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THE LONG-TIME BEHAVIOR OF WEIGHTED p-LAPLACIAN EQUATIONS

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Abstract. In this work we study weighted p-Laplacian equations in a bounded domain with a variable and generally non-smooth diffusion coefficient having at most a finite number of zeroes. The main attention is focused on the case that the diffusion coefficient $a(x)$ in such equations satisfies the inequality $\liminf_{x\to z}|x-z|^{-p}a(x)>0$ for every $z\in\overline{\Omega}$. We show the existence of weak solutions and global attractors in $L^2(\Omega)$, $L^q(\Omega)$ $(q\geq 2)$ and $D_0^{1,p}(\Omega)$, respectively.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ $(n\geq 2)$. We consider weighted p-Laplacian equations

$$
(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)+f(u)=g & \text{in } \Omega\times\mathbb{R}^+,\\ u=0 & \text{on } \partial\Omega\times\mathbb{R}^+,\\ u(x,0)=u_0 & \text{in } \Omega, \end{cases}
$$

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where $1 < p < n$, and $f \in C^1(\Omega)$ satisfying

$$f'(s) \geq -l, \tag{1.2}$$

$$C_1|s|^q - C_0 \leq f(s)s \leq C_2|s|^q + C_0, \quad q \geq 2 \tag{1.3}$$

for any $s \in \mathbb{R}$. In order to handle applications to media which possibly are somewhere 'perfect' insulators (see [7]) we allow the coefficient $a(\,\cdot\,)$ to vanish somewhere. Therefore, the problem (1.1) is considered as being degenerated. The degeneracy of problem (1.1) is studied in the sense that the measurable, nonnegative diffusion coefficient $a(x)$ is allowed to have at most a finite number of zeroes at some points, and we assume that $a\colon \Omega \to \mathbb{R}$ satisfies the following assumption:

($\mathcal{H}_p$) $a \in L^1_{\rm loc}(\Omega)$ and $\liminf_{x\to z} |x-z|^{-p}a(x) > 0$, for every $z \in \overline{\Omega}$.

Recently, motivated by [5], where a semilinear degenerate elliptic problem was studied, the authors in [1]–[3], [9]–[11], [13], [14] considered the existence of global attractors for some classes of degenerate evolutionary equations under the assumption that $a \in L^1_{\rm loc}(\Omega)$, for some $\alpha \in (0,2)$, satisfies

$$\liminf_{x\to z} |x-z|^{-\alpha}a(x) > 0, \quad \text{for every } z \in \overline{\Omega}. \tag{1.4}$$

Typically, Anh and Ke in [3] have studied the existence of weak solutions and global attractors for degenerate p-Laplacian equation whenever $\alpha \in (0,p)$.

In this paper, we mainly consider degenerate p-Laplacian equation in the case of $\alpha = p$, that is, the weighted function a satisfies assumption ($\mathcal{H}_p$). In this case, $D^{1,p}_0(\Omega)$ (see the definition in Section 2) is compactly embedded only in the space $L^r(\Omega)$ $(1 < r < p)$ but not in $L^2(\Omega)$ or $L^p(\Omega)$ $(p \geq 2)$, which gives rise the additional difficulty due to the lack of compactness. Firstly, for the existence of solutions to our problem, Galerkin method seems so inconvenient to deal with parabolic equations because the inverse of the prime operator is not always compact in Hilbert space $L^2(\Omega)$. Secondly, it is well known that if we want to prove the existence of global attractor in $L^p(\Omega)$ we need to verify that the semigroup associated with (1.1) has some kind of compactness in $L^p(\Omega)$. However, there is no corresponding compact embedding result for this case. Hence, we can obtain the compact attractor only in $L^r(\Omega)$ $(1 < r < p)$ but not $L^2(\Omega)$ or $L^p(\Omega)$ by uniformly compact methods.

For our problem, we firstly give the corresponding embedding theorem, then prove the existence of weak solutions based on the singular perturbation method. Furthermore, we discuss the compact attractors of the weak solutions in $L^2(\Omega)$ and $L^q(\Omega)$ by use of asymptotic a priori estimate method introduced in [21] and combining with the existence of absorbing set in $D^{1,p}_0(\Omega)$ which is compactly embedding only in $L^r(\Omega)$ $(1 < r < p)$. Finally, we show the existence of global attractor in $D_0^{1,p}(\Omega)$ by verifying that the semigroup of weak solutions is asymptotically compact.

2. Preliminary results

In this section, we introduce some of the basic results on functional spaces, and then review briefly some necessary concepts and theorems on global attractors. In order to study problem (1.1) we introduce the weighted Sobolev space $D_0^{1,p}(\Omega)$ defined as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{D_0^{1,p}} = \bigg(\int_\Omega a(x)|\nabla u|^p\,dx\bigg)^{1/p}.$$

Next proposition, which is easily proved by using similar arguments as in [3], refers to continuous and compact inclusion of $D_0^{1,p}(\Omega)$.

Proposition 2.1. *Let Ω be bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ and $a \in L^1_{\rm loc}(\Omega)$ satisfy* (1.4) *for some $\alpha \in (0,p]$. Then the following embeddings hold*:

(a) *$D_0^{1,p}(\Omega)$ is continuously embedded in $W_0^{1,pn/(n+\alpha)}(\Omega)$*;
(b) *$D_0^{1,p}(\Omega)$ is continuously embedded in $L^{p^*_\alpha}(\Omega)$*;
(c) *$D_0^{1,p}(\Omega)$ is compactly embedded in $L^r(\Omega)$ as $1 \leq r < p^*_\alpha = pn/(n-p+\alpha)$.*

Remark 2.2. $p^*_\alpha \geq p$ when $\alpha \in (0,p)$, $p^*_\alpha = p$ when $\alpha = p$, which plays the role of the critical exponent in the Sobolev embeddings.

