Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 1, 2019, 383–406
DOI: 10.12775/TMNA.2019.056

NONLOCAL SCHRÖDINGER EQUATIONS FOR INTEGRO-DIFFERENTIAL OPERATORS WITH MEASURABLE KERNELS

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Abstract. In this paper we investigate the existence of positive solutions for the problem

$$-\mathcal{L}_K u + V(x)u = f(u)$$

in $\mathbb{R}^N$, where $-\mathcal{L}_K$ is an integro-differential operator with measurable kernel K. Under apropriate hypotheses, we prove by variational methods that this equation has a nonnegative solution.

1. Introduction

In this paper we consider the class of integro-differential Schrödinger equations

$$-\mathcal{L}_K u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \tag{P}$$

where $-\mathcal{L}_K$ is an integro-differential operator, given by

$$-\mathcal{L}_K u(x) = 2 \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} (u(y) - u(x))K(x-y)\,dy$$

and K satisfies general properties. This study leads both to nonlocal and to nonlinear difficulties. For example, we cannot benefit from the s-harmonic extension of Caffarelli and Silvestre (see [11]) or commutator properties (see [29]).

2010 *Mathematics Subject Classification.* Primary 35J60; Secondary 35J10.
Key words and phrases. Integro-differential operator; nonlocal Schrödinger equation; asymptotic potential.
The second author was supported in part by CNPq/Brazil 306082/2017-9.

The study of nonlocal operators is important because it intervenes in a quantity of applications and models. For example, we mention their use in phase transition models (see [1], [10]), image reconstruction problems (see [24]), obstacle problem, optimization, finance, stratified materials, anomalous diffusion, crystal dislocation, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and Lévi process (see [9]).

This paper was motivated by [3]. In this paper the authors studied the existence of positive solutions for the problem

$$\begin{cases} -\Delta u + V(x)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where V and f are Hölder continuous functions, V is nonnegative and f has a subcritical or critical growth. Our purpose is to study a similar problem when the laplacian operator is replaced by the operator $-\mathcal{L}_K$. In this case, we have difficulties because our operator is nonlocal and some methods used in [3] cannot be used.

Several papers have studied the problem (P) when $K(x) = C_{N,s}|x|^{-N-2s}$, where

$$C_{N,s} = \bigg(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\, d\xi \bigg)^{-1},$$

that is, when $-\mathcal{L}_K$ is the fractional laplacian operator (see [18]); we will mention some of these papers. In [5], the author has proved the existence of positive solutions of (P) when V is a constant small enough. Also, in [27], the problem was studied when f is asymptotically linear and V is constant. In [37], the authors have studied the problem (P) when $V \in C^N(\mathbb{R}^N, \mathbb{R})$, V is positive and

$$\lim_{|x|\to\infty} V(|x|) \in (0, \infty].$$

In [43], the authors have studied (P) when V and f are asymptotically periodic. When $V = 1$, Felmer et al. have studied the existence, the regularity and the qualitative properties of ground states solutions for problem (P) (see [22]). In [40], the authors have shown the existence of solutions for (P) when $V \in C^N(\mathbb{R}^N, \mathbb{R})$ and there exists $r_0 > 0$ such that, for any $M > 0$,

$$\operatorname{meas}(\{x \in B_{r_0}(y); V(x) \le M\}) \to 0 \quad \text{as } |y| \to \infty.$$

In [29] the problem (P) was studied when $V \in C^1(\mathbb{R}^N, \mathbb{R})$,

$$\liminf_{|x|\to\infty} V(x) \ge V_\infty,$$

where V_∞ is constant, and $f \in C^1(\mathbb{R}^N, \mathbb{R})$. By method of the Nehari manifold, Secchi has showed that the problem (P) has a solution if $V \le V_\infty$, but V is not identically equal to V_∞, where V_∞ is a constant. Also in [29], Secchi has obtained the existence of ground state solutions of (P) for general $s \in (0,1)$ when $V(x) \to \infty$ as $|x| \to \infty$. In [42], the authors obtain the existence of a sequence of radial and non radial solutions for the problem (P) when V and f are radial functions. Some other interesting studies by variational methods of the problem (P) can be found in [4], [7], [12]–[14], [16], [21], [25], [26], [28], [30], [34], [35], [38] and [41]. Many of them use strong tools that we cannot use here in our problem, as the s-harmonic extension and commutator properties.

In the literature, interesting conditions on V have been studied. Motivated by the above papers, especially by [3], we will assume hypotheses about f and V analogous to the hypotheses assumed in [3]. We will assume that the potential V satisfies:

(V_1) $\inf_{x\in\mathbb{R}^N} V(x) > 0$.

(V_2) $V(x) \leq V_\infty$ for some constant $V_\infty > 0$ and for all $x \in B_1(0) = \{x \in \mathbb{R}^N : |x| < 1\}$.

(V_3) There are $R > 0$ and $\Lambda > 0$ such that $V(x) \geq \Lambda$ for all $|x| \geq R$.

Also, we will assume that $f \in C(\mathbb{R},\mathbb{R})$ is a function satisfying:

(f_1) $|f(s)| \leq c_0|s|^{p-1}$, for some constant $c_0 > 0$ and $p \in (2, 2_s^*)$, where $2_s^* = 2N/(N-2s)$ and $N > 2s$.

(f_2) There is $\theta > 2$ such that $\theta F(t) \leq tf(t)$ for all $t > 0$, where

$$F(t) = \int_0^t f(s)\,ds.$$

(f_3) $f(t) > 0$ for all $t > 0$ and $f(t) = 0$ for all $t < 0$.

The kernel $K\colon \mathbb{R}^N \to (0,\infty)$ is a measurable function satisfying:

(K_1) $K(x) = K(-x)$ almost everywere in $\mathbb{R}^N$.

(K_2) There are $\lambda > 0$ and $s \in (0,1)$ such that $\lambda \leq K(x)|x|^{N+2s}$ almost everywere in $\mathbb{R}^N$.

(K_3) $\gamma K \in L^1(\mathbb{R}^N)$, where $\gamma(x) = \min\{|x|^2, 1\}$.

Note that, if $K = C_{N,s}|x|^{-(N+2s)}$ then the operator $-\mathcal{L}_K$ is the fractional laplacian, $(-\Delta)^s$.

Our main result is the Theorem 5.2. It states that if V satisfies (V_1)–(V_3) and f satisfies (f_1)–(f_3), then there is $\Lambda^* = \Lambda^*(V_\infty, \theta, p, c_0, s) > 0$ such that if $\Lambda > \Lambda^*$, then the problem (P) has a nonnegative nontrivial solution.

Our paper is organized as follows. In Section 2 we will present some properties of the space in which we will study the problem (P). The functional associated with the problem (P) does not have the mountain pass geometry. Therefore, inspired by the idea of [3], we define an auxiliary problem. We show in Proposition 3.7 that the functional associated with the auxiliary problem has the mountain pass geometry and it satisfies the Palais–Smale condition. The argument used in [3] does not work well in our case because our operator is nonlocal. To show Proposition 3.7, we need to prove some technical lemmas: Lemmas 3.2–3.6. We emphasize that the techniques used in [3] could not be adapted for our case, therefore we use a new technique. In Section 4 we will prove a general estimate for weak solutions of

$$-\mathcal{L}_K u + b(x)u = g(x,u),$$

where $b \geq 0$, $|g(x,t)| \leq h(x)|t|$ and $h \in L^q(\mathbb{R}^N)$ with $q > N/2s$. We will show that there is $M = M(q, \|h\|_{L^q(\mathbb{R}^N)})$ such that, the solution u satisfies

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq M \|u\|_{L^{2_s^*}(\mathbb{R}^N)}.$$

This estimate will be obtained in Proposition 4.5. In [2], using the s-harmonic extension of [11], the authors have showed the same estimate when $-\mathcal{L}_K$ is the fractional laplacian operator. In our case, we cannot use the s-harmonic extension, because we don't have a analogously version of this result for general operators. The strategy of the proof is to define special functions through the mean value theorem (see equations (4.1) and (4.2)). The Lemmas 4.3 and 4.4 are technical lemmas and they show that there is a order between these two functions. This order and other properties are fundamental in the proof of inequality (4.13), consequently in the proof Proposition 4.5. To the best of our knowledge, we emphasize that this general estimate obtained in Proposition 4.5 is new in the literature. As an application of the estimate obtained in Section 4, we prove the Theorem 5.2. This is a new result in the literature.

