B_r^+ = \{z \in E^+ : \|z\| = r\}.$$

(b) *For any finite dimensional subspace $\mathscr{M} \subset E^+$, there exist constants $C = C_{\mathscr{M}} > 0$ and $R = R_{\mathscr{M}} > 0$, both independent of ε, such that*

$$\sup \Phi_\varepsilon(\mathscr{M} \oplus E^-) < C \quad \text{and} \quad \sup \Phi_\varepsilon(\mathscr{M} \oplus E^- \setminus B_R) < 0.$$

Lemma 2.5. *Palais–Smale sequence for Φ_ε is bounded independent of the choice for $\varepsilon > 0$.*

Lemma 2.6. *For all $\varepsilon > 0$, $\mathscr{N}_\varepsilon$ is a smooth manifold; and there exist $\theta > 0$ independent of ε such that, for any $z \in \mathscr{N}_\varepsilon$,*

$$\|z\| \geq \theta \quad \text{and} \quad I_\varepsilon(z) \geq \theta.$$

Moreover, critical points of I_ε constrained on $\mathscr{N}_\varepsilon$ are free critical points of I_ε in E^+.

Lemma 2.7. *Let c_ε, c_0 denote the minimax levels of Φ_ε and Φ_0 deduced by the linking structure:*

$$c_\varepsilon = \inf_{e \in E^+ \setminus \{0\}} \sup_{z \in \mathbb{R}e \oplus E^-} \Phi_\varepsilon(z) \quad \text{and} \quad c_0 = \inf_{e \in E^+ \setminus \{0\}} \sup_{z \in \mathbb{R}e \oplus E^-} \Phi_0(z).$$

Then we have

(a) $c_\varepsilon = \inf\limits_{z \in \mathscr{N}_\varepsilon} I_\varepsilon(z)$;

(b) *$c_\varepsilon \leq c_0 + o_\varepsilon(1)$ as $\varepsilon \to 0$ provided that c_0 is attained.*

The above listed results could be referred as geometric properties which are basically derived from the formulation of Φ_ε and Proposition 2.1. A general discussion of the properties of Φ_ε and its reduction couple $(\ell_\varepsilon, I_\varepsilon)$ in an abstract setting can be found in [10, Section 4].

Remark 2.8. It is worth pointing out that $\mathscr{N}_\varepsilon$ is the graph of a C^1 function m_ε defined on S_1^+ by

$$m_\varepsilon(z) = t_\varepsilon(z)z \quad \text{for } z \in S_1^+,$$

$t_\varepsilon(z)$ being the unique positive number which realizes the maximum of the function $t \mapsto I_\varepsilon(tz)$ and that $t_\varepsilon \colon S_1^+ \to \mathbb{R}$ is a C^1 function. It can be also seen that

$$\mathscr{N}_\varepsilon = \big\{t_\varepsilon(z)z : z \in E^+ \setminus \{0\},\ 0 < t_\varepsilon(z) < \infty\big\}.$$

For the sake of a multiplicity result, let us consider a critical point theorem involving strongly indefinite character. And before stating our abstract result, let first remark that Φ_ε (being as a even functional) not only admits the group action of $\mathbb{Z}_2$ (i.e. the multiplication by ± 1) but also the group action of the time-axis translation. This is due to the fact that the RD system (2.1) or (2.3) is invariant by translations in time (the coefficients in the RD systems are time-independent). Hence, Φ_ε is invariant under the action of $\mathbb{R}$, a noncompact Lie group.

In the sequel, we will write it briefly by $\mathscr{G}$-invariant if a functional Φ admits a action of some group $\mathscr{G}$. Recall that a sequence $\{z_n\} \subset E$ is called to be a $(\mathrm{PS})_c$-sequence for a functional $\Phi \in C^1(E, \mathbb{R})$ if $\Phi(z_n) \to c$ and $\Phi'(z_n) \to 0$. We remark that if Φ is $\mathscr{G}$-invariant then $\{g_n z_n\}$ is also a $(\mathrm{PS})_c$-sequence for any $\{g_n\} \subset \mathscr{G}$ provided that $\{z_n\}$ is a $(\mathrm{PS})_c$-sequence. And for any two elements z_1 and z_2, by $\mathscr{G}$-distinct we mean $z_1 \neq g z_2$ for all $g \in \mathscr{G}$.

Definition 2.9. We say that a $\mathscr{G}$-invariant functional $\Phi \in C^1(E, \mathbb{R})$ satisfies the $\mathscr{G}$-$(\mathrm{PS})_c$-condition if every $(\mathrm{PS})_c$-sequence has a subsequence which converges after an accordingly $\mathscr{G}$-action:

$$\left.\begin{aligned} \Phi(z_n) &\to c \\ \Phi'(z_n) &\to 0 \end{aligned}\right\} \Rightarrow g_n z_n \to z \in E \text{ along a subsequence for some } \{g_n\} \subset \mathscr{G}.$$

For the abstract setting, let us consider a splitting $E = X \oplus Y$ of E into complete subspaces X and Y with associated projectors P^X and P^Y. We write $z^X := P^X z$ and $z^Y := P^Y z$ for $z \in E$. In addition to the norm topology we need the topology $\mathcal{T}$ on E which is the product of the norm topology in X and the weak topology in Y. In particular, $z_n \xrightarrow{\mathcal{T}} z$ provided that $z_n^X \to z^X$ and $z_n^Y \rightharpoonup z^Y$. On bounded subsets of E the topology $\mathcal{T}$ coincides with the metrizable topology considered by Bartsch and Ding [4] and for Hilbert spaces by Kryszewski and Szulkin [13]. And denoted by $\mathcal{T}_{w^*}$ the weak*-topology of E^*.

For a functional Φ and real numbers a, b we write $\Phi^b := \{z \in E : \Phi(z) \le b\}$, $\Phi_a := \{z \in E : \Phi(z) \ge a\}$ and $\Phi_a^b := \Phi^b \cap \Phi_a$. The following assumptions will be needed.

(Φ1) $\Phi \in C^1(E, \mathbb{R})$, $\Phi\colon (E, \mathcal{T}) \to \mathbb{R}$ *is upper semi-continuous, and* $\Phi'\colon (\Phi_a, \mathcal{T}) \to (E^*, \mathcal{T}_{w^*})$ *is continuous for every* $a \in \mathbb{R}$.

(Φ2) *There exists* $r > 0$ *with* $\rho := \inf \Phi(S_r^X) > \Phi(0) = 0$ *where* $S_r^X := \{z \in X : \|z\| = r\}$.

(Φ3) *There exists a finite dimensional subspace* $X_0 \subset X$ *and* $R > r$ *such that for* $E_0 := X_0 \oplus Y$ *and* $B_0 := \{z \in E_0 : \|z\| \le R\}$ *there holds* $d := \sup \Phi(E_0) < \infty$ *and* $\sup \Phi(E_0 \setminus B_0) < \inf \Phi(B_r^X)$*, where* $B_r^X := \{z \in X : \|z\| \le r\}$.

In the case in which Φ is even, i.e. Φ admits a $\mathbb{Z}_2$-action, let us introduce the notations as was introduced in [4, 9]: for $c \in \mathbb{R}$, denoted by $\mathcal{M}(\Phi^c)$ the collection of maps $h\colon \Phi^c \to E$ satisfying

(i) h *is* $\mathcal{T}$*-continuous and odd*;

(ii) $h(\Phi^a) \subset \Phi^a$ *for all* $a \in [\rho, d]$;

(iii) *each* $z \in \Phi^c$ *has a* $\mathcal{T}$*-open neighbourhood* $O \subset E$ *such that* $(\mathrm{id} - h)(O \cap \Phi^c)$ *is contained in a finite dimensional linear subspace.*

Then a pseudo-index of Φ^c can be defined by

$$\psi(c) := \min \big\{ \mathrm{gen}(h(\Phi^c) \cap S_r^X) : h \in \mathcal{M}(\Phi^c) \big\} \in \mathbb{N} \cup \{\infty\} \tag{2.8}$$

where $\mathrm{gen}(\,\cdot\,)$ denotes the classical $\mathbb{Z}_2$-genus.

Let us emphasize here the situation that Φ may admits a group action of $\mathscr{G}$ in addition to the $\mathbb{Z}_2$-action is a considerable issue in our abstract settings. If $\mathscr{G}$ is a compact group, then a generalized index theory can be employed as was introduced by Rabinowitz [22] (see also [2] for related material). However, in general, the group $\mathscr{G}$ may be noncompact and quotient space $E/\mathscr{G}$ will no longer be a linear space or a manifold. Therefore classical results can not be applied. One way to treat such $\mathscr{G} \times \mathbb{Z}_2$-invariant problem is to separate these two group actions. For such, in the sequel, let us introduce the following concept.

Definition 2.10. Let E be a Hilbert space (or a Banach space) associated with a group action of $\mathscr{G} \times \mathbb{Z}_2$. We say that $\mathscr{G}$ is separated with respect to $\mathbb{Z}_2$ if and only if, for all closed $\mathbb{Z}_2$-invariant subset $A \subset E$, the $\mathscr{G}$-orbit $\mathscr{G}(A) := \{gz : g \in \mathscr{G},\ z \in A\}$ is closed and $\mathrm{gen}(A) = \mathrm{gen}(\mathscr{G}(A))$.

Let us remark that, for a separated $\mathscr{G}$-action, there holds $\mathscr{G}(z_1) \cap \mathscr{G}(z_2) = \emptyset$ for any $\mathscr{G}$-distinct elements z_1 and z_2. Hence, for a closed $\mathbb{Z}_2$-invariant subset A, $\mathscr{G}(A)$ is the disjoint union of all the $\mathscr{G}$-orbits of its elements.

Then our general existence result comes as follows:

Theorem 2.11. *Let Φ be a even functional satisfying* (Φ1)–(Φ3). *If Φ also admits another separated group action for some noncompact group $\mathscr{G}$ and satisfies the $\mathscr{G}$-$(\mathrm{PS})_c$-condition for every $c \in [\rho, d]$. Then Φ has at least $n := \dim X_0$ pairs of $\mathscr{G}$-distinct critical points. In particular, if Φ has only finitely many $\mathscr{G}$-distinct critical points in Φ_ρ^d, then*

$$c_i = \inf\{c : \psi(c) \geq i\} \in [\rho, d], \quad i = 1, \ldots, n \tag{2.9}$$

are critical values satisfying $\rho \leq c_1 < \ldots < c_n \leq d$.

Remark 2.12. Theorem 2.11 generalizes Theorem 4.6 of [4] (see also [9, p. 31]) in the sense that Φ is allowed to carry a $\mathscr{G}$-invariant property for some noncompact group action. This generalization permits us to treat such as RD systems of the form (2.1). Setting $X = E^+$ and $Y = E^-$, it follows from the definition, Proposition 2.1 and Lemma 2.4 that the functional Φ_ε is even and satisfies (Φ1)–(Φ3). And it is also evident that the group action of time-axis translation is separated with respect to $\mathbb{Z}_2$-action. We postpone the proof of Theorem 2.11 to Appendix B.

