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NONLINEAR PERIODIC SYSTEMS WITH UNILATERAL CONSTRAINTS

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Dedicated to the memory of Ioan I. Vrabie

Abstract. We consider a general periodic system driven by a nonlinear, nonhomogeneous differential operator, with a maximal monotone term which is not defined everywhere. Using a topological approach based on Leray–Schauder alternative principle, we show the existence of a periodic solution.

1. Introduction

In this paper, we study the existence of solutions for the following periodic system

$$\text{(P)}\qquad \begin{cases} a(u'(t))' \in A(u(t)) + f(t,u(t),u'(t)) & \text{for a.a. } t \in T := [0,b], \\ u(0) = u(b), \quad u'(0) = u'(b). \end{cases}$$

In this problem, $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is a continuous, strictly monotone (hence maximal monotone too) map which satisfies certain polynomial growth conditions. As a special case, the differential operator $u \to a(u')'$ incorporates the vector p-Laplacian $u \to (|u'|^{p-2}u')'$, where $|\cdot|$ denotes the $\mathbb{R}^N$ norm. However, we stress that a is not in general homogeneous. On the right-hand side of (P),

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$A\colon \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map and $D(A) = \{x \in \mathbb{R}^N : A(x) \neq \emptyset\}$ needs not to be all of $\mathbb{R}^N$. In this way problem (P) includes also systems with unilateral constraints (differential variational inequalities). The perturbation $f\colon T \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $t \to f(t,x,y)$ is measurable and for almost all $t \in T$, $(x,y) \to f(t,x,y)$ is continuous). We impose on $f(t,x,y)$ some general growth restrictions and unilateral conditions for $|x|, |y|$ big. The presence of the multivalued maximal monotone term A and the dependence of f on the derivative u', make problem (P) nonvariational. Therefore our approach is topological based on the fixed point theory. More precisely, we use the Leray–Schauder alternative principle.

In the past periodic systems were studied assuming $A \equiv 0$ and that the function $f(t,x,y)$ satisfied the Hartman or the Nagumo–Hartman condition (see Hartman [4], Knobloch [5]). A condition of this kind is very convenient because it produces an a priori bound for the solutions of the problem. We refer also to the works of Knobloch and Schmitt [6], Manasevich and Mawhin [7], Mawhin [9]. We mention that problems with maximal monotone terms (unilateral constraints), both finite and infinite dimensional, can be found in the book of Vrabie [10].

2. Mathematical background – hypotheses

Let X be a reflexive Banach space. By X^* we denote the topological dual of X and by $\langle \cdot , \cdot \rangle$ the duality brackets for the pair (X^*, X). Given $A\colon X \to 2^{X^*}$, the *graph of* A is the set

$$\mathrm{Gr}(A) = \{(u,u^*) \in X \times X^* : u^* \in A(u)\}.$$

We say that A is *monotone* if

$$\langle u^* - v^*, u - v\rangle \geq 0 \quad \text{for all } (u,u^*), (v,v^*) \in \mathrm{Gr}(A).$$

We say that A is *strictly monotone* if the above inequality is strict when $u \neq v$. The map A is *maximal monotone* if $\mathrm{Gr}(A)$ is maximal with respect to the inclusion among the graphs of all monotone maps. This is equivalent to the following condition:

$$\text{if } \langle u^* - v^*, u - v\rangle \geq 0 \quad \text{for all } (u,u^*) \in \mathrm{Gr}(A), \text{ then } (v,v^*) \in \mathrm{Gr}(A).$$

By $D(A)$ we denote the domain of A, that is, the set

$$D(A) = \{u \in X : A(u) \neq \emptyset\}.$$

If $A\colon X \to 2^{X^*}$ is maximal monotone, then it is easy to check that $\mathrm{Gr}(A)$ is sequentially closed in $X_w \times X^*$ and in $X \times X^*_w$. Here by X_w (resp. X^*_w) we denote the space X (resp. X^*) furnished with the weak topology.

When the ambient space is a Hilbert space, then we introduce some useful single valued approximations of the identity and of A. So, suppose that H is a Hilbert space of norm $\|\cdot\|$. We identify H with its dual (that is, $H = H^*$ by the Riesz–Frechet theorem). Given $A\colon H \to 2^H$ and $\lambda > 0$, we introduce the following single-valued maps

$$J_\lambda := (I + \lambda A)^{-1} \quad \text{(the \textit{resolvent} of } A),$$
$$A_\lambda := \frac{1}{\lambda}(I - J_\lambda) \quad \text{(the \textit{Yosida approximation} of } A).$$

The next proposition summarizes the properties of these maps.

Proposition 2.1. *If $A\colon H \to 2^H$ is a maximal monotone map and $\lambda > 0$, then*:

(a) *$J_\lambda\colon H \to H$ is nonexpansive, that is,*
$$\|J_\lambda(u) - J_\lambda(v)\| \le \|u - v\| \quad \textit{for all } u, v \in H;$$
(b) *$A_\lambda(u) \in A(J_\lambda(u))$ for all $u \in H$;*
(c) *A_λ is $1/\lambda$ Lipschitz, that is,*
$$\|A_\lambda(u) - A_\lambda(v)\| \le \frac{1}{\lambda}\|u - v\| \quad \textit{for all } u, v \in H;$$
(d) *$\|A_\lambda(u)\| \le \|A^0(u)\| = \min\{\|u^*\| : u^* \in A(u)\}$ and $A_\lambda(u) \to A^0(u)$ as $\lambda \to 0^+$ for all $u \in D(A)$;*
(e) *$\overline{D(A)}$ is convex and $J_\lambda(x) \to \mathrm{proj}(u; \overline{D(A)})$ as $\lambda \to 0^+$ for all $u \in H$.*