2. Preliminaries

Consider $s \in (0,1)$. We denote by $H^s(\mathbb{R}^N)$ the fractional Sobolev space, defined as

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dx\, dy < \infty \right\}.$$

The space $H^s(\mathbb{R}^N)$ is a Hilbert space with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dx\, dy \right)^{1/2}.$$

We define X as the linear space of functions $u\colon \mathbb{R}^N \to \mathbb{R}$ in $L^2(\mathbb{R}^N)$ such that

$$(x,y) \mapsto (u(x)-u(y))\sqrt{K(x-y)}$$

is in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$. The function

$$\|u\|_X := \left(\int_{\mathbb{R}^N} u^2\, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x)-u(y))^2 K(x-y)\, dx\, dy \right)^{1/2}$$

defines a norm in X and $(X, \|\cdot\|_X)$ is a Hilbert space. By (K_2), the space X is continuously embedded in $H^s(\mathbb{R}^N)$. Therefore, X is continuously embedded in $L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*]$, where $2_s^* = 2N/(N-2s)$. If $\Omega \subset \mathbb{R}^N$, we define

$$X_0(\Omega) = \{u \in X : u = 0 \text{ in } \Omega^c\}.$$

The space $X_0(\Omega)$ is a Hilbert space with the norm

$$\|u\|_{X_0(\Omega)} := \bigg(\int_\Omega u^2\,dx + \int_Q (u(x)-u(y))^2 K(x-y)\,dx\,dy\bigg)^{1/2},$$

where $Q = (\Omega^c \times \Omega^c)^c$ (see Lemma 7 in [31]). It is continuously embedded in $H_0^s(\mathbb{R}^N)$. For definition and properties of $H_0^s(\mathbb{R}^N)$ we indicate [18].

In the problem (P), we will consider the space E defined as

$$E = \bigg\{u \in X : \int_{\mathbb{R}^N} V(x)u^2\,dx < \infty\bigg\}. \tag{2.1}$$

The space E is a Hilbert space with the norm

$$\|u\| := \bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))^2 K(x-y)\,dx\,dy + \int_{\mathbb{R}^N} V(x)u^2\,dx\bigg)^{1/2}.$$

Let $A, B \subset \mathbb{R}^N$ be measurable and let $u, v \in X$. We will denote

$$[u,v]_{A\times B} := \int_A\int_B (u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy$$

and we will denote $[u,v]_{\mathbb{R}^N\times\mathbb{R}^N}$ by $[u,v]$. If $u, v \in C_0^\infty(\mathbb{R}^N)$ then

$$(-\mathcal{L}_K u, v)_{L^2(\mathbb{R}^N)} = [u,v].$$

Therefore, we say that $u \in E$ is a solution for the problem (P) if

$$[u,v] + \int_{\mathbb{R}^N} V(x)uv\,dx = \int_{\mathbb{R}^N} f(u)v\,dx \quad \text{for all } v \in E. \tag{2.2}$$

The Euler–Lagrange functional associated with (P) is given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u)\,dx, \quad \text{where } F(t) = \int_0^t f(s)\,ds.$$

From the hypotheses about f, we have $F \in C^1(E, \mathbb{R})$ and

$$I'(u)v = [u,v] + \int_{\mathbb{R}^N} V(x)uv\,dx - \int_{\mathbb{R}^N} f(u)v\,dx. \tag{2.3}$$

By equations (2.2) and (2.3), u is a solution for the problem (P) if and only if u is a critical point of I.

We will denote by $B_r(y) = \{x \in \mathbb{R}^N : |x-y| < r\}$ and $B_r^c(y) = [B_r(y)]^c$. Define $I_0 \colon X_0(B_1(0)) \longrightarrow \mathbb{R}$ by

$$I_0(u) =: \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))^2 K(x-y)\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx,$$

where V_∞ is the constant of (V_2). The functional I_0 has the mountain pass geometry. We will denote by d the mountain pass level associated with I_0, that is

$$d = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_0(\gamma(t)), \tag{2.4}$$

where

$$\Gamma = \{\gamma \in C([0,1], X_0(\Omega)) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}, \tag{2.5}$$

with e have fixed and verifying $I_0(e) < 0$. Note that d depends only on V_∞, θ and f.

3. An auxiliary problem

We will modify the problem (P). We will define an auxiliary problem, as in [3]. But, as the operator $-\mathcal{L}_K$ is nonlocal, we cannot use the same ideas of [3] to prove that the functional associated with the auxiliary problem satisfies the Palais–Smale condition. Therefore, we will use other techniques.

For $k = 2\theta/(\theta - 2)$ we consider

$$\widetilde{f}(x,t) := \begin{cases} f(t) & \text{if } kf(t) \le V(x)t, \\ \dfrac{V(x)}{k}\, t & \text{if } kf(t) > V(x)t, \end{cases}$$

and

$$g(x,t) := \begin{cases} f(t) & \text{if } |x| \le R, \\ \widetilde{f}(x,t) & \text{if } |x| > R, \end{cases} \tag{3.1}$$

and define the auxiliary problem

$$\begin{cases} -\mathcal{L}_K u + V(x)u = g(x,u) & \text{in } \mathbb{R}^N, \\ u \in E. \end{cases}$$

We have that, for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$,

(1) $\widetilde{f}(x,t) \le f(t)$;
(2) $g(x,t) \le V(x)t/k$, if $|x| \ge R$;
(3) $G(x,t) = F(t)$, if $|x| \le R$;
(4) $G(x,t) \le V(x)t^2/2k$ if $|x| > R$,

where

$$G(x,t) = \int_0^t g(x,s)\, ds.$$

The Euler–Lagrange functional associated with the auxiliary problem is given by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} G(x,u)\, dx.$$

The functional $J \in C^1(E,\mathbb{R})$ and

$$J'(u)v = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy + \int_{\mathbb{R}^N} V(x)uv\,dx - \int_{\mathbb{R}^N} g(x,u)v\,dx.$$

The functional J has the mountain pass geometry. Then, there is a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that

$$J'(u_n)\to 0 \quad\text{and}\quad J(u_n)\to c, \tag{3.2}$$

where $c>0$ is the mountain pass level associated with J, that is

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \tag{3.3}$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0)=0 \text{ and } \gamma(1)=e\}$ and e is the function fixed in (2.5). By definition

$$c \le d \tag{3.4}$$

uniformly in $R>0$ (see (2.4)).

Lemma 3.1. *The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E.*

Proof. By (f_2),

$$\begin{aligned} J(u)-\frac{1}{\theta}J'(u)u &= \left(\frac{\theta-2}{4\theta}\right)\|u\|^2 + \frac{1}{2k}\|u\|^2 + \int_{\mathbb{R}^N}\frac{1}{\theta}\,g(x,u)u - G(x,u)\,dx \\ &\ge \left(\frac{\theta-2}{4\theta}\right)\|u\|^2 + \frac{1}{2k}\|u\|^2 + \int_{|x|>R}\frac{1}{\theta}\,g(x,u)u - \frac{1}{2k}\int_{|x|>R}V(x)u^2\,dx \\ &\ge \left(\frac{1}{2k}\right)\|u\|^2. \end{aligned}$$

Therefore

$$|J(u)| + |J'(u)u| \ge \left(\frac{\theta-2}{4\theta}\right)\|u\|^2, \tag{3.5}$$

for all $u\in E$. This last inequality ensures that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E. □

The next results (Lemmas 3.2–3.6) are technical. We will use these lemmas to prove that the functional J satisfies the Palais–Smale condition (Proposition 3.7).

