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FIXED POINT INDEX THEORY FOR PERTURBATION OF EXPANSIVE MAPPINGS BY k-SET CONTRACTIONS

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Abstract. In this work, we develop a fixed point index theory for the sum of k-set contractions and expansive mappings with constant $h > 1$ when $0 \leq k < h-1$ as well as in the limit case $k = h-1$. After computing this new index, several fixed point theorems and recent results are derived, including Krasnosel'skii type theorems. Two examples of application illustrate the theoretical results.

1. Introduction

Starting from the Krasnosel'skiĭ fixed point theorem (KFPT for short) [22], the fixed point theory for sums of operators developed promptly and has been widely extended to various types of nonlinear mappings (see, e.g. [10], [29], [36]) in theory as well as in applications to many problems in nonlinear sciences. KFPT (1958) concerns the sum of a contraction and a compact mapping and turns out to be a generalization of Banach's contraction mapping principle (1922) and Schauder's fixed point theorem (1930) [33]. However, its proof uses both of these important results. It states that the sum $T+F$ has at least one fixed point in D whenever the mappings $T, F\colon D \to E$ satisfy the following conditions:

(a) for all $x, y \in D$, $T(x) + F(y) \in D$.
(b) T is a contraction.

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(c) F is compact, continuous.

The proof of this result lies in the fact that if E is a linear vector space, $F \subset E$ a nonempty subset, and $g\colon F \to E$ is a contraction, then the mapping $I-g\colon F \to F$ is a homeomorphism [34, Lemma 2.9].

Earlier, the concept of measure of noncompactness has been employed by Darbo [12] and later by Sadovski [32] to introduce the notions of k-set contractions and condensing mappings, respectively and then to derive some fixed point theorems for these classes of operators. One of the most important feature of these results is that the sum of a contraction and a completely continuous mapping turned out to be a strict k-set contraction (with respect to some measure of noncompactness), extending by the way Krasnosel'skiĭ's fixed point theorem.

Later many researchers have been interested in the extension of the above theorem in various directions by modifying assumptions (a)–(c), or even the underlying space E; we cite [10], [18], [35], [36].

Another fixed point theorem established by Krasnosel'skiĭ in 1964 is the cone expansion and compression theorem (see, e.g. [20], [24], [23]) for mappings that need not be the sum of operators but act on some cones of Banach spaces. The latter theorem has been recently deeply improved too; see [3], [4], [17], [25], [26], [30] and references therein.

This second existence result plays a key role in the study of positive fixed points for various classes of nonlinear operators posed in some ordered Banach spaces. Indeed the positivity of solutions of nonlinear equations, especially ordinary, partial differential equations, and integral equations is a very important issue in applications, where a positive solution may represent a density, temperature, velocity, ...

The positivity condition can be mathematically described by introducing a cone $\mathcal{P}$ in some Banach space E, that is a closed convex subset such that $\alpha\mathcal{P} \subset \mathcal{P}$ for all positive real number α and $\mathcal{P}\cap(-\mathcal{P}) = \{0\}$. Notice that a cone $\mathcal{P}$ induces a partial ordering $\leq$ in E defined by $x \leq y$ if and only if $y - x \in \mathcal{P}$. We will denote by $\mathcal{P}^* = \mathcal{P} \setminus \{0\}$ and $E^* = E \setminus \{0\}$ the punctured cone and space, respectively. We say that $x < y$ if $y - x \in \mathcal{P}^*$ and $x \not\leq y$ if $y - x \notin \mathcal{P}$. A cone $\mathcal{P}$ is called normal if there exists a positive constant N such that, for all $x, y \in \mathcal{P}$, we have

$$x \leq y \ \Rightarrow\ \|x\| \leq N\|y\|.$$

The least positive constant N is called the normal constant of $\mathcal{P}$.

This work is part of generalizations leading to fixed point theory for sums of operators. More precisely, our aim is to construct a generalized fixed point index for operators that are sums of the form $T + F$, where T is an expansive operator and F is a k-set contraction. For this we will appeal to the fixed point index theory for strict set contractions (see [1], [27], [28]). Then we derive several existence results for the nonlinear equation $x = Tx + Fx$, where $x \in \mathcal{P}$, extending by the way Krasnosel'skii and Sadovski fixed point theorems for classes of operators that can be expressed as sums of nonlinear contractions and expansive mappings. Moreover, using some approximation arguments, we show that the theory still holds for the limit case concerning the important class of mappings F that are 1-set contractions. These mappings include, as particular cases, sums of nonexpansive and compact mappings. Note that the topological degree for these special mappings was first introduced by Petryshyn in [31] (see also [14] and [21] for a survey on the fixed point theory for 1-set contractions).