Remarks 2.2. We know that when $A\colon H \to 2^H$ is maximal monotone, then for every $u \in D(A)$, $A(u)$ is nonempty, closed and convex. Therefore it is proximinal (that is, it has the best approximation property, which means that given any $v^* \in H$, we can find $\widehat{u}^* \in A(u)$ such that

$$\|v^* - \widehat{u}^*\|_* = d(v^*, A(u)) = \inf\{\|v^* - u^*\| : u^* \in A(u)\}.$$

Moreover, when $v^* = 0$, the strict convexity of H (a consequence of the parallelogram law), implies that this best approximation element $\widehat{u}^*$ denoted by $A^0(u)$ is unique. The map $u \to A^0(u)$ is known as the "minimal section of A". Similarly, since $\overline{D(A)} \subseteq H$ is convex, given $u \in H$, by $\mathrm{proj}(u, \overline{D(A)})$ we denote the unique best approximation of u from $\overline{D(A)}$. If $D(A) = H$, then $J_\lambda(u) \to u$ for all $u \in H$ as $\lambda \to 0^+$ and so, we can think of J_λ as an approximation of the identity. For more details on these and related issues we refer to Gasinski and Papageorgiou [3] and Vrabie [10].

Let X, Y be two Banach spaces and $G\colon X \to Y$. We introduce the following topological notions for G:

(a) We say that G is *compact,* if it is continuous and maps bounded sets into relatively compact sets.
(b) We say that G is *completely continuous,* if
$$u_n \xrightarrow{w} u \quad \text{in } X \;\Rightarrow\; G(u_n) \to G(u) \quad \text{in } Y.$$

Here and in what follows $\xrightarrow{w}$ denotes weak convergence.

In general, these concepts are distinct. Indeed, let $X = Y = l^1$ and let $G = I =$ the identity map. Then by the Schur property, G is completely continuous, but since l^1 is infinite dimensional, it cannot be compact. However, if X is reflexive, then complete continuity implies compactness. Moreover, if in addition G is linear, then the two notions are equivalent.

Next we recall the Leray–Schauder alternative principle which we will use in the analysis of problem (P); see e.g. [3, p. 627].

Theorem 2.3. *If X is a Banach space, $G\colon X \to X$ is compact and*

$$K := \{u \in X : u = \theta G(u) \text{ for some } 0 < \theta < 1\},$$

then one of the following statements holds:

(a) *K is unbounded*;
(b) *G has a fixed point.*

In the analysis of problem (P) we will use the space

$$W^{1,p}_{\text{per}}\big((0,b);\mathbb{R}^N\big) := \big\{u \in W^{1,p}\big((0,b);\mathbb{R}^N\big) : u(0) = u(b)\big\}, \quad 1 < p < \infty.$$

By $\|\cdot\|$ we denote the norm of this space which is defined by

$$\|u\| = (\|u\|_p^p + \|u'\|_p^p)^{1/p} \quad \text{for all } u \in W^{1,p}_{\text{per}}\big((0,b);\mathbb{R}^N\big),$$

where $\|\cdot\|_p$ denotes the L^p-norm. In the sequel, for notational economy, we will write $W^{1,p}_N = W^{1,p}_{\text{per}}((0,b);\mathbb{R}^N)$. Also, given a measurable function $g\colon T \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ (for example, a Carathéodory function), by N_g we denote the Nemytski operator corresponding to g, defined by

$$N_g(u)(\cdot) = g(\cdot, u(\cdot), u'(\cdot)) \quad \text{for all } u \in W^{1,p}_N.$$

Now we introduce the hypotheses on the data of (P).

H(a) $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is a map such that $a(y) = a_0(|y|)y$ with $a_0\colon (0,+\infty) \to (0,+\infty)$ such that $ta_0 \to 0^+$ as $t \to 0^+$ and

(i) a is continuous, strictly monotone (hence maximal monotone too);
(ii) $(a(y),y)_{\mathbb{R}^N} \geq C_0|y|^p$ for all $y \in \mathbb{R}^N$, with $C_0 > 0$, $2 \leq p < +\infty$;
(iii) $|a(y)| \leq C_1(1 + |y|^{p-1})$ for all $y \in \mathbb{R}^N$, with $C_1 > 0$.

Remarks 2.4. The above hypotheses are general and include the case of the vector p-Laplacian which corresponds to the map

$$y \to |y|^{p-2}y, \quad \text{for all } y \in \mathbb{R}^N.$$

Other possibilities are the maps

$$y \to |y|^{p-2}y + |y|^{q-2}y, \quad 2 \leq q < p, \quad \text{for all } y \in \mathbb{R}^N;$$
$$y \to (1 + |y|^2)^{(p-2)/2}y, \quad \text{for all } y \in \mathbb{R}^N.$$

The restriction $2 \le p$ (see hypothesis H(a) (ii)) is needed because in general we have $D(A) \ne \mathbb{R}^N$ (see hypothesis H(A) below). If $D(A) = \mathbb{R}^N$, then we can have $1 < p < \infty$. Finally note that $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is a homeomorphism.

The hypothesis on the multivalued term A is the following:

H(A) $A\colon \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map such that $0 \in A(0)$.

REMARK 2.5. We do not require $D(A) = \mathbb{R}^N$. This way we incorporate in our framework systems with inequality constraints.

