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ON THE LINEARIZATION OF VECTOR FIELDS ON A TORUS WITH PRESCRIBED FREQUENCY

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Abstract. In this paper we are mainly concerned with the linearization of the flow with prescribed frequency for analytic perturbation of constant vector fields on a torus under weaker non-degeneracy condition and non-resonant condition. As is well known the perturbation of constant vector fields may induce a shift of frequency, when Kolmogorov's non-degeneracy condition is violated. By introducing external parameters and using the polynomial structure to truncate, we prove that if the frequency mapping has the nonzero Brouwer's topological degree at some non-resonant frequency, then the conjugated vector fields will have a linear flow with this frequency.

1. Introduction and main results

KAM theory was founded by A.N. Kolmogorov, V.I. Arnold and J. Moser [1], [10], [12] as a powerful tool to deal with the problem of small denominator and small perturbation of conservative systems. Due to its importance in dynamical systems, KAM theory has been extensively applied to Hamiltonian, reversible, volume preserving, and general (dissipative) systems.

In this paper we consider a vector field with parameters $X = N + P$, where $N = \omega(\xi)$ denotes a constant vector on the n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, describing

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uniform rotation motions with frequency $\omega(\xi) = (\omega_1(\xi), \ldots, \omega_n(\xi))$, $P(\theta;\xi)$ is a small perturbation, the parameters ξ vary on some bounded closed connected domain Π. Suppose the frequency $\omega(\xi)$ satisfies

$$|\langle k, \omega(\xi)\rangle| \geq \frac{\alpha}{\Delta(|k|)}, \quad \text{for all } 0 \neq k \in \mathbb{Z}^n, \tag{1.1}$$

where $\alpha > 0$, Δ is continuous increasing unbounded function $\Delta\colon [1,\infty) \to [1,\infty)$ such that $\Delta(1) = 1$ and

$$\int_1^\infty \frac{\ln \Delta(t)}{t^2}\, dt < \infty.$$

The condition (1.1) is usually called Brjuno–Rüssmann's non-resonant condition. When the frequency $\omega(\xi)$ satisfies the *Kolmogorov's non-degeneracy* condition

$$\operatorname{Rank}\left(\frac{\partial \omega}{\partial \xi}\right) = n, \quad \text{for all } \xi \in \Pi, \tag{1.2}$$

and non-resonant condition (1.1), Pöschel [14] proved that if the perturbation P is sufficiently small, and if we are allowed to add a small correctional n-vector to adjust frequency, then X is conjugated to ω, i.e. for any given frequency in the image of the frequency mapping, which satisfies the Brjuno–Rüssmann's non-resonant condition (1.1), the conjugated vector field still has a linear flow with this frequency. Bounemoura and Fischler [3], [4] used rational approximations to obtain the persistence of invariant tori with prescribed frequency for Hamiltonian systems.

Recently, in [22] we proved that the results on the linearization of vector field in [14] also hold under weaker non-degeneracy condition. Namely there exists $\xi \in \Pi$ such that

$$\operatorname{Rank}\left\{\omega(\xi),\ \frac{\partial^\beta \omega}{\partial \xi^\beta} \ \middle|\ \text{for all } \beta \in \mathbb{Z}_+^n,\ |\beta| \leq n-1\right\} = n, \tag{1.3}$$

but the frequency of the flow of conjugated vector field may undergo some drifts. For the relevant results of weaker non-degeneracy condition, we refer to [15]–[18], [21] and the references therein. In fact, under weaker non-degeneracy condition [22] we can only get the linearization of vector fields, there is no information on the persistence of the frequency of the flow of conjugated vector field.

In this paper we are mainly concerned with the persistence of the frequency of the flow of conjugated vector field. The method of KAM iteration in [14] closely depends on the Kolmogorov's non-degeneracy condition, which ensures that the frequency of each step remains the same. When the Kolmogorov's non-degeneracy condition is violated, the frequency of linear flow may undergo some drifts. So the method in [14] can not be directly applied. By an improved KAM iteration with parameters and introducing external parameters, we will prove that if the frequency mapping has nonzero Brouwer's topological degree at some Brjuno–Rüssmann's non-resonant frequency, then the linear flow with this frequency will persist under small perturbation.

In order to state our results, we now describe the setting more precisely. Usually, by $\mathbb{Z}$ and $\mathbb{Z}_+$ we denote the sets of integers and positive integers. Define

$$D(s) = \{\theta \in \mathbb{C}^n/2\pi\mathbb{Z}^n \mid |\mathrm{Im}\,\theta_j| \le s,\ j = 1, \dots, n\}$$

and a complex neighbourhood of Π,

$$\Pi_h = \{\xi \in \mathbb{C}^n \mid \mathrm{dist}(\xi, \Pi) \le h\}.$$

Suppose that the function $f(\theta;\xi)$ is real analytic on $D(s) \times \Pi_h$. We expand $f(\theta;\xi)$ into the Fourier series with respect to θ,

$$f(\theta;\xi) = \sum_{k\in\mathbb{Z}^n} f_k(\xi) e^{i\langle k,\theta\rangle},$$

and define

$$\|f\|_{D(s)\times\Pi_h} = \sum_{k\in\mathbb{Z}^n} \|f_k\|_h e^{s|k|},$$

where $\|f_k\|_h = \sup\limits_{\xi\in\Pi_h} |f_k(\xi)|$.

Finally, with any approximation function Δ we define two other functions $\Lambda(\tau) = \tau\Delta(\tau)$ and $\Lambda^{-1}(t) = \sup\{\tau \ge 1 \mid \Lambda(\tau) \le t\}$ for $t \ge \Lambda(1)$. The following theorem is the main result of this paper.