Remark 2.13. Theorem 2.11 holds true for more general classes of symmetries, typical example is Φ admits a group action of $\mathscr{G} \times G$ where G is some compact group and $\mathscr{G}$ is separated. Typical examples of G include the case of a finite group such as $\mathbb{Z}_p$ for a prime number p and a compact Lie group such as S^1, $SU(2)$, et al. In all these cases, there exists an index i: $\{A \subset E : A \text{ is } G\text{-invariant}\} \to \mathbb{N} \cup \{\infty\}$ for G-action satisfying the monotonicity, subadditivity, super-variance as well as a dimension property: $i(F \setminus \{0\}) = c \cdot \dim F$ for any finite-dimensional G-invariant linear subspace $F \subset E$. We refer the readers to [2], [5], [7] for a discussion of group actions, index theories, examples and applications.

3. Variational framework for superlinear systems

In what follows, we consider weak solutions to the equation

$$\mathscr{L} z + \omega z = \lambda |z|^{p-2} z + \kappa |z|^{4/N} z \qquad \text{on } \mathbb{R} \times \mathbb{R}^N \tag{3.1}$$

belonging to the class $\mathscr{B}^2$ with ω being a $2K \times 2K$ symmetric constant matrix with its eigenvalues $\sigma(\omega) \subset (-1, 1)$, $\lambda > 0$ and $\kappa \geq 0$. As before, let us introduce the associated functional

$$\Phi(z) = \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{1}{2}\iint \omega z \cdot z \, dt \, dx - \frac{\lambda}{p}|z|_p^p - \frac{\kappa}{2^*}|z|_{2^*}^{2^*}$$

on $E = E^+ \oplus E^-$. Denote by (ℓ, I) the reduction couple for Φ and let $\mathscr{N} = \{u \in E^+ \setminus \{0\} : I'(u)[u] = 0\}$. Then, by the same arguments used in [1], [10], we get that $\mathscr{N}$ is a smooth manifold of codimension 1 in E^+, and $\mathscr{N}$ is diffeomorphic to S_1^+ by a C^1 diffeomorphism. Particularly, the function $t \mapsto I(tu)$ attains its unique critical point $t = t(u) > 0$ for each $u \in E^+ \setminus \{0\}$, and $t\colon S_1^+ \to \mathbb{R}$ is a C^1 function. If denoted by

$$\gamma(\omega, \lambda, \kappa) \equiv \inf_{e \in E^+ \setminus \{0\}} \sup_{u \in \mathbb{R}e \oplus E^-} \Phi(u),$$

it can be also seen that $\gamma(\omega, \lambda, \kappa) = \inf_{\mathscr{N}} I > 0$. We will write $\gamma(\omega, \lambda)$ for simplicity in the case when $\kappa \equiv 0$.

The main result of this section is the following:

PROPOSITION 3.1. *Let $\nu(\omega) = \min\{\nu : \nu \in \sigma(\omega)\}$ and set $\nu_* = \min\{\nu(\omega), 0\}$. Then $\gamma(\omega, \lambda, \kappa)$ is attained provided that*

$$(1 + \nu_*)^{-(N+2)/2} \cdot \kappa^{N/2} \cdot \gamma(\omega, \lambda, \kappa) < \frac{S^{(N+2)/2}}{N+2} \tag{3.2}$$

where S denotes the best constant for the embedding $E \hookrightarrow L^{2^*}$.

This proposition yields information about the competing effect of ω and λ against κ. In the case κ vanishes, (3.2) is satisfied automatically, and hence $\gamma(\omega, \lambda)$ is always attained. Before proving Proposition 3.1, we begin with some preliminary materials. Let us first consider the following functional

$$\mathcal{F}_\omega : E \setminus \{0\} \to \mathbb{R}, \quad z \mapsto \frac{1}{|z|_{2^*}^2} \biggl(\|z^+\|^2 - \|z^-\|^2 + \iint \omega z \cdot z \, dt \, dx \biggr),$$

and the minimax scheme

$$T_\omega = \inf_{z \in E^+ \setminus \{0\}} \sup_{w \in E^-} \mathcal{F}_\omega(z + w).$$

Denoted by $\mathscr{F}\colon L^2 \to L^2$ the Fourier transform, then in Fourier domain $\xi = (\xi_0, \ldots, \xi_N)$ we have

$$\|z\|^2 = \iint \biggl(\xi_0^2 + \biggl(1 + \sum_{k=1}^N \xi_k^2 \biggr)^2 \biggr)^{1/2} |\mathscr{F} z(\xi)|^2 \, d\xi.$$

Remark that S denotes the best constant for the embedding $E \hookrightarrow L^{2^*}$, i.e. $S|z|_{2^*}^2 \le \|z\|_2^2$, and that $\nu(\omega)$ is the smallest eigenvalue of ω, we soon infer that, for any $z \in E^+ \setminus \{0\}$,

$$\begin{aligned} \sup_{w \in E^-} \mathcal{F}_\omega(z + w) \ge \mathcal{F}_\omega(z) &= \frac{1}{|z|_{2^*}^2} \biggl(\|z\|^2 + \iint \omega z \cdot z \, dt \, dx \biggr) \\ &\ge \frac{1}{|z|_{2^*}^2} \iint \biggl[\biggl(\xi_0^2 + \biggl(1 + \sum_{k=1}^N \xi_k^2 \biggr)^2 \biggr)^{1/2} + \nu(\omega) \biggr] \bigl| \mathscr{F} z(\xi) \bigr|^2 \, d\xi. \end{aligned}$$

Taking into account that

$$\inf_{|\xi|>0} \frac{\left(\xi_0^2 + \left(1+\sum_{k=1}^N \xi_k^2\right)^2\right)^{1/2} + \nu(\omega)}{\left(\xi_0^2 + \left(1+\sum_{k=1}^N \xi_k^2\right)^2\right)^{1/2}} = \begin{cases} 1 & \text{if } \nu(\omega)\ge 0, \\ 1+\nu(\omega) & \text{if } \nu(\omega)<0, \end{cases}$$

we have

$$T_\omega \ge (1+\nu_*)S \quad \text{with } \nu_* = \min\{\nu(\omega),0\}. \tag{3.3}$$

Next, let us consider the equation

$$\mathscr{L} z + \omega z = |z|^{4/N} z \quad \text{on } \mathbb{R}\times\mathbb{R}^N \tag{3.4}$$

and the corresponding functional

$$\widehat{\Phi}(z) = \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{1}{2}\iint \omega z\cdot z\,dt\,dx - \frac{1}{2^*}|z|_{2^*}^{2^*}$$

on $E = E^+\oplus E^-$. Denote by $(\widehat{\ell},\widehat{I})$ the reduction couple for $\widehat{\Phi}$ and set $\widehat{\mathscr{N}} = \{u\in E^+\setminus\{0\} : \widehat{I}'(u)[u]=0\}$. We have $\widehat{\mathscr{N}}$ is a smooth manifold of codimension 1 in E^+, and $\widehat{\mathscr{N}}$ is diffeomorphic to S_1^+ by a C^1 diffeomorphism. Particularly, the function $t\mapsto \widehat{I}(tu)$ attains its unique critical point $\widehat{t} = \widehat{t}(u)>0$ for each $u\in E^+\setminus\{0\}$. It is also standard to see that $\widehat{\gamma}_\omega := \inf_{\widehat{\mathscr{N}}} \widehat{I} > 0$, and in particular, we have

Lemma 3.2. $T_\omega = \big((N+2)\widehat{\gamma}_\omega\big)^{2/(N+2)}$.

Proof. We sketch the proof as follows: Let $z\in E^+\setminus\{0\}$ be fixed, and set $\pi_z(\,\cdot\,) = \mathcal{F}_\omega(z+\,\cdot\,)$, then elementary calculation shows that any critical point $w\in E^-$ for π_z satisfies $\pi_z''(w)[\eta,\eta]<0$ for all $\eta\in E^-$. Hence, π_z has a unique critical point in E^- which realize its maximum (if there exists).

For any $z\in\widehat{\mathscr{N}}$, we have

$$\|z\|^2 - \big\|\widehat{\ell}(z)\big\|^2 + \iint \omega\cdot\big(z+\widehat{\ell}(z)\big)\cdot\big(z+\widehat{\ell}(z)\big)\,dt\,dx - \big|z+\widehat{\ell}(z)\big|_{2^*}^{2^*} = 0,$$

and hence $\pi_z\big(\widehat{\ell}(z)\big) = \big|z+\widehat{\ell}(z)\big|_{2^*}^{2^*-2}$. Moreover, it is standard to check that $\pi_z'\big(\widehat{\ell}(z)\big)[w]=0$ for all $w\in E^-$. Thus, we have $\big|z+\widehat{\ell}(z)\big|_{2^*}^{2^*-2} = \max_{w\in E^-}\pi_z(w)$, i.e. $\widehat{\ell}(z)$ is the unique critical point for π_z.

Now, using the fact $\mathcal{F}_\omega(z) = \mathcal{F}_\omega(tz)$ for all $z\in E$ and $t>0$, we can conclude

$$\begin{aligned} T_\omega &= \inf_{z\in S_1^+}\sup_{w\in E^-}\mathcal{F}_\omega(z+w) = \inf_{z\in\widehat{\mathscr{N}}}\sup_{w\in E^-}\mathcal{F}_\omega(z+w) \\ &= \inf_{z\in\widehat{\mathscr{N}}}\big|z+\widehat{\ell}(z)\big|_{2^*}^{2^*-2} = \inf_{z\in\widehat{\mathscr{N}}}\big((N+2)\widehat{I}(z)\big)^{2/(N+2)} = \big((N+2)\widehat{\gamma}_\omega\big)^{2/(N+2)} \end{aligned}$$

as is desired. For further reference, we mention that the uniqueness of $\widehat{\ell}(z)$ as critical point for π_z also implies that $\widehat{\ell}(tz) = t\widehat{\ell}(z)$ for all $z\in E^+\setminus\{0\}$. □

Now, we give the proof of the proposition.

Proof of Proposition 3.1. We only give the proof when $\kappa > 0$ since it is much easier for the case $\kappa = 0$. Let $\{z_n\} \subset \mathscr{N}$ be a minimizing sequence for I. It is no difficult to check that $\{w_n = z_n + \ell(z_n)\}$ is bounded in E. Then by Lion's result (see [16]) it follows that $\{w_n\}$ is either vanishing or non-vanishing.

If $\{w_n\}$ is non-vanishing then we are done, so let us assume contrarily that $\{w_n\}$ is vanishing. Then $|w_n|_s \to 0$ for all $s \in (2, 2^*)$. And thus we have

$$\kappa^{N/2}\Phi(w_n) = \widehat{\Phi}\big(\kappa^{N/4}w_n\big) + o_n(1) \le \widehat{\Phi}(\widehat{w}_n) + o_n(1) \le \kappa^{N/2}\Phi(w_n) + o_n(1),$$

where we used the notation $\widehat{w}_n := \widehat{t}_n u_n + \widehat{\ell}\big(\widehat{t}_n u_n\big)$ with $\widehat{t}_n = \widehat{t}(u_n)$ being bounded and such that $\widehat{t}_n u_n \in \widehat{\mathscr{N}}$ (the last inequality comes from the fact that $\widehat{\ell}(tz) = t\widehat{\ell}(z)$ guarantees $\{\widehat{w}_n\}$ is vanishing).