In the present paper we only consider the case of $\alpha = p$, that is, a satisfies the hypothesis $(\mathcal{H}_p)$.

We now review briefly the basic results on the existence of global attractors, see [4], [6], [15], [18], [21] for more details.

Definition 2.3. Semigroup $\{S(t)\}_{t\geq 0}$ on Banach space X is called asymptotically compact if for any bounded sequence $\{x_n\}_{n=1}^\infty$ and $t_n \geq 0$, $t_n \to \infty$ as $n \to \infty$, $\{S(t_n)x_n\}_{n=1}^\infty$ has a convergent subsequence in X.

Theorem 2.4. *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on $L^p(\Omega)$ $(p \geq 1)$, which is a continuous or weak continuous semigroup on $L^q(\Omega)$ for some $q \leq p$ and have a global attractor in $L^q(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded. Then $\{S(t)\}_{t\geq 0}$ has a global attractor in $L^p(\Omega)$ if and only if*

(a) *$\{S(t)\}_{t\geq 0}$ has a bounded absorbing set B_0 in $L^p(\Omega)$, and*
(b) *for any $\varepsilon > 0$ and $T = T(\varepsilon, B)$, such that*

$$\int_{\Omega(|S(t)u_0| \geq M)} |S(t)u_0|^p\,dx < \varepsilon \quad \textit{for any } u_0 \in B \textit{ and } t \geq T.$$

Theorem 2.5. *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on $L^p(\Omega)$ $(p \geq 1)$ and $\{S(t)\}_{t\geq 0}$ have a bounded absorbing set in $L^p(\Omega)$. Then, for any $\varepsilon > 0$ and any bounded subset $B \subset L^p(\Omega)$, there exist positive constants $T = T(B)$ and $M = M(\varepsilon)$*

such that $m(\Omega(|S(t)u_0| \geq M)) \leq \varepsilon$ *for any* $t \geq T$ *and* $u_0 \in B$*, where* $m(e)$ (*sometimes we also write it as* $|e|$) *denotes the Lebesgue measure of* $e \subset \Omega$ *and* $\Omega(|u| \geq M) \triangleq \{x \in \Omega \mid |u(x)| \geq M\}$.

THEOREM 2.6. *For any* $\varepsilon > 0$*, the bounded subset* B *of* $L^p(\Omega)$ $(p \geq 1)$ *has a finite* ε*-net in* $L^p(\Omega)$ *if there exists a positive constant* $M = M(\varepsilon)$*, which depends on* ε*, such that*

(a) B *has a finite* $(3M)^{(q-p)/q}(\varepsilon/2)^{p/q}$*-net in* $L^q(\Omega)$ *for some* $q \geq 1$;

(b) $\displaystyle\int_{\Omega(|u|\geq M)} |u|^p\,dx \leq 2^{-(2p+2)/p}\varepsilon$ *for any* $u \in B$.

3. Existence and uniqueness of the global solution

In this section, we show the existence and the uniqueness of the global solutions for (1.1). Let $D^{-1,p'}(\Omega)$ be the dual space of $D_0^{-1,p}(\Omega)$, where p' is the conjugate of p, i.e. $1/p + 1/p' = 1$. We denote $\Omega_T = \Omega \times [0,T]$, $V = L^p(0,T;D_0^{1,p}(\Omega)) \cap L^q(\Omega_T)$ and $V^* = L^{p'}(0,T;D^{-1,p'}(\Omega)) + L^{q'}(\Omega_T)$, respectively, where q' is the conjugate exponent of q, i.e. $1/q + 1/q' = 1$. For a convenience, hereafter let $\|\cdot\|_p$ be the norm of $L^p(\Omega)$ $(p \geq 1)$, $|u|$ be the modular (or absolute value) of u, C be the arbitrary positive constant, which may be different from line to line and even in the same line.

DEFINITION 3.1. A function $u(x,t)$ is called a weak solution of (1.1) on [0,T] if and only if $u \in C([0,T];L^2(\Omega)) \cap L^p(0,T;D_0^{1,p}(\Omega)) \cap L^q(0,T;L^q(\Omega))$ and $u|_{t=0} = u_0$ almost everywhere in Ω such that

$$\int_{\Omega_T} \bigg(\frac{\partial u}{\partial t}\,\xi + a|\nabla u|^{p-2}\nabla u\nabla\xi + f(u)\xi\bigg)dx\,dt = \int_{\Omega_T} g\xi\,dx\,dt$$

holds for all test functions $\xi \in V$.

The following lemma makes the initial condition in problem (1.1) meaningful.

LEMMA 3.2 ([3]). *Assume* $u \in V$ *and* $du/dt \in V^*$*. Then* $u \in C([0,T];L^2(\Omega))$.

Let

$$\begin{aligned} L_{p,a}u &= -\mathrm{div}(a(x)|\nabla u|^{p-2}\,\nabla u), & u \in D_0^{1,p}(\Omega),\\ L_{p,a_\varepsilon}u &= -\mathrm{div}\big(a_\varepsilon(x)\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{(p-2)/2}\,\nabla u_\varepsilon\big), & u \in D_0^{1,p}(\Omega). \end{aligned}$$

We now establish the existence and uniqueness of the problem (1.1).