The hypotheses on the perturbation $f(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ are the following:

H(f) $f\colon T \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying:

(i) there exist $\beta_1\colon T \times \mathbb{R}^+ \to \mathbb{R}^+$ and $\beta_2\colon T \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t,x,y)| \le \beta_1(t,|x|) + \beta_2(t,|x|)|y|^{q-1}$$

for almost all $t \in T$, all x, $y \in \mathbb{R}^N$, with $1 < q < p$ and, for every $M > 0$, we have

$$\sup\{\beta_1(t,s) : 0 \le s \le M\} \le \gamma_{1,M}(t), \quad \text{for a.a. } t \in T,$$
$$\sup\{\beta_2(t,s) : 0 \le s \le M\} \le \gamma_{2,M}(t), \quad \text{for a.a. } t \in T,$$

where $\gamma_{1,M} \in L^{p'}(T)$ and $\gamma_{2,M} \in L^\infty(T)$ (where $1/p + 1/p' = 1$);

(ii) there exists a function $\eta \in L^\infty(T)$ such that $0 \le \eta(t)$ for almost all $t \in T$, $\eta \ne 0$, and for every $\varepsilon > 0$, there exist $M_\varepsilon > 0$ and $\widehat{C}_\varepsilon > 0$ such that

$$(f(t,x,y),x)_{\mathbb{R}^N} \ge [\eta(t) - \varepsilon]|x|^p - \widehat{C}_\varepsilon |y|^{q-1}|x|$$

for almost all $t \in T$, all $|x|$, $|y| \ge M_\varepsilon$.

REMARK 2.6. Consider the function

$$f(t,x,y) = \eta(t)g(x) + \widehat{C}|y|^{q-1} + k(t)x$$

where $\eta \in L^\infty(T)$, $\eta(t) \ge 0$ for almost all $t \in T$, $\eta \ne 0$, $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ and satisfies

$$\liminf_{|x|\to\infty} \frac{(g(x),x)_{\mathbb{R}^N}}{|x|^p} \ge \mu > 0, \quad \widehat{C} \in \mathbb{R}^N \text{ and } k \in L^{p'}(T),\ k(t) \ge 0 \text{ for a.a. } t \in T.$$

This function satisfies hypotheses H(f) above.

3. Existence theorem

Let $g \in L^{p'}(T, \mathbb{R}^N)$. We first consider the following periodic problem

$$\begin{cases} -a(u'(t))' + |u(t)|^{p-2}u(t) = g(t) & \text{for a.a. } t \in T := [0,b], \\ u(0) = u(b), \quad u'(0) = u'(b). \end{cases} \tag{3.1}$$

Proposition 3.1. *If hypotheses* H(a) *hold and* $g \in L^{p'}(T,\mathbb{R}^N)$*, then problem* (3.1) *admits a unique solution* $\widehat{u} \in C^1(T,\mathbb{R}^N)$.

Proof. Let $G\colon W_N^{1,p} \to \big(W_N^{1,p}\big)^*$ be the nonlinear map defined by

$$\langle G(u),h\rangle = \int_0^b (a(u'),h')_{\mathbb{R}^N}\,dt \quad \text{for all } u,h \in W_N^{1,p}.$$

Hypotheses H(a) imply that G is continuous, monotone, hence maximal monotone too. In addition, let

$$\xi_p\colon L^p\big(T,\mathbb{R}^N\big) \to L^{p'}\big(T,\mathbb{R}^N\big) = \big(L^p\big(T,\mathbb{R}^N\big)\big)^*$$

be defined by

$$\xi_p(u)(\,\cdot\,) = |u(\,\cdot\,)|^{p-2}u(\,\cdot\,).$$

This map is continuous and strictly monotone, hence maximal monotone too. Then the map $V = G+\xi_p\colon W_N^{1,p} \to \big(W_N^{1,p}\big)^*$ is continuous and strictly monotone, hence maximal monotone too. Also for all $u \in W_N^{1,p}$, we have

$$\langle V(u),u\rangle \geq C_0\|u'\|_p^p + \|u\|_p^p$$

(see hypothesis H(a) (ii)), hence V is coercive.

Invoking Corollary 3.2.32, p. 320 of Gasinski and Papageorgiou [3], we infer that V is surjective. So, we can find $\widehat{u} \in W_N^{1,p} \subseteq C\big(T,\mathbb{R}^N\big)$ such that $V(\widehat{u}) = g$, therefore

$$\begin{cases} -a(\widehat{u}'(t))' + |\widehat{u}(t)|^{p-2}\widehat{u}(t) = g(t) & \text{for a.a. } t \in T := [0,b], \\ \widehat{u}(0) = \widehat{u}(b). \end{cases} \tag{3.2}$$

Moreover, the strict monotonicity of V implies that this solution is unique. From (3.2) we see that $(a(u'))' \in L^{p'}(T,\mathbb{R}^N)$. Also, since $\widehat{u}' \in L^p(T,\mathbb{R}^N)$, from hypothesis H$(a)$ (iii) we see that $a(\widehat{u}') \in L^{p'}(T,\mathbb{R}^N)$. It follows that $a(\widehat{u}') \in W_N^{1,p'} \subseteq C\big(T,\mathbb{R}^N\big)$. Recalling that a is a homeomorphism, we infer that $\widehat{u}' \in C\big(T,\mathbb{R}^N\big)$ and so, we conclude that $\widehat{u} \in C^1(T,\mathbb{R}^N)$. Finally, it follows that $\widehat{u}'(0) = \widehat{u}'(b)$. □