By the above observation, and $\Phi(w_n) = I(z_n) = \gamma(\omega, \lambda, \kappa) + o_n(1)$, we easily deduce from Lemma 3.2 and (3.3) that

$$\begin{aligned}\kappa^{N/2} \cdot \gamma(\omega, \lambda, \kappa) + o_n(1) &= \widehat{\Phi}(\widehat{w}_n) \\ &\ge \widehat{\gamma}_\omega = \frac{1}{N+2} T_\omega^{(N+2)/2} \ge (1+\nu_*)^{(N+2)/2} \frac{S^{(N+2)/2}}{N+2},\end{aligned}$$

which contradict to (3.2). Therefore we have $\{w_n\}$ is non-vanishing, and this ends the proof. □

Recalling the notations introduced in the previous section, let us go back to our model problems by denoting ω_ν the constant matrix $\nu \cdot \mathrm{id}$ or $\nu \mathcal{J}_0$, and consider

$$\mathscr{L}z + \omega_\nu z = \lambda |z|^{p-2} z + \kappa |z|^{4/N} z \quad \text{on } \mathbb{R} \times \mathbb{R}^N \tag{3.5}$$

with the associated functional

$$\Phi_{\nu\lambda\kappa}(z) = \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{1}{2} \iint \omega_\nu z \cdot z \, dt \, dx - \frac{\lambda}{p}|z|_p^p - \frac{\kappa}{2^*}|z|_{2^*}^{2^*}$$

and minimal energy $\gamma(\omega_\nu, \lambda, \kappa)$. We then end this section by concluding

Lemma 3.3. *Let $\nu_1, \nu_2 \in (-1, 1)$, $\lambda_1, \lambda_2 > 0$ and $\kappa_1, \kappa_2 \ge 0$. If* $\min\{\nu_2 - \nu_1, \lambda_1 - \lambda_2, \kappa_1 - \kappa_2\} \ge 0$ *and either $(\omega_{\nu_1}, \lambda_1, \kappa_1)$ or $(\omega_{\nu_2}, \lambda_2, \kappa_2)$ satisfies* (3.2), *then $\gamma(\omega_{\nu_1}, \lambda_1, \kappa_1) \le \gamma(\omega_{\nu_2}, \lambda_2, \kappa_2)$. And, if in additional* $\max\{\nu_2 - \nu_1, \lambda_1 - \lambda_2, \kappa_1 - \kappa_2\} > 0$, *then $\gamma(\omega_{\nu_1}, \lambda_1, \kappa_1) < \gamma(\omega_{\nu_2}, \lambda_2, \kappa_2)$.*

Proof. We only prove the second statement.

Case 1. $\omega_\nu = \nu \cdot \mathrm{id}$. As mentioned before, let us introduce the reduction couple (ℓ_j, I_j) for $\Phi_{\nu_j\lambda_j\kappa_j}$, $j = 1, 2$, and $m_j \colon S^+ \to \mathscr{N}_j$, $j = 1, 2$, to be the C^1 diffeomorphism between $S^+ := \{z \in E^+ : \|z\| = 1\}$ and $\mathscr{N}_j := \{z \in E^+ \setminus \{0\} : I_j'(z)[z] = 0\}$.

Let $z \in S^+$ be arbitrary; then we have

$$
\begin{aligned}
\gamma(\omega_{\nu_1},\lambda_1,\kappa_1) &\le I_1\big(m_1(z)\big) \\
&\le I_2\big(m_1(z)\big) + \frac{\nu_1-\nu_2}{2}\big|m_1(z)+\ell_1(m_1(z))\big|_2^2 \\
&\quad - \frac{\lambda_1-\lambda_2}{p}\big|m_1(z)+\ell_1(m_1(z))\big|_p^p - \frac{\kappa_1-\kappa_2}{2^*}\big|m_1(z)+\ell_1(m_1(z))\big|_{2^*}^{2^*} \\
&\le I_2\big(m_2(z)\big) + \frac{\nu_1-\nu_2}{2}\big|m_1(z)+\ell_1(m_1(z))\big|_2^2 \\
&\quad - \frac{\lambda_1-\lambda_2}{p}\big|m_1(z)+\ell_1(m_1(z))\big|_p^p - \frac{\kappa_1-\kappa_2}{2^*}\big|m_1(z)+\ell_1(m_1(z))\big|_{2^*}^{2^*}.
\end{aligned} \tag{3.6}
$$

If $(\omega_{\nu_2},\lambda_2,\kappa_2)$ satisfies (3.2), then it is clear that $\gamma(\omega_{\nu_2},\lambda_2,\kappa_2)$ is achieved. Now, we can fix $z \in S^+$ be such that $I_2\big(m_2(z)\big) = \gamma(\omega_{\nu_2},\lambda_2,\kappa_2)$, by (3.6) we have

$$
\begin{aligned}
\gamma(\omega_{\nu_1},\lambda_1,\kappa_1) &\le \gamma(\omega_{\nu_2},\lambda_2,\kappa_2) + \frac{\nu_1-\nu_2}{2}\big|m_1(z)+\ell_1(m_1(z))\big|_2^2 \\
&\quad - \frac{\lambda_1-\lambda_2}{p}\big|m_1(z)+\ell_1(m_1(z))\big|_p^p - \frac{\kappa_1-\kappa_2}{2^*}\big|m_1(z)+\ell_1(m_1(z))\big|_{2^*}^{2^*}.
\end{aligned}
$$

Therefore, by $\max\{\nu_2-\nu_1,\lambda_1-\lambda_2,\kappa_1-\kappa_2\} > 0$, we have $\gamma(\omega_{\nu_1},\lambda_1,\kappa_1) < \gamma(\omega_{\nu_2},\lambda_2,\kappa_2)$.

Otherwise if $(\omega_{\nu_1},\lambda_1,\kappa_1)$ satisfies (3.2), then let us consider a sequence $\{z_n\} \subset S^+$ such that

$$I_2\big(m_2(z_n)\big) \to \gamma(\omega_{\nu_2},\lambda_2,\kappa_2).$$

Then the sequence $\{m_2(z_n)\}$ is bounded. If $\{m_2(z_n)\}$ is vanishing, as was argued in Proposition 3.1, we have

$$\gamma(\omega_{\nu_1},\lambda_1,\kappa_1) < \frac{((1+\nu_{1*})S)^{(N+2)/2}}{\kappa_1^{N/2}(N+2)} \le \frac{((1+\nu_{2*})S)^{(N+2)/2}}{\kappa_2^{N/2}(N+2)} \le \gamma(\omega_{\nu_2},\lambda_2,\kappa_2),$$

where $\nu_{j*} = \min\{\nu_j,\, 0\}$ for $j = 1,2$. And hence the proof is completed.

Case 2. $\omega_\nu = \nu\mathcal{J}_0$. In this situation, let us consider the scaling transform $t \mapsto (1+\nu)t$, $x \mapsto \sqrt{1+\nu}\,x$. Then system (3.5) is equivalent to

$$\mathscr{L}z = \frac{\lambda}{1+\nu}|z|^{p-2}z + \frac{\kappa}{1+\nu}|z|^{4/N}z$$

with the minimal energy denoted by $\overline{\gamma}(\omega_\nu,\lambda,\kappa)$. Remark that, by the scaling transformation, we always have

$$\gamma(\omega_\nu,\lambda,\kappa) = (1+\nu)^{(N+4)/2}\overline{\gamma}(\omega_\nu,\lambda,\kappa). \tag{3.7}$$

At this stage, we can go back to Case 1, and the monotonicity of $\overline{\gamma}$ will directly imply our desired conclusion for $\gamma(\omega_\nu,\lambda,\kappa)$ which completes the proof. □

REMARK 3.4. Lemma 3.3 provides a criterion for checking (3.2). Indeed, since $\kappa \geq 0$, we always have $\gamma(\omega_\nu, \lambda, \kappa) \leq \gamma(\omega_\nu, \lambda)$. Hence, we can set an upper bound for κ as

$$\widehat{\kappa}(\nu, \lambda) = \frac{((1 - |V|_\infty)S)^{(N+2)/N}}{((N+2)\gamma(\omega_\nu, \lambda))^{2/N}}$$

such that $(\omega_\nu, \lambda, \kappa)$ satisfies (3.2) provided that $0 \leq \kappa < \widehat{\kappa}(\nu, \lambda)$. Furthermore, we can assert form the relations (3.6) and (3.7) that $\widehat{\kappa}(\nu, \lambda)$ is decreasing in ν and increasing in λ. And hence for fixed $\nu \in (-1, 1)$ and $\lambda > 0$, let $\kappa \in [0, \widehat{\kappa}(\nu, \lambda))$, then the triple $(\omega_{\nu'}, \lambda', \kappa)$ will satisfy (3.2) provided that $\nu' \leq \nu$ and $\lambda' \geq \lambda$.

4. Palais–Smale condition

In order to apply the abstract multiplicity result stated in Theorem 2.11, we need Φ_ε to satisfy a compactness condition. However, due to the non-compactness of the Sobolev embedding $E \hookrightarrow L^{2^*}_{\mathrm{loc}}$, it is not difficult to see that such a condition is not fulfilled in general. Nevertheless, recalling Φ_ε admits a group action of the time-axis translation and denoting such action by $\mathscr{G} := \mathbb{R}$, we can recover the compactness in the sense of $\mathscr{G}$-$(\mathrm{PS})_c$-condition holds below some energy threshold, which related to the minimal energy "at infinity". For ease of notation, let us set $\overline{\kappa} = \max_{x \in \mathbb{R}^N} Q(x)$. Then, inspired by the priori bound $\widehat{\kappa}(\nu, \lambda)$ for the factor κ in Remark 3.4, our compactness result can be stated as follows.

PROPOSITION 4.1. *Assume* (A1) *and* (A2). *Suppose that* $\overline{\kappa} < \widehat{\kappa}(\nu_\infty, \lambda_\infty)$. *Then, for any* $\varepsilon > 0$, *the functional* Φ_ε *satisfies* $\mathscr{G}$-$(\mathrm{PS})_c$-*condition in the sublevel* $\{z \in E : \Phi_\varepsilon(z) < \overline{\gamma}_\infty\}$, *where* $\overline{\gamma}_\infty := \gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa})$.

Let us mention here that for the case Q vanishes identically we can infer that $(\omega_{\nu_\infty}, \lambda_\infty)$ satisfies (3.2) automatically, and hence Φ_ε will satisfy the $\mathscr{G}$-$(\mathrm{PS})_c$-condition with no more additional assumptions except $c < \gamma_\infty(\omega_{\nu_\infty}, \lambda_\infty)$. Before proving Proposition 4.1, let us remark that the following lemma will be the key ingredient in the proofs.

PROOF OF PROPOSITION 4.1. To begin with, let $c < \overline{\gamma}_\infty$ and let $\{z_n\}$ be a Palais–Smale sequence for Φ_ε at level c, namely

$$\Phi_\varepsilon(z_n) = c + o_n(1), \qquad \Phi'_\varepsilon(z_n) = o_n(1).$$

By Lemma 2.5, $\{z_n\}$ is bounded (independent of ε). Remark that, for any $\{g_n\} \subset \mathscr{G}$, the time-axis translated sequence $\{g_n z_n\}$ is also a $(\mathrm{PS})_c$-sequence. Hence, up to a subsequence, it has a associated weak limit $z \in E$ and we have to prove that $g_n z_n \to z$ in E for some $\{g_n\} \subset \mathscr{G}$.