Consider $r>R$, $A=\{x\in\mathbb{R}^N : r<\|x\|<2r\}$ and $\eta\colon\mathbb{R}^N\to\mathbb{R}$ a function such that $\eta=1$ in $B_{2r}^c(0)$, $\eta=0$ in $B_r(0)$, $0\le\eta\le 1$ and $|\nabla\eta|<2/r$. Note that

$$(B_r(0)\times B_r(0))^c = (B_r^c(0)\times\mathbb{R}^N)\cup(B_r(0)\times B_r^c). \tag{3.6}$$

We will decompose

$$B_r^c(0) \times \mathbb{R}^N = (A \times \mathbb{R}^N) \cup (B_{2r}^c(0) \times B_r(0)) \cup (B_{2r}^c(0) \times A) \cup (B_{2r}^c(0) \times B_{2r}^c(0)) \tag{3.7}$$

and

$$B_r(0) \times B_r^c(0) = (B_r(0) \times A) \cup (B_r(0) \times B_{2r}^c(0)). \tag{3.8}$$

Lemma 3.2. *We have that*

$$\int_{B_r(0)} \int_{B_{2r}^c(0)} (u_n(x) - u_n(y))(\eta(x)u_n(x) - \eta(y)u_n(y))K(x-y)\,dx\,dy + \int_{B_{2r}^c(0)} \int_{B_r(0)} (u_n(x) - u_n(y))(\eta(x)u_n(x) - \eta(y)u_n(y))K(x-y)\,dx\,dy \geq -\int_{B_r(0)} \int_{B_{2r}^c(0)} u_n(y)^2 K(x-y)\,dx\,dy.$$

Proof.

$$\begin{aligned}
&\int_{B_r(0)} \int_{B_{2r}^c(0)} (u_n(x) - u_n(y))(\eta(x)u_n(x) - \eta(y)u_n(y))K(x-y)\,dx\,dy \\
&\quad + \int_{B_{2r}^c(0)} \int_{B_r(0)} (u_n(x) - u_n(y))(\eta(x)u_n(x) - \eta(y)u_n(y))K(x-y)\,dx\,dy \\
&= 2\int_{B_r(0)} \int_{B_{2r}^c(0)} (u_n(x) - u_n(y))u_n(x)K(x-y)\,dx\,dy \\
&= \int_{B_r(0)} \int_{B_{2r}^c(0)} (u_n(x) - u_n(y))^2 K(x-y)\,dx\,dy \\
&\quad + \int_{B_r(0)} \int_{B_{2r}^c(0)} u_n(x)^2 - u_n(y)^2 K(x-y)\,dx\,dy \\
&\geq -\int_{B_r(0)} \int_{B_{2r}^c(0)} u_n(y)^2 K(x-y)\,dx\,dy. \qquad \square
\end{aligned}$$

Lemma 3.3. *Let $\varepsilon > 0$. There is $r_0 > 1$ that depends on ε, such that if $r > r_0$ then*

$$\int_{B_r(0)} \int_{B_{2r}^c(0)} u_n(y)^2 K(x-y)\,dx\,dy < \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

Proof. For each $y \in B_r(0)$, $B_r(y) \subset B_{2r}(0)$. Then

$$\int_{B_{2r}^c(0)} K(x-y)\,dx \leq \int_{B_r^c(y)} K(x-y)\,dx = \int_{B_r^c(0)} K(z)\,dz. \tag{3.9}$$

By Lemma 3.1, there is $L > 0$ such that $\|u_n\|^2_{L^2(\mathbb{R}^N)} < L$ for all $n \in \mathbb{N}$. By (K_3), there is $r_0 > 1$ such that

$$\int_{B_r^c(0)} K(z)\,dz < \frac{\varepsilon}{L},$$

for all $r > r_0$. Then, by (3.9),

$$\int_{B_r(0)} \int_{B_{2r}^c(0)} u_n(y)^2 K(x-y)\,dx\,dy = \int_{B_r(0)} u_n(y)^2 \int_{B_{2r}^c(0)} K(x-y)\,dx\,dy$$
$$\leq \int_{B_r(0)} u_n(y)^2 \int_{B_r^c(0)} K(z)\,dz\,dy = \int_{B_r^c(0)} K(z)\,dz \int_{B_r(0)} u_n(y)^2\,dy \leq \varepsilon,$$

for all $n \in \mathbb{N}$ and $r > r_0$. □

LEMMA 3.4. *There are constants $K_1 > 0$ and $K_2 > 0$ such that*

$$\int_A \int_{\mathbb{R}^N} |u_n(y)||u_n(x) - u_n(y)||\eta(x) - \eta(y)|K(x-y)\,dx\,dy$$
$$\leq \left(\frac{K_1}{r} + K_2\right) \|u_n\|_{L^2(A)} [u_n, u_n]^{1/2}.$$

PROOF. Note that

$$\int_{\mathbb{R}^N} |\eta(x) - \eta(y)|^2 K(x-y)\,dx = \int_{\mathbb{R}^N} |\eta(z+y) - \eta(y)|^2 K(z)\,dz$$
$$= \int_{B_1(0)} |\eta(z+y) - \eta(y)|^2 K(z)\,dz + \int_{B_1^c(0)} |\eta(z+y) - \eta(y)|^2 K(z)\,dz$$
$$\leq \frac{4}{r^2} \int_{B_1(0)} |z|^2 K(z)\,dz + 4 \int_{B_1^c(0)} K(z)\,dz \leq \frac{4}{r^2} P_1 + 4P_2,$$

where

$$P_1 = \int_{B_1(0)} |z|^2 K(z)\,dz \quad \text{and} \quad P_2 = \int_{B_1^c(0)} K(z)\,dz.$$

Let $K_1 = 2\sqrt{P_1}$ and $K_2 = 2\sqrt{P_2}$. Then, by the Hölder inequality,

$$\int_A \int_{\mathbb{R}^N} |u_n(y)||(u_n(x) - u_n(y))||(\eta(x) - \eta(y))|K(x-y)\,dx\,dy \tag{3.10}$$
$$\leq \left(\frac{2\sqrt{P_1}}{r} + 2\sqrt{P_2}\right) \int_A |u_n(y)| \left(\int_{\mathbb{R}^N} |(u_n(x) - u_n(y))|^2 K(x-y)\,dx\right)^{1/2} dy$$
$$\leq \left(\frac{K_1}{r} + K_2\right) \|u_n\|_{L^2(A)} [u_n, u_n]^{1/2}. \square$$

LEMMA 3.5. *For the same constants $K_1 > 0$ and $K_2 > 0$ of Lemma 3.4, we have*

$$\int_{B_r(0)} \int_A |u_n(x) - u_n(y)||\eta(x)u_n(x) - \eta(y)u_n(y)|K(x-y)\,dx\,dy$$
$$\leq \left(\frac{K_1}{r} + K_2\right) \|u_n\|_{L^2(A)} [u_n, u_n]^{1/2}.$$

Proof. Indeed,

$$\begin{aligned}\int_{B_r(0)}\int_A |u_n(x)-u_n(y)||\eta(x)u_n(x)-\eta(y)u_n(y)|K(x-y)\,dx\,dy \\ = \int_{B_r(0)}\int_A |u_n(x)||u_n(x)-u_n(y)||\eta(x)|K(x-y)\,dx\,dy \\ = \int_A\int_{B_r(0)} |u_n(x)||u_n(x)-u_n(y)||\eta(x)-\eta(y)|K(x-y)\,dy\,dx \\ = \int_A\int_{B_r(0)} |u_n(y)||u_n(y)-u_n(x)||\eta(y)-\eta(x)|K(y-x)\,dx\,dy.\end{aligned}$$

By (K_1) and Lemma 3.4, we conclude the proof of this lemma. □

Lemma 3.6. *We have that*

$$-\int_{B^c_{2r}(0)}\int_A u_n(y)(u_n(x)-u_n(y))(\eta(x)-\eta(y))K(x-y)\,dx\,dy \le \bigg(\frac{K_1}{r}+K_2\bigg)\|u_n\|_{L^2(A)}[u_n,u_n]^{1/2}.$$

Proof.

$$\begin{aligned}-\int_{B^c_{2r}(0)}\int_A u_n(y)(u_n(x)-u_n(y))(\eta(x)-\eta(y))K(x-y)\,dx\,dy \\ = \int_{B^c_{2r}(0)}\int_A (u_n(x)-u_n(y))^2(\eta(x)-\eta(y))K(x-y)\,dx\,dy \\ - \int_{B^c_{2r}(0)}\int_A u_n(x)(u_n(x)-u_n(y))(\eta(x)-\eta(y))K(x-y)\,dx\,dy \\ = \int_{B^c_{2r}(0)}\int_A (u_n(x)-u_n(y))^2(\eta(x)-1)K(x-y)\,dx\,dy \\ - \int_{B^c_{2r}(0)}\int_A u_n(x)(u_n(x)-u_n(y))(\eta(x)-\eta(y))K(x-y)\,dx\,dy \\ \le -\int_{B^c_{2r}(0)}\int_A u_n(x)(u_n(x)-u_n(y))(\eta(x)-\eta(y))K(x-y)\,dx\,dy \\ = -\int_{B^c_{2r}(0)}\int_A u_n(x)(u_n(y)-u_n(x))(\eta(y)-\eta(x))K(x-y)\,dx\,dy \\ \le \int_{B^c_{2r}(0)}\int_A |u_n(x)||u_n(y)-u_n(x)||\eta(y)-\eta(x)|K(x-y)\,dx\,dy \\ = \int_A\int_{B^c_{2r}(0)} |u_n(x)||u_n(y)-u_n(x)||\eta(y)-\eta(x)|K(y-x)\,dy\,dx \\ \le \bigg(\frac{K_1}{r}+K_2\bigg)\|u_n\|_{L^2(A)}[u_n,u_n]^{1/2}.\end{aligned}$$

In the last inequality, we have used Lemma 3.4 and (K_1). □

The next proposition ensures the existence of a solution at the level c for the auxiliary problem (see (3.3)). We will prove that the functional J satisfies the Palais–Smale condition. We cannot proceed as in [3], because our operator is nonlocal.