THEOREM 1.1. *Suppose the vector field $X = N + P = \omega(\xi) + P(\theta;\xi)$ is real analytic on $D(s) \times \Pi_h$. Let $\omega_0 = \omega(\xi_0)$, where $\xi_0 \in \Pi$. Suppose that ω_0 satisfies the Brjuno–Rüssmann's non-resonant condition*

$$|\langle k, \omega_0\rangle| \ge \frac{\alpha}{\Delta(|k|)}, \quad \text{for all } 0 \ne k \in \mathbb{Z}^n,$$

and the Brouwer's degree of the frequency mapping $\omega(\xi)$ at ξ_0 on Π is not zero, i.e. $\deg(\omega(\xi), \Pi, \omega_0) \ne 0$. Then there exists a sufficiently small constant $\varepsilon > 0$, such that if

$$\|P\|_{D(s)\times\Pi_h} = \varepsilon < \frac{h}{16} \le \frac{\alpha}{200\Lambda(\tau)},$$

where τ is so large that

$$3\int_\tau^\infty \frac{\ln\Lambda(t)}{t^2}\,dt < \frac{s}{2},$$

there exists a real analytic diffeomorphism Φ_{ω_0} such that $\Phi_{\omega_0}^*(N+P) = N_*$, and at least one $\xi_* \in \Pi$ such that the conjugated vector field N_* at $\xi = \xi_*$ has a linear flow with ω_0 as its frequency.

REMARK 1.2. An example corresponding to the above theorem is $\omega(\xi) = \omega_0 + (\xi_1^3, \dots, \xi_n^3)$. At $\xi = 0$, $\omega(\xi)$ is degenerate in the Kolmogorov's sense. The previous KAM theorems in the context of vector field can not tell whether the conjugated vector field has a linear flow with ω_0 at its frequency.

Remark 1.3. In paper [23], [24] we consider the reversible systems with normal degenerate equilibrium point, and prove the existence of at least one n-dimensional invariant torus with ω_0 as its frequency, when ω_0 satisfies the Diophantine condition:

$$|\langle k, \omega_0\rangle| \geq \frac{\alpha}{|k|^\nu}, \quad \text{for all } 0 \neq k \in \mathbb{Z}^n,$$

where $\nu > n-1$. By the methods in this paper, the above results can be generalized to the Brjuno–Rüssmann's non-resonant condition. For the latest results of more weaker non-resonant condition, we refer to [2], [9] and the references therein.

2. Proof of the main results

In this section we will prove our Theorem 1.1. It is effective to introduce an artificial external parameter γ and consider the vector field

$$X = N + \gamma + P, \tag{2.1}$$

where $N = \omega(\xi), P = P(\theta;\xi)$. The vector field (2.1) with $\gamma = 0$ returns to the original vector field.

The idea of introducing external parameters was proposed by Herman [8] and heavily employed later on by others in [5]–[7], [11], [13], [19], [20]. We will first give a KAM theorem for vector field (2.1) with parameters (ξ,γ) and then prove Theorem 1.1.

Let $\mu = \max\limits_{\xi_1,\xi_2\in\Pi_h} |\omega(\xi_1) - \omega(\xi_2)|$, and define

$$B(\omega,\mu) = \{\gamma \in \mathbb{C}^n \mid \operatorname{dist}(\gamma,\omega) < \mu\}.$$

Let $M = \Pi_h \times B(0, 2\mu+1)$. The vector field $X = \omega(\xi) + \gamma + P(\theta;\xi)$ is real analytic on $D(s) \times M$.

Suppose $\omega_0 = \omega(\xi_0)$ satisfies the Brjuno–Rüssmann's non-resonant condition

$$|\langle k, \omega_0\rangle| \geq \frac{\alpha}{\Delta(|k|)}, \quad \text{for all } k \in \mathbb{Z}^n \setminus \{0\},$$

where the function Δ satisfies

$$\int_1^\infty \frac{\ln \triangle(t)}{t^2}\, dt < \infty.$$

Let $d = \alpha/2\tau\Delta(\tau)$. Then, for all $\omega \in B(\omega_0, d)$, it follows that

$$|\langle k, \omega\rangle| \geq \frac{\alpha}{2\Delta(|k|)}, \quad 0 < |k| \leq \tau.$$

Let $q = (1 - a + a^2 b)(1+b)e^a$, where $0 < a < 1$, $0 < b \leq 1/2$ are positive constants.

Theorem 2.1. *There exists a small* $\varepsilon > 0$ *such that if* $\|P\|_{D(s)\times M} \le \varepsilon$, *then we have an analytic curve* $\{\Gamma_{\omega_0} : \gamma = \gamma(\xi),\ \xi \in \Pi\} \subset M$, *which is determined implicitly by the equation*

$$\omega(\xi) + \gamma + \widehat{N}_*(\xi, \gamma) = \omega_0,$$

where

$$\big|\widehat{N}_*(\xi, \gamma)\big| \le \frac{a\varepsilon}{1-q}, \qquad \big|\widehat{N}_{*\xi}(\xi, \gamma)\big| + \big|\widehat{N}_{*\gamma}(\xi, \gamma)\big| \le \frac{1}{2}, \tag{2.2}$$

and a parameterized mapping

$$\Phi_{\omega_0}(\,\cdot\,; \xi, \gamma) \colon D\left(\frac{s}{2}\right) \to D(s), \quad (\xi, \gamma) \in \Gamma_{\omega_0},$$

where Φ_{ω_0} *is* C^∞*-smooth in* (ξ, γ) *on* Γ_{ω_0} *in the sense of Whitney and analytic in* θ *on* $D(s/2)$, *such that for each* $(\xi, \gamma) \in \Gamma_{\omega_0}$, *we have*

$$\Phi^*_{\omega_0}(N(\xi) + \gamma + P(\theta; \xi)) = \Phi^*_{\omega_0}(\omega(\xi) + \gamma + P(\theta; \xi)) = \omega_0.$$