Before proceeding with the theoretical results of this paper, we recall that as early as in the 70's, F. Browder [9, Chapter 13] introduced a topological degree for a class of compact perturbation of strongly accretive mappings. The construction of the degree was based on the Leray–Schauder degree together with the perturbation of maximal monotone operators. An interesting fact in the study of accretive operators is that they may lead to the construction of contractive mappings (see [7]). In this respect, we also mention Chen, Ha, and Cho [11] who discussed the fixed point index for operator sums of the form $-A+K$, where $A\colon D(A) \subset \mathcal{P} \to 2^{\mathcal{P}}$ is an accretive operator with $(I + A)(D(A)) = \mathcal{P}$ and K is a strict set contraction. As consequences, they have deduced some solvability results for the operator inclusion $x \in -Ax + Kx$, where $x \in \mathcal{P}$.

In this section, we will also collect some notations, definitions, and auxiliary results we need throughout this paper. Of particular importance for our purpose is the fixed point index for k-set contractions and its properties. Then we will present our main contributions in the subsequent sections. We will consider separately two cases: firstly the case of the sum $T + F$, where T is expansive and F is a k-set contraction is treated in Section 2. Then in Section 3, we will discuss the limit case where F is a $(h - 1)$-set contraction. Sections 2 and 3 are also concerned with the computation of the fixed point index. Some fixed point theorems are further derived as consequences in Section 4. The paper ends with two examples of application to nonlinear integral equations in Section 5 illustrating the abstract results obtained in this work.

Let E be a Banach space. A mapping $f\colon E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets. A subset $X \subset E$ is called a retract of E if there exists a continuous map $\rho\colon E \to X$ such that $\rho(x) = x$ for all $x \in X$. Then ρ is called a retraction. It is easy to see that, in any topological space, every retract is closed. Conversely, every closed convex subset of a locally convex topological vector space is a retract of this space.

The concept of k-set contraction is related to that of the Kuratowski measure of noncompactness (MNC for short) (1930) which we recall for the sake of completeness.

Definition 1.1. Let E be a real Banach space and Ω_E be the class of all bounded subsets of E. The Kuratowski measure of noncompactness $\alpha : \Omega_E \to [0,+\infty)$ is defined by

$$\alpha(V) = \inf\bigg\{\delta > 0 \ \bigg|\ V = \bigcup_{i=1}^{n} V_i \text{ and } \operatorname{diam}(V_i) \leq \delta, \text{ for all } i = 1,\ldots,n\bigg\},$$

where $\operatorname{diam}(V_i) = \sup\{\|x-y\|_E,\ x,y \in V_i\}$ is the diameter of V_i.

For the main properties of measures of noncompactness, we recommend, e.g. [5], [6] and [13].

Definition 1.2. Let E be a Banach space and $A\colon D \subset E \to E$ a continuous mapping which maps bounded subsets of E into bounded subsets.

(a) A is said a k-set contraction if there exists a constant $k \geq 0$ such that $\alpha(A(V)) \leq k\alpha(V)$, for every bounded $V \subset D$.
(b) A is a strict k-set contraction if $k < 1$.

The proofs of our results involve the fixed point index for strict k-set contractions whose basic properties are collected in the following lemma. For the proof we refer the reader to [19, Theorem 1.3.5] or [1], [13], [20].