PROPOSITION 3.7. *Suppose that f and V satisfy* (V_1), (f_1)–(f_3). *Then the functional J satisfies the Palais–Smale condition.*

PROOF. By Lemma 3.1 the Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E. We can suppose that $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in E to $u \in E$. By the properties of K we have that $\eta u_n \in X$ and $\|\eta u_n\| \leq \|u_n\|$ (see Lemma 5.1 in [18]). Then, the sequence $\{\eta u_n\}_{n\in\mathbb{N}}$ is bounded in X. Therefore $J'(u_n)(\eta u_n) = o_n(1)$, that is,

$$[u_n, \eta u_n] + \int_{\mathbb{R}^N} V(x) u_n^2 \eta \, dx = \int_{\mathbb{R}^N} g(x, u_n) \eta u_n \, dx + o_n(1).$$

But, note that $[u_n, \eta u_n] = [u_n, \eta u_n]_{B_r(0)\times B_r^c(0) + [u_n,\eta u_n]_{B_r^c(0)\times B_r(0)}}$, because $\eta = 0$ in $B_r(0)$. By (3.6), (3.7) and (3.8) we have

$$\begin{aligned}
&[u_n, \eta u_n]_{A\times\mathbb{R}^N} + [u_n, \eta u_n]_{B_{2r}^c(0)\times A} + [u_n, \eta u_n]_{B_{2r}^c(0)\times B_{2r}^c(0)} \\
&\quad + [u_n, \eta u_n]_{B_{2r}^c(0)\times B_r(0)} + [u_n, \eta u_n]_{B_r(0)\times B_{2r}^c(0)} + [u_n, \eta u_n]_{B_r(0)\times A} \\
&\qquad + \int_{\mathbb{R}^N} V(x) u_n^2 \eta \, dx = \int_{\mathbb{R}^N} g(x, u_n) \eta u_n \, dx + o_n(1).
\end{aligned}$$

By Lemma 3.2 and $[u_n, \eta u_n]_{B_{2r}^c(0)\times B_{2r}^c(0)} = [u_n, u_n]_{B_{2r}^c(0)\times B_{2r}^c(0)} \geq 0$ (because $\eta = 1$ in $B_{2r}^c(0)$), we have

$$\begin{aligned}
&[u_n, \eta u_n]_{A\times\mathbb{R}^N} + [u_n, \eta u_n]_{B_{2r}^c(0)\times A} + \int_{\mathbb{R}^N} V(x) u_n^2 \eta \, dx \\
&\quad \leq \int_{\mathbb{R}^N} g(x, u_n) \eta u_n \, dx + \int_{B_r(0)} \int_{B_{2r}^c(0)} u_n(y)^2 K(x-y) \, dx \, dy \\
&\qquad - [u_n, \eta u_n]_{B_r(0)\times A} + o_n(1).
\end{aligned}$$

If C and D are measurable subsets of $\mathbb{R}^N$ and $u \in E$, then

$$\begin{aligned}
[u, \eta u]_{C\times D} &= \int_C \int_D (u(x) - u(y))(\eta u(x) - \eta u(y)) K(x-y) \, dx \, dy \\
&= \int_C \int_D \eta(x)(u(x) - u(y))^2 K(x-y) \, dx \, dy \\
&\quad + \int_C \int_D u(y)(u(x) - u(y))(\eta(x) - \eta(y)) K(x-y) \, dx \, dy.
\end{aligned}$$

Thereby,

$$\begin{aligned}
&\int_A \int_{\mathbb{R}^N} \eta(x)(u_n(x) - u_n(y))^2 K(x-y) \, dx \, dy + \\
&\quad + \int_{B_{2r}^c(0)} \int_A \eta(x)(u_n(x) - u_n(y))^2 K(x-y) \, dx \, dy + \int_{\mathbb{R}^N} V(x) u_n^2 \eta \, dx
\end{aligned}$$

$$\begin{aligned}\le \int_{\mathbb{R}^N} g(x,u_n)\eta u_n dx &+ \int_{B_r(0)}\int_{B_{2r}^c(0)} u_n(y)^2K(x-y)\,dx\,dy - [u_n,\eta u_n]_{B_r(0)\times A}\\ &- \int_A\int_{\mathbb{R}^N} u_n(y)(u_n(x)-u_n(y))(\eta(x)-\eta(y))K(x-y)\,dx\,dy\\ &- \int_{B_{2r}^c(0)}\int_A u(y)(u_n(x)-u_n(y))(\eta(x)-\eta(y))K(x-y)\,dx\,dy + o_n(1).\end{aligned}$$

From Lemmas 3.4–3.6, we obtain constants $K_1, K_2 > 0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N} V(x)u_n^2\eta\,dx\\ &\quad\le \int_A\int_{\mathbb{R}^N} \eta(x)(u_n(x)-u_n(y))^2K(x-y)\,dx\,dy\\ &\qquad+ \int_{B_{2r}^c(0)}\int_A \eta(x)(u_n(x)-u_n(y))^2K(x-y)\,dx\,dy + \int_{\mathbb{R}^N} V(x)u_n^2\eta\,dx\\ &\quad\le \int_{\mathbb{R}^N} g(x,u_n)\eta u_n\,dx + \int_{B_r(0)}\int_{B_{2r}^c(0)} u_n(y)^2K(x-y)\,dx\,dy\\ &\qquad+ \left(\frac{K_1}{r}+K_2\right)\|u_n\|_{L^2(A)}[u_n,u_n]^{1/2} + o_n(1).\end{aligned}$$

By (2), (f_3) and $r > R$ we have

$$\int_{\mathbb{R}^N} g(x,u_n)\eta u_n\,dx \le \frac{1}{k}\int_{\mathbb{R}^N} \eta V(x)u_n^2\,dx.$$

Thereby,

$$\begin{aligned}\left(1-\frac{1}{k}\right)\int_{\mathbb{R}^N} V(x)u_n^2\eta\,dx \le \int_{B_r(0)}\int_{B_{2r}^c(0)} u_n(y)^2K(x-y)\,dx\,dy&\\ + \left(\frac{K_1}{r}+K_2\right)\|u_n\|_{L^2(A)}[u_n,u_n]^{1/2} + o_n(1).&\end{aligned}$$

By Lemma 3.1, there is $C_1 > 0$ such that $\|u_n\| \le C_1$. Then, for some constant $C > 0$

$$\begin{aligned}&\left(1-\frac{1}{k}\right)\int_{|x|>2r} V(x)u_n^2\,dx\\ &\quad\le \int_{B_r(0)}\int_{B_{2r}^c(0)} u_n(y)^2K(x,y)\,dx\,dy + C\left(\frac{1}{r}+1\right)\|u_n\|_{L^2(A)} + o_n(1).\end{aligned}\tag{3.11}$$