Therefore, the conjugated vector field of (2.1) *has a linear flow with* ω_0 *as its frequency.*

Now we first use Theorem 2.1 to prove Theorem 1.1. In fact, we only need to prove that the external parameter γ has at least one zero point so that the vector field (2.1) returns to the original vector field. By the estimates (2.2) and using the implicit function theorem, the equation

$$\omega(\xi) + \gamma + \widehat{N}_*(\xi, \gamma) = \omega_0$$

determines an analytic curve $\gamma(\xi) = \omega_0 - \omega(\xi) + \widehat{\gamma}(\xi)$, satisfying

$$|\widehat{\gamma}(\xi)| \le \frac{a\varepsilon}{1-q}, \qquad \big|\widehat{\gamma}_\xi(\xi)\big| \le \frac{2a\varepsilon}{1-q}.$$

By assumption $\deg(\omega_0 - \omega(\xi), \Pi, 0) \neq 0$, if ε is sufficiently small, we have

$$\deg(\gamma(\xi), \Pi, 0) = \deg(\omega_0 - \omega(\xi), \Pi, 0) \neq 0.$$

Thus we have at least one $\xi_* \in \Pi$ such that $\gamma(\xi_*) = 0$. Therefore, the conjugated vector field $\Phi^*_{\omega_0}(X(\theta; \xi_*)) = \Phi^*_{\omega_0}(X(\theta; \xi_*, \gamma(\xi_*)))$ has a linear flow with ω_0 as its frequency.

Now we begin to prove Theorem 2.1, its detailed proof consists of KAM step, setting the parameters and iteration, and convergence of iteration. The idea is to use the method of introducing external parameter to have a good control of frequency drift, so that we can obtain a Cantor-like family of analytic curves in KAM iteration, on which the frequency remains the same and satisfies the Brjuno–Rüssmann's non-resonant condition. Every KAM iteration is carried out in the neighbourhood of one curve, the radius of neighbourhood gradually tends to zero. When the radius of neighbourhood shrinks to zero, the family of curves can converge to the desired curve, on which the frequency is prescribed and satisfies the Brjuno–Rüssmann's non-resonant condition.

2.1. KAM step. First we describe our linear iterative scheme with respect to (2.1) for one KAM step. Suppose we are now in the n-th step, and in what follows the quantities without subscripts refer to those at the n-th step, while the quantities with subscripts "+" denote the corresponding ones at the $(n+1)$-th step. We will use the same notation c to indicate different constants, which are independent of iterative process.

Suppose at the n-th step, the vector field is written as

$$X = N(\xi,\gamma) + P(\theta;\xi,\gamma), \tag{2.3}$$

where the constant vector $N(\xi,\gamma) = \omega(\xi,\gamma) = \omega(\xi)+\gamma+\widehat{N}(\xi,\gamma)$. We summarize one KAM step in the following lemma.

Lemma 2.2. *Consider the real analytic vector field* (2.3) *on* $D(s)\times M$. *Let* $0<\sigma<s/2$ *and* $\tau\geq 1$. *Then define* $a = 1-e^{-\tau\sigma}$. *Let* $\omega_0 = \omega(\xi_0)$ *be the prescribed Brjuno–Rüssmann's non-resonant frequency, i.e.*

$$|\langle k,\omega_0\rangle| \geq \frac{\alpha}{\Delta(|k|)}, \quad \textit{for all } 0<|k|\leq\tau. \tag{2.4}$$

Let

$$T = \max_{\xi\in\Pi_h}\left|\frac{\partial\omega}{\partial\xi}\right|, \qquad h = \frac{\alpha}{2\Lambda(\tau)T}.$$

Suppose that

$$\|P\|_{D(s)\times M} \leq \varepsilon < \frac{b\alpha}{2\Lambda(\tau)}, \tag{2.5}$$

where $0<b\leq 1/2$ *is a positive constant, and the function* $\widehat{N}(\xi,\gamma)$ *satisfies*

$$\big|\widehat{N}_\xi(\xi,\gamma)\big| + \big|\widehat{N}_\gamma(\xi,\gamma)\big| \leq \frac{1}{2}, \quad \textit{for all } (\xi,\gamma)\in M, \tag{2.6}$$

such that the equation

$$\omega(\xi)+\gamma+\widehat{N}(\xi,\gamma) = \omega_0$$

defines implicitly an analytic curve $\Gamma:\gamma=\gamma(\xi)$, $\xi\in\Pi_h\to\gamma(\xi)\in B(0,2\mu+1)$, *satisfying* $\Gamma = \{(\xi,\gamma(\xi))|\xi\in\Pi_h\}\subset M$. *Moreover, for* $d=\alpha/2\tau\Delta(\tau)$, $L = 2+\max_{\xi\in\Pi_h}|\omega_\xi(\xi)|$, *we define* $\delta = d/L$ *such that*

$$B(\Gamma,\delta) = \big\{(\xi',\gamma')\in\Pi_h\times\mathbb{C}^n \,\big|\, |\xi'-\xi|+|\gamma'-\gamma(\xi)|\leq\delta, (\xi,\gamma)\in\Gamma\big\}\subset M.$$