Lemma 1.3. *Let X be a retract of a Banach space E. For every bounded open subset $U \subset X$ and every strict k-set contraction $f\colon \overline{U} \to X$ without fixed point on the boundary ∂U, there exists uniquely one integer $i\,(f,U,X)$ satisfying the following conditions*:

(a) (Normalization) *If $f\colon \overline{U} \to U$ is a constant map, then $i(f,U,X) = 1$.*
(b) (Additivity) *For any pair of disjoint open subsets U_1, U_2 in U such that f has no fixed point on $\overline{U}\backslash(U_1 \cup U_2)$, we have*

$$i(f,U,X) = i(f,U_1,X) + i(f,U_2,X),$$

where $i(f,U_j,X) := i(f|_{\overline{U_j}},U_j,X)$, $j = 1,2$.
(c) (Homotopy Invariance) *The index $i(h(x,t),U,X)$ does not depend on the parameter $t \in [0,1]$, where*
  (i) *$h\colon [0,1] \times \overline{U} \to X$ is continuous and $h(t,x)$ is uniformly continuous in t with respect to $x \in \overline{U}$,*
  (ii) *$h(t,\,\cdot\,)\colon \overline{U} \to X$ is a strict k-set contraction, where k does not depend on $t \in [0,1]$,*
  (iii) *$h(t,x) \neq x$, for every $t \in [0,1]$ and $x \in \partial U$.*

(d) (Preservation) *If Y is a retract of X and $f(\overline{U}) \subset Y$, then*

$$i\,(f, U, X) = i\,(f, U \cap Y, Y),$$

where $i\,(f, U \cap Y, Y) := i(f|_{\overline{U \cap Y}}, U, Y)$.

(e) (Excision property). *Let $V \subset U$ an open subset such that f has no fixed point in $\overline{U} \backslash V$. Then*

$$i(f, U, X) = i(f, V, X).$$

(f) (Solvability). *If $i(f, U, X) \neq 0$, then f has a fixed point in U.*

The following results are direct consequences of this definition.

Proposition 1.4. *Let X be a closed convex of a Banach space E and $U \subset X$ a bounded open subset with $0 \in U$. Assume that $A\colon \overline{U} \to X$ is a strict k-set contraction that satisfies the Leray–Schauder boundary condition*:

$$Ax \neq \lambda x, \quad \textit{for all } x \in \partial U \textit{ and for all } \lambda \geq 1.$$

Then $i(A, U, X) = 1$.

Corollary 1.5. *Let $\mathcal{P}$ be a cone of a Banach space E and $U \subset \mathcal{P}$ a bounded open subset with $0 \in U$. Assume that $A\colon \overline{U} \to \mathcal{P}$ is a strict k-set contraction satisfying $\|Ax\| \leq \|x\|$ and $Ax \neq x$ for all $x \in \partial U$. Then the fixed point index $i(A, U, \mathcal{P}) = 1$.*

Proposition 1.6 ([19, Corollary 1.3.1]). *Let X be a closed convex of a Banach space E and $U \subset X$ a nonempty bounded open convex subset of X. Assume that $A\colon \overline{U} \to X$ is a strict set contraction such that $A(\overline{U}) \subset U$. Then $i(A, U, X) = 1$.*

Proposition 1.7 ([19, Theorem 1.3.8]). *Let X be a closed convex of a Banach space E and $U \subset X$ be a bounded open subset. Assume that $A\colon \overline{U} \to X$ is a strict k-set contraction. If there exists $u_0 \in X$, $u_0 \neq 0$, such that $\lambda u_0 \in X$, for all $\lambda \geq 0$ and*

$$x - Ax \neq \lambda u_0, \quad \textit{for all } x \in \partial U \textit{ and for all } \lambda \geq 0,$$

then the fixed point index $i(A, U, X) = 0$.

Remark 1.8.

(a) Proposition 1.4 remains true even if the operator A is a 1-set contraction (see [21, Theorem 3]).
(b) Proposition 1.7 remains true even if the operator A is a semi-closed ($I - A$ closed) 1-set contraction and X is a wedge in E (see [21, Theorem 4]).
(c) At least for compact mappings, the properties of the fixed point index remain valid in the more general setting of a translate of a cone $\mathcal{K} = \mathcal{P} + \theta$ ($\theta \in E$) (see [16]).

To establish our results, we need the following technical one concerning expansive mappings.