Let $\varepsilon > 0$. By Lemma 3.3, we can take r, large enough, such that

$$\int_{|x|>2r} V(x)u_n^2\,dx \le \frac{\varepsilon(k-1)}{3k} + C\left(\frac{1}{r}+1\right)\|u_n\|_{L^2(A)} + o_n(1),\tag{3.12}$$

for all $n \in \mathbb{N}$. Also, we can assume that

$$\|u\|_{L^2(A)} < \frac{\varepsilon(k-1)}{6\,C(1/r+1)k}.\tag{3.13}$$

By property (2) of g

$$g(x,u_n)u_n \le \frac{V(x)}{k}\,u_n^2,$$

for all x, with $|x| > 2r > R$. Therefore, by (3.11)

$$\int_{|x|>2r} g(x,u_n)u_n\,dx \le \frac{\varepsilon}{3} + C\left(\frac{1}{r}+1\right)\frac{k}{k-1}\|u_n\|_{L^2(A)} + o_n(1). \tag{3.14}$$

The space E is continuously embedded in $H^s(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N)$ is compactly embedded in $L^p(A)$ for $p \in [2, 2_s^*)$. Hence E is compactly embedded in $L^p(A)$ for $p \in [2, 2_s^*)$. By the weak convergence of $\{u_n\}_{n\in\mathbb{N}}$, we obtain that $\|u_n\|_A \to \|u\|_A$. By (3.13), for n large enough,

$$\|u_n\|_{L^2(A)} < \frac{\varepsilon(k-1)}{6\,C(1/r+1)k}.$$

By (3.14) we can take $n_1 \in \mathbb{N}$ such that if $n > n_1$ then

$$\int_{|x|>2r} g(x,u_n)u_n\,dx \le \frac{5\varepsilon}{6}.$$

Note that, we can suppose that $r > 0$ satisfies

$$\int_{|x|>2r} g(x,u)u\,dx \le \frac{\varepsilon}{12}.$$

By the compact embedding of E in $L^q(B_{2r}(0))$ for $q \in [2, 2_s^*)$, $\{u_n\}_{n\in\mathbb{N}}$ converges for u in $L^q(B_{2r}(0))$ for $q \in [2, 2_s^*)$. By definition of g and the Lebesgue dominated convergence theorem

$$\int_{|x|\le 2r} g(x,u_n)u_n\,dx \to \int_{|x|\le 2r} g(x,u)u\,dx$$

Then, we take $n_0 \in \mathbb{N}$ with $n_0 > n_1$ and such that if $n > n_0$ then

$$\left|\int_{|x|\le 2r} g(x,u_n)u_n\,dx - \int_{|x|\le 2r} g(x,u)u\,dx\right| < \frac{\varepsilon}{12}.$$

Thereby, for $n > n_0$ we have

$$\left|\int_{\mathbb{R}^N} g(x,u_n)u_n\,dx - \int_{\mathbb{R}^N} g(x,u)u\,dx\right| < \varepsilon,$$

that is,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} g(x,u_n)u_n\,dx = \int_{\mathbb{R}^N} g(x,u)u\,dx.$$

From $J'(u_n)u_n = o_n(1)$,

$$\frac{1}{2}\|u_n\|^2 = \int_{\mathbb{R}^N} g(x,u_n)u_n\,dx + o_n(1).$$

Then

$$\frac{1}{2}\|u_n\|^2 \to \int_{\mathbb{R}^N} g(x,u)u\,dx = \frac{1}{2}\|u\|^2,$$

because $J'(u)u = 0$. We conclude that $\|u_n\| \to \|u\|$ and therefore, the weak convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u ensure that $\{u_n\}_{n\in\mathbb{N}}$ converges to u in E. □

Corollary 3.8. *Suppose* (V_1), (f_1)–(f_3). *Then, there is $u \in X$ such that $J(u) = c$ and $J'(u) = 0$. Moreover, $u \geq 0$ almost everywere in $\mathbb{R}^N$.*

Proof. By 3.2 and Proposition 3.7, there is $u \in X$ such that $J(u) = c$ and $J'(u) = 0$. Let $A = \{x \in \mathbb{R}^N : |x| > R\} \cap \{x \in \mathbb{R}^N : u(x) < 0\}$. If $x \in A$, then $g(x, u(x)) = V(x)/ku(x)$ and if $x \in A^c$, then $g(x, u) \geq 0$. We have

$$0 \geq [u, u^-] + \int_{A^c} V(x)uu^- = \left(\frac{1}{k} - 1\right)\int_A V(x)uu^- + \int_{A^c} g(x, u)u^-\,dx \geq 0$$

where $u^-(x) = \max\{-u(x), 0\}$. Then $[u, u^-] = 0$. This implies that $u^- = 0$ (see proof of Lemma 4.1 in [20]). □

As a consequence of inequalities 3.4 and 3.5 we have the following proposition.

Proposition 3.9. *If V and f satisfies* (V_1), (V_2), (f_1)–(f_3), *then the solution u of the auxiliary problem satisfies $\|u\|^2 \leq 2kd$ uniformly in $R > 0$.*

4. L^∞ estimate for solution of auxiliary problem

In this section, we will prove a Brezis-Kato type estimate. We will prove that, assuming some hypotheses, there is $M > 0$ such that the solution of the problem

$$-\mathcal{L}_K v + b(x)v = g(x, v) \quad \text{in } \mathbb{R}^N$$

satisfies $\|u\|_{L^\infty(\mathbb{R}^N)} \leq M\|u\|_{L^{2^*_s}(\mathbb{R}^N)}$ and M does not depend on $\|u\|$ (see Proposition 4.5). We emphasize that, to the best of our knowledge, this result is being presented for the first time in the literature. In [2], the authors have showed this result when the operator $-\mathcal{L}_K$ is the fractional laplacian operator, that is, when $K(x) = C_{N,s}|x|^{-N-2s}$. But, in our case, we cannot use the same technique used in [2], because we do not have a version of the s-harmonic extension for general integro-differential operators. Therefore, we present an another technique.

Remark 4.1. Let $\beta > 1$. Define the real functions

$$\begin{aligned} m(x) &:= (\lambda - 1)(x^\beta + x^{-\beta}) - \lambda(x^{\beta-1} + x^{1-\beta}) + 2, \\ p(x) &:= (\lambda - 1)(x^\beta + x^{-\beta}) + \lambda(x^{\beta-1} + x^{1-\beta}) - 2, \end{aligned}$$

where $\lambda := \beta^2/(2\beta - 1)$. Then $m(x) \geq 0$ and $p(x) \geq 0$ for all $x > 0$. Indeed, defining the function $g(x) = x^{\beta+1}(\beta-1)m'(x)/\beta(\lambda-1)$ we have $g(1) = g'(1) = 0$ and $g''(x) > 0$ for all $x > 1$. Then, $m'(x) > 0$ for all $x > 1$. From $m(1) = 0$ and $m(x) = m(x^{-1})$ for all $x > 0$, we conclude that $m(x) \geq 0$ for all $x > 0$. From $p(1) = 0$, $p(x) = p(x^{-1})$ for all $x \neq 0$ and $p'(x) > 0$ for all $x > 1$, we conclude that $p(x) \geq 0$ for all $x > 0$.

Let $\beta > 1$. Define

$$f(x) = x|x|^{2(\beta-1)} \quad \text{and} \quad g(x) = x|x|^{\beta-1}.$$

The functions f and g are continuous and differentiable for all $x \in \mathbb{R}$. Consider $x, y \in \mathbb{R}$ with $x \neq y$. By the mean value theorem, there are $\theta_1(x,y)$, $\theta_2(x,y) \in \mathbb{R}$ such that

$$f'(\theta_1(x,y)) = \frac{f(x) - f(y)}{x - y} \tag{4.1}$$

and

$$g'(\theta_2(x,y)) = \frac{g(x) - g(y)}{x - y}, \tag{4.2}$$

that is

$$(2\beta - 1)|\theta_1(x,y)|^{2(\beta-1)} = \frac{x|x|^{2(\beta-1)} - y|y|^{2(\beta-1)}}{x - y}, \tag{4.3}$$

$$\beta|\theta_2(x,y)|^{(\beta-1)} = \frac{x|x|^{(\beta-1)} - y|y|^{(\beta-1)}}{x - y}. \tag{4.4}$$

Implying that

$$|\theta_1(x,y)| = \left(\frac{1}{2\beta - 1} \frac{x|x|^{2(\beta-1)} - y|y|^{2(\beta-1)}}{x - y}\right)^{1/2(\beta-1)}, \tag{4.5}$$

$$|\theta_2(x,y)| = \left(\frac{1}{\beta} \frac{x|x|^{(\beta-1)} - y|y|^{(\beta-1)}}{x - y}\right)^{1/(\beta-1)}. \tag{4.6}$$

We will consider $\theta_1(x,x) = \theta_2(x,x) = 0$ for all $x \in \mathbb{R}$.