Next define

$$s_+ = s-\sigma, \qquad h_+ = h-\frac{\delta}{2}, \qquad \Lambda_+ = \frac{\Lambda(\tau)}{\lambda q},$$
$$\tau_+ = \Lambda^{-1}(\Lambda_+), \qquad d_+ = \frac{\alpha}{2\tau_+\Delta(\tau_+)}, \qquad a = 1-e^{-\tau_+\sigma_+},$$

where $\lambda \geq 2$ is a fixed constant. Then, there exists

$$(2.7)\quad M_+ = \left\{ (\xi', \gamma') \in \Pi_{h_+} \times \mathbb{C}^n \;\middle|\; |\xi' - \xi| + |\gamma' - \gamma(\xi)| \leq \frac{\delta}{2},\ (\xi, \gamma) \in \Gamma \right\} \subset M$$

such that for any $(\xi, \gamma) \in M_+$, there exists a transformation $\Phi(\,\cdot\,; \xi, \gamma)\colon D(s_+) \to D(s)$, which conjugates the vector field (2.3) *to $X_+ = N_+(\xi, \gamma) + P_+(\theta; \xi, \gamma)$, with the new frequency $N_+(\xi, \gamma) = \omega_+(\xi, \gamma) = \omega(\xi) + \gamma + \widehat{N}_+(\xi, \gamma)$, $\widehat{N}_+(\xi, \gamma) = \widehat{N}(\xi, \gamma) + P^0(\xi, \gamma)$, the new perturbation P_+ and the drift term $P^0(\xi, \gamma)$ satisfying $\|P_+\|_{D(s_+)\times M_+} \leq \varepsilon_+ = q\varepsilon$, and*

$$(2.8)\qquad |P^0(\xi, \gamma)| \leq a\varepsilon, \qquad |P^0_\xi(\xi, \gamma)| + |P^0_\gamma(\xi, \gamma)| \leq \frac{2a\varepsilon}{\delta},$$

where $q = (1 - a + a^2 b)(1 + b)e^a$, $0 < a < 1$, $0 < b \leq 1/2$ are positive constants, $P^0(\xi, \gamma)$ denotes the average value of truncation term. Moreover, the mapping Φ has the estimates

$$|\Phi - \mathrm{id}|_{D(s_+)\times M_+} \leq 2\Lambda(\tau)\alpha^{-1}\sigma\varepsilon, \quad |D\Phi - \mathrm{Id}|_{D(s_+)\times M_+} \leq 2a\Lambda(\tau)\alpha^{-1}\varepsilon.$$

Thus, if

$$(2.9)\qquad \frac{2a\varepsilon}{\delta} \leq \frac{1}{4},$$

the equation $\omega(\xi) + \gamma + \widehat{N}(\xi, \gamma) + P^0(\xi, \gamma) = \omega_0$ defines implicitly an analytic curve $\Gamma_+ : \gamma_+ = \gamma_+(\xi)\colon \xi \in \Pi_{h_+} \to \gamma_+(\xi) \in B(0, 2\mu + 1)$ with $h_+ = h - \delta/2$, satisfying

$$(2.10)\qquad |\gamma_+(\xi) - \gamma(\xi)| \leq 2a\varepsilon \leq \frac{1}{2}\delta,$$

and

$$(2.11)\qquad \Gamma_+ = \{(\xi, \gamma_+(\xi)) \mid \xi \in \Pi_{h_+}\} \subset M_+.$$

If

$$(2.12)\qquad \delta_+ \leq \frac{1}{4}\delta,$$

then we have $B(\Gamma_+, \delta_+) \subset M_+$.

PROOF. We divide the proof into several parts. In the following the notation $\|\cdot\|_{s;h}$ indicates the norm $\|\cdot\|_{D(s)\times\Pi_h}$ for simplicity.

A. *Truncation.* As P is real analytic, we expand P as Fourier series $P = \sum_k P_k(\xi)e^{i\langle k,\theta\rangle}$. Let $P = \widetilde{P} + \widehat{P}$ with

$$\widehat{P} = \sum_{|k|>\tau} P_k e^{i\langle k,\theta\rangle} + (1 - a)\sum_{|k|\leq\tau} P_k e^{|k|\sigma} e^{i\langle k,\theta\rangle}.$$

In view of $e^{-\tau\sigma} = 1 - a$, we have

$$\|\widehat{P}\|_{s-\sigma;h} \leq (1 - a)\|P\|_{s;h} \leq (1 - a)\varepsilon.$$

On the other hand, the polynomial rest

$$\widetilde{P} = \sum_{|k|\le\tau} \widetilde{P}_k e^{i\langle k,\theta\rangle}, \qquad \widetilde{P}_k = \big(1-(1-a)e^{|k|\sigma}\big)P_k,$$

is bounded. Indeed, with $\widetilde{\sigma} = \sigma(1-a)/a$,

$$\begin{aligned}\|\widetilde{P}\|_{s+\widetilde{\sigma};h} &= \sum_{0\le|k|\le\tau} \big(1-(1-a)e^{|k|\sigma}\big)\|P_k\|_h e^{|k|(s+\widetilde{\sigma})} \\ &\le \sup_{0\le t\le\tau}\big(1-(1-a)e^{t\sigma}\big)e^{t\widetilde{\sigma}} \sum_{0\le|k|\le\tau}\|P_k\|_h e^{|k|s} = a\varepsilon,\end{aligned} \tag{2.13}$$

as the function under the sup is monotonically decreasing for $0\le t\le\tau$ and equals a at $t=0$.