Definition 1.9. Let (X,d) be a metric space and D be a subset of X. The mapping $T\colon D\to X$ is said to be expansive if there exists a constant $h>1$ such that

$$d(Tx,Ty)\geq h\,d(x,y),\quad \text{for all } x,y\in D.$$

Lemma 1.10 ([35, Lemma 2.1]). *Let $(X,\|\cdot\|)$ be a linear normed space and $D\subset X$. Assume that the mapping $T\colon D\to X$ is expansive with constant $h>1$. Then the inverse of $I-T\colon D\to(I-T)(D)$ exists and*

$$\big\|(I-T)^{-1}x-(I-T)^{-1}y\big\|\leq\frac{1}{h-1}\|x-y\|,\quad \textit{for all } x,y\in(I-T)(D).$$

Proposition 1.11. *Let (X,d) be a complete metric space and D be a closed subset of X. Assume that the mapping $T\colon D\to X$ is expansive and $D\subset T(D)$, then there exists a unique point $x^*\in D$ such that $Tx^*=x^*$.*

In case of a Banach space, it can be observed that the operator $T^{-1}\colon T(D)\to D$ is a surjective $1/h$-contraction. Proposition 1.11 then follows from Banach's contraction principle.

In all what follows, $\mathcal{P}$ will refer to a cone in a Banach space E, Ω is a subset of $\mathcal{P}$, and U is a bounded open subset of $\mathcal{P}$. For some constant $r>0$, we will denote the conical shell by $\mathcal{P}_r=\mathcal{P}\cap\mathcal{B}_r$, where $\mathcal{B}_r=\{x\in E:\|x\|<r\}$ is the open ball centered at the origin with radius r.

Assume that $T\colon\Omega\to E$ is an expansive mapping with constant $h>1$ and $F\colon\overline{U}\to E$ is a k-set contraction. By Lemma 1.10, the operator $(I-T)^{-1}$ is $1/(h-1)$-Lipschtzian on $(I-T)(\Omega)$.

2. The case where F is a k-set contraction with $0\leq k<h-1$

2.1. Definition of a fixed point index. Suppose that $0\leq k<h-1$,

$$F(\overline{U})\subset(I-T)(\Omega), \tag{2.1}$$

$$x\neq Tx+Fx,\quad \text{for all } x\in\partial U\cap\Omega. \tag{2.2}$$

Then $x\neq(I-T)^{-1}Fx$, for all $x\in\partial U$ and the mapping $(I-T)^{-1}F\colon\overline{U}\to\mathcal{P}$ is a strict $k/(h-1)$-set contraction. Indeed, $(I-T)^{-1}F$ is continuous and bounded; and for any bounded set B in U, we have

$$\alpha\big(((I-T)^{-1}F)(B)\big)\leq\frac{1}{h-1}\,\alpha(F(B))\leq\frac{k}{h-1}\,\alpha(B).$$

By Lemma 1.3, the fixed point index $i((I-T)^{-1}F,U,\mathcal{P})$ is well defined. Thus we put

Definition 2.1. $i_*(T+F,U\cap\Omega,\mathcal{P})=i((I-T)^{-1}F,U,\mathcal{P})$.

We call this integer *the generalized fixed point index* of the sum $T+F$ on $U\cap\Omega$ with respect to the cone $\mathcal{P}$. Notice that, for $U\cap\Omega=\emptyset$, the index $i_*(T+F,U\cap\Omega,\mathcal{P})=0$.

Remark 2.2. If $T\colon E\to E$ and $F\colon\overline{U}\to E$, then the condition $F(\overline{U})\subset(I-T)(\Omega)$ can be replaced by $F(\overline{U})\subset(I-T)(E)$ both with

$$(y=Ty+Fx,\ x\in\overline{U})\ \Rightarrow\ y\in\mathcal{P}.$$

Indeed, for every $x\in\overline{U}$, there exists $y\in E$ such that $(I-T)y=Fx$, i.e. $y=Ty+Fx$. Hence $y=(I-T)^{-1}Fx\in\mathcal{P}$.

Theorem 2.3. *The fixed point index defined in Definition* 2.1 *satisfies the following properties*:

(a) (Normalization) *If* $U=\mathcal{P}_r$, $0\in\Omega$ *and* $Fx=z_0\in\mathcal{B}(-T0,(h-1)r)\cap\mathcal{P}$ *for all* $x\in\overline{\mathcal{P}_r}$, *then*

$$i_*(T+F,\mathcal{P}_r\cap\Omega,\mathcal{P})=1.$$

(b) (Additivity) *For any pair of disjoint open subsets* U_1, U_2 *in* U *such that* $T+F$ *has no fixed point on* $(\overline{U}\backslash(U_1\cup U_2))\cap\Omega$, *we have*

$$i_*(T+F,U\cap\Omega,\mathcal{P})=i_*(T+F,U_1\cap\Omega,\mathcal{P})+i_*(T+F,U_2\cap\Omega,\mathcal{P}),$$

where $i_*(T+F,U_j\cap\Omega,X):=i_*(T+F|_{\overline{U_j}},U_j\cap\Omega,\mathcal{P})$, $j=1,2$.