Let $\widehat{a}\colon D(\widehat{a}) \subseteq L^p(T,\mathbb{R}^N) \to L^{p'}(T,\mathbb{R}^N)$ be defined by

$$\widehat{a}(u)(\,\cdot\,) = -a(u'(\,\cdot\,))', \tag{3.3}$$

for all $u \in D(\widehat{a}) = \big\{y \in C^1(T,\mathbb{R}^N) : a(y') \in W_N^{1,p'}, y(0) = y(b), y'(0) = y'(b)\big\}$. We have the following result for this map:

Proposition 3.2. *If hypotheses* H(a) *hold, then the map*

$$\widehat{a}\colon D(\widehat{a}) \subseteq L^p(T,\mathbb{R}^N) \to L^{p'}(T,\mathbb{R}^N)$$

defined by (3.3) *is maximal monotone.*

Proof. From Proposition 3.1 we know that

$$R(\widehat{a}+\xi_p) = L^{p'}(T,\mathbb{R}^N). \tag{3.4}$$

Also, the map $\widehat{a}$ is monotone. Indeed, let $u,\ v \in D(\widehat{a})$ and let $\langle \cdot\,,\,\cdot\rangle_{p,p'}$ be the duality brackets for the pair $\big(L^p\big(T,\mathbb{R}^N\big), L^{p'}\big(T,\mathbb{R}^N\big)\big)$. We have

$$\begin{aligned}
&\langle \widehat{a}(u)-\widehat{a}(v), u-v\rangle_{p,p'}\\
&\quad= \int_0^b \big(-a(u')'+a(v')', u-v\big)_{\mathbb{R}^N}\,dt\\
&\quad= \int_0^b \big(a(u')-a(v'), u'-v'\big)_{\mathbb{R}^N}\,dt \quad \text{(by integration by parts)}\\
&\quad\geq 0,
\end{aligned}$$

hence $\widehat{a}$ is monotone.

Suppose that $v \in L^p(T,\mathbb{R}^N)$, $v^* \in L^{p'}(T,\mathbb{R}^N)$ and assume that

$$\langle \widehat{a}(u)-v^*, u-v\rangle_{p,p'} \geq 0 \quad \text{for all } u \in D(\widehat{a}). \tag{3.5}$$

From (3.4) we know that there exists $u_1 \in D(\widehat{a})$ such that

$$\widehat{a}(u_1)+\xi_p(u_1) = v^*+\xi_p(v). \tag{3.6}$$

Using (3.6) in (3.5) with $u = u_1 \in D(\widehat{a})$, we obtain

$$0 \leq \langle \xi_p(v)-\xi_p(u_1), u_1-v\rangle_{p,p'},$$

hence $u_1 = v$ (recall that ξ_p is strictly monotone), therefore $\widehat{a}(u_1) = v^*$ (see (3.6)). So, $(v,v^*) \in \mathrm{Gr}(\widehat{a})$ and we conclude that $\widehat{a}$ is maximal monotone. □

For $\lambda > 0$, we next consider the following approximation to problem (P):

$$(\mathrm{P}_\lambda)\qquad \begin{cases} a(u'(t))' = A_\lambda(u(t))+f(t,u(t),u'(t)) & \text{for a.a. } t \in T := [0,b],\\ u(0)=u(b), \quad u'(0)=u'(b). \end{cases}$$

Proposition 3.3. *If hypotheses* H(a), H(A), H(f) *hold and* $\lambda > 0$, *then problem* (P_λ) *has a solution* $u_\lambda \in C^1(T,\mathbb{R}^N)$.

Proof. Let $\widehat{A}_\lambda\colon L^p(T,\mathbb{R}^N) \to L^{p'}(T,\mathbb{R}^N)$ be defined by

$$\widehat{A}_\lambda(u)(\,\cdot\,) = A_\lambda(u(\,\cdot\,)).$$

Recall that $1 < p' \leq 2 \leq p$. We consider the map

$$L_\lambda\colon D(\widehat{a}) \subseteq L^p(T,\mathbb{R}^N) \to L^{p'}(T,\mathbb{R}^N)$$

defined by

$$L_\lambda(u) = \widehat{a}(u)+\xi_p(u)+\widehat{A}_\lambda(u) \quad \text{for all } u \in D(\widehat{a}).$$

Using Theorem 3.2.41 of Gasinski and Papageorgiou [3, p. 328,], we have

$$L_\lambda \text{ is maximal monotone.} \tag{3.7}$$

Also, since A_λ is monotone and $A_\lambda(0) = 0$, via hypothesis H(a) (ii) we see that

$$L_\lambda \text{ is coercive.} \tag{3.8}$$

From (3.7), (3.8) and Corollary 3.2.31 of Gasinski and Papageorgiou [3, p. 319], it follows that L_λ is surjective. Evidently L_λ is strictly monotone (recall that ξ_p is so). Hence, the inverse map

$$L_\lambda^{-1} \colon L^{p'}\big(T, \mathbb{R}^N\big) \to D(\widehat{a}) \subseteq L^p\big(T, \mathbb{R}^N\big)$$

is well defined. Recall that $D(\widehat{a}) \subseteq C^1\big(T, \mathbb{R}^N\big)$.

Claim 1. $L_\lambda^{-1} \colon L^{p'}(T, \mathbb{R}^N) \to C^1(T, \mathbb{R}^N)$ is completely continuous.