THEOREM 3.3. *Assume* $\Omega \subset \mathbb{R}^n$ *is a bounded open domain with smooth boundary,* f *satisfies* (1.2)–(1.3) *and* $g \in L^2(\Omega)$*. Then, for any* $u_0 \in L^2(\Omega)$ *and* $T > 0$*, there exists a unique solution* u *of* (1.1) *which satisfies*

$$u \in C([0,T];L^2(\Omega)) \cap L^p(0,T;D_0^{1,p}(\Omega)) \cap L^q(\Omega_T).$$

The mapping $u_0 \mapsto u(t)$ *is continuous in* $L^2(\Omega)$.

PROOF. For any $0 < \varepsilon < 1$, we choose $u_{\varepsilon,0} \in C_c^\infty(\Omega)$ such that $\|u_{\varepsilon,0}\|_\infty$ are uniformly bounded with respect to ε, and $u_{\varepsilon,0} \to u_0$ in $L^2(\Omega)$ as $\varepsilon \to 0$.

Consider the problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\big(a_\varepsilon(x)\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{(p-2)/2}\nabla u_\varepsilon\big) + f(u_\varepsilon) = g & \text{in } \Omega \times \mathbb{R}^+, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u_\varepsilon(x,0) = u_{\varepsilon 0} & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $a_\varepsilon(x) = a(x) + \varepsilon$ for $x \in \Omega$.

According to the classical theory on parabolic equations (see for example [4], [6], [17], [18]), the problem (3.1) admits a unique weak solution

$$u_\varepsilon \in C([0,T]; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap L^q(\Omega_T) \quad \text{with } \frac{\partial u}{\partial t} \in L^2(\Omega_T).$$

Here u_ε is called a weak solution of the problem (3.1), if

$$\int_0^T \int_\Omega \bigg(\frac{\partial u_\varepsilon}{\partial t}\varphi + a_\varepsilon\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{(p-2)/2}\nabla u_\varepsilon \nabla\varphi + f(u_\varepsilon)\varphi\bigg)\,dx\,dt = \int_0^T \int_\Omega g\varphi\,dx\,dt, \tag{3.2}$$

for $\varphi \in C_0^\infty(\Omega_T)$ and $u_\varepsilon|_{t=0} = u_{\varepsilon,0}$ almost everywhere in Ω.

We do some estimates on u_ε. Multiplying (3.1) by u_ε and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|_2^2 + \int_\Omega a_\varepsilon\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{(p-2)/2}|\nabla u_\varepsilon|^2 + \int_\Omega f(u_\varepsilon)u_\varepsilon = \int_\Omega g u_\varepsilon\,dx. \tag{3.3}$$

We can use (1.3) and the Hölder inequality to write

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|_2^2 + \int_\Omega a(x)|\nabla u_\varepsilon|^p\,dx + \frac{C_1}{2}\int_\Omega |u_\varepsilon|^q\,dx \le C_0|\Omega| + \frac{C}{2C_1}\|g\|_2^2. \tag{3.4}$$

where $|\Omega| = \int_\Omega dx$. The Gronwall inequality implies, for any $T \in \mathbb{R}$,

$$u_\varepsilon \text{ is uniformly bounded in } L^\infty(0,T,L^2(\Omega)) \text{ with respect to } \varepsilon.$$

Integrating (3.3) and (3.4) both sides between 0 and T and using the Young inequality, we may get by a standard procedure(see for example [4], [8], [17] and [18]) that

$$\int_{\Omega_T} a_\varepsilon\bigg(|\nabla u_\varepsilon|^2 + \varepsilon\bigg)^{(p-2)/2}|\nabla u_\varepsilon|^2\,dx\,dt \le C,$$

$$\int_{\Omega_T} a|\nabla u_\varepsilon|^p\,dx\,dt \le C \quad \text{and} \quad \int_{\Omega_T} |u_\varepsilon|^q\,dx\,dt \le C$$

with C independent of ε. Noting (1.3), we obtain

$$\begin{aligned}\|f(u_\varepsilon)\|^{q'}_{L^{q'}(\Omega_T)} &= \int_0^T \bigg(\int_\Omega |f(u_\varepsilon)|^{q'}\,dx\bigg)\,dt \\ &\leq C\int_0^T \bigg(\int_\Omega \big(1+|u_\varepsilon|^{q-1}\big)^{q'}\,dx\bigg)dt \leq C\int_0^T \bigg(\int_\Omega 1+|u_\varepsilon|^{q'(q-1)}\,dx\bigg)dt.\end{aligned}$$

So we have that $f(u_\varepsilon)$ is uniform bounded in $L^{q'}(0,T;L^{q'}(\Omega))$ with respect to ε.

We now extract a weakly convergent subsequence, denoted also by u_ε for convenience, with

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{in } L^2(0,T;D_0^{1,p}(\Omega)),\\ u_\varepsilon &\rightharpoonup u \quad \text{in } L^q(0,T;L^q(\Omega)),\\ f(u_\varepsilon) &\rightharpoonup \chi \quad \text{in } L^{q'}(0,T;L^{q'}(\Omega)),\\ L_{p,a_\varepsilon}u_\varepsilon &\rightharpoonup \vartheta \quad \text{in } L^{p'}(0,T;D^{-1,p'}(\Omega)).\end{aligned} \tag{3.5}$$

Since $f\in C(\mathbb{R})$, it follows that $f(u_\varepsilon)\rightharpoonup f(u)$ in $L^{q'}(0,T;L^{q'}(\Omega))$.