(c) (Homotopy Invariance) *The fixed point index* $i_*(T+H(t,\cdot\,),U\cap\Omega,\mathcal{P})$ *does not depend on the parameter* $t\in[0,1]$ *whenever*
   - (i) $H\colon[0,1]\times\overline{U}\to E$ *is continuous and* $H(t,x)$ *is uniformly continuous in* t *with respect to* $x\in\overline{U}$,
   - (ii) $H([0,1]\times\overline{U})\subset(I-T)(\Omega)$,
   - (iii) $H(t,\ :\ )\colon\overline{U}\to E$ *is a* l*-set contraction with* $0\le l<h-1$ *and* l *does not depend on* $t\in[0,1]$,
   - (iv) $Tx+H(t,x)\neq x$, *for all* $t\in[0,1]$ *and* $x\in\partial U\cap\Omega$.

(d) (Solvability) *If* $i_*(T+F,U\cap\Omega,\mathcal{P})\neq0$, *then* $T+F$ *has a fixed point in* $U\cap\Omega$.

Proof. Properties (b), (c) and (d) follow directly from Definition 2.1 and the corresponding properties of the fixed point index for strict k-set contractions (see Lemma 1.3). We only check that if $U=\mathcal{P}_r$ then

$$i((I-T)^{-1}z_0,U,\mathcal{P})=1.$$

For this, we show that $y_0:=(I-T)^{-1}z_0\in\mathcal{P}_r\cap\Omega$. We have $F(\overline{\mathcal{P}}_r)=\{z_0\}\subset(I-T)(\Omega)$, which gives $y_0\in\Omega$ and since T is an expansive operator with $h>1$ and $F(\overline{\mathcal{P}_r})\subset\mathcal{B}(-T0,(h-1)r)\cap\mathcal{P}$, Lemma 1.10 guarantees that

$$\|(I-T)y_0+T0\|=\|(I-T)y_0-(I-T)0\|\ge(h-1)\|y_0\|.$$

Hence

$$(h-1)\|y_0\| \leq \|(I-T)y_0 + T0\| = \|z_0 - (-T0)\| < (h-1)r,$$

that is $y_0 = (I-T)^{-1}z_0 \in \mathcal{P}_r$. By property (a) in Lemma 1.3, we deduce that

$$i((I-T)^{-1}z_0, \mathcal{P}_r, \mathcal{P}) = 1.$$

Therefore $i_*(T + z_0, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$, which completes the proof. □

Remark 2.4. Theorem 2.3 still holds if instead of the cone $\mathcal{P}$, we consider a retract X of E. In this case, the conical shell $\mathcal{P}_r$ is replaced by $X \cap \mathcal{B}_r$.

Next, we compute the fixed point index for the class of mappings under consideration.

2.2. Computation of the fixed point index.

Proposition 2.5. *Assume that $T\colon \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $F\colon \overline{\mathcal{P}_r} \to E$ is a k-set contraction with $0 \leq k < h-1$, $F(\partial\mathcal{P}_r \cap \Omega) \subset \mathcal{P}$, and $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If*

$$0 \in \Omega, \quad \|T0\| < (h-1)r \quad \text{and} \quad Fx \ngeq x - Tx, \quad \text{for all } x \in \partial\mathcal{P}_r \cap \Omega,$$

then $i_*(T + F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

Proof. Consider the homotopic deformation $H\colon [0,1] \times \overline{\mathcal{P}_r} \to E$ defined by $H(t,x) = tFx$. The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t,\,\cdot\,)$ is a k-set contraction for each t and the mapping $T + H(t,\,\cdot\,)$ has no fixed point on $\partial\mathcal{P}_r \cap \Omega$. Otherwise, there would exist some $x_0 \in \partial\mathcal{P}_r \cap \Omega$ and $t_0 \in [0,1]$ such that $x_0 = Tx_0 + t_0Fx_0$. Consider two cases:

Case 1. If $t_0 = 0$, then $Tx_0 = x_0$ and

$$(h-1)\|x_0\| \leq \|(I-T)x_0 + T0\| = \|T0\| < (h-1)r,$$

which is a contradiction.