B. *Extension of the small divisor estimates.* First note that the mapping

$$\omega\colon (\xi,\gamma)\in B(\Gamma,\delta)\to\omega(\xi,\gamma)\in B(\omega_0,d)$$

is well defined. In fact, for any $(\xi',\gamma')\in B(\Gamma,\delta)$, there exists $(\xi,\gamma(\xi))\in\Gamma$ such that

$$|\xi'-\xi|+|\gamma'-\gamma(\xi)|\le\delta,$$

so that

$$\begin{aligned}|\omega(\xi',\gamma')-\omega_0| &= |\omega(\xi')+\gamma'+\widehat{N}(\xi',\gamma')-(\omega(\xi)+\gamma+\widehat{N}(\xi,\gamma))| \\ &\le (2+\max_{\xi\in\Pi_h}|\omega_\xi(\xi)|)\delta\le d.\end{aligned}$$

Moreover, we can extend the small divisor condition (2.4) to the neighbourhood $B(\Gamma,\delta)$, i.e. for any $(\xi',\gamma')\in B(\Gamma,\delta)$, we have

$$\big|\langle k,\omega(\xi',\gamma')\rangle\big| \ge \frac{\alpha}{2\Delta(|k|)}, \quad \text{for all } 0<|k|\le\tau. \tag{2.14}$$

In fact, for any $(\xi',\gamma')\in B(\Gamma,\delta)$, there exists $(\xi,\gamma(\xi))\in\Gamma$ such that $|\xi'-\xi|+|\gamma'-\gamma(\xi)|\le\delta$. So it follows that

$$\big|\langle k,\omega(\xi',\gamma')-\omega_0\rangle\big|\le|k|d\le\frac{\alpha}{2\Delta(\tau)}, \quad \text{for all } 0<|k|\le\tau.$$

Note that ω_0 satisfies the Brjuno–Rüssmann's non-resonant condition, this prove the claim (2.14).

C. *Construction of the transformation.* The coordinate transformation Φ is generated by the time-1 map of the flow F_t of a vector field F, which satisfies that

$$\begin{aligned}\Phi^*(N+P) &= F_t^*\big(N+\widetilde{P}+\widehat{P}\big)\big|_{t=1} \\ &= N+[N,F]+\int_0^1(1-t)F_t^*[[N,F],F]\,dt+\widetilde{P}+\int_0^1 F_t^*\big[\widetilde{P},F\big]\,dt+F_1^*\widehat{P},\end{aligned}$$

where $[\,\cdot\,,\,\cdot\,]$ is the Lie bracket of two vector fields.

The point is to find F which solves the homological equation

$$[N,F]+\widetilde{P}=P^0, \tag{2.15}$$

where P^0 denotes the mean value of $\widetilde{P}$. By (2.14), for any $\xi\in\Pi_h$ the linearized equation (2.15) is solved by

$$F=\sum_{0<|k|\le\tau}\frac{\widetilde{P}_k}{i\langle k,\omega(\xi,\gamma)\rangle}\,e^{i\langle k,\theta\rangle}.$$

Then with $a=1-e^{-\tau\sigma}\le\tau\sigma$, the estimates (2.13) and (2.14), we get

$$\|F\|_{s+\widetilde{\sigma}}\le 2\triangle(\tau)\alpha^{-1}\|\widetilde{P}\|_{s+\widetilde{\sigma}}\le 2\triangle(\tau)\alpha^{-1}a\varepsilon=2\Lambda(\tau)\sigma\alpha^{-1}\varepsilon. \tag{2.16}$$

By hypothesis (2.5), $\|F\|_{s+\widetilde{\sigma}}\le\sigma$, so the vector field F generates a flow F_t, which satisfies for $0\le t\le 1$,

$$F_t\colon D(s-2\sigma)\to D(s-\sigma),\qquad \|F_t-id\|_{s-2\sigma}\le 2\Lambda(\tau)\sigma\alpha^{-1}\varepsilon.$$

Therefore, the vector field X is transformed into $X_+=N_++P_+$, where

$$N_+=N+P^0=\omega(\xi)+\gamma+\widehat{N}(\xi,\gamma)+P^0(\xi,\gamma)$$

is a new constant vector field,

$$\begin{aligned}P_+=\int_0^1(1-t)F_t^*[[N,F],F]\,dt+\int_0^1 F_t^*[\widetilde{P},F]\,dt+F_1^*\widehat{P}\\ =\int_0^1 F_t^*[P_t,F]\,dt+F_1^*\widehat{P}\end{aligned}$$

is the new perturbation of the constant vector field N_+, $P_t=t\widetilde{P}+(1-t)P^0$.

D. *Construction of parameters domains.* Let M_+ be defined by (2.7), it follows that M_+ is closed and $M_+\subset B(\Gamma,\delta)\subset M$, $\operatorname{dist}(M_+,\partial M)\ge\delta/2$, where ∂M is the boundary of M.

Let $P^0(\xi,\gamma)=[\widetilde{P}(\theta;\xi,\gamma)]$. By Cauchy's estimates, the estimates (2.8) hold. Set $\widehat{N}_+(\xi,\gamma)=\widehat{N}(\xi,\gamma)+P^0(\xi,\gamma)$. By the implicit function theorem, if

$$\left|\frac{\partial\widehat{N}_+}{\partial\gamma}\right|\le\frac{1}{2},\quad\text{for all }(\xi,\gamma)\in M,$$

which will be verified in (2.17), the equation

$$\omega(\xi)+\gamma+\widehat{N}_+(\xi,\gamma)=\omega_0$$

defines implicitly an analytic curve

$$\Gamma_+:\gamma_+=\gamma_+(\xi):\xi\in\Pi_{h_+}\to\gamma_+(\xi)\in B(0,2\mu+1).$$

Note that γ_+ and γ satisfy

$$\omega(\xi)+\gamma_++\widehat{N}_+(\xi,\gamma_+)=\omega(\xi)+\gamma+\widehat{N}(\xi,\gamma)=\omega_0.$$

Then it follows that

$$\begin{aligned}|\gamma_+(\xi)-\gamma(\xi)| &\le \big|\widehat{N}_+(\xi,\gamma_+(\xi))-\widehat{N}(\xi,\gamma(\xi))\big|\\ &\le \big|\widehat{N}(\xi,\gamma_+(\xi))-\widehat{N}(\xi,\gamma(\xi))\big|+|P^0(\xi,\gamma_+(\xi))|\\ &\le \frac{1}{2}|\gamma_+(\xi)-\gamma(\xi)|+a\varepsilon\end{aligned}$$

Hence the conclusions (2.10) and (2.11) hold. By (2.12), we have $B(\Gamma_+,\delta_+)\subset M_+$.