Let $g_n \xrightarrow{w} g$ in $L^{p'}(T, \mathbb{R}^N)$. Let $u_n = L_\lambda^{-1}(g_n)$ for $n \in \mathbb{N}$, and $u = L_\lambda^{-1}(g)$. We have $L_\lambda(u_n) = g_n$ for all $n \in \mathbb{N}$, hence

$$\widehat{a}(u_n) + \xi_p(u_n) + \widehat{A}_\lambda(u_n) = g_n, \quad u_n \in D(\widehat{a}), \quad \text{for all } n \in \mathbb{N},$$

therefore

$$C_0\|u_n'\|_p^p + \|u_n\|_p^p \leq \|g_n\|_{p'}\|u_n\|_p$$

(recall that $(A_\lambda(x), x)_{\mathbb{R}^N} \geq 0$ for all $x \in \mathbb{R}^N$) and we derive $\|u_n\|^p \leq C_2\|u_n\|$ for some $C_2 > 0$ and for all $n \in \mathbb{N}$, therefore

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_N^{1,p} \text{ is bounded.} \tag{3.9}$$

For almost all $t \in T$ and all $n \in \mathbb{N}$ we have

$$-a(u_n'(t))' + |u_n(t)|^{p-2}u_n(t) + A_\lambda(u_n(t)) = g_n(t),$$

therefore

$$\{a(u_n')'\}_{n\in\mathbb{N}} \subseteq L^{p'}\big(T, \mathbb{R}^N\big) \text{ is bounded.} \tag{3.10}$$

Hypothesis H(a) (iii) and (3.9) imply that

$$\{a(u_n')\}_{n\in\mathbb{N}} \subseteq L^{p'}\big(T, \mathbb{R}^N\big) \text{ is bounded.} \tag{3.11}$$

Then, from (3.10) and (3.11), it follows that

$$\{a(u_n')\}_{n\in\mathbb{N}} \subseteq W^{1,p'}\big(T, \mathbb{R}^N\big) \text{ is bounded,}$$

hence

$$\{a(u_n')\}_{n\in\mathbb{N}} \subseteq C\big(T, \mathbb{R}^N\big) \text{ is relatively compact} \tag{3.12}$$

(recall that $W^{1,p'}\big(T, \mathbb{R}^N\big) \subseteq C\big(T, \mathbb{R}^N\big)$ compactly). We know that a is a homeomorphism. Let $\widehat{\eta} \colon C\big(T, \mathbb{R}^N\big) \to C\big(T, \mathbb{R}^N\big)$ be defined by

$$\widehat{\eta}(u)(\,\cdot\,) = a^{-1}(u(\,\cdot\,)) \quad \text{for all } u \in C\big(T, \mathbb{R}^N\big).$$

Evidently $\widehat{\eta}$ is continuous and bounded (that is, maps bounded sets to bounded sets). Then, from (3.12) we infer that

$$\{u_n'\}_{n\in\mathbb{N}} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.} \tag{3.13}$$

In addition by (3.9) and the compact embedding of $W^{1,p}(T,\mathbb{R}^N)$ into $C(T,\mathbb{R}^N)$, we conclude that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq C(T,\mathbb{R}^N) \text{ is relatively compact.} \tag{3.14}$$

From (3.13) and (3.14) we infer that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq C^1(T,\mathbb{R}^N) \text{ is relatively compact.}$$

So, we may assume that (along a subsequence)

$$u_n \to \widehat{u} \quad \text{in } C^1(T,\mathbb{R}^N), \text{ as } n\to\infty. \tag{3.15}$$

Note that $(u_n, g_n) \in \mathrm{Gr}(L_\lambda)$ for all $n \in \mathbb{N}$. Since L_λ is maximal monotone, we know that $\mathrm{Gr}(L_\lambda)$ is sequentially closed in $L^p(T,\mathbb{R}^N) \times L^{p'}(T,\mathbb{R}^N)_w$. Therefore $(\widehat{u}, g) \in \mathrm{Gr}(L_\lambda)$, hence $L_\lambda(\widehat{u}) = g$. Hence, for the original sequence, we have

$$u_n \to \widehat{u} = L_\lambda^{-1}(g) \quad \text{in } C^1(T,\mathbb{R}^N), \text{ as } n\to\infty.$$

This proves Claim 1.

Now let $\widehat{N} \colon W^{1,p}_N \to L^{p'}(T,\mathbb{R}^N)$ be defined by

$$\widehat{N}(u)(\cdot) = -N_f(u)(\cdot) + \xi_p(u)(\cdot) \quad \text{for all } u \in W^{1,p}_N.$$

CLAIM 2. $\widehat{N} \colon W^{1,p}_N \to L^{p'}(T,\mathbb{R}^N)$ is continuous.

Consider a sequence $u_n \to u$ in $W^{1,p}_N$. Then $u_n \to u$ in $C(T,\mathbb{R}^N)$ and so

$$\|u_n\|_{C(T,\mathbb{R}^N)} \le M \quad \text{for some } M>0, \text{ all } n\in\mathbb{N}.$$

Then hypothesis H(f) (i) implies that

$$|f(t,u_n(t),u_n'(t))| \le \gamma_{1,M}(t) + \gamma_{2,M}(t)(1+|u_n'(t)|^{p-1}) \tag{3.16}$$

for all $t \in T$, all $n \in \mathbb{N}$. We may assume that

$$\begin{cases} u_n(t) \to u(t) & \text{for all } t\in T, \\ u_n'(t) \to u'(t) & \text{for a.a. } t\in T, \\ |u_n'(t)| \le \varphi(t) & \text{for a.a. } t\in T, \text{ all } n\in\mathbb{N} \text{ with } \varphi\in L^p(T). \end{cases} \tag{3.17}$$

From (3.17) it follows that

$$f(t,u_n(t),u_n'(t)) \to f(t,u(t),u'(t)) \quad \text{for a.a. } t\in T. \tag{3.18}$$

Then (3.16)–(3.18) and Vitali's theorem (the extended dominated convergence theorem; see Gasinski and Papageorgiou [3, p. 901]) imply that

$$N_f(u_n) \to N_f(u) \quad \text{in } L^{p'}(T,\mathbb{R}^N), \text{ as } n\to\infty.$$

Also, the continuity of ξ_p implies that

$$\xi_p(u_n) \to \xi_p(u) \quad \text{in } L^{p'}(T,\mathbb{R}^N), \text{ as } n\to\infty.$$

We conclude that

$$\widehat{N}(u_n) \to \widehat{N}(u) \quad \text{in } L^{p'}\big(T, \mathbb{R}^N\big), \text{ as } n \to \infty,$$

that is, $\widehat{N}(\,\cdot\,)$ is continuous. This proves Claim 2.