Case 2. If $t_0 \in (0,1]$, then $Fx_0 \geq t_0Fx_0 = x_0 - Tx_0$, which contradicts our assumption.

From the invariance under homotopy and the normalization property of the index fixed point of the sum $T + F$, we deduce that

$$i_*(T + F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = i_*(T + 0, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1,$$

which completes the proof. □

Remark 2.6. The results of Proposition 2.5 does not hold if $F(\partial\mathcal{P}_r \cap \Omega) \not\subset \mathcal{P}$. Indeed, let $\mathcal{P}$ be a cone in a Banach space E with non-empty interior and $r > 0$. We choose a point x_1 in the interior of $\mathcal{P}$ with $\operatorname{dist}(x_1, E \setminus \mathcal{P}) > r$

and $\Omega = \mathcal{P}_{2\|x_1\|}$. With $T = 2I$ and $F\colon \overline{\mathcal{P}}_r \ni x \mapsto -x_1 \in -\mathcal{P} \subset E$, all assumptions of the latter proposition are satisfied. However $0 \in -\mathcal{P} \subset E \setminus \mathcal{P}$ implies that $i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 0$.

Corollary 2.7. *Assume that $T\colon \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $F\colon \overline{\mathcal{P}_r} \to E$ is a k-set contraction with $0 \le k < h-1$, $F(\partial\mathcal{P}_r \cap \Omega) \subset \mathcal{P}$, and $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$, for all $t \in [0,1]$. If $0 \in \Omega$, $\|T0\| < (h-1)r$, and*

$$Tx + Fx < x, \quad \text{for all } x \in \partial\mathcal{P}_r \cap \Omega,$$

then $T+F$ has a fixed point in $\mathcal{P}_r \cap \Omega$.

Proof. Since F and T satisfy the assumptions of Proposition 2.5, then

$$i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1.$$

Corollary 2.7 then follows from the existence property of the fixed point index in Theorem 2.3. □

Proposition 2.8. *Assume that $T\colon \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $F\colon \overline{\mathcal{P}_r} \to E$ is a k-set contraction with $0 \le k < h-1$ and $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$, for all $t \in [0,1]$. If $0 \in \Omega$, $\|T0\| < (h-1)r$, and*

$$Fx \neq \lambda(x - Tx), \quad \text{for all } x \in \partial\mathcal{P}_r \cap \Omega \text{ and } \lambda \ge 1, \tag{2.3}$$

then the fixed point index $i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

Proof. Define the homotopic deformation $H\colon [0,1] \times \overline{\mathcal{P}_r} \to E$ by $H(t,x) = tFx$. The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t,\,\cdot\,)$ is a k-set contradiction for each t and the mapping $T + H(t,\,\cdot\,)$ has no fixed point on $\partial\mathcal{P}_r \cap \Omega$. Otherwise, there would exist some $x_0 \in \partial\mathcal{P}_r \cap \Omega$ and $t_0 \in [0,1]$ such that $x_0 = Tx_0 + t_0 F x_0$. We may distinguish between two cases:

Case 1. If $t_0 = 0$, then $Tx_0 = x_0$ and

$$(h-1)\|x_0\| \le \|(I-T)x_0 + T0\| = \|T0\| < (h-1)r,$$

which contradicts $x_0 \in \partial\mathcal{P}_r$.

Case 2. If $t_0 \in (0,1]$, then $Fx_0 = (x_0 - Tx_0)/t_0$, where $1/t_0 \ge 1$, leading again to a contradiction with the hypothesis (2.3).

By properties (a) and (d) of the fixed point index in Theorem 2.3, we deduce that

$$i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = i_*(T+0, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1. \qquad \square$$

The following result is an immediate consequence of Proposition 2.8:

Corollary 2.9. *Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$, $F\colon \overline{\mathcal{P}_r} \to E$ *is a* k*-set contraction with* $0 \le k < h-1$ *and* $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$, *for all* $t \in [0,1]$. *If* $0 \in \Omega$, $\|T0\| < (h-1)r$ *and*

$$\|Fx\| + \|Tx\| < \|x\|, \quad \text{for all } x \in \partial\mathcal{P}_r \cap \Omega.$$

Then $T + F$ *has a fixed point in* $\mathcal{P}_r \cap \Omega$.