Now we show that u is a weak solution of problem (1.1). Multiplying (3.1) by φ and let $\varepsilon\to 0^+$ to derive

$$\int_0^T\int_\Omega \frac{\partial u}{\partial t}\varphi + \vartheta\nabla\varphi + f(u)\varphi\,dx\,dt = \int_0^T\int_\Omega g\varphi\,dx\,dt \quad \text{for } \varphi\in C_0^\infty(\Omega_T).$$

Therefore, to obtain the existence we need only to prove

$$\int_0^T\int_\Omega \vartheta\nabla\varphi\,dx\,dt = \int_0^T\int_\Omega a|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx\,dt, \quad \varphi\in C_0^\infty(\Omega_T) \tag{3.6}$$

for $C_0^\infty(\Omega_T)$ is dense in V. From (3.3) we can obtain

$$\begin{aligned}\int_{\Omega_T} & a_\varepsilon\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{(p-2)/2}|\nabla u_\varepsilon|^2\,d\,xdt \\ &= -\int_{\Omega_T}\frac{\partial u_\varepsilon}{\partial t}u_\varepsilon\,dx\,dt - \int_{\Omega_T} f(u_\varepsilon)u_\varepsilon\,dx\,dt + \int_{\Omega_T} gu_\varepsilon\,dx\,dt.\end{aligned}$$

Let $v\in C_0^\infty(\Omega_T)$. It is obvious that

$$\int_0^T\int_\Omega a_\varepsilon\big(\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{(p-2)/2}\nabla u_\varepsilon - \big(|\nabla v|^2+\varepsilon\big)^{(p-2)/2}\nabla v\big)\cdot(\nabla u_\varepsilon-\nabla v)\,dx\,dt \geq 0.$$

Therefore,

$$\begin{aligned} -\int_{\Omega_T}\frac{\partial u_\varepsilon}{\partial t}u_\varepsilon\,dx\,dt &- \int_{\Omega_T} a_\varepsilon\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{(p-2)/2}\nabla u_\varepsilon\nabla v\,dx\,dt \\ &- \int_{\Omega_T} a\big(|\nabla v|^2+\varepsilon\big)^{(p-2)/2}\nabla v\big(\nabla u_\varepsilon-\nabla v\big)\,dx\,dt\end{aligned}$$

$$+\varepsilon\int_{\Omega_T}\big(|\nabla v|^2+\varepsilon\big)^{(p-2)/2}\nabla v\big(\nabla u_\varepsilon-\nabla v\big)\,dx\,dt$$
$$-\int_{\Omega_T}f(u_\varepsilon)u_\varepsilon\,dx\,dt+\int_{\Omega_T}gu_\varepsilon\,dx\,dt\ge 0.$$

Letting $\varepsilon\to 0^+$ in the above inequality and noticing that

$$\varepsilon\bigg|\int_{\Omega_T}\big(|\nabla v|^2+\varepsilon\big)^{(p-2)/2}\nabla v\big(\nabla u_\varepsilon-\nabla v\big)\,dx\,dt\bigg|$$
$$\le\varepsilon\int_{\Omega_T}\big(|\nabla v|^2+\varepsilon\big)^{p-2/2}|\nabla v|\,|\nabla u_\varepsilon|\,dx\,dt+\varepsilon\int_{\Omega_T}\big(|\nabla v|^2+\varepsilon\big)^{p/2}\,dx\,dt\to 0$$

as $\varepsilon\to 0^+$, we arrive at

$$-\int_{\Omega_T}\frac{\partial u}{\partial t}\,u\,dx\,dt-\int_{\Omega_T}\vartheta\nabla v\,dx\,dt$$
$$-\int_{\Omega_T}a|\nabla v|^{p-2}\nabla v\big(\nabla u-\nabla v\big)\,dx\,dt-\int_\Omega f(u)u\,dx\,dt+\int_\Omega gu\,dx\,dt\ge 0. \tag{3.7}$$

On the other hand, choosing $\varphi=u$ in (3.2) leads to

$$\int_0^T\int_\Omega\vartheta\nabla u\,dx\,dt=-\int_{\Omega_T}\frac{\partial u_\varepsilon}{\partial t}\,u_\varepsilon-\int_0^T\int_\Omega f(u)u\,dx\,dt+\int_0^T\int_\Omega gu\,dx\,dt. \tag{3.8}$$

Then, it follows from (3.7) and (3.8) that

$$\int_0^T\int_\Omega\big(\vartheta-a|\nabla v|^{p-2}\nabla v\big)(\nabla u-\nabla v)\,dx\,dt\ge 0.$$

Choosing $v=u-\lambda\varphi$ with $\lambda>0$ in the above inequality, we get

$$\int_0^T\int_\Omega\big(\vartheta-a|\nabla(u-\lambda\varphi)|^{p-2}\big)\nabla(u-\lambda\varphi)\nabla\varphi\,dx\,dt\ge 0,$$

which implies by letting $\lambda\to 0^+$ that

$$\int_0^T\int_\Omega\big(\vartheta-a|\nabla u|^{p-2}\nabla u\big)\nabla\varphi\,dx\,dt\ge 0.$$

If we choose $\lambda<0$, we achieve the inequality with opposite sign. Thus

$$\int_0^T\int_\Omega\big(\vartheta-a|\nabla u|^{p-2}\nabla u\big)\nabla\varphi\,dx\,dt=0,$$

which lead to (3.6). And then $u\in C([0,T];L^2(\Omega))$ follows from Lemma 3.2.