E. *Estimates of the new perturbation.* To estimate P_+ we note that

$$\|P_t\|_{s+\widetilde{\sigma}}\le t\big\|\widetilde{P}\big\|_{s+\widetilde{\sigma}}+(1-t)\|P^0\|_{s+\widetilde{\sigma}}\le a\varepsilon$$

and

$$(s+\widetilde{\sigma})-(s-\sigma)=\frac{1-a}{a}\,\sigma+\sigma=\frac{\sigma}{a}.$$

Thus Lemma A.1 of Appendix and the definition $b=2\Lambda(\tau)\alpha^{-1}\varepsilon$ yield

$$\|[P_t,F]\|_{s-\sigma}\le\frac{a}{\sigma}\|P_t\|_{s+\widetilde{\sigma}}\|F\|_{s+\widetilde{\sigma}}\le 2a^2\Lambda(\tau)\alpha^{-1}\varepsilon^2=a^2b\varepsilon.$$

In view of (2.16), we can apply Lemma A.2 of Appendix with $r=s-\sigma$ and $\eta=1/a$ to obtain

$$\int_0^1\|F_t^*[P_t,F]\|_{s-2\sigma}\,dt\le(1+b)e^a\|[P_t,F]\|_{s-\sigma}\le a^2b(1+b)e^a\varepsilon.$$

Similarly,

$$\big\|F_1^*\widehat{P}\|_{s-2\sigma}\le(1+b)e^a\big\|\widehat{P}\big\|_{s-\sigma}\le(1-a)(1+b)e^a\varepsilon.$$

Both estimates together yield the stated estimates of P_+. □

2.2. Setting the parameters and iteration. Now we choose some suitable parameters so that the above iteration can go on infinitely. We can always choose $0<a<1$ and $0<b\le 1/2$ so that

$$q=\big(1-a+a^2b\big)(1+b)e^a<1,$$

and we can even make q sufficiently small as we wish. Then it suffices to choose for ε and Λ geometric sequences with the same base q, namely

$$\varepsilon_n=\varepsilon_0q^n,\qquad \Lambda_n=\frac{\Lambda_0}{(\lambda q)^n},$$

where $\lambda\ge 2$ is a constant such that $\lambda q\le 1/4$, we assume $\Lambda_0\ge\Lambda(1)=\Delta(1)$. Next, let

$$\tau_n=\Lambda^{-1}(\Lambda_n)=\sup\{\tau_n\mid\Lambda(\tau_n)\le\Lambda_n\},$$

and define other parameters through

$$1-a=e^{-\tau_n\sigma_n},\qquad s_n=s_{n-1}-\sigma_{n-1},$$

$$d_n=\frac{\alpha}{2\tau_n\Delta(\tau_n)},\qquad \delta_n=\frac{d_n}{L}=\frac{\alpha}{2L\Lambda_n},\qquad h_n=h_{n-1}-\frac{\delta_n}{2},$$

where $L = 2 + \max_{\xi\in\Pi_h} |\omega_\xi(\xi)|$. As we will see in a moment, s_n has a positive limit for Λ_0 sufficiently large. Denote

$$M_n = \left\{ (\xi', \gamma') \in \Pi_{h_n} \times \mathbb{C}^n \;\middle|\; |\xi' - \xi| + |\gamma' - \gamma(\xi)| \le \frac{\delta_{n-1}}{2},\ (\xi, \gamma) \in \Gamma_{n-1} \right\}$$

and $D_n = D(s_n)$ for simplicity. Moreover, for any $(\xi', \gamma') \in B(\Gamma_n, \delta_n)$, there exists $(\xi, \gamma(\xi)) \in \Gamma_n$ such that $|\xi' - \xi| + |\gamma' - \gamma(\xi)| < \delta_n = d_n/L$, hence, in view of $\Lambda(\tau_n) = \tau_n \triangle(\tau_n)$,

$$\begin{aligned} |\langle k, \omega_n(\xi', \gamma') - \omega_0\rangle| &\le |k||\omega(\xi') + \gamma' + \widehat{N}_n(\xi', \gamma') - (\omega(\xi) + \gamma + \widehat{N}_n(\xi, \gamma))| \\ &\le |k| d_n \le \frac{\alpha}{2\Delta(\tau_n)}, \end{aligned}$$

for all $0 < |k| \le \tau_n$. As ω_0 satisfies (2.4), the following divisor admit the lower bound

$$\begin{aligned} |\langle k, \omega_n(\xi', \gamma')\rangle| &\ge |\langle k, \omega_0\rangle| - |\langle k, \omega_n(\xi', \gamma') - \omega_0\rangle| \\ &\ge \frac{\alpha}{\triangle(|k|)} - \frac{\alpha}{2\triangle(\tau_n)} \ge \frac{\alpha}{2\triangle(|k|)}, \end{aligned}$$

for all $0 < |k| \le \tau_n$. Thus the small divisor condition (2.4) also holds in the neighbourhood $B(\Gamma_n, \delta_n)$.