Proposition 2.10. *Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$, $F\colon \overline{\mathcal{P}_r} \to E$ *is a* k*-set contraction with* $0 \le k < h-1$, *and* $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$, *for all* $t \in [0,1]$. *If* $0 \in \Omega$, $\|T0\| < (h-1)r$, *and*

$$\|Fx\| \le \|x - Tx\| \quad \text{and} \quad Tx + Fx \ne x \quad \text{for all } x \in \partial\mathcal{P}_r \cap \Omega,$$

then the fixed point index $i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

Proof. It is sufficient to prove that the condition in the proposition implies the analogous one in Proposition 2.8. For this, assume by contradiction that some $x_0 \in \partial\mathcal{P}_r \cap \Omega$ and $\lambda_0 \ge 0$ exist and satisfy $Fx_0 = \lambda_0(x_0 - Tx_0)$. Then two cases are discussed separately:

Case 1. If $\lambda_0 = 1$, then $Tx_0 + Fx_0 = x_0$ and a contradiction is reached.

Case 2. If $\lambda_0 > 1$, then $\|Fx_0\| = \lambda_0\|x_0 - Tx_0\| > \|x_0 - Tx_0\|$, whence a contradiction. □

Proposition 2.11. *Let* U *be a bounded open subset of* $\mathcal{P}$ *with* $0 \in U$. *Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$, $F\colon \overline{U} \to E$ *is a* k*-set contraction with* $0 \le k < h-1$, *and* $F(\overline{U}) \subset (I-T)(\Omega)$. *If*

$$Fx \ne (I-T)(\lambda x), \quad \text{for all } x \in \partial U \cap \Omega \text{ and } \lambda \ge 1,$$

then the fixed point index $i_*(T+F, U \cap \Omega, \mathcal{P}) = 1$.

Proof. The mapping $(I-T)^{-1}F\colon \overline{U} \to \mathcal{P}$ is a strict $k/(h-1)$-set contraction and it is readily seen that the following condition of Leray–Schauder type is satisfied

$$(I-T)^{-1}Fx \ne \lambda x, \quad \text{for all } x \in \partial U \text{ and } \lambda \ge 1.$$

In fact, if there exist $x_0 \in \partial U$ and $\lambda_0 \ge 1$ such that $(I-T)^{-1}Fx_0 = \lambda_0 x_0$. Then $Fx_0 = (I-T)(\lambda_0 x_0)$, which contradicts our assumption. Our claim then follows from Definition 2.1 and Proposition 1.4. □

Proposition 2.12. *Let* U *be a bounded open subset of* $\mathcal{P}$ *with* $0 \in U \cap \Omega$. *Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$, $F\colon \overline{U} \to E$ *is a* k*-set contraction with* $0 \le k < h-1$ *and* $F(\overline{U}) \subset (I-T)(\Omega)$. *If*

$$\|Fx + T0\| \le (h-1)\|x\| \quad \text{and} \quad Tx + Fx \ne x, \quad \text{for all } x \in \partial U \cap \Omega, \tag{2.4}$$

then the fixed point index $i_*(T+F, U \cap \Omega, \mathcal{P}) = 1$.

PROOF. According to Lemma 1.10, we can see that $(I-T)^{-1}F\colon \overline{U} \to \mathcal{P}$ is a strict $k/(h-1)$-set contraction. From the inclusion $F(\overline{U}) \subset (I-T)(\Omega)$, for each $x \in \overline{U}$, we can find some $y \in \Omega$ such that $Fx = y - Ty$. In what follows, we check that the condition of Corollary 1.5 is satisfied. For each $x \in \overline{U}$, $(I-T)^{-1}Fx \in \Omega$ and

$$T\big((I-T)^{-1}Fx\big) + Fx = (I-T)^{-1}Fx,$$

which implies that

$$\big\|T\big((I-T)^{-1}Fx\big) - T0\big\| \leq \big\|(I-T)^{-1}Fx\big\| + \|Fx + T0\|.$$

T being expansive with constant h, we have

$$\big\|T((I-T)^{-1}Fx) - T0\big\| \geq h\|(I-T)^{-1}Fx\|.$$

Therefore

$$\big\|(I-T)^{-1}Fx\big\| \leq \frac{1}{h-1}\|Fx + T0\|. \tag{2.5}$$

From (2.5) and assumption (2.4), we conclude that for all $x \in \partial U$,

$$\|(I-T)^{-1}Fx\| \leq \frac{1}{h-1}\|Fx + T0\| \leq \|x\|.$$

Our claim then follows from Definition 2.1 and Corollary 1.5. □

The following result is as straightforward consequence of Proposition 1.6.