Now we will show that $u(0)=u_0$. Choosing some $\varphi\in C^1([0,T];D_0^{1,p}(\Omega)\cap L^q(\Omega))$ with $\varphi(T)=0$ as a test function and integrating by parts in the t variable we have

$$\int_0^T-\langle u,\varphi'\rangle+\big\langle-\operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big),\varphi\big\rangle\,ds$$
$$+\int_0^T\langle f(u(s)),\varphi\rangle\,ds=\int_0^T\langle g(x),\varphi\rangle\,ds+(u(0),\varphi(0)).$$

Doing the same in the above approximations, we get

$$\int_0^T -\langle u_\varepsilon, \varphi'\rangle + \big\langle -\operatorname{div}(a_\varepsilon(x)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{(p-2)/2}\nabla u_\varepsilon), \varphi\big\rangle\, ds$$
$$+\int_0^T \langle f(u_\varepsilon(s)),\varphi\rangle ds = \int_0^T \langle g(x),\varphi\rangle + (u_\varepsilon(0),\varphi(0)).$$

By taking limits we conclude that

$$\int_0^T -\langle u, \varphi'\rangle + \big\langle -\operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big), \varphi\big\rangle\, ds$$
$$+\int_0^T \langle f(u(s)),\varphi\rangle\, ds = \int_0^T \langle g(x),\varphi\rangle\, ds + (u_0,\varphi(0)),$$

since $u_{\varepsilon 0}\to u_0$. Thus $u(0)=u_0$.

Finally, we prove the uniqueness and the continuous dependence on u_0. Let u_0 and v_0 be in L^2 and consider $w(t)=u(t)-v(t)$. Then

$$\frac{\partial w}{\partial t} - \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + \operatorname{div}(a(x)|\nabla v|^{p-2}\nabla v) + f(u) - f(v) = 0$$

and $w(0)=u_0-v_0$. Multiplying by w and integrating over Ω gives

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + \big\langle -\operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) + \operatorname{div}\big(a(x)|\nabla v|^{p-2}\nabla v\big), u-v\big\rangle$$
$$+\langle f(u)-f(v), u-v\rangle = 0.$$

Using (1.2), we obtain

$$\frac{d}{dt}\|w\|^2 \le l\|w\|_2^2.$$

Integrating this gives the uniqueness and the continuous dependence on initial conditions. □

Now we can use these solutions to define a semigroup $\{S(t)\}_{t\ge 0}$ on $L^2(\Omega)$ by setting $S(t)u_0=u(t)$, which is continuous on u_0 in $L^2(\Omega)$. In what follows, we always assume that f satisfies (1.2)–(1.3) and $g\in L^2(\Omega)$, and $\{S(t)\}_{t\ge 0}$ is the semigroup generated by the weak solutions of (1.1) with initial data $u_0\in L^2(\Omega)$.

4. Existence of global attractors

4.1. Existence of absorbing sets.

Theorem 4.1. *The semigroup $\{S(t)\}_{t\ge0}$ possesses a bounded absorbing set in $L^2(\Omega)$, $L^q(\Omega)$ and $D_0^{1,p}(\Omega)$ respectively, i.e. for any bounded subset B in $L^2(\Omega)$, there exists a constant $T(\|u_0\|_2)$, such that*

$$\|u(t)\|_2^2 \le \rho_0 \quad \textit{and} \quad \|u(t)\|_q^q + \int_\Omega a(x)|\nabla u(t)|^p\, dx \le \rho_1,$$

for all $t\ge T$ and $u_0\in B$, where $u(t)=S(t)u_0$.

PROOF. Let $F(s) = \int_0^s f(\tau)\, d\tau$, from (1.3), we deduce that

$$\widetilde{C}_1|s|^q - k \le F(s) \le k + \widetilde{C}_2|s|^q.$$

So,

$$\widetilde{C}_1 \int_\Omega |u|^q\, dx - k|\Omega| \le \int_\Omega F(u)\, dx \le k|\Omega| + \widetilde{C}_2 \int_\Omega |u|^q\, dx. \tag{4.1}$$

Multiplying (1.1) by u and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \int_\Omega a(x)|\nabla u|^p\, dx + \int_\Omega f(u)u\, dx = \int_\Omega gu\, dx.$$

Using (1.3) and the Hölder inequality to deduce that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \int_\Omega a(x)|\nabla u|^p\, dx + \frac{C_1}{2}\int_\Omega |u|^q\, dx \le \frac{1}{2}\|g\|_2^2 + C_0|\Omega|.$$

So we have

$$\frac{d}{dt}\|u\|_2^2 + C\bigg(\int_\Omega a(x)|\nabla u|^p\, dx + \int_\Omega |u|^q\, dx\bigg) \le C(\|g\|_2, |\Omega|). \tag{4.2}$$

Noticing that $q \ge 2$, we deduce

$$\frac{d}{dt}\|u\|_2^2 + C\int_\Omega |u|^2\, dx \le C(\|g\|_2, |\Omega|).$$

By the Gronwall lemma, we obtain the existence of absorbing set in $L^2(\Omega)$, i.e. there exist ρ_0 and $T_0 = T_0(\|u_0\|_2) > 0$ such that

$$\|u(t)\|_2^2 \le \rho_0 \quad \text{for } t \ge T_0.$$

Taking $t \ge T_0$ and integrating (4.2) on $[t, t+1]$, we have

$$\int_t^{t+1}\bigg(\int_\Omega a(x)|\nabla u|^p\, dx + \int_\Omega |u|^q\, dx\bigg) ds \le C(\|g\|_2, |\Omega|, \rho_0), \quad \text{for all } t \ge T_0.$$

Then, by (4.1), we can deduce that, for all $t \ge T_0$,

$$\int_t^{t+1}\bigg(\int_\Omega a(x)|\nabla u|^p\, dx + \int_\Omega F(u)\, dx\bigg) ds \le C(\|g\|_2, |\Omega|, \rho_0). \tag{4.3}$$