Let $K_\lambda = \big\{u \in W^{1,p}_N : u = \theta L_\lambda^{-1}\widehat{N}(u),\ 0 < \theta < 1\big\}$.

Claim 3. $K_\lambda \subseteq W^{1,p}_N$ is bounded.

Let $u \in K_\lambda$. Then, for some $\theta \in (0,1)$, we have $L_\lambda(u/\theta) = \widehat{N}(u)$, hence

$$\widehat{a}\bigg(\frac{1}{\theta}\,u\bigg) + \frac{1}{\theta^{p-1}}\,\xi_p(u) + \widehat{A}_\lambda\bigg(\frac{1}{\theta}\,u\bigg) = -N_f(u) + \xi_p(u). \tag{3.19}$$

Hypotheses H(f) (i), (ii) imply that for a given $\varepsilon > 0$, we can find $C_3 = C_3(\varepsilon) > 0$ and $\mu \in L^{p'}(T)$ such that

$$(-f(t,x,y), x)_{\mathbb{R}^N} \le [-\eta(t) + \varepsilon]|x|^p + C_3|y|^{q-1}|x| + \mu(t) \tag{3.20}$$

for almost all $t \in T$, all $x, y \in \mathbb{R}^N$. On (3.19) we act with u. Using hypothesis H(a) (ii) and (3.20), we obtain

$$\begin{aligned} &\frac{C_0}{\theta^{p-1}}\|u'\|_p^p + \frac{1}{\theta^{p-1}}\|u\|_p^p \\ &\qquad\le \int_0^b [-\eta(t) + \varepsilon]|u|^p\,dt + C_3\int_0^b |u'|^{q-1}|u|\,dt + C_4 \end{aligned} \tag{3.21}$$

with $C_4 = \|\mu\|_1 > 0$, hence

$$C_0\|u'\|_p^p + [1-\varepsilon]\|u\|_p^p + \int_0^b \eta(t)|u|^p\,dt \le C_3\int_0^b |u'|^{q-1}|u|\,dt + C_4 \tag{3.22}$$

(recall that $0 < \theta < 1$). Using Young's inequality with $\varepsilon > 0$ (see Gasinski and Papageorgiou [3, p. 913]), we obtain

$$C_3\int_0^b |u'|^{q-1}|u|\,dt \le C_5\|u'\|^\tau + \varepsilon\|u\|_p^p \tag{3.23}$$

for some $\tau < p$ and some $C_5 = C_5(\varepsilon) > 0$ (recall that $\theta < 1$ and $q < p$).

We return to (3.22) and use (3.23). Then

$$C_0\|u'\|_p^p + [1-2\varepsilon]\|u\|_p^p + \int_0^b \eta(t)|u|^p\,dt \le C_6[\|u'\|^\tau + 1] \tag{3.24}$$

for some $C_6 > 0$. Also, reasoning as in the proof of Lemma 1 of Aizicovici, Papageorgiou and Staicu [1], we conclude that

$$C_6\|u'\|_p^p + \int_0^b \eta(t)|u|^p\,dt \ge C_7\|u\|^p \quad \text{for some } C_7 > 0.$$

Using this in (3.24) and choosing $\varepsilon \in (0, 1/2)$ we finally arrive at

$$\|u\|^p \le C_8[\|u\|^\tau + 1] \quad \text{for some } C_8 > 0.$$

Since $\tau < p$, it follows that $K_\lambda \subseteq W^{1,p}_N$ is bounded. This proves Claim 3.

Claims 1 and 2 imply that $L_\lambda^{-1}\widehat{N}\colon W_N^{1,p} \to W_N^{1,p}$ is continuous. In addition, since $\widehat{N}$ maps bounded sets into bounded sets, it follows by Claim 1 that $L_\lambda^{-1}\widehat{N}$ maps bounded sets into relatively compact sets. Hence $L_\lambda^{-1}\widehat{N}$ is compact. Combining this with Claim 3 and Theorem 2.3 (the Leray–Schauder alternative principle), we see that we can find $u_\lambda \in D(A)$ such that

$$u_\lambda = L_\lambda^{-1}\widehat{N}(u_\lambda).$$

Hence $u_\lambda \in C^1(T, \mathbb{R}^N)$ is a solution of (P_λ). □

Letting $\lambda \to 0^+$ we will now produce a solution of problem (P).

Theorem 3.4. *If hypotheses* H(a), H(A), H(f) *hold, then problem* (P) *has a solution* $\widetilde{u} \in C^1(T, \mathbb{R}^N)$.