Without loss of generality, for arbitrary $\{g_n\} \subset \mathscr{G}$, let us denote $\overline{z}_n := g_n z_n$ and z to be its weak limit (here and subsequently, one should always keep in mind that such weak limit will change correspondingly when $\{g_n\}$ is replaced by other elements). For simplicity, we set $\zeta_n = \overline{z}_n - z$, then $\zeta_n \rightharpoonup 0$ in E and $\|\overline{z}_n^{\pm}\|^2 = \|z^{\pm}\|^2 + \|\zeta_n^{\pm}\|^2 + o_n(1)$. By using the Brézis–Lieb type result (see for example [26]), we have

$$\iint W_\varepsilon(x)|\overline{z}_n|^p\,dt\,dx = \iint W_\varepsilon(x)|z|^p\,dtdx + \iint W_\varepsilon(x)|\zeta_n|^p\,dt\,dx,$$

and

$$\iint Q_\varepsilon(x)|\overline{z}_n|^{2^*}\,dtdx = \iint Q_\varepsilon(x)|z|^{2^*}\,dtdx + \iint Q_\varepsilon(x)|\zeta_n|^{2^*}\,dtdx.$$

Thus,

$$\Phi_\varepsilon(\overline{z}_n) = \Phi_\varepsilon(z) + \Phi_\varepsilon(\zeta_n) + o_n(1), \tag{4.1}$$

and

$$\Phi'_\varepsilon(\overline{z}_n)[\overline{z}_n] = \Phi'_\varepsilon(z)[z] + \Phi'_\varepsilon(\zeta_n)[\zeta_n] + o_n(1).$$

Obviously, $\Phi'_\varepsilon(z) = 0$. Therefore, $\Phi'_\varepsilon(\zeta_n)[\zeta_n] = o_n(1)$.

Claim 4.2. *$\Phi'_\varepsilon(\zeta_n) \to 0$ as $n \to \infty$.*

In fact, let $\varphi \in E$ with $\|\varphi\| \le 1$ be arbitrary. We have

$$\begin{aligned} \Phi'_\varepsilon(\overline{z}_n)[\varphi] &= \langle \overline{z}_n^+, \varphi^+\rangle - \langle \overline{z}_n^-, \varphi^-\rangle + \iint \mathcal{V}_\varepsilon(x)\overline{z}_n \cdot \varphi\,dt\,dx - \Psi'_\varepsilon(\overline{z}_n)[\varphi] \\ &= \langle \zeta_n^+ + z^+, \varphi^+\rangle - \langle \zeta_n^- + z^-, \varphi^-\rangle \\ &\quad + \iint \mathcal{V}_\varepsilon(x)(\zeta_n + z)\cdot\varphi\,dt\,dx - \Psi'_\varepsilon(\zeta_n + z)[\varphi] \\ &= \langle \zeta_n^+, \varphi^+\rangle - \langle \zeta_n^-, \varphi^-\rangle + \langle z^+, \varphi^+\rangle - \langle z^-, \varphi^-\rangle \\ &\quad + \iint \mathcal{V}_\varepsilon(x)\zeta_n\cdot\varphi\,dt\,dx + \iint \mathcal{V}_\varepsilon(x)z\cdot\varphi\,dt\,dx \\ &\quad - \iint W_\varepsilon(x)|\zeta_n|^{p-2}\zeta_n\cdot\varphi\,dt\,dx - \iint W_\varepsilon(x)|z|^{p-2}z\cdot\varphi\,dt\,dx \\ &\quad - \iint Q_\varepsilon(x)|\zeta_n + z|^{\frac{4}{N}}(\zeta_n + z)\cdot\varphi\,dt\,dx + o_n(\|\varphi\|). \end{aligned} \tag{4.2}$$

Here the estimate for the subcritical part

$$\iint W_\varepsilon(x)|\overline{z}_n|^{p-2}\overline{z}_n\cdot\varphi - \iint W_\varepsilon(x)|\zeta_n|^{p-2}\zeta_n\cdot\varphi - \iint W_\varepsilon(x)|z|^{p-2}z\cdot\varphi = o_n(\|\varphi\|)$$

follows from a standard argument. To estimate the last integral in (4.2), we set

$$\psi_n := |\zeta_n + z|^{4/N}(\zeta_n + z) - |\zeta_n|^{4/N}\zeta_n - |z|^{4/N}z.$$

It is not difficult to see that exists a constant $C > 0$ independent of n such that

$$|\psi_n| \le C|\zeta_n|^{4/N}\cdot|z| + C|\zeta_n|\cdot|z|^{4/N}.$$

Then, using the Egorov theorem on any bounded domains, we have

$$\iint |\psi_n| \cdot |\varphi| \, dt \, dx = o_n(\|\varphi\|) \quad \text{as } n \to \infty.$$

And hence,

$$\Phi_\varepsilon'(\overline{z}_n)[\varphi] = \Phi_\varepsilon'(\zeta_n)[\varphi] + \Phi_\varepsilon'(z)[\varphi] + o_n(\|\varphi\|) = \Phi_\varepsilon'(\zeta_n)[\varphi] + o_n(\|\varphi\|)$$

where we have used the fact $\Phi_\varepsilon'(z) = 0$. The above estimation shows $\Phi_\varepsilon'(\zeta_n) \to 0$ as $n \to \infty$ as was claimed.

Now to show that $\|\zeta_n\| \to 0$ as $n \to \infty$, the concentration compactness argument will be used.

Claim 4.3. *There exists a sequence $\{g_n\} \subset \mathscr{G}$ such that $\{\zeta_n\}$ is vanishing.*

Assuming Claim 4.3 for the moment, we have $\zeta_n \to 0$ in L^q for $q \in (2, 2^*)$. And it follows from Claim 4.2 that

$$\|\zeta_n\|^2 + \iint \mathcal{V}_\varepsilon(x)\zeta_n \cdot (\zeta_n^+ - \zeta_n^-) \, dt \, dx = \iint Q_\varepsilon(x)|\zeta_n|^{4/N}\zeta_n \cdot (\zeta_n^+ - \zeta_n^-) \, dt \, dx + o_n(1).$$

By noting that $|V|_\infty < 1$, we immediately obtain that

$$(1 - |V|_\infty)\|\zeta_n\|^2 \le \left(\iint Q_\varepsilon(x)|\zeta_n|^{2^*} \, dt \, dx \right)^{(N+4)/(2(N+2))} \cdot \left(\iint Q_\varepsilon(x)|\zeta_n^+ - \zeta_n^-|^{2^*} \, dt \, dx \right)^{1/2^*} + o_n(1).$$

It should be pointed out that, since $Q(x) \le \overline{\kappa} := \max Q$, there holds

$$\left(\iint Q_\varepsilon(x)|z|^{2^*} dt \, dx \right)^{2/2^*} \le \overline{\kappa}^{2/2^*} \cdot S^{-1} \cdot \|z\|^2, \quad \text{for all } z \in E$$

where S is the best constant for the embedding $E \hookrightarrow L^{2^*}$. Then, we deduce

$$(1 - |V|_\infty) \cdot S \cdot \overline{\kappa}^{-2/2^*} \le \left(\iint Q_\varepsilon(x)|\zeta_n|^{2^*} dt \, dx \right)^{4/(2(N+2))} + o_n(1) \tag{4.3}$$

provided that $\zeta_n \not\to 0$ in L^{2^*}. On the other hand, by (4.1) and Claim 4.2, we have

$$\begin{aligned} \gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa}) = \overline{\gamma}_\infty &> \Phi_\varepsilon(\zeta_n) - \frac{1}{2}\, \Phi_\varepsilon'(\zeta_n)[\zeta_n] \\ &= \frac{1}{N+2} \iint Q_\varepsilon(x)|\zeta_n|^{2^*} dt \, dx + o_n(1) \end{aligned}$$

as $n \to \infty$. Thus, by (4.3), we can get

$$\gamma(\omega_{\nu_\infty}, \lambda_\infty) \ge \gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa}) > \frac{\big((1 - |V|_\infty)S\big)^{(N+2)/22}}{N+2} \cdot \overline{\kappa}^{-N/2}$$

which contradicts to the fact $\overline{\kappa} < \widehat{\kappa}(\nu_\infty, \lambda_\infty)$. Therefore, we have $\zeta_n \to 0$ in L^{2^*} and the compactness for $\{\overline{z}_n\}$ follows.

Now it remains to prove Claim 4.3. Suppose, contrary to our claim, that the $\mathscr{G}$-dependent sequence $\{\zeta_n\}$ is non-vanishing for any $\{g_n\} \subset \mathscr{G}$. Then there exists $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^N$ and $R, \delta > 0$ such that

$$\int_{t_n-R}^{t_n+R} \int_{B_R(x_n)} |\zeta_n|^2 \, dt \, dx \geq \delta.$$

Without loss of generality we can assuming $\{t_n\}$ is bounded, and hence we have $|x_n| \to \infty$ as $n \to \infty$ since $\zeta_n \to 0$ in L^2_{loc}. At this point, let us assume $V_\varepsilon(x_n) \to \widetilde{\nu}$, $W_\varepsilon(x_n) \to \widetilde{\lambda}$ and $Q_\varepsilon(x_n) \to \widetilde{\kappa}$ as $n \to \infty$. Consider the translated functions $\widetilde{z}_n(t,x) = \zeta_n(t, x + x_n)$, it follows that $\widetilde{z}_n \rightharpoonup z_0$ in E where z_0 is a non-trivial solution to

$$\mathscr{L} z + \omega_{\widetilde{\nu}} z = \widetilde{\lambda} |z|^{p-2} z + \widetilde{\kappa} |z|^{4/N} z.$$

And then, we can argue similarly as in (4.1) to obtain

$$\gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa}) > c \geq \Phi_\varepsilon(\zeta_n) \geq \Phi_{\widetilde{\nu}\widetilde{\lambda}\widetilde{\kappa}}(\widetilde{z}_n) + o_n(1) \geq \gamma\big(\omega_{\widetilde{\nu}}, \widetilde{\lambda}, \widetilde{\kappa}\big) + o_n(1).$$

Since we have $\widetilde{\nu} \geq \nu_\infty$, $\widetilde{\lambda} \leq \lambda_\infty$ and $\widetilde{\kappa} \leq \overline{\kappa}$, the above inequality becomes absurd. This completes the proof. □

5. Proof of the mian results

Now, we are ready to prove our main results. We emphasize here that, according to the threshold found in Proposition 4.1, we only need to check the minimax levels introduced in Section 2 are below $\overline{\gamma}_\infty := \gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa})$. Since V, W, Q are functions on $\mathbb{R}^N$, according to Remark 3.4, we can precisely fix Q in a range as

$$0 \leq Q(x) < \widehat{\kappa}\big(|V|_\infty, \inf W\big), \quad \text{for all } x \in \mathbb{R}^N. \tag{5.1}$$

Note that $\widehat{\kappa}\big(|V|_\infty, \inf W\big) \leq \widehat{\kappa}(\nu_\infty, \lambda_\infty)$, we have $(\omega_{V(y)}, W(y), Q(y))$ satisfies (3.2) for all $y \in \mathbb{R}^N$.