Remark 4.2. Note that $|\theta_1(x,y)| = |\theta_1(y,x)|$ and $|\theta_2(x,y)| = |\theta_2(y,x)|$ for all $x, y \in \mathbb{R}$.

Lemma 4.3. *With the same notation, if $x \neq 0$ then $|\theta_1(x,0)| \geq |\theta_2(x,0)|$.*

Proof. By (4.5) and (4.6), we have

$$|\theta_1(x,0)| = \left(\frac{1}{2\beta - 1} \frac{x|x|^{2(\beta-1)}}{x}\right)^{1/2(\beta-1)} = \left(\frac{1}{2\beta - 1}\right)^{1/2(\beta-1)} |x|,$$

$$|\theta_2(x,0)| = \left(\frac{1}{\beta} \frac{x|x|^{\beta-1}}{x}\right)^{1/(\beta-1)} = \left(\frac{1}{\beta}\right)^{1/(\beta-1)} |x|.$$

Thereby, $|\theta_1(x,0)| \geq |\theta_2(x,0)|$. □

Lemma 4.4. *If $x, y \in \mathbb{R}$, then $|\theta_1(x,y)| \geq |\theta_2(x,y)|$.*

Proof. If $x = 0$ or $y = 0$ then the inequality was proved by Lemma 4.3 and Remark 4.2. The case $x = y$ is trivial. We can suppose that $x \neq y$, $x \neq 0$ and $y \neq 0$. By (4.5) and (4.6) we have that $|\theta_1(x,y)| \geq |\theta_2(x,y)|$ if and only if

$$\left(\frac{1}{2\beta - 1} \frac{x|x|^{2(\beta-1)} - y|y|^{2(\beta-1)}}{x - y}\right)^{1/2(\beta-1)} \geq \left(\frac{1}{\beta} \frac{x|x|^{\beta-1} - y|y|^{\beta-1}}{x - y}\right)^{1/(\beta-1)}.$$

This last inequality is true if and only if

$$\frac{1}{2\beta-1}\frac{x|x|^{2(\beta-1)}-y|y|^{2(\beta-1)}}{x-y}\geq\frac{1}{\beta^2}\bigg(\frac{x|x|^{\beta-1}-y|y|^{\beta-1}}{x-y}\bigg)^2.$$

This last inequality occurs if and only if

$$\lambda(x-y)(x|x|^{2(\beta-1)}-y|y|^{2(\beta-1)})\geq(x|x|^{\beta-1}-y|y|^{\beta-1})^2,$$

that is

$$\lambda(|x|^{2\beta}-xy|y|^{2(\beta-1)}-yx|x|^{2(\beta-1)}+|y|^{2\beta})\geq|x|^{2\beta}-2xy|x|^{\beta-1}|y|^{\beta-1}+|y|^{2\beta}.$$

But, if $x\neq0$ and $y\neq0$, then the last inequality is equivalent to

$$\lambda\bigg[\bigg(\frac{|x|}{|y|}\bigg)^\beta-\bigg(xy\frac{|y|^{\beta-2}}{|x|^\beta}\bigg)-\bigg(\frac{xy|x|^{\beta-2}}{|y|^\beta}\bigg)+\bigg(\frac{|y|}{|x|}\bigg)^\beta\bigg]\geq\bigg(\frac{|x|}{|y|}\bigg)^\beta-2\frac{x}{|x|}\frac{y}{|y|}+\bigg(\frac{|y|}{|x|}\bigg)^\beta. \tag{4.7}$$

We will prove that (4.7) is true. If $x\cdot y>0$, then

$$\begin{aligned}
&\lambda\bigg[\bigg(\frac{|x|}{|y|}\bigg)^\beta-\bigg(xy\frac{|y|^{\beta-2}}{|x|^\beta}\bigg)-\bigg(\frac{xy|x|^{\beta-2}}{|y|^\beta}\bigg)+\bigg(\frac{|y|}{|x|}\bigg)^\beta\bigg]\\
&\qquad-\bigg(\frac{|x|}{|y|}\bigg)^\beta+2\frac{x}{|x|}\frac{y}{|y|}-\bigg(\frac{|y|}{|x|}\bigg)^\beta\\
&\quad=\lambda\bigg[\bigg(\frac{|x|}{|y|}\bigg)^\beta-\bigg(\frac{|y|}{|x|}\bigg)^{\beta-1}-\bigg(\frac{|x|}{|y|}\bigg)^{\beta-1}+\bigg(\frac{|y|}{|x|}\bigg)^\beta\bigg]\\
&\qquad-\bigg(\frac{|x|}{|y|}\bigg)^\beta+2-\bigg(\frac{|y|}{|x|}\bigg)^\beta\\
&\quad=(\lambda-1)\bigg[\bigg(\frac{|x|}{|y|}\bigg)^\beta+\bigg(\frac{|x|}{|y|}\bigg)^{-\beta}\bigg]-\lambda\bigg[\bigg(\frac{|x|}{|y|}\bigg)^{\beta-1}+\bigg(\frac{|x|}{|y|}\bigg)^{-\beta+1}\bigg]+2\\
&\quad=m\bigg(\frac{|x|}{|y|}\bigg).
\end{aligned}$$

If $x\cdot y<0$, then

$$\begin{aligned}
&\lambda\bigg[\bigg(\frac{|x|}{|y|}\bigg)^\beta-\bigg(xy\frac{|y|^{\beta-2}}{|x|^\beta}\bigg)-\bigg(\frac{xy|x|^{\beta-2}}{|y|^\beta}\bigg)+\bigg(\frac{|y|}{|x|}\bigg)^\beta\bigg]\\
&\qquad-\bigg(\frac{|x|}{|y|}\bigg)^\beta+2\frac{x}{|x|}\frac{y}{|y|}-\bigg(\frac{|y|}{|x|}\bigg)^\beta\\
&\quad=\lambda\bigg[\bigg(\frac{|x|}{|y|}\bigg)^\beta+\bigg(\frac{|y|}{|x|}\bigg)^{\beta-1}+\bigg(\frac{|x|}{|y|}\bigg)^{\beta-1}+\bigg(\frac{|y|}{|x|}\bigg)^\beta\bigg]\\
&\qquad-\bigg(\frac{|x|}{|y|}\bigg)^\beta-2-\bigg(\frac{|y|}{|x|}\bigg)^\beta
\end{aligned}$$

$$= (\lambda - 1)\bigg[\bigg(\frac{|x|}{|y|}\bigg)^{\beta} + \bigg(\frac{|x|}{|y|}\bigg)^{-\beta}\bigg] + \lambda\bigg[\bigg(\frac{|x|}{|y|}\bigg)^{\beta-1} + \bigg(\frac{|x|}{|y|}\bigg)^{-\beta+1}\bigg] - 2$$
$$= p\bigg(\frac{|x|}{|y|}\bigg).$$

By Remark 4.1, we have that $m(|x|/|y|) \geq 0$ and $p(|x|/|y|) \geq 0$. This proves the inequality (4.7) and Lemma 4.4. □

Our main result of this section will establish an important estimate involving the $L^\infty(\mathbb{R}^N)$ norm of the solution u of the auxiliary problem. It states that:

PROPOSITION 4.5. *Let $h \in L^q(\mathbb{R}^N)$ with $q > N/2s$, and $v \in E \subset X$ be a weak solution of*

$$-\mathcal{L}_K v + b(x)v = g(x, v) \quad \textit{in } \mathbb{R}^N,$$

where g is a continuous functions satisfying $|g(x,s)| \leq h(x)|s|$ for $s \geq 0$, b is a positive function in $\mathbb{R}^N$ and E is definded as in (2.1)*. Then, there is a constant $M = M(q, \|h\|_{L^q(\mathbb{R}^N)})$ such that*

$$\|v\|_{L^\infty(\mathbb{R}^N)} \leq M\|v\|_{L^{2_s^*}(\mathbb{R}^N)}.$$

PROOF. Let $\beta > 1$. For any $n \in \mathbb{N}$ we define $A_n = \{x \in \mathbb{R}^N; |v(x)|^{\beta-1} \leq n\}$ and $B_n := A_n^c$. Consider

$$f_n(t) := \begin{cases} t|t|^{2(\beta-1)} & \text{if } |t|^{\beta-1} \leq n, \\ n^2 t & \text{if } |t|^{\beta-1} > n, \end{cases} \quad \text{and} \quad g_n(t) := \begin{cases} t|t|^{(\beta-1)} & \text{if } |t|^{\beta-1} \leq n, \\ nt & \text{if } |t|^{\beta-1} > n. \end{cases}$$