Proof. Let $\lambda_n \to 0^+$ and let $u_n = u_{\lambda_n}$ be the solution of problem (P_{λ_n}) (see Proposition 3.3). We have

$$\widehat{a}(u_n) + \widehat{A}_{\lambda_n}(u_n) + N_f(u_n) = 0 \quad \text{in } L^{p'}(T, \mathbb{R}^N), \text{ for all } n \in \mathbb{N}. \tag{3.25}$$

By using integration by parts, H(a) (iii) and that

$$t(\widehat{A}_{\lambda_n}(x), x)_{\mathbb{R}^N} \geq 0 \quad \text{for all } x \in \mathbb{R}^N,$$

we obtain

$$\begin{aligned} C_0\|u_n'\|_p^p &\leq \int_0^b (-f(t, u_n, u_n'), u_n)_{\mathbb{R}^N}\,dt \\ &\leq \int_0^b [-\eta(t) + \varepsilon]|u_n(t)|^p\,dt + C_3\int_0^b |u_n'|^{q-1}|u_n|\,dt + \|\mu\|_1 \end{aligned}$$

(see (3.20), hence, as before, using Young's inequality and Lemma 1 of [1], we obtain

$$[C_9 - \varepsilon C_{10}]\|u_n\|^p \leq C_{11}[1 + \|u_n\|^\tau] \quad \text{with } C_9, C_{10}, C_{11} > 0,\ 1 < \tau < p.$$

Choosing $\varepsilon \in (0, C_9/C_{10})$ and recalling that $\tau < p$, we obtain

$$\{u_n\}_{n\geq 1} \subseteq W_N^{1,p} \quad \text{is bounded.} \tag{3.26}$$

On (3.25) we act with $\widehat{A}_{\lambda_n}(u_n) \in L^p(T, \mathbb{R}^N)$. Noting that

$$\big|\widehat{A}_{\lambda_n}(u_n)(t)\big| = |A_{\lambda_n}(u_n(t))| \leq \frac{1}{\lambda_n}|u_n(t)| \quad \text{for all } t \in T, \text{ all } n \in \mathbb{N},$$

we have

$$\begin{aligned} &\int_0^b (-a(u_n'(t))', A_{\lambda_n}(u_n))_{\mathbb{R}^N}\,dt + \|A_{\lambda_n}(u_n)\|_2^2 \\ &\qquad\qquad \leq \int_0^b |N_f(u_n)||A_{\lambda_n}(u_n)|\,dt. \end{aligned} \tag{3.27}$$

From Proposition 2.1 we know that A_{λ_n} is Lipschitz continuous. Hence by Rademacher's theorem (see Gasinski and Papageorgiou [3, p. 56]), A_{λ_n} is differentiable almost everywhere on $\mathbb{R}^N$. Recall that $u_n \in C^1(T, \mathbb{R}^N)$. So, the map $t \to A_{\lambda_n}(u_n(t))$ is differentiable almost everywhere on T and

$$\frac{d}{dt} A_{\lambda_n}(u_n(t)) = A'_{\lambda_n}(u_n(t))u'_n(t) \quad \text{for a.a. } t \in T$$

(chain rule). Here $A'_{\lambda_n}(u_n(t))u'_n(t)$ is interpreted to be zero when $u'_n(t) = 0$ (even if A_{λ_n} is not differentiable); see Marcus and Mizel [8]. Moreover, by the monotonicity of A_{λ_n} (see Proposition 2.1), at every point of differentiability $x \in \mathbb{R}^N$, we have

$$(y, A'_{\lambda_n}(x)y)_{\mathbb{R}^N} \geq 0 \quad \text{for all } y \in \mathbb{R}^N. \tag{3.28}$$

Performing an integration by parts, we have

$$\begin{aligned}\int_0^b (-a(u'_n)', A_{\lambda_n}(u_n))_{\mathbb{R}^N}\, dt &= \int_0^b \bigg(a(u'_n), \frac{d}{dt} A_{\lambda_n}(u_n)\bigg)_{\mathbb{R}^N} dt \\ &= \int_0^b a_0(|u'_n|)(u'_n, A'_{\lambda_n}(u_n)u'_n)_{\mathbb{R}^N}\, dt \quad \text{(see hypotheses H}(a)) \\ &\geq 0\end{aligned} \tag{3.29}$$

for all $n \in \mathbb{N}$ (see (3.28)). Returning to (3.27) and using (3.29) and the Cauchy–Schwarz inequality (recall that $1 < p' \leq 2 \leq p$), we obtain

$$\|\widehat{A}_{\lambda_n}(u_n)\|_2 \leq C_{12} \quad \text{for some } C_{12} > 0, \text{ all } n \in \mathbb{N}. \tag{3.30}$$

So, we may assume that

$$\widehat{A}_{\lambda_n}(u_n) \xrightarrow{w} k \quad \text{in } L^2\big(T, \mathbb{R}^N\big) \quad \big(\text{hence in } L^{p'}\big(T, \mathbb{R}^N\big) \text{ too}\big). \tag{3.31}$$

As before (see the proof of Proposition 3.3, Claim 1), using (3.26) and (3.25), we obtain that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq C^1(T, \mathbb{R}^N) \text{ is relatively compact.}$$

Hence we may assume that

$$u_n \to \widetilde{u} \quad \text{in } C^1(T, \mathbb{R}^N), \text{ as } n \to \infty. \tag{3.32}$$

We have

$$N_f(u_n) \to N_f(\widetilde{u}) \quad \text{in } L^{p'}(T, \mathbb{R}^N), \text{ as } n \to \infty. \tag{3.33}$$