5.1. The minimal energy solution. Let us first go back to Lemma 2.7, and we can find c_ε is the minimal energy level for Φ_ε and, particularly, $c_0 = \gamma(\omega_{V(0)}, W(0), Q(0))$. Note that it is not difficult to see that c_0 is attained and hence we have $c_\varepsilon \leq c_0 + o_\varepsilon(1)$. By noting that $\gamma(\omega_{V(0)}, W(0), Q(0)) < \gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa})$ due to Lemma 3.3, we thus conclude that Φ_ε will satisfy the $\mathscr{G}$-(PS)-condition at level c_ε for all small $\varepsilon > 0$ thanks to Proposition 4.1. And therefore, we shall have the existence of a critical point for Φ_ε which lies in the minimal energy level associated to the equation (2.3).

Let us remark here that if z_ε is a minimal energy solution to (2.3), then taking into account that the energy estimate $0 < c_\varepsilon \leq c_0 + o_\varepsilon(1)$, it is possible to prove that the family $\{z_\varepsilon\}$ has a concentration behavior. We will divide this proof into three steps.

Step I. For $y \in \mathbb{R}^N$ being fixed arbitrarily, let us set $V_\varepsilon^y(x) = V_\varepsilon(x + y/\varepsilon)$, $W_\varepsilon^y(x) = W_\varepsilon(x + y/\varepsilon)$ and $Q_\varepsilon^y(x) = Q_\varepsilon(x + y/\varepsilon)$. Then, we have $V_\varepsilon^y(0) = V(y)$, $W_\varepsilon^y(0) = W(y)$ and $Q_\varepsilon^y(0) = Q(y)$. Now, let us consider the transformed equation

$$\mathscr{L}z + \mathcal{V}_\varepsilon^y(x)z = W_\varepsilon^y(x)|z|^{p-2}z + Q_\varepsilon^y(x)|z|^{4/N}z \tag{5.2}$$

where $\mathcal{V}_\varepsilon^y(x) =$ either $V_\varepsilon^y(x)$ or $V_\varepsilon^y(x)\mathcal{J}_0$. Denoted by Φ_ε^y the associated energy functional to (5.2) and Φ_0^y the energy functional defined correspondingly by $V(y)$, $W(y)$ and $Q(y)$, let us set

$$c_\varepsilon(y) = \inf_{e\in E^+\setminus\{0\}} \sup_{z\in\mathbb{R}e\oplus E^-} \Phi_\varepsilon^y(z) \quad \text{and} \quad c_0(y) = \inf_{e\in E^+\setminus\{0\}} \sup_{z\in\mathbb{R}e\oplus E^-} \Phi_0^y(z).$$

As was mentioned at the beginning of this section, we can immediately conclude the fact that $c_0(y) = \gamma(\omega_{V(y)}, W(y), Q(y))$ is always attained. Then, repeated application of Lemma 2.7 enables us to have $c_\varepsilon(y) \leq c_0(y) + o_\varepsilon(1)$ for each $y \in \mathbb{R}^N$.

Since a trivial verification would show that

$$c_\varepsilon(y) = \inf_{e\in E^+\setminus\{0\}} \sup_{z\in\mathbb{R}e\oplus E^-} \Phi_\varepsilon^y(z) \equiv \inf_{e\in E^+\setminus\{0\}} \sup_{z\in\mathbb{R}e\oplus E^-} \Phi_\varepsilon(z) = c_\varepsilon$$

for all $y \in \mathbb{R}^N$, we thus have actually have that

$$c_\varepsilon \leq \min_{y\in\mathbb{R}^N} c_0(y) + o_\varepsilon(1) = \min_{y\in\mathbb{R}^N} \gamma(\omega_{V(y)}, W(y), Q(y)) + o_\varepsilon(1).$$

Step II. Due to the fact that the family $\{z_\varepsilon\}$ is bounded and non-vanishing, let us choose $(t_\varepsilon, x_\varepsilon) \in \mathbb{R} \times \mathbb{R}^N$ and $R, \delta > 0$ be such that

$$\int_{t_\varepsilon - R}^{t_\varepsilon + R} \int_{B_R(x_\varepsilon/\varepsilon)} |z_\varepsilon|^2 \, dt \, dx \geq \delta.$$

Set $w_\varepsilon(t,x) = z_\varepsilon(t + t_\varepsilon, x + x_\varepsilon/\varepsilon)$, we then have w_ε become a minimal energy solution to the equation

$$\mathscr{L}z + \mathcal{V}_\varepsilon^{x_\varepsilon}(x)z = W_\varepsilon^{x_\varepsilon}(x)|z|^{p-2}z + Q_\varepsilon^{x_\varepsilon}(x)|z|^{4/N}z. \tag{5.3}$$

Suppose now $V(x_\varepsilon) \to V_\infty$, $W(x_\varepsilon) \to W_\infty$ and $Q(x_\varepsilon) \to Q_\infty$ as $\varepsilon \to 0$. By the Hölder continuity, we then have

$$V_\varepsilon^{x_\varepsilon}(x) \to V_\infty, \quad W_\varepsilon^{x_\varepsilon}(x) \to W_\infty, \quad Q_\varepsilon^{x_\varepsilon}(x) \to Q_\infty \quad \text{as } \varepsilon \to 0$$

uniformly on bounded sets of x. Remark that, if by w we denote the weak limit of w_ε, then we immediately have $w \neq 0$. And hence, a multiplication by compactly supported functions in (5.3) will generally imply us that w solves

$$\mathscr{L}z + \omega_{V_\infty}z = W_\infty|z|^{p-2}z + Q_\infty|z|^{4/N}z$$

with its critical level being estimated by

$$\begin{aligned}\Phi_{V_\infty W_\infty Q_\infty}(w) &= \left(\frac{1}{2}-\frac{1}{p}\right)W_\infty|w|_p^p + \left(\frac{1}{2}-\frac{1}{2^*}\right)Q_\infty|w|_{2^*}^{2^*} \\ &\geq \gamma(\omega_{V_\infty}, W_\infty, Q_\infty) \geq \min_{y\in\mathbb{R}^N}\gamma(\omega_{V(y)}, W(y), Q(y)).\end{aligned}$$

On the other hand, by Fatou's lemma and Step I, we have

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{p}\right)W_\infty|w|_p^p + \left(\frac{1}{2}-\frac{1}{2^*}\right)Q_\infty|w|_{2^*}^{2^*} \\ &\leq \liminf_{\varepsilon\to 0}\left(\frac{1}{2}-\frac{1}{p}\right)\iint W_\varepsilon^{x_\varepsilon}(x)|w_\varepsilon|^p\,dt\,dx + \left(\frac{1}{2}-\frac{1}{2^*}\right)\iint Q_\varepsilon^{x_\varepsilon}(x)|w_\varepsilon|^{2^*}\,dt\,dx \\ &= \liminf_{\varepsilon\to 0} c_\varepsilon(x_\varepsilon) = \liminf_{\varepsilon\to 0} c_\varepsilon \leq \limsup_{\varepsilon\to 0} c_\varepsilon \leq \min_{y\in\mathbb{R}^N}\gamma(\omega_{V(y)}, W(y), Q(y)).\end{aligned}$$

Therefore, we can conclude

$$\lim_{\varepsilon\to 0} c_\varepsilon = \lim_{\varepsilon\to 0} c_0(x_\varepsilon) = \min_{y\in\mathbb{R}^N}\gamma(\omega_{V(y)}, W(y), Q(y)). \tag{5.4}$$

Remark that by the Hölder continuity of V, W and Q, jointly with a similar estimate like (3.6), we deduce that $\gamma(\omega_{V(\,\cdot\,)}, W(\,\cdot\,), Q(\,\cdot\,))$ is also Hölder continuous. Hence $\gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa}) > \gamma(\omega_{V(0)}, W(0), Q(0)) = c_0$ implies $\{x_\varepsilon\}$ is bounded in $\mathbb{R}^N$. And due to

$$c_0(y) = \gamma(\omega_{V(y)}, W(y), Q(y)) > \min_{y\in\mathbb{R}^N}\gamma(\omega_{V(y)}, W(y), Q(y))$$

provided that $\operatorname{dist}(y, \mathscr{C}) > 0$, thus we conclude $\lim_{\varepsilon\to 0}\operatorname{dist}(x_\varepsilon, \mathscr{C}) = 0$ from (5.4).

Actually, by (5.4) and the Brézis–Lieb type result, we also get $|w_\varepsilon - w|_{2^*} \to 0$ and then $|w_\varepsilon^\pm - w^\pm|_{2^*} \to 0$ by Proposition 2.1. And using $w_\varepsilon^\pm - w^\pm$ as test functions in (5.3), we finally get $w_\varepsilon \to w$ in E as $\varepsilon \to 0$.

Step III. According to Step II, let us assume without loss of generality that $x_\varepsilon \to y_0 \in \mathscr{C}$ as $\varepsilon \to 0$. Then we have $V(x_\varepsilon) \to V(y_0)$, $W(x_\varepsilon) \to W(y_0)$ and $Q(x_\varepsilon) \to Q(y_0)$ as $\varepsilon \to 0$ and w solves the limit equation

$$\mathscr{L}z + \mathcal{V}(y_0)z = W(y_0)|z|^{p-2}z + Q(y_0)|z|^{4/N}z, \ \text{ where } \mathcal{V}(y_0) = \omega_{V(y_0)}. \tag{5.5}$$

Hence, by (5.3), we have

$$\begin{aligned}\mathscr{L}(w_\varepsilon - w) = W_\varepsilon^{x_\varepsilon}(x)|w_\varepsilon|^{p-2}w_\varepsilon - W(y_0)|w|^{p-2}w + Q_\varepsilon^{x_\varepsilon}(x)|w_\varepsilon|^{4/N}w_\varepsilon \\ - Q(y_0)|w|^{4/N}w - \big(\mathcal{V}_\varepsilon^{x_\varepsilon}(x)w_\varepsilon - \mathcal{V}(y_0)w\big).\end{aligned}$$

Using the fact $w_\varepsilon \to w$ in E and the uniform L^∞ estimate (see Appendix, Lemma A.2), it is easy to check that $|L(w_\varepsilon - w)|_2 \to 0$ as $\varepsilon \to 0$. And therefore, we have $w_\varepsilon \to w$ in $\mathscr{B}^2$ as $\varepsilon \to 0$.

Next, let us remark that: for $w_\varepsilon = (w_\varepsilon^1, w_\varepsilon^2) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^{2K}$ solves (5.5), if denoted by $\widehat{w}_\varepsilon(t,x) = \big(w_\varepsilon^1(t,x), w_\varepsilon^2(-t,x)\big)$, it is clear that $\widehat{w}_\varepsilon$ satisfies a equation of the form

$$\partial_t \widehat{w}_\varepsilon - \Delta_x \widehat{w}_\varepsilon + \widehat{w}_\varepsilon = \widehat{f}_\varepsilon(t,x) \quad \text{in } \mathbb{R}\times\mathbb{R}^N.$$

By virtue of Lemma A.2, we have $\widehat{f}_\varepsilon \in L^q$ for all $q \geq 2$. And according to the interpolation theory, we infer that $w_\varepsilon \to w$ in $\mathscr{B}^r$ for all $q \geq 2$. So, we get $\widehat{f}_\varepsilon \to \widehat{f}_0$ in L^q for some f_0 and all $q \geq 2$. Then an trivial application of [10, Corollary A.4] would show that $|\widehat{w}_\varepsilon(t,x)| \to 0$ uniformly as $|(t,x)| \to \infty$, which yields the uniformly decay property of $\{w_\varepsilon\}$.