On the other hand, multiplying (1.1) by u_t, we obtain

$$\frac{d}{dt}\bigg(\frac{1}{p}\int_\Omega a(x)|\nabla u|^p + \int_\Omega F(u)\bigg) dx \le C(\|g\|_2, |\Omega|). \tag{4.4}$$

Therefore, from (4.3) and (4.4), using the uniform Gronwall lemma, we get

$$\int_\Omega a(x)|\nabla u|^p\, dx + \int_\Omega F(u)\, dx \le C(\|g\|_2, |\Omega|, \rho_0).$$

Thanks to (4.1), this inequality implies that, for all $t \ge T_0 + 1$,

$$\int_\Omega a(x)|\nabla u|^p\, dx + \int_\Omega |u|^q\, dx \le C(\|g\|_2, |\Omega|, \rho_0).$$

Taking $T = T_0 + 1$ and $\rho_1 = C(\|g\|_2, |\Omega|, \rho_0)$, we complete the proof of Theorem 4.1. □

4.2. Global attractors in $L^2(\Omega)$ and $L^q(\Omega)$. Our goal in this subsection is to deal with the existence of global attractor in $L^2(\Omega)$ and $L^q(\Omega)$, respectively. In view of the definition of global attractor (see [4], [6], [17], [18]) we need to verify $\{S(t)\}_{t\geq 0}$ is compact in $L^2(\Omega)$ and $L^q(\Omega)$, however, which we can not obtain by the uniformly compact method for lack of the corresponding Sobolev compact embedding theorem. Here we notice that $D^1_0(\Omega)$ can be embedding in to $L^r(\Omega)$ and finally obtain the compactness of the attractor in $L^2(\Omega)$ and $L^q(\Omega)$ using the asymptotic a priori estimate method introduced in [21].

THEOREM 4.2. *The semigroup $\{S(t)\}_{t\geq 0}$ generated by* (1.1) *with initial data $u_0 \in L^2(\Omega)$ has a global attractor $\mathscr{A}_2$. That is, $\mathscr{A}_2$ is compact, invariant in $L^2(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $L^2(\Omega)$.*

PROOF. In order to prove the existence of global attractor in $L^2(\Omega)$, we need to verify that the semigroup associated with (1.1) has some kind of compactness in $L^2(\Omega)$. Now we distinguish two cases to verify it.

Firstly, we consider the case $p > 2$. From Proposition 2.1, it is easy to know that $\{S(t)\}_{t\geq 0}$ is compact in $L^2(\Omega)$ for $D^{1,p}_0(\Omega)$ is compactly embedded in $L^2(\Omega)$, so there exist a global attractor in $L^2(\Omega)$.

In the case $1 < p \leq 2$ the proof is more complicated. Firstly, we give a priori estimate for the unbounded part of modular $|u|$ for the solution u of (1.1) in L^2-norm. For any fixed $\varepsilon > 0$, there exists $\delta > 0$ such that for any $e \subset \Omega$ and $m(e) \leq \delta$

$$\int_e |g(x)|^2\,dx < \varepsilon. \tag{4.5}$$

Moreover, from Theorem 2.5 and 4.1, we know that there exist $T = T(B,\varepsilon)$ and $M_1 = M(\varepsilon)$ such that

$$m\big(\Omega(|u(t)| \geq M_1)\big) \leq \min\{\varepsilon,\delta\} \quad \text{for } u_0 \in B \text{ and } t \geq T. \tag{4.6}$$

In addition, thanks to (1.3), we know $f(s) \geq 0$ when $s > (C_0/C_1)^{1/q}$. In the following we assume $M = \max\big\{M_1, (C_0/C_1)^{1/q}\big\}$ and $t \geq T$.

Multiplying (1.1) by $(u-M)_+$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|(u-M)_+\|_2^2 + \int_\Omega a(x)|\nabla(u-M)_+|^p\,dx + \int_\Omega f(u)(u-M)_+\,dx = \int_\Omega g(u-M)_+\,dx$$

where $(u-M)_+$ denotes the positive part of $u-M$, that is

$$(u-M)_+ = \begin{cases} u-M & \text{if } u \geq M,\\ 0 & \text{if } u \leq M.\end{cases}$$

Set $\Omega_1 = \Omega(u(t) \geq M)$. Notice that $f(u) \geq C_1 u^{q-1} - C_3$ and $u - M \leq u$ as $u \geq M$, from the Cauchy inequality and the Hölder inequality, we deduce that

$$\frac{d}{dt}\|(u-M)_+\|_2^2 + C\int_{\Omega_1}(u-M)^q dx \leq C\int_{\Omega_1}|g|^2\,dx, \quad \text{as } t \geq T. \tag{4.7}$$

Combining with (4.5)–(4.6) and $L^q \hookrightarrow L^2(\Omega)(q \geq 2)$, we get

$$\frac{d}{dt}\|(u-M)_+\|_2^2 + C\int_{\Omega_1}(u-M)^2\,dx \leq C\varepsilon.$$

We apply the Gronwall lemma to infer

$$\|(u-M)_+\|_2^2 = \int_{\Omega_1}|u-M|^2\,dx \leq C\varepsilon.$$

Let $\Omega_2 = \Omega\ (u \geq 2M)$, we have

$$\int_{\Omega_2}|u|^2\,dx \leq C\varepsilon. \tag{4.8}$$

Replacing $(u-M)_+$ with $(u+M)_-$ and using the same method as above, we obtain

$$\int_{\Omega(u\leq -2M)}|u|^2\,dx \leq C\varepsilon. \tag{4.9}$$

Hence, assertions (4.8) and (4.9) yield

$$\int_{\Omega(|u(t)|\geq M)}|u(t)|^2\,dx < C\varepsilon \quad \text{for any } u_0 \in B \text{ and } t \geq T,$$

where the constant C is independent of ε and B.