Therefore, passing to the limit as $n \to \infty$ in (3.25) and using (3.31)–(3.33), we obtain

$$\widehat{a}(\widetilde{u}) + k + N_f(\widetilde{u}) = 0.$$

We now complete the proof of the theorem by showing that

$$k(t) \in A(\widetilde{u}(t) \quad \text{for almost all } t \in T.$$

To this end, note that

$$J_{\lambda_n}(u_n(t)) + \lambda_n A_{\lambda_n}(u_n(t)) = u_n(t) \quad \text{for all } t \in T, \text{ all } n \in \mathbb{N},$$

hence

$$\widehat{J}_{\lambda_n}(u_n) + \lambda_n \widehat{A}_{\lambda_n}(u_n) = u_n$$

with

$$\widehat{J}_{\lambda_n}(u)(\,\cdot\,) = J_{\lambda_n}(u_n(\,\cdot\,)) \quad \text{for all } u \in W_N^{1,p},$$

therefore

$$\|\widehat{J}_{\lambda_n}(u_n) - u_n\|_2 = \lambda_n \|\widehat{A}_{\lambda_n}(u_n)\|_2 \leq C_{12}\lambda_n \quad \text{for all } n \in \mathbb{N}$$

(see (3.30)), and we conclude that $\widehat{J}_{\lambda_n}(u_n) \to \widetilde{u}$ in $L^2(T, \mathbb{R}^N)$ (see (3.32)).

From Proposition 2.1 we know that

$$A_{\lambda_n}(u_n(t)) \in A(J_{\lambda_n}(u_n(t))) \quad \text{for all } t \in T, \text{ all } n \in \mathbb{N},$$

therefore

$$(\widehat{J}_{\lambda_n}(u_n), \widehat{A}_{\lambda_n}(u_n)) \in \mathrm{Gr}(\widehat{A}) \quad \text{for all } n \in \mathbb{N},$$

where $\widehat{A}$ is the lifting of A on $L^2(T, \mathbb{R}^N)$, that is,

$$\widehat{A}(u) = \big\{v \in L^2\big(T, \mathbb{R}^N\big) : v(t) \in A(u(t)) \text{ for a.a. } t \in T\big\}.$$

We know that $\widehat{A}$ is maximal monotone on $L^2(T, \mathbb{R}^N)$ (see e.g. Aizicovici, Papageorgiou and Staicu [2, Lemma 1]). Therefore

$$\mathrm{Gr}(\widehat{A}) \subseteq L^2(T, \mathbb{R}^N) \times L^2(T, \mathbb{R}^N)_w \text{ is sequentially closed.}$$

Hence, from (3.32) and (3.31), it follows that $(\widetilde{u}, k) \in \mathrm{Gr}(\widehat{A})$, hence $k(t) \in A(\widetilde{u}(t))$ for almost all $t \in T$. We conclude that $\widetilde{u} \in C^1(T, \mathbb{R}^N)$ is a solution of problem (P). □

4. An example

Let $C = \mathbb{R}^N_+ = \big\{x = (x_k)_{k=1}^N \in \mathbb{R}^N : x_k \geq 0 \text{ for all } k = 1, \ldots, N\big\}$ and let i_C be the indicator function of C, that is

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C = \mathbb{R}^N_+, \\ +\infty & \text{otherwise.} \end{cases}$$

We know that i_C is proper, convex, lower semicontinuous (that is, $i_C \in \Gamma_C(\mathbb{R}^N)$; see Gasinski and Papageorgiou [3, p. 488]). We set

$$A(x) = \partial i_C(x) = N_C(x),$$

where ∂ stands for subdifferential in the sense of convex analysis and $N_C(x)$ is the normal cone to C at x.

Recall that

$$\begin{aligned} N_C(x) &= \big\{x^* \in \mathbb{R}^N : (x^*, c-x)_{\mathbb{R}^N} \le 0 \text{ for all } c \in C\big\} \\ &= \big\{x^* \in \mathbb{R}^N : (x^*, x)_{\mathbb{R}^N} = \sigma(x^*, C) := \sup\big\{(x^*, c) : c \in \mathbb{R}^N\big\}\big\}. \end{aligned}$$

Evidently, if $x \in \operatorname{int}(C)$, then $N_C(x) = \{0\}$ and if $x \notin C = \operatorname{dom} i_C$ then $N_C(x) = \emptyset$ (see Gasinski and Papageorgiou [3, p. 526]). We have

$$D(A) = C = \mathbb{R}^N_+$$

and

$$A(x) = \begin{cases} \{0\} & \text{if } x = (x_k)_{k=1}^N \in \operatorname{int}(\mathbb{R}^N_+) \\ & \text{(that is } x_k > 0 \text{ for all } k = 1, \dots, N), \\ -\mathbb{R}^N_+ \cap \{x\}^\perp & \text{if } x = (x_k)_{k=1}^N \in \partial\mathbb{R}^N_+ \\ & \text{(that is } x_k = 0 \text{ for some } k = 1, \dots, N). \end{cases}$$

Then problem (P) is equivalent to the following differential inequality

$$\begin{cases} a(u'(t))' = f(t, u(t), u'(t)) & \text{a.e. on } \big\{t \in T : u(t) \in \operatorname{int}\big(\mathbb{R}^N_+\big)\big\}, \\ a(u'(t))' \le f(t, u(t), u'(t)) & \text{a.e. on } \big\{t \in T : u(t) \in \partial\mathbb{R}^N_+\big\}, \\ (f(t, u(t), u'(t)) - a(u'(t))', u(t))_{\mathbb{R}^N} = 0 & \text{for a.a. } t \in T, \\ u(t) \in \mathbb{R}^N_+ & \text{for } t \in T,\ u(0) = u(b), u'(0) = u'(b). \end{cases} \tag{4.1}$$

Using Theorem 3.4, we can state the following existence result for problem (4.1):

Theorem 4.1. *If hypotheses* H(a), H(f) *hold, then problem* (4.1) *admits a solution* $\widetilde{u} \in C^1(T, \mathbb{R}^N)$.

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