LEMMA 2.3 (Iterative Lemma). *Consider the real analytic vector field $X_0 = N_0 + P_0 = \omega_0(\xi) + P_0(\theta; \xi)$, satisfying*

$$\|P_0\|_{D_0 \times M_0} \le \varepsilon_0 \le \min\left\{ \frac{b\alpha}{2\Lambda_0}, \frac{\alpha}{16 L a \Lambda_0} \right\},$$

with Λ_0 sufficiently large. Then for each $n \ge 1$ there exists a parameter and coordinate transformation

$$\Phi^n = \Phi_0 \circ \ldots \circ \Phi_n \colon D(s_n) \times M_n \to D(s_0) \times M_0,$$

which conjugates the vector field X_0 to $X_n = N_n + P_n = \omega_n(\xi, \gamma) + P_n(\theta; \xi)$ such that

$$\|P_n\|_{D_n \times M_n} \le \varepsilon_n = \varepsilon_0 q^n.$$

Moreover,

$$|\Phi^{n+1} - \Phi^n|_{D_{n+1} \times M_{n+1}} \le c\Lambda(\tau_{n+1})\alpha^{-1}\sigma_{n+1}\varepsilon_{n+1}.$$

PROOF. By induction and the definition of ε_n, Λ_n and δ_n, we have

$$\varepsilon_n \le \frac{b\alpha}{2\Lambda_n(\tau_n)}, \qquad \frac{2a\varepsilon_n}{\delta_n} \le \frac{1}{4}, \qquad \frac{\delta_{n+1}}{\delta_n} \le \frac{1}{4}, \quad \text{for all } n \ge 0,$$

i.e. the assumptions (2.5), (2.9) and (2.12) holds for all $n \ge 0$. The last assumption (2.6) will be verified in (2.17). Thus the proof follows by applying

Lemma 2.2 repeatedly. In the following we only give the estimate of Φ^n. First by Lemma 2.2, we observe that

$$|\Phi_n - \mathrm{id}|_{D_n\times M_n} \le 2\Lambda(\tau_n)\alpha^{-1}\sigma_n\varepsilon_n,$$
$$|D\Phi_n - \mathrm{Id}|_{D_n\times M_n} \le 2a\Lambda(\tau_n)\alpha^{-1}\varepsilon_n,$$

where D denotes the Jacobian with respect to θ. Then we have

$$\begin{aligned}|\Phi^{n+1}-\Phi^n|_{D_{n+1}\times M_{n+1}} &= |\Phi^n\circ\Phi_{n+1}-\Phi^n|_{D_{n+1}\times M_{n+1}}\\ &\le |D\Phi^n|_{D_n\times M_n}|\Phi_{n+1}-id|_{D_{n+1}\times M_{n+1}}\\ &\le c\Lambda(\tau_{n+1})\alpha^{-1}\sigma_{n+1}\varepsilon_{n+1},\end{aligned}$$

provided that $D\Phi^n$ is uniformly bounded on $D_n\times M_n$. In fact by induction we have $D\Phi^n = D\Phi_0\circ\ldots\circ D\Phi_n$, with the Jacobians evaluated at different points, and

$$\begin{aligned}|D\Phi^n| = |D\Phi_0\circ\ldots\circ D\Phi_n| &\le \prod_{n\ge0}\big(1+2a\Lambda(\tau_n)\alpha^{-1}\varepsilon_n\big)\\ &\le \prod_{n\ge0}\bigg(1+\frac{2a\Lambda_0\varepsilon_0}{\alpha}\frac{1}{\lambda^n}\bigg) \le \exp\bigg\{\frac{2a\lambda\Lambda_0\varepsilon_0}{(\lambda-1)\alpha}\bigg\} \le e^{ab\lambda/(\lambda-1)},\end{aligned}$$

which is uniformly bounded. □

2.3. Convergence of iteration. We first verify that the sequences of s_i tend to a positive limit. Indeed, let $t=\Lambda^{-1}(\Lambda_0(\lambda q)^{-x})$, we have

$$\sum_{i\ge0}\frac{1}{\tau_i} \le \int_0^\infty \frac{dx}{\Lambda^{-1}(\Lambda_0(\lambda q)^{-x})} = \frac{1}{\ln(\lambda q)^{-1}}\int_{\tau_0}^\infty \frac{d\Lambda(t)}{t\Lambda(t)}.$$

Integrating by parts and requiring $\Lambda(\tau_0)\ge(\lambda q)^{-1}$ we get

$$\sum_{i\ge0}\frac{1}{\tau_i} \le \frac{1}{\ln(\lambda q)^{-1}}\int_{\tau_0}^\infty \frac{\ln\Lambda(t)}{t^2}\,dt.$$

It follows that

$$\sum_{i\ge0}\sigma_i = \sum_{i\ge0}\frac{\ln(1-a)^{-1}}{\tau_i} \le \frac{\ln(1-a)}{\ln\lambda q}\int_{\tau_0}^\infty \frac{\ln\Lambda(t)}{t^2}\,dt.$$

Hence by choosing τ_0 sufficiently large, we can achieve that $\sum_{i\ge0}\sigma_i\le s/2$ and thus $s_i\to s_*\ge s/2$. By Iterative Lemma 2.3, Φ^i satisfies

$$|\Phi^{i+1}-\Phi^i|_{D_{i+1}\times M_{i+1}} \le c\Lambda(\tau_{i+1})\alpha^{-1}\sigma_{i+1}\varepsilon_{i+1}.$$