By collecting all the results proved in Steps I–III, we actually proved that $\widetilde{z}_\varepsilon(t,x) = z_\varepsilon(t,x/\varepsilon)$ is a minimal energy solution to the multi-component incongruent RD system

$$\partial_t z = \mathbf{D}\Delta_x z + M(x)z + \big(W(x)|z|^{p-2} + Q(x)|z|^{4/N}\big)\beta z$$

for all small $\varepsilon > 0$, and $|\widetilde{z}_\varepsilon(t,\cdot)|$ has a maximum point x_ε which converge to a suitable $y_0 \in \mathscr{C}$. Moreover, we have

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \|\widetilde{z}_\varepsilon(t,\,\cdot\,)\|_{L^\infty(\mathbb{R}^N\setminus B_{\varepsilon R}(x_\varepsilon))} = 0.$$

After a translation in time-axis if necessary, the rescaled functions $w_\varepsilon(t,x) = z_\varepsilon(t, x + x_\varepsilon/\varepsilon)$ will converge in $\mathscr{B}^2$ (and hence in $\mathscr{B}^r$, $r \geq 2$) to a minimal energy solution of the limit system

$$\partial_t z = \mathcal{J}\Delta_x z + M(y_0)z + \big(W(y_0)|z|^{p-2} + Q(y_0)|z|^{4/N}\big)\beta z,$$

and this finishes the whole characterization of the asymptotic behavior of the minimal energy solution to our original problem.

5.2. More distinct solutions. By virtue of Lemma 2.4, in order to show the existence of other critical points, it is sufficient to construct a finite dimensional subspace $X_0 \subset E^+$ for Φ_ε such that

$$d := \sup \Phi_\varepsilon(X_0 \oplus E^-) < \overline{\gamma}_\infty = \gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa}). \tag{5.6}$$

Thus, since the $\mathscr{G}$-$(PS)_c$-condition is satisfied in the sublevel $\{z \in E : \Phi_\varepsilon(z) < \overline{\gamma}_\infty\}$, the change of pseudo-index (2.8) will imply the existence of at least $n := \dim X_0$ distinct critical points in the energy range $(0, d\,]$.

For simplify notation, we set $\nu_0 = V(0)$, $\lambda_0 = W(0)$ and $\kappa_0 = Q(0)$. Since we have assumed $Q(0) = \sup_{x\in\mathbb{R}^N} Q(x)$, we get $\kappa_0 = \overline{\kappa}$. In what follows, let us consider

$$\mathscr{L}z + \omega_{\nu_\infty} z = \lambda_\infty |z|^{p-2}z + \overline{\kappa}|z|^{4/N}z,$$
$$\mathscr{L}z + \omega_{\nu_0} z = \lambda_0 |z|^{p-2}z + \kappa_0|z|^{4/N}z,$$

with the minimal energy $\gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa})$ and $\gamma(\omega_{\nu_0}, \lambda_0, \kappa_0)$, respectively. And for a clear expression, let us rewrite the above equations as

$$-\beta\partial_t z + \mathcal{J}_0(-\Delta_x)z + (\mathcal{J}_0 + \omega_{\nu_\infty})z = \lambda_\infty|z|^{p-2}z + \overline{\kappa}|z|^{4/N}z \tag{5.7}$$

and

$$-\beta\partial_t z + \mathcal{J}_0(-\Delta_x)z + (\mathcal{J}_0 + \omega_{\nu_0})z = \lambda_0|z|^{p-2}z + \kappa_0|z|^{4/N}z. \tag{5.8}$$

Considering the scaling transform $w(t,x) = bz(at,\sqrt{a}x)$ with $a,b>0$, we get that (5.7) is equivalent to

$$-\beta\partial_t w + \mathcal{J}_0(-\Delta_x)w + a\cdot(\mathcal{J}_0 + \omega_{\nu_\infty})w = \frac{a\lambda_\infty}{b^{p-2}}|w|^{p-2}w + \frac{a\overline{\kappa}}{b^{4/N}}|w|^{4/N}w \tag{5.9}$$

Now, let us take $a = (1+\nu_0)/(1+\nu_\infty)$, $b = a^{N/4}$ and denote $\widetilde{\gamma}(\omega_{\nu_\infty},\lambda_\infty,\overline{\kappa})$ the minimal energy for (5.9), then we have

$$\widetilde{\gamma}(\omega_{\nu_\infty},\lambda_\infty,\overline{\kappa}) = \frac{b^2}{a^{N/2}}\gamma(\omega_{\nu_\infty},\lambda_\infty,\overline{\kappa}) = \gamma(\omega_{\nu_\infty},\lambda_\infty,\overline{\kappa}).$$

Moreover, by the choice of a and b, we also get

$$a\cdot(\mathcal{J}_0 + \omega_{\nu_\infty}) \geq (\mathcal{J}_0 + \omega_{\nu_0}) \quad \text{for the quadratic part,} \tag{5.10}$$

and

$$\frac{a\lambda_\infty}{b^{p-2}} < \lambda_0, \quad \frac{a\overline{\kappa}}{b^{4/N}} = \overline{\kappa} \quad \text{for the nonlinear coefficients.} \tag{5.11}$$

Here in (5.10), for two matrices A and B, by $A \geq B$ we mean $A-B$ is positive definite. Denoted by $\lambda' = a\lambda_\infty/b^{p-2}$, we are then led to the stage to consider the equation

$$-\beta\partial_t w + \mathcal{J}_0(-\Delta_x)w + (\mathcal{J}_0 + \omega_{\nu_0})w = \lambda'|w|^{p-2}w + \overline{\kappa}|w|^{4/N}w \tag{5.12}$$

and its minimal energy $\gamma(\omega_{\nu_0},\lambda',\overline{\kappa})$. Comparing (5.8) and (5.12), remark that we have the relations $\kappa_0 = \overline{\kappa}$ and

$$\gamma(\omega_{\nu_0},\lambda,\overline{\kappa}) = \lambda^{-\frac{2}{p-2}}\cdot\gamma\bigg(\omega_{\nu_0},1,\frac{\overline{\kappa}}{\lambda^{4/(N(p-2))}}\bigg), \quad \text{for all } \lambda > 0$$

jointly with (5.10) and (5.11) we soon deduce that

$$\begin{aligned}\frac{\gamma(\omega_{\nu_\infty},\lambda_\infty,\overline{\kappa})}{\gamma(\omega_{\nu_0},\lambda_0,\kappa_0)} &= \frac{\widetilde{\gamma}(\omega_{\nu_\infty},\lambda_\infty,\overline{\kappa})}{\gamma(\omega_{\nu_0},\lambda_0,\kappa_0)} \geq \frac{\gamma(\omega_{\nu_0},\lambda',\overline{\kappa})}{\gamma(\omega_{\nu_0},\lambda_0,\overline{\kappa})} \\ &= \bigg(\frac{\lambda_0}{\lambda'}\bigg)^{2/(p-2)}\cdot\frac{\gamma\Big(\omega_{\nu_0},1,\dfrac{\overline{\kappa}}{\lambda'^{4/N(p-2)}}\Big)}{\gamma\Big(\omega_{\nu_0},1,\dfrac{\overline{\kappa}}{\lambda_0^{4/N(p-2)}}\Big)}.\end{aligned}$$

Let us emphasize here that the above estimate become an equality when ω_ν is in the form of $\omega_\nu = \nu\mathcal{J}_0$. Now, let us set $n, n' \in \mathbb{N}$ to be the largest integers satisfying

$$n < \bigg(\frac{\lambda_0}{\lambda'}\bigg)^{2/(p-2)} \quad \text{and} \quad n' < \bigg(\frac{\lambda_0}{\lambda'}\bigg)^{2/(p-2)}\cdot\frac{\gamma\Big(\omega_{\nu_0},1,\dfrac{\overline{\kappa}}{\lambda'^{4/N(p-2)}}\Big)}{\gamma\Big(\omega_{\nu_0},1,\dfrac{\overline{\kappa}}{\lambda_0^{4/N(p-2)}}\Big)},$$

then we can infer from $\lambda' < \lambda_0$ that $n \geq 1$. We should remark that n and n' are not equal in general because

$$\gamma\left(\omega_{\nu_0}, 1, \frac{\overline{\kappa}}{\lambda'^{4/N(p-2)}}\right) < \gamma\left(\omega_{\nu_0}, 1, \frac{\overline{\kappa}}{\lambda_0^{4/N(p-2)}}\right),$$

which implies $n \geq n'$. Nevertheless, we can still conclude from the continuity of γ in the third variable that: in addition to (5.1), if we consider the function Q in a properly smaller range then n and n' will be equal as an invariant constant with respect to the factor $\overline{\kappa}$. At this moment, there holds

$$\gamma(\omega_{\nu_\infty}, \lambda_\infty, \overline{\kappa}) > n \cdot \gamma(\omega_{\nu_0}, \lambda_0, \kappa_0). \tag{5.13}$$

Next, recalling the notations introduced in the previous subsection, let us take $w \in E$ be the minimal energy solution of

$$\mathscr{L} z + \mathcal{V}(y_0) z = W(y_0)|z|^{p-2} z + Q(y_0)|z|^{4/N} z$$

such that w is also the limit of the rescaled sequence w_ε defined in Step II. For $r > 0$ let us choose a cut-off function $\eta_r \colon \mathbb{R}^N \to [0,1]$ such that $\eta_r(x) \equiv 1$ for $|x| \leq r$ and $\eta_r(x) \equiv 0$ for $|x| \geq r+1$. Set $w_r = \eta_r \cdot w$, then we have $\|w_r - w\| \to 0$ as $r \to \infty$, particularly, $\{w_r\}$ is a (PS)-sequence for $\Phi_0^{y_0}$ at the minimal energy level $c_0(y_0) = \gamma(\omega_{V(y_0)}, W(y_0), Q(y_0))$.