Note that f_n and g_n are continuous functions and they are differentiable at all points with the exception on $n^{1/(\beta-1)}$ and $-n^{1/(\beta-1)}$ and their derivatives are limited. Then f_n and g_n are Lipschitz continuous. Therefore, setting

$$v_n := f_n \circ v \quad \text{and} \quad w_n := g_n \circ v$$

we have that $v_n, w_n \in E$. Note that

$$[v, v_n] = \int_{A_n}\int_{A_n} (v_n(x) - v_n(y))(v(x) - v(y)K(x - y)\,dx\,dy$$
$$+ \int_{B_n}\int_{B_n} (v_n(x) - v_n(y))(v(x) - v(y)K(x - y)\,dx\,dy + 2[v, v_n]_{A_n \times B_n}.$$

By (4.1), if $x, y \in A_n$ then

$$v_n(x) - v_n(y) = f_n'(\theta_1(x, y))(v(x) - v(y)),$$

where $\theta_1(x, y) = \theta_1(v(x), v(y))$. Therefore

$$(4.8) \quad [v, v_n] = \int_{A_n}\int_{A_n} (2\beta - 1)|\theta_1(x, y)|^{2(\beta-1)}(v(x) - v(y))^2 K(x - y)\,dx\,dy$$
$$+ n^2 \int_{B_n}\int_{B_n} (v(x) - v(y))^2 K(x - y)\,dx\,dy + 2[v, v_n]_{A_n \times B_n}.$$

Analogously, by (4.2)

$$[w_n,w_n]=\int_{A_n}\int_{A_n}\beta^2|\theta_2(x,y)|^{2(\beta-1)}(v(x)-v(y))^2K(x-y)\,dx\,dy + n^2\int_{B_n}\int_{B_n}(v(x)-v(y))^2K(x-y)\,dx\,dy+2[w_n,w_n]_{A_n\times B_n},$$

where $\theta_2(x,y)=\theta_2(v(x),v(y))$. By Lemma 4.4,

$$[w_n,w_n]\le\int_{A_n}\int_{A_n}\beta^2|\theta_1(x,y)|^{2(\beta-1)}(v(x)-v(y))^2K(x-y)\,dx\,dy + n^2\int_{B_n}\int_{B_n}(v(x)-v(y))^2K(x-y)\,dx\,dy+2[w_n,w_n]_{A_n\times B_n}.$$

This implies that

$$[w_n,w_n]+\int_{\mathbb{R}^N}b(x)w_n^2\,dx-[v,v_n]-\int_{\mathbb{R}^N}b(x)vv_n\,dx \le(\beta-1)^2\int_{A_n}\int_{A_n}|\theta_1(x,y)|^{2(\beta-1)}(v(x)-v(y))^2K(x-y)\,dx\,dy + 2[w_n,w_n]_{A_n\times B_n}-2[v,v_n]_{A_n\times B_n}. \tag{4.9}$$

By (4.8), we have

$$[v,v_n]+\int_{\mathbb{R}^N}b(x)vv_n\,dx-2[v,v_n]_{A_n\times B_n} \ge(2\beta-1)\int_{A_n}\int_{A_n}|\theta_1(x,y)|^{2(\beta-1)}(v(x)-v(y))^2K(x-y)\,dx\,dy, \tag{4.10}$$

because $b(x)vv_n=b(x)w_n^2\ge0$. Replacing (4.10) in (4.9) we obtain

$$[w_n,w_n]+\int_{\mathbb{R}^N}b(x)w_n^2\,dx-[v,v_n]-\int_{\mathbb{R}^N}b(x)vv_n\,dx \le\frac{(\beta-1)^2}{2\beta-1}\bigg([v,v_n]+\int_{\mathbb{R}^N}b(x)vv_n\,dx\bigg) +2[w_n,w_n]_{A_n\times B_n}+\bigg(-2-\frac{2(\beta-1)^2}{2\beta-1}\bigg)[v,v_n]_{A_n\times B_n},$$

that is

$$\begin{aligned}[w_n,w_n]&+\int_{\mathbb{R}^N}b(x)w_n^2\,dx\\ &\le\bigg(\frac{(\beta-1)^2}{2\beta-1}+1\bigg)\bigg([v,v_n]+\int_{\mathbb{R}^N}b(x)vv_n\,dx\bigg)\\ &\quad+2[w_n,w_n]_{A_n\times B_n}+\bigg(-2-\frac{2(\beta-1)^2}{2\beta-1}\bigg)[v,v_n]_{A_n\times B_n}\\ &=\frac{\beta^2}{2\beta-1}\bigg([v,v_n]+\int_{\mathbb{R}^N}bvv_n\,dx\bigg)\end{aligned}$$

$$
\begin{aligned}
&\quad + 2[w_n, w_n]_{A_n\times B_n} + \Big(-2 - \frac{2(\beta-1)^2}{2\beta-1}\Big)[v, v_n]_{A_n\times B_n}\\
&\le \beta \int_{\mathbb{R}^N} g(x,v)v_n\,dx\\
&\quad + 2[w_n, w_n]_{A_n\times B_n} + \Big(-2 - \frac{2(\beta-1)^2}{2\beta-1}\Big)[v, v_n]_{A_n\times B_n}.
\end{aligned}
$$

In short,

$$
\begin{aligned}
(4.11)\quad [w_n, w_n] + \int_{\mathbb{R}^N} b(x)w_n^2\,dx \le \beta \int_{\mathbb{R}^N} g(x,v)v_n\,dx + 2[w_n, w_n]_{A_n\times B_n}\\
+ \Big(-2 - \frac{2(\beta-1)^2}{2\beta-1}\Big)[v, v_n]_{A_n\times B_n}.
\end{aligned}
$$

But, if $n \in \mathbb{N}$ and

$$C = 2 + \frac{2(\beta-1)^2}{2\beta-1},$$

then a simple calculation shows that the function

$$r(s,t) = 2(ns - t|t|^{\beta-1})^2 - C(s-t)(n^2 s - t|t|^{2(\beta-1)}),$$

satisfies

$$(4.12)\qquad r(s,t) \le 0 \quad \text{for all } |s| > n^{1/(\beta-1)} \text{ and } |t| \le n^{1/(\beta-1)}.$$

Taking $s = v(x)$ and $t = v(y)$ for $x \in B_n$ and $y \in A_n$ and replacing in (4.12) we obtain

$$2(w_n(x) - w_n(y))^2 - C(v(x) - v(y))(v_n(x) - v_n(y)) \le 0.$$

Hence

$$2[w_n, w_n]_{A_n\times B_n} + \Big(-2 - \frac{2(\beta-1)^2}{2\beta-1}\Big)[v, v_n]_{A_n\times B_n} \le 0.$$

By (4.11),

$$(4.13)\qquad [w_n, w_n] + \int_{\mathbb{R}^N} b(x)w_n^2\,dx \le \beta \int_{\mathbb{R}^N} g(x,v)v_n\,dx.$$

Let $S > 0$ be the best constant verifying $\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \le S[u,u]^2$, for all $u \in H^s(\mathbb{R}^N)$ (see theorem 6.5 in [18]), that is

$$(4.14)\qquad S = \sup_{u\in H^s(\mathbb{R}^N)} \frac{\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)}}{[u,u]^2}.$$

The critical inequality (theorem 6.5 in [18]) ensures the existence of S. By (4.13)

$$\begin{aligned}\bigg(\int_{A_n}|w_n|^{2_s^*}\,dx\bigg)^{2/2_s^*} &\leq \bigg(\int_{\mathbb{R}^N}|w_n|^{2_s^*}dx\bigg)^{2/2_s^*}\\ &\leq S[w_n,w_n]^2 \leq S\|w_n\|^2 \leq S\beta\int_{\mathbb{R}^N} g(x,v(x))v_n\,dx\\ &\leq S\beta\int_{\mathbb{R}^N} h(x)w_n^2\,dx \leq S\beta\|h\|_{L^q(\mathbb{R}^N)}\|w_n\|^2_{L^{2q/(q-1)}(\mathbb{R}^N)}.\end{aligned}$$