Note that $D_*=D(s_*)$, $M_*=\bigcap_{i\ge0}M_i$, and $\Phi_{\omega_0}=\lim_{i\to\infty}\Phi^i$. Thus the mappings Φ^i converge uniformly on

$$\bigcap_{i\ge0}D_i\times M_i = D(s_*)\times M_*,$$

to a mapping Φ_{ω_0}, which is real analytic on $D(s_*)$ and uniformly continuous on M_*. Moreover, we have

$$|\Phi_{\omega_0} - id|_{D(s_*)\times M_*} \le e^{ab\lambda/(\lambda-1)} \frac{\lambda\Lambda_0\varepsilon_0\sigma}{(\lambda-1)\alpha}.$$

Let $h_* = h - (1/2)\sum_{i=0}^{\infty} \delta_i$. It follows that $h_* \ge h - 2\delta_0/3$. By the definition of δ_0, we can choose τ_0 sufficiently large such that $\delta_0 \le h$. Therefore $h_* \ge h/3$, and $\Pi_{h_*} \subset \bigcap_{i\ge 0} \Pi_{h_i}$.

By iteration we have $\widehat{N}_i(\xi,\gamma) = \sum_{k=0}^{i-1} P_k^0(\xi,\gamma)$. We now prove the convergence of $\widehat{N}_i(\xi,\gamma)$. Combining with the estimates for P_k^0, we have that for all $(\xi,\gamma) \in B(\Gamma_i,\delta_i) \subset M_i$,

$$\left|\widehat{N}_i(\xi,\gamma)\right| \le \sum_{k=0}^{i-1} a\varepsilon_k \le \frac{a\varepsilon_0}{1-q}.$$

By Cauchy's estimate, for $(\xi,\gamma) \in M_i$,

$$\left|\widehat{N}_{i\xi}(\xi,\gamma)\right| + \left|\widehat{N}_{i\gamma}(\xi,\gamma)\right| \le \sum_{k=0}^{i-1} \frac{2a\varepsilon_k}{\delta_k} = \sum_{k=0}^{i-1} \frac{4La\varepsilon_0\Lambda_0}{\lambda^k} \le \frac{4\lambda La\varepsilon_0\Lambda_0}{\lambda-1} \le \frac{\lambda}{4(\lambda-1)}.$$

Thus, by $\lambda \ge 2$, we have

$$\left|\widehat{N}_{i\xi}(\xi,\gamma)\right| + \left|\widehat{N}_{i\gamma}(\xi,\gamma)\right| \le \frac{1}{2}, \quad \text{for all } (\xi,\gamma) \in M_i, \tag{2.17}$$

the assumption (2.6) holds.

Let $\widehat{N}_* = \lim_{i\to\infty} \widehat{N}_i$. Then, for all $(\xi,\gamma) \in M_*$, we have

$$|\widehat{N}_*(\xi,\gamma)| \le \frac{a\varepsilon_0}{1-q}, \qquad \left|\widehat{N}_{*\xi}(\xi,\gamma)\right| + \left|\widehat{N}_{*\gamma}\right| \le \frac{1}{2}.$$

Similarly, we can prove the convergence of $\gamma_i(\xi)$ on Π_{h_*}. In fact, we can choose τ_0 sufficiently large such that $2a\varepsilon_i/\delta_i \le 1/4$, for all $i \ge 0$. Then for $j \ge i$, if follows that

$$|\gamma_j(\xi) - \gamma_i(\xi)| \le \sum_{k=i}^{j-1} 2a\varepsilon_k \le \frac{\delta_i}{2}.$$

Let $\gamma_*(\xi) = \lim_{i\to\infty} \gamma_i(\xi)$. Then we have $|\gamma_*(\xi) - \gamma(\xi)| \le \delta_i/2$, which implies that $\Gamma^* = \{(\xi,\gamma_*(\xi)) \mid \xi \in \Pi_{h_*}\} \subset M_i$, and therefore $\Gamma^* \subset M_* = \bigcap_{i\ge 0} M_i$. Moreover, for any $(\xi,\gamma) \in \Gamma^*$, we have $\omega(\xi) + \gamma + \widehat{N}_*(\xi) = \omega_0$. Therefore, for any $(\xi,\gamma) \in \Gamma^*$ with ω_0 satisfying the Brjuno–Rüssmann's non-resonant condition, the transformation $\Phi_{\omega_0}(\,\cdot\,;\xi,\gamma)$ conjugates the vector field (2.1) to $X_* = \omega_0$.

For the statement of the theorem, we choose $a = 99/100, b = 1/100$ and $\lambda = 2$, which results in

$$q \approx 0.0538, \qquad \frac{\ln(1-q)}{\ln \lambda q} \approx 2.0661, \qquad \lambda q \leq \frac{1}{4}.$$

Thus this completes the proof of Theorem 1.1. □

Appendix A

In this section we formulate some lemmas which have been used in the previous section. For detailed proofs we refer to [14].

Lemma A.1. *Let U and V be analytic vector fields on the torus $\mathbb{T}^n$. Then, for $0 < r < \min\{u, v\}$,*

$$\big\|[U, V]\big\|_r \leq \frac{1}{e}\bigg(\frac{1}{u-r} + \frac{1}{v-r}\bigg)\|u\|_u\|v\|_v.$$

Lemma A.2. *Suppose the vector fields F and V are analytic on the torus $\mathbb{T}^n$. If $b = \sigma^{-1}\|F\|_{r+\eta\sigma} \leq 1/2$ with $0 < \sigma < r$ and $\eta > 0$, then*

$$\big\|F_t^* V\big\|_{r-\sigma} \leq (1 + bt)e^{1/\eta}\|V\|_r, \quad 0 \leq t \leq 1.$$

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