Now, let us take some $x_r \in \mathbb{R}^N$ with $|x_r| = 2(r+1)$ and set $x_{rj} = (j-1)x_r$ for $j = 1, \ldots, n$, where $n \in \mathbb{N}$ is fixed in (5.13). Define $w_{rj}(t,x) = w_r(t, x - x_{rj})$, then it is all clear that for r large enough $\{w_{rj}^+\}_{j=1}^n$ is linearly independent. Indeed, for constants c_j such that

$$\varphi^+ = \sum_{j=1}^n c_j w_{rj}^+ = 0 \quad \text{where } \varphi := \sum_{j=1}^n c_j w_{rj},$$

we can deduce that

$$\begin{aligned}
0 &\geq -\|\varphi^-\|^2 + \iint \mathcal{V}(y_0)\varphi^- \cdot \varphi^- \, dt\, dx \\
&\geq \|\varphi^+\|^2 - \|\varphi^-\|^2 - |V|_\infty |\varphi|_2^2 \\
&= \sum_{j=1}^n c_j^2 \cdot \left(\|w_{rj}^+\|^2 - \|w_{rj}^-\|^2 - |V|_\infty |w_{rj}|_2^2\right) \\
&= \left(\|w_r^+\|^2 - \|w_r^-\|^2 - |V|_\infty |w_r|_2^2\right) \sum_{j=1}^n c_j^2
\end{aligned}$$

which implies $c_j = 0$ for all j thanks to the fact $\|w_r^+\|^2 - \|w_r^-\|^2 - |V|_\infty |w_r|_2^2$ is strictly positive for large r. Having disposed of this preliminary step, we can now set

$$X_j^r = \mathbb{R}\, w_{rj}^+ = \operatorname{span}\{w_{rj}^+\} \quad \text{and} \quad X^r = \bigoplus_{j=1}^n X_j^r.$$

Then let $x_\varepsilon \in \mathbb{R}^N$ be the maximum point of w_ε found in the previous subsection, by using the Brézis-Lieb type result again, we can obtain

$$
\begin{aligned}
\max_{z\in X^r\oplus E^-} \Phi_\varepsilon^{x_\varepsilon}(z) &\le \max_{z\in X^r\oplus E^-} \Phi_0^{y_0}(z) + o_\varepsilon(1) \\
&\le \sum_{j=1}^n \max_{z\in X_j^r\oplus E^-} \Phi_0^{y_0}(z) + o_r(1) + o_\varepsilon(1) \\
&= \sum_{j=1}^n \max_{z\in X_1^r\oplus E^-} \Phi_0^{y_0}(z) + o_r(1) + o_\varepsilon(1) \\
&= n\cdot \gamma(\omega_{V(y_0)}, W(y_0), Q(y_0)) + o_r(1) + o_\varepsilon(1)
\end{aligned}
\tag{5.14}
$$

where the first inequality can be derived from the Arzelà–Ascoli theorem (for related details, we refer the readers to [10, Corollary 4.4]). Therefore, by setting $w_{rj\varepsilon}(t,x) = w_{rj}(t, x - x_\varepsilon/\varepsilon)$,

$$
X_{j\varepsilon}^r = \mathbb{R}w_{rj\varepsilon}^+ = \operatorname{span}\{w_{rj\varepsilon}^+\} \quad \text{and} \quad X_\varepsilon^r = \bigoplus_{j=1}^n X_{j\varepsilon}^r,
$$

we can conclude from (5.13) and (5.14) that, for $r_0 > 0$ fixed large enough, there exists $\varepsilon_0 > 0$ such that $X_0 := X_{\varepsilon_0}^{r_0} \subset E^+$ satisfies (5.6) for all $\varepsilon \in (0, \varepsilon_0]$. And then the proof is hereby completed.

Appendix A. Regularity results

We devote this appendix to For this purpose we set $B_\rho := \{x \in \mathbb{R}^N : |x| < \rho\}$ for any $\rho > 0$. Recall

$$
\mathscr{B}^r = W^{1,r}\big(\mathbb{R}, L^r\big(\mathbb{R}^N, \mathbb{R}^{2M}\big)\big) \cap L^r\big(\mathbb{R}, W^{2,r}\big(\mathbb{R}^N, \mathbb{R}^{2M}\big)\big) \quad \text{for } r \ge 1
$$

denotes the Banach space equipped with the norm $\|\cdot\|_{\mathscr{B}^r}$ defined in (1.8) and $L^r := L^r\big(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M}\big)$ is equipped with the usual L^r norm. The operator $\mathscr{L}$ is defined by $\mathscr{L} = -\beta\partial_t + \mathcal{J}_0(-\Delta_x + 1)$ in Section 2. Let us give the following fundamental result in the study of the system in the form of (2.3). Recall $E := \mathscr{D}(|\mathscr{L}|^{1/2})$ is the Hilbert space equipped with the norm $\|\cdot\|$. Denote $\mathscr{M}_{2K\times 2K}(\mathbb{R})$ by the space of all $2K\times 2K$ real matrixes equipped with the usual vector norm. In order to give our key regularity result for critical nonlinearities, let us first list the following regularity result for subcritical cases.

Lemma A.1. *For $N \ge 2$, let $M \in L^\infty\big(\mathbb{R}\times\mathbb{R}^N, \mathscr{M}_{2M\times 2M}(\mathbb{R})\big)$ and $H\colon \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{2M} \to \mathbb{R}$ satisfy*

$$
|\nabla_z H(t,x,z)| \le |z| + c|z|^{p-1}
$$

for some $c > 0$ and $p \in (2, 2^*)$. If $z \in E$ is a weak solution to

$$
\mathscr{L}z + M(t,x)z = \nabla_z H(t,x,z),
$$

then $z \in \mathscr{B}^r$ for all $r \ge 2$ and $\|z\|_{\mathscr{B}^r} \le C\big(\|M\|_\infty, \|z\|, c, p, r\big)$.

The proof of Lemma A.1 can be found in [9] (see Lemma 8.6, p. 149). Now we are ready to give our key result:

Lemma A.2. *For $N \geq 2$, let $M \in L^\infty\big(\mathbb{R}\times\mathbb{R}^N, \mathscr{M}_{2M\times 2M}(\mathbb{R})\big)$ and $h\colon \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{2M}\to\mathbb{R}$ satisfy*

$$|h(t,x,z)| \leq c\big(1+|z|^{4/N}\big) \tag{A.1}$$

for some $c>0$. If $z\in E$ is a weak solution to

$$\mathscr{L}z + M(t,x)z = h(t,x,z)z, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^N \tag{A.2}$$

then $z\in L^\infty$ and $\|z\|_\infty \leq C(\|M\|_\infty, \|z\|, c)$. Moreover, $z\in\mathscr{B}^r$ for all $r\geq 2$ and $\|z\|_{\mathscr{B}^r} \leq C(\|M\|_\infty, \|z\|, c, r)$.

Proof. Our proof starts with the observation that if we have proved $z\in L^\infty$ then the $\mathscr{B}^r$-estimate follows as a direct application of Lemma A.1. For this end, let us denote $\Gamma_\rho := (-\rho^2,\rho^2)\times B_\rho$ for $\rho>0$, and set

$$\Gamma_\rho\big(\overrightarrow{x}\big) := \big(-\rho^2+t, \rho^2+t\big)\times B_\rho(x) \quad \text{with } \overrightarrow{x} := (t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

Now fix $\overrightarrow{x}\in\mathbb{R}\times\mathbb{R}^N$, and let $\overline{\rho}\in C_c^\infty\big(\Gamma_2\big(\overrightarrow{x}\big)\big)$ be arbitrary. Choose $\overline{\eta}\in C_c^\infty\big(\Gamma_2\big(\overrightarrow{x}\big)\big)$ such that $\overline{\eta}\equiv 1$ on $\operatorname{supp}\overline{\rho}$. We have that, by denoting $D=-\beta\partial_t+\mathcal{J}_0(-\Delta_x)$,

$$D(\overline{\rho}z) = \overline{\rho}Dz + R(\overline{\rho},z) = \overline{\eta}\cdot\overline{\rho}Dz + R(\overline{\rho},z), \tag{A.3}$$

where $R(\overline{\rho},z) = -\partial_t\overline{\rho}\cdot\beta z - \Delta_x\overline{\rho}\cdot\mathcal{J}_0 z - 2\mathcal{J}_0\nabla_x\overline{\rho}\cdot\nabla_x z$. Noting that, for a weak solution z, there holds

$$Dz = -\mathcal{J}_0 z - M(t,x)z + h(t,x,z)z,$$

then, we may rewrite (A.3) as

$$R(\overline{\rho},z) = \mathscr{L}(\overline{\rho}z) - T_z(\overline{\rho}z) \tag{A.4}$$

where T_z is a linear multiplier defined by

$$T_z(w) = \overline{\eta}\cdot\big[-M(t,x)+h(t,x,z)\big]\,w.$$

Remark that $z\in E$, we have $z\in L^2\cap L^{2^*}$ and $|\nabla_x z|\in L^2$, and hence $R(\overline{\rho},z)\in L^2\big(\Gamma_2\big(\overrightarrow{x}\big)\big)$. In the sequel, we want to improve this estimate iteratively: We first begin the proof for $N\geq 3$ and assume that we have already obtained $z\in L^q\big(\Gamma_2\big(\overrightarrow{x}\big)\big)$ and $R(\overline{\rho},z)\in L^q\big(\Gamma_2\big(\overrightarrow{x}\big)\big)$ for some $q\in[2,(N+2)/2)$.

Let us consider the map

$$T_z\colon \mathscr{B}^q\big(\Gamma_2\big(\overrightarrow{x}\big)\big)\to L^q\big(\Gamma_2\big(\overrightarrow{x}\big)\big),$$

then it follows form embedding results for t-anisotropic Sobolev spaces (see [27]) that and the above linear multiplier T_z is well defined and, by using Minkowski's and Hölder's inequalities, the operator norm can be estimated as

$$\|T_z\|_{\mathscr{B}^q\to L^q} \le C_1\Big(|z|^{4/N}_{L^{2^*}(B)} + |B|^{2/(N+2)}\Big)$$

for some constant C_1 (depends on $|M|_\infty$, c, q), where $B := \operatorname{supp}\overline{\eta}$. Thus, when $|B|$ is fixed small enough, we shall assert that $\mathscr{L} - T_z$ is invertible. Therefore, by (A.4), there is a unique solution $w \in \mathscr{B}^q\big(\Gamma_2(\overrightarrow{x})\big)$ to

$$\mathscr{L}w - T_z(w) = R(\overline{\rho}, z) \quad \text{in } \Gamma_2(\overrightarrow{x})$$

which vanishes on the boundary of $\Gamma_2(\overrightarrow{x})$.

On the other hand, we also have a well defined map

$$T_z\colon L^q\big(\Gamma_2(\overrightarrow{x})\big) \to \mathscr{B}^{q/(q-1)}\big(\Gamma_2(\overrightarrow{x})\big)^*.$$

and the operator norm is estimated as before:

$$\|T_z\|_{L^q\to(\mathscr{B}^q)^*} \le C_2\Big(|z|^{4/N}_{L^{2^*}(B)} + |B|^{2/(N+2)}\Big).$$

for some constant C_2 (depends on $|M|_\infty$, c, q). And thus, for small B, there exists uniquely $\widetilde{w} \in L^q\big(\Gamma_2(\overrightarrow{x})\big)$ to the equation

$$\mathscr{L}\widetilde{w} - T_z\widetilde{w} = R(\overline{\rho}, z). \tag{A.5}$$

Notice that we have assumed $z \in L^q\big(\Gamma_2(\overrightarrow{x})\big)$ solves (A.4), hence $\widetilde{w} = \overline{\rho}z$. Using the embedding $\mathscr{B}^q\big(\Gamma_2(\overrightarrow{x})\big) \hookrightarrow L^q\big(\Gamma_2(\overrightarrow{x})\big)$, we have $w \in \mathscr{B}^q\big(\Gamma_2(\overrightarrow{x})\big)$ is also a L^q-solution to (A.5). And thus, by the uniqueness, we obtain $w = \overline{\rho}z$ and $\overline{\rho}z \in \mathscr{B}^q\big(\Gamma_2(\overrightarrow{x})\big)$ provided that $B = \operatorname{supp}\overline{\eta}$ is small. Since $\overline{\rho}$ and $\overline{\eta}$ arbitrary (under the assumption that $\operatorname{supp}\overline{\eta}$ is small and $\overline{\eta} \equiv 1$ on $\operatorname{supp}\overline{\rho}$), this implies that $z \in \mathscr{B}^q\big(\Gamma_1(\overrightarrow{x})\big)$. Furthermore, due to the arbitrariness of $\overrightarrow{x} \in \mathbb{R}\times\mathbb{R}^N$, we have $z \in B^q_{\mathrm{loc}}\big(\mathbb{R}\times\mathbb{R}^N\big)$.