But, we have that $|w_n(x)|\leq |v(x)|^\beta$ for all $x\in B_n$ and $|w_n(x)|=|v(x)|^\beta$ for all $x\in A_n$. Thereby,

$$\bigg(\int_{A_n}|v|^{\beta 2_s^*}\,dx\bigg)^{2/2_s^*} \leq S\beta\|h\|_{L^q(\mathbb{R}^N)}\bigg(\int_{\mathbb{R}^N}|v|^{2q\beta/(q-1)}\,dx\bigg)^{(q-1)/q}.$$

By the monotone convergence theorem,

$$\|v\|_{L^{2_s^*\beta}(\mathbb{R}^N)} \leq (\beta S\|h\|_{L^q(\mathbb{R}^N)})^{1/2\beta}\|v\|_{L^{2\beta q_1}(\mathbb{R}^N)} \tag{4.15}$$

where $q_1=q/(q-1)$. Define $\eta:=2_s^*/(2q_1)$, and note that $\eta>1$. When $\beta=\eta$ we have that $2\beta q_1=2_s^*$. Then, by (4.15)

$$\|v\|_{L^{2_s^*\eta}(\mathbb{R}^N)} \leq (\eta S\|h\|_{L^q(\mathbb{R}^N)})^{1/2\eta}\|v\|_{L^{2_s^*}(\mathbb{R}^N)}. \tag{4.16}$$

Taking $\beta=\eta^2$ in (4.15) we obtain

$$\|v\|_{L^{2_s^*\eta^2}(\mathbb{R}^N)} \leq \eta^{1/\eta^2}(S\|h\|_{L^q(\mathbb{R}^N)})^{1/2\eta^2}\|v\|_{L^{2_s^*\eta}(\mathbb{R}^N)}. \tag{4.17}$$

By (4.16) and (4.17) we have

$$\|v\|_{L^{2_s^*\eta^2}(\mathbb{R}^N)} \leq \eta^{1/\eta^2+1/2\eta}(S\|h\|_{L^q(\mathbb{R}^N)})^{1/2\eta^2+1/2\eta}\|v\|_{L^{2_s^*}(\mathbb{R}^N)}.$$

Inductively, we can prove that

$$\begin{aligned}&\|v\|_{L^{2_s^*\eta^m}(\mathbb{R}^N)}\\ &\qquad\leq \eta^{1/2\eta+1/\eta^2+\ldots+m/2\eta^m}(S\|h\|_{L^q(\mathbb{R}^N)})^{1/2\eta+1/2\eta^2+\ldots+1/2\eta^m}\|v\|_{L^{2_s^*}(\mathbb{R}^N)}\end{aligned}$$

for all $m\in\mathbb{N}$. But,

$$\sum_{m=1}^{\infty}\frac{m}{2\eta^m}=\frac{\eta}{2(\eta-1)^2}\quad\text{and}\quad\sum_{m=1}^{\infty}\frac{1}{2\eta^m}=\frac{1}{2(\eta-1)}.$$

Thereby, for all $m>0$,

$$\|v\|_{L^{2_s^*\eta^m}(\mathbb{R}^N)} \leq \eta^{\eta/2(\eta-1)^2}(S\|h\|_{L^q(\mathbb{R}^N)})^{1/2(\eta-1)}\|v\|_{L^{2_s^*}(\mathbb{R}^N)}.$$

Recalling that $\|v\|_{L^\infty(\mathbb{R}^N)}=\lim_{n\to\infty}\|v\|_{L^p(\mathbb{R}^N)}$ and that $\eta>1$ we have that

$$\|v\|_{L^\infty(\mathbb{R}^N)}\leq M\|v\|_{L^{2_s^*}(\mathbb{R}^N)}$$

for $M=\eta^{\eta/2(\eta-1)^2}(S\|h\|_{L^q(\mathbb{R}^N)})^{1/2(\eta-1)}$ and $\eta=N(q-1)/(q(N-2s))$.

We conclude the proof of Proposition 4.5 noting that M depends only on q and $\|h\|_{L^q(\mathbb{R}^N)}$. □

5. Solution for Problem (P)

In this section, we prove the main result (Theorem 5.2). By Corollary 3.8, there is $u \in E$ such that $J(u) = c$ and $J'(u) = 0$. We have the following estimate for $\|u\|_{L^\infty(\mathbb{R}^N)}$.

Lemma 5.1. *The solution u of the auxiliary problem satisfies*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le M(2Skd)^{1/2},$$

where d and S are defined respectively in (2.4) *and* (4.14), $k = 2\theta/(\theta - 2)$, *and θ is defined in* (f_1).

Proof. Consider the functions

$$H(x,t) = \begin{cases} f(t) & \text{if } |x| < R \text{ or } f(t) \le \dfrac{V(x)}{k}\, t, \\ 0 & \text{if } |x| \ge R \text{ and } f(t) > \dfrac{V(x)}{k}\, t, \end{cases}$$

and

$$b(x) = \begin{cases} V(x) & \text{if } |x| < R \text{ and } f(u) \le \dfrac{V(x)}{k}\, u, \\ \left(1 - \dfrac{1}{k}\right) V(x) & \text{if } |x| \ge R \text{ and } f(u) > \dfrac{V(x)}{k} k u. \end{cases}$$

Note that the function u is solution of

$$\begin{cases} -\mathcal{L}_K u + b(x) u = H(x,u) & \text{in } \mathbb{R}^N, \\ u \in E. \end{cases}$$

By (f_1), $|H(x,t)| \le c_0 |t|^{p-1}$ for $p \in (2, 2_s^*)$. Thereby, $|H(x,u)| \le h(x)|u|$, where $h(x) = c_0 |u|^{p-2}$. Note that $h \in L^{2_s^*/(p-2)}(\mathbb{R}^N)$ and

$$\|h\|_{L^q(\mathbb{R}^N)} \le C(2kSd)^{(p-2)/2_s^*},$$

where $q = 2_s^*/(p-2)$. The number p satisfies

$$p < 2_s^* = 2 + \frac{2s}{N}\, 2_s^*, \quad \text{then} \quad q = \frac{2_s^*}{p-2} > \frac{N}{2s}.$$

By Proposition 4.5 and sobolev embedding

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le M \|u\|_{L^{2_s^*}(\mathbb{R}^N)} \le M S^{1/2} \|u\|,$$

where $M = M(q, \|h\|_{L^q(\mathbb{R}^N)})$. By Proposition 3.9, we have

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le M(2kSd)^{1/2}. \tag{5.1}$$

This concludes the proof. □

Theorem 5.2. *Suppose that V satisfies conditions* (V_1)–(V_3) *and that f satisfies* (f_1)–(f_3). *There is $\Lambda^* = \Lambda^*(V_\infty, \theta, p, c_0, S) > 0$ such that if $\Lambda > \Lambda^*$ in* (V_3), *then the problem* (P) *has a nonnegative nontrivial solution.*

Proof. Let u be a weak solution of the auxiliary problem. Let $|x| \geq R$. If $u(x) = 0$ then by denfition $f(u(x)) = g(x, u(x))$. If $u(x) > 0$ then

$$\begin{aligned}\frac{f(u(x))}{u(x)} &\leq c_0 |u|^{p-2} \leq c_0 \|u\|^{p-2}_{L^\infty(\mathbb{R}^N)} \\ &= \frac{c_0 \|u\|^{p-2}_{L^\infty(\mathbb{R}^N)}}{\Lambda}\Lambda \leq \frac{k^{p/2} c_0 M^{p-2} (2Sd)^{(p-2)/2}}{\Lambda} \frac{V(x)}{k}.\end{aligned}$$

Define $\Lambda^* = k^{p/2} c_0 M^{p-2} (2Sd)^{(p-2)/2}$. If $\Lambda > \Lambda^*$ then

$$\frac{f(u(x))}{u(x)} \leq \frac{V(x)}{k}.$$

By definition of g we have $g(x, u(x)) = f(u(x))$. Then $g(x, u(x)) = f(u(x))$ for all $x \in \mathbb{R}^N$. Therefore, u is a nonnegative and nontrivial solution of (P). □

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Manuscript received August 4, 2018

accepted April 12, 2019

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