Let B_0 be the absorbing set in $D_0^{1,p}(\Omega)$, we can consider our problem only in B_0. From Proposition 2.1 we know $D_0^{1,p}(\Omega)$ is compactly embedded into $L^r(\Omega)$ for some $1 < r < p^*$, so B_0 is compact in $L^r(\Omega)$, and B_0 has a finite ε-net in $L^r(\Omega)$. According to Theorem 2.6 we see B_0 is compact in $L^2(\Omega)$, hence, Theorem 4.1 implies the existence of attractor in $L^2(\Omega)$. □

We now prove the existence of global attractors for $\{S(t)\}_{t\geq 0}$ in $L^q(\Omega)$. We firstly give a priori estimate for the unbounded part of modular $|u|$ for the solution u of (1.1) in L^q-norm.

THEOREM 4.3. *For any $\varepsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist $T = T(B,\varepsilon)$ and $M = M(\varepsilon)$ such that*

$$\int_{\Omega(|u|\geq M)}|u|^q\,dx \leq C\varepsilon \quad \textit{for all } t \geq T \textit{ and } u_0 \in B,$$

where the constant C is independent of ε and B.

Proof. Let $t \geq T$. Integrating (4.7) on $[t, t+1]$ and combining with (1.3) and Theorem 4.2, we have

$$\int_t^{t+1} \left(\int_\Omega |u - M|^q \, dx \right) dt \leq C\varepsilon. \tag{4.10}$$

On the other hand, multiplying (1.1) by $(u - M)_+^{q-1}$ and integrating over Ω_1, we get

$$\frac{1}{q} \frac{d}{dt} \int_{\Omega_1} |u - M|^q \, dx + (q-1) \int_{\Omega_1} a(x) |\nabla u|^p (u - M)^{q-2} \, dx \\ + \int_{\Omega_1} f(u)(u - M)^{q-1} \, dx = \int_{\Omega_1} g(u - M)_+ \, dx.$$

In view of (1.3), we have

$$\frac{1}{q} \frac{d}{dt} \int_{\Omega_1} |u - M|^q dx + (q-1) \int_{\Omega_1} a(x) |\nabla u|^p (u - M)^{q-2} \, dx \\ + C \int_{\Omega_1} |u - M|^{2q-2} \, dx \leq C\varepsilon,$$

where we used the Hölder inequality and the Cauchy inequality. Hence,

$$\frac{d}{dt} \int_{\Omega_1} |u - M|^q \, dx \leq C\varepsilon. \tag{4.11}$$

Using (4.10) and (4.11), the uniform Gronwall lemma leads to

$$\int_{\Omega_1} |u - M|^q \, dx \leq C\varepsilon,$$

where the constant C is independent of ε and B. Thus

$$\int_{\Omega(u \geq 2M)} |u|^q \, dx \leq C\varepsilon. \tag{4.12}$$

Repeating the same steps above, just taking $(u + M)_-$ and $(u + 2M)_-^{q-1}$ instead of $(u - M)_+$ and $(u - 2M)_+^{q-1}$, respectively, we deduce that

$$\int_{\Omega(u \leq -2M)} |u|^q \, dx \leq C\varepsilon. \tag{4.13}$$

Combining (4.12) with (4.13), we obtain that

$$\int_{\Omega(|u| \geq 2M)} |u|^q \, dx \leq C\varepsilon. \qquad \square$$

Theorems 2.4, 4.1 and 4.3 lead to the existence of global attractor in $L^q(\Omega)$.

Theorem 4.4. *The semigroup $\{S(t)\}_{t \geq 0}$ generated by the weak solution of equations* (1.1) *has a global attractor $\mathscr{A}_q$ in $L^q(\Omega)$, i.e. $\mathscr{A}_q$ compact, invariant in $L^q(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $L^q(\Omega)$.*

4.3. Global attractor in $D_0^{1,p}(\Omega)\cap L^q(\Omega)$. In this subsection, we prove the existence of a global attractor in $D_0^{1,p}(\Omega)$. At first, we will give some a priori estimates about u_t endowed with L^2-norm.

THEOREM 4.5. *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$, $g\in L^2(\Omega)$, and f satisfies (1.2) and (1.3). Then, for any bounded subset of $L^2(\Omega)$, there exists a positive constant $T=T(B)$ such that*

$$\|u_t\|_2^2\le C \quad \textit{for } u_0\in B \textit{ and } s\ge T,$$

where $u_t(s)=d\,S(t)u_0/dt|_{t=s}$ and C is independent of B.

PROOF. From Lemma 5.1 in [19] we know that for any bounded subset of $L^2(\Omega)$, there exists a positive constant $T=T(B)$ such that

$$\|u_{\varepsilon_t}\|_2^2\le C \quad \text{for all } u_0\in B \text{ and } s\ge T,$$

where u_ε is solution to (3.1). Thus, we have

$$u_{\varepsilon_t}\rightharpoonup h(s) \quad \text{in } L^2(\Omega) \text{ for any } s\ge T \quad \text{and} \quad \|h(s)\|_2^2\le C \quad \text{for all } s\ge T.$$

On the other hand, from the proof of Theorem 3.3, we know that $u_{\varepsilon t}\rightharpoonup u_t$ in V^*. By the uniqueness of limit, we have

$$\|u_t\|_2^2=\|h(s)\|_2^2\le C \quad \text{for all } s\ge T. \qquad \square$$

In order to verify that the semigroup $\{S(t)\}_{t\ge0}$ is asymptotically compact in $D_0^{1,p}(\Omega)$ we need the following simple property of weighted p-Laplacian:

PROPOSITION 4.6. *The operator $L_{a,p}\colon D_0^{1,p}(\Omega)\to D^{-1,p'}(\Omega)$ is strong monotone, i.e. for any $u,v\in D_0^{1,p}(\Omega)$, there exists a positive constant δ such that*

$$\langle L_{p,a}u_1-L_{p,a}u_2,u_1-u_2\rangle\ge\delta\|u_1-u_2\|_{D_0^{1,p}}.$$

PROOF. We need to prove the strong monotonicity. From the Lemma A.0.5 in [16], we know

$$\big((|x|^{p-2}x-|y|^{p-2}y,x-y)\big)\ge\delta|x-y|^p,$$

where $((\,\cdot\,,\,\cdot\,))$ is the standard scalar product in $\mathbb{R}^n$. So

$$\begin{aligned}\langle L_{p,a}u_1-L_{p,a}u_2,u_1-u_2\rangle&=\int_\Omega a(x)\big((|\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2,\nabla u_1-\nabla u_2)\big)\\&\ge\delta\|u_1-u_2\|_{D_0^{1,p}}. \ \square\end{aligned}$$

THEOREM 4.7. *The semigroup $\{S(t)\}_{t\ge0}$ generated by the weak solution of equations (1.1) has a global attractor $\mathscr{A}$ in $D_0^{1,p}(\Omega)\cap L^q(\Omega)$, i.e. $\mathscr{A}$ compact, invariant in $D_0^{1,p}(\Omega)\cap L^q(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $D_0^{1,p}(\Omega)\cap L^q(\Omega)$.*

Proof. Firstly, we prove that the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically compact in $D_0^{1,p}(\Omega)$. Let B_0 be a bounded absorbing set in $D_0^{1,p}(\Omega)$ obtained in Theorem 4.1, then we need only to show that for any sequence $\{u_{0n}\}_{n=1}^{\infty}\subset B_0$

$$\{u_n(t_n)\}_{n=1}^{\infty} \text{ is precompact in } D_0^{1,p}(\Omega), \tag{4.14}$$

where $u_n(t_n)=S(t_n)u_{0n}$.

Thanks to Theorems 4.2 and 4.4, we know that $\{u_n(t_n)\}_{n=1}^{\infty}$ is precompact in $L^2(\Omega)$ and $L^q(\Omega)$, so we can assume that the subsequence $\{u_{nk}(t_{nk})\}_{k=1}^{\infty}$ is Cauchy sequence in $L^2(\Omega)$ and $L^q(\Omega)$.

In the following, we prove that $\{u_{nk}(t_{nk})\}_{k=1}^{\infty}$ is Cauchy sequence in $D_0^{1,p}(\Omega)$. Noting Proposition 4.6, we have

$$\begin{aligned}
&\delta\|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\|_{D_0^{1,p}}\\
&\leq\langle L_{p,a}u_{n_k}(t_{n_k})-L_{p,a}u_{n_j}(t_{n_j}),u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\rangle\\
&=\bigg\langle -\frac{d}{dt}u_{n_k}(t_{n_k})-f(u_{n_k}(t_{n_k}))+\frac{d}{dt}u_{n_j}(t_{n_j})+f(u_{n_j}(t_{n_j})),u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\bigg\rangle\\
&\leq\int_\Omega\bigg|\frac{d}{dt}u_{n_k}(t_{n_k})-\frac{d}{dt}u_{n_j}(t_{n_j})\bigg||u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})|\\
&\quad+\int_\Omega\big|f(u_{n_k}(t_{n_k}))-f(u_{n_j}(t_{n_j}))\big|\,\big|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\big|\\
&\leq\bigg\|\frac{d}{dt}u_{n_k}(t_{n_k})-\frac{d}{dt}u_{n_j}(t_{n_j})\bigg\|_2\big\|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\big\|_2\\
&\quad+C\big(1+\|u_{n_k}(t_{n_k})\|_q^q+\|u_{n_j}(t_{n_j})\|_q^q\big)\big\|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\big\|_q,
\end{aligned}$$

which, combining with Theorem 4.4 and 4.5, yields (4.14) immediately. Set

$$\mathscr{A}=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}S(t)B_0}^{D_0^{1,p}\cap L^q},$$

which consists of all the limit points of the orbit of B_0, i.e.

$$\mathscr{A}=\big\{y:\exists\, t_n\to\infty,\ x_n\in B_0 \text{ with } S(t_n)x_n\to y \text{ in } D_0^{1,p}(\Omega)\cap L^q(\Omega)\big\}.$$

On the other hand, from Theorem 4.2, we know there exists $y^*\in L^2(\Omega)$ such that

$$S(t_n)x_n\to y^* \quad \text{in } L^2(\Omega) \text{ as } n\to\infty.$$

By the uniqueness of the limit, we get $y=y^*$, and $\mathscr{A}=\mathscr{A}_2$. Using Theorem 4.2 again, we know $\mathscr{A}$ is invariant. So $\mathscr{A}$ is the global attractor in $D_0^{1,p}(\Omega)\cap L^q(\Omega)$, which is compact, invariant in $D_0^{1,p}(\Omega)\cap L^q(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $D_0^{1,p}(\Omega)\cap L^q(\Omega)$. □

Remark 4.8. From the procedure of the proof of Theorem 4.7, obviously, we know that $\mathcal{A}=\mathcal{A}_2=\mathcal{A}_q$.

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