Using the embedding result, we obtain $z \in L^{q'}_{\mathrm{loc}}$, $|\nabla_x z| \in L^{q'}_{\mathrm{loc}}$ and $R(\overline{\rho}, z) \in L^{q'}\big(\Gamma_2(\overrightarrow{x})\big)$ for $q' := (N+2)q/(N+2-q)$. Repeating this process, we shall prove that $z \in \mathscr{B}^{(N+2)/2}_{\mathrm{loc}}\big(\mathbb{R}\times\mathbb{R}^N\big)$. It should be point out that, in the iterative process, we have the initial data $z, |\nabla_x z| \in L^2$. Therefore, by the interior estimates (see for example [14]) and $\mathscr{B}^{(N+2)/2}_{\mathrm{loc}}\big(\mathbb{R}\times\mathbb{R}^N\big) \hookrightarrow \bigcap_{q\ge2} L^q_{\mathrm{loc}}$, we can conclude that $z \in L^\infty$ and

$$\|z\|_\infty \le C\big(\|M\|_\infty, \|z\|, c\big).$$

This completes the proof for the case $N \ge 3$.

Next, let us assume $N = 2$. For $1 < q < 2$, as argued before, we have

$$T_z\colon \mathscr{B}^q\big(\Gamma_2(\overrightarrow{x})\big) \to L^q\big(\Gamma_2(\overrightarrow{x})\big)$$

is well-defined and its operator norm is estimated as

$$\|T_z\|_{\mathscr{B}^q\to L^q} \le C_1\big(|z|^2_{L^4(B)} + |B|^{1/2}\big)$$

for some constant C_1 (depends on $|M|_\infty$, c, q). And thus, if $\operatorname{supp}\overline{\eta} = B$ is fixed small, there exists a unique solution $w \in \mathscr{B}^q\big(\Gamma_2(\overrightarrow{x})\big)$ to the equation

$$\mathscr{L}w - T_z(w) = R(\overline{\rho}, z) \quad \text{in } \Gamma_2(\overrightarrow{x})$$

vanishing on the boundary of $\Gamma_2(\overrightarrow{x})$. Meanwhile, we also have

$$T_z \colon L^4\big(\Gamma_2(\overrightarrow{x})\big) \to \mathscr{B}^{4/3}\big(\Gamma_2(\overrightarrow{x})\big)^*$$

is well-defined and its operator norm can be estimated as

$$\|T_z\|_{L^4 \to (\mathscr{B}^{4/3})^*} \le C_2\big(|z|^2_{L^4(B)} + |B|^{1/2}\big).$$

for some constant C_2 (depends on $|M|_\infty$, c, q). Therefore, for small B, there exists a unique solution $\widetilde{w} \in L^4\big(\Gamma_2(\overrightarrow{x})\big)$ to the equation

$$\mathscr{L}\widetilde{w} - T_z\widetilde{w} = R(\overline{\rho}, z).$$

Since we already have $z \in E \hookrightarrow L^4$ and $R(\overline{\rho}, z) \in L^2\big(\Gamma_2(\overrightarrow{x})\big)$, the same conclusion can be drawn here that $w = \overline{\rho}z = \widetilde{w}$ is a $\mathscr{B}^q$-solution for all $q \in [\frac{4}{3}, 2)$. Therefore, we can conclude $z \in \cap_{q\ge 2} L^q_{loc}$. Once this is proved, together with the interior estimates, we have $z \in L^\infty$ and hence

$$\|z\|_\infty \le C\big(\|M\|_\infty, \|z\|, c\big).$$

In summary, we are here to complete the proof for all $N \ge 2$ by invoking Lemma A.1. □

Appendix B. Proof of Theorem 2.11

Recall all the notations introduced in Section 2 and, for a subset $\mathcal{S} \subset E$ and $\sigma > 0$, let us denote the σ-neighbourhood of $\mathcal{S}$ as

$$U_\sigma(\mathcal{S}) := \Big\{ z \in E : \inf_{w \in \mathcal{S}} \|z - w\| < \sigma \Big\},$$

we remind the reader the following definition of (PS)-attractor (see [9, Chapter 3]):

Definition B.1. A subset $\mathscr{A} \subset E$ is said to be a $(\mathrm{PS})_c$-attractor for Φ if any $(\mathrm{PS})_c$-sequence approaches the σ-neighbourhood of $\mathscr{A} \cap \Phi^{c+\delta}_{c-\delta}$ for any $\sigma, \delta > 0$:

$$\left.\begin{array}{r} \Phi(z_n) \to c \\ \Phi'(z_n) \to 0 \end{array}\right\} \Rightarrow z_n \in U_\sigma\big(\mathscr{A} \cap \Phi^{c+\delta}_{c-\delta}\big) \text{ for all } n \text{ suitably large.}$$

And given $\mathcal{I} \subset \mathbb{R}$, $\mathscr{A} \subset E$ is said to be a $(\mathrm{PS})_{\mathcal{I}}$-attractor if $\mathscr{A}$ is a $(\mathrm{PS})_c$-attractor for all $c \in \mathcal{I}$.

For later arguments we introduce a comparison function $\psi_l \colon [0, l] \to \mathbb{N} \cup \{\infty\}$: let $l > 0$ and

$$\mathcal{M}_0(\Phi^l) := \big\{ h \in \mathcal{M}(\Phi^l) : h \text{ is a homeomorphism from } \Phi^l \text{ to } h(\Phi^l) \big\},$$

for $c \in [0,l]$, define

$$\psi_l(c) := \min\big\{\mathrm{gen}\big(h(\Phi^c)\cap S_r^X\big) : h \in \mathcal{M}_0(\Phi^l)\big\}.$$

Note that due to $\mathcal{M}_0(\Phi^l) \subset \mathcal{M}(\Phi^l) \hookrightarrow \mathcal{M}(\Phi^c)$ via the restriction $h \mapsto h|_{\Phi^c}$ we have $\psi(c) \le \psi_l(c)$ for all $c \in [0,l]$.

Now, we can sketch the proof as follows: If $c_i \in [\rho,d]$ as was defined in (2.9) is not a critical value, then for any sufficiently small $\overline{\delta} > 0$ we have

$$\inf\Big\{\|\Phi'(z)\| : z \in \Phi_{c_i-\overline{\delta}}^{c_i+\overline{\delta}}\Big\} > 0.$$

By virtue of the Deformation Theorem (a version developed for strongly indefinite functionals can be found in [9, Theorem 3.2]), we can infer the existence of $\eta \in C\big([0,1]\times\Phi^{c_i+\delta},\Phi^{c_i+\delta}\big)$ for some $\delta \in (0,\overline{\delta})$ such that $g := \eta(1,\,\cdot\,) \in \mathcal{M}(\Phi^{c_i+\delta})$ and $g(\Phi^{c_i+\delta}) \subset \Phi^{c_i-\delta}$. This is impossible since we can deduce that

$$\begin{aligned}\psi(c_i-\delta) &= \min\big\{\mathrm{gen}(h(\Phi^{c_i-\delta})\cap S_r^X) :\ h \in \mathcal{M}(\Phi^{c_i-\delta})\big\}\\ &\ge \min\big\{\mathrm{gen}(h\circ g(\Phi^{c_i+\delta})\cap S_r^X) :\ h \in \mathcal{M}(\Phi^{c_i-\delta})\big\}\\ &\ge \min\big\{\mathrm{gen}(h(\Phi^{c_i+\delta})\cap S_r^X) :\ h \in \mathcal{M}(\Phi^{c_i+\delta})\big\} = \psi(c_i+\delta),\end{aligned}$$

and the monotonicity of the $\mathbb{Z}_2$-genus implies $\psi(c_i-\delta) = \psi(c_i+\delta)$.

If Φ has only finitely many $\mathscr{G}$-distinct critical points in Φ_ρ^d, then thanks to the $\mathscr{G}$-(PS)-condition we have

$$\mathscr{A} := \big\{gz : g \in \mathscr{G} \text{ and } z \in \Phi_\rho^d \text{ such that } \Phi'(z) = 0\big\}$$

is a $(\mathrm{PS})_{\mathcal{I}}$-attractor with $\mathcal{I} := [\rho,d]$. Moreover, we also have that $\mathscr{A}/\mathscr{G}$ is the critical set of Φ in Φ_ρ^d which is finite. Hence, we deduce that

$$\inf\big\{\|P^Xz - P^Xw\| : z,w \in \mathscr{A} \text{ and } P^Xz, P^Xw \text{ are } \mathscr{G}\text{-distinct}\big\} > 0.$$

For $\sigma > 0$ small we then have that $U_\sigma(P^X\mathscr{A}) \subset X$ is the union of disjoint σ-$\mathscr{G}$-orbits around the elements of $P^X\mathscr{A}$. This, jointly with the fact the $\mathscr{G}$-action is separated, implies that $\mathrm{gen}(U_\sigma) = \mathrm{gen}(U_\sigma(P^X\mathscr{A})) = \mathrm{gen}(P^X\mathscr{A}) = 1$ where $U_\sigma := U_\sigma(P^X\mathscr{A})\times Y$. Let $\eta \in C\big([0,1]\times\Phi^d,\Phi^d\big)$ be a deformation deduced from Theorem 3.5 a) in [9]. For $\delta > 0$ small enough the map $g := \eta(1,\,\cdot\,)$ satisfies $g(\Phi^{c_i+\delta}) \subset \Phi^{c_i-\delta}\cup U_\sigma$. Let $l = d+1$ and choose $h_0 \in \mathcal{M}_0(\Phi^l)$ such that $\psi_l(c_i-\delta) = \mathrm{gen}(h_0(\Phi^{c_i-\delta})\cap S_r^X)$. And consequently,

$$\begin{aligned}\psi_l(c_i+\delta) &= \min\big\{\mathrm{gen}(h(\Phi^{c_i+\delta})\cap S_r^X) :\ h \in \mathcal{M}_0(\Phi^l)\big\}\\ &\le \mathrm{gen}(h_0\circ g(\Phi^{c_i+\delta})\cap S_r^X) \le \mathrm{gen}(h_0(\Phi^{c_i-\delta}\cup U_\sigma)\cap S_r^X)\\ &\le \mathrm{gen}(h_0(\Phi^{c_i-\delta})\cap S_r^X) + \mathrm{gen}(h_0(U_\sigma))\\ &= \mathrm{gen}(h_0(\Phi^{c_i-\delta})\cap S_r^X) + 1 = \psi_l(c_i-\delta)+1.\end{aligned}$$

It is then immediate that $\psi_l(c_i+\delta) = \psi_l(c_i-\delta)+1$, otherwise one would get a contradiction to the definition of c_i. And finally, the fact $\psi_l(c_i) \ge i$ implies $\rho \le c_1 < \ldots < c_n \le d$ which completes the proof. □

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