PROPOSITION 2.13. *Assume that $T\colon \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $F\colon \overline{\mathcal{P}_r} \to E$ is a k-set contraction with $0 \leq k < h-1$ and $F(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$. If further $(I-T)^{-1}F(\overline{\mathcal{P}_r}) \subset \mathcal{P}_r$, then the fixed point index $i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.*

As a particular case, we get

COROLLARY 2.14. *Assume that $T\colon \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $F\colon \overline{\mathcal{P}_r} \to E$ is a k-set contraction with $0 \leq k < h-1$, and $F(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$. If $0 \in \Omega$ and*

$$\|Fx + T0\| < (h-1)r, \quad \text{for all } x \in \overline{\mathcal{P}_r}, \tag{2.6}$$

then the fixed point index $i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

PROOF. From (2.5) and Assumption (2.6), for any $x \in \overline{\mathcal{P}_r}$, we conclude that

$$\big\|(I-T)^{-1}Fx\big\| \leq \frac{1}{h-1}\|Fx + T0\| < r,$$

which implies that $(I-T)^{-1}F(\overline{\mathcal{P}_r}) \subset \mathcal{P}_r$. □

A particular situation in Corollary 2.14 is given by

Corollary 2.15. *Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 2$, $F\colon \overline{\mathcal{P}_r} \to E$ *is a* k*-set contraction with* $0 \le k < h - 1$, r *is sufficiently large and* $F(\overline{\mathcal{P}_r}) \subset (I - T)(\Omega)$*. If* $0 \in \Omega$ *and*

$$\|Fx\| \le \|x\|, \quad \text{for all } x \in \overline{\mathcal{P}_r}, \tag{2.7}$$

then $T + F$ *has at least one fixed point in* $\mathcal{P}_r \cap \Omega$.

Proof. We have the estimates:

$$\|Fx + T0\| \le \|Fx\| + \|T0\| \le \|x\| + \|T0\| \le r + \|T0\| \le (h-1)r,$$

for $r > \|T0\|/(h-2)$. By Corollary 2.14, $i_*(T + F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$. As a consequence, $T + F$ has a fixed point in $\mathcal{P}_r \cap \Omega$. □

Proposition 2.16. *Let* U *be a bounded open subset of* $\mathcal{P}$*. Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$, $F\colon \overline{U} \to E$ *is a* k*-set contraction with* $0 \le k < h - 1$*, and* $F(\overline{U}) \subset (I - T)(\Omega)$*. If there exists* $u_0 \in \mathcal{P}^*$ *such that*

$$Fx \neq (I - T)(x - \lambda u_0), \quad \text{for all } \lambda \ge 0 \text{ and } x \in \partial U \cap (\Omega + \lambda u_0), \tag{2.8}$$

then the fixed point index $i_*(T + F, U \cap \Omega, \mathcal{P}) = 0$.

Proof. The mapping $(I-T)^{-1}F\colon \overline{U} \to \mathcal{P}$ is a strict $k/(h-1)$-set contraction and for some $u_0 \in \mathcal{P}^*$ this operator satisfies

$$x - (I - T)^{-1}Fx \neq \lambda u_0, \quad \text{for all } x \in \partial U \text{ and for all } \lambda \ge 0.$$

By Definition 2.1 and Proposition 1.7, we deduce that

$$i_*(T + F, U \cap \Omega, \mathcal{P}) = i((I - T)^{-1}F, U, \mathcal{P}) = 0,$$

proving our claim. □

Proposition 2.17. *Let* U *be a bounded open subset of* $\mathcal{P}$*. Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$, $F\colon \overline{U} \to E$ *a* k*-set contraction with* $0 \le k < h - 1$*, and* $F(\overline{U}) \subset (I - T)(\Omega)$*. Suppose further that there exists* $u_0 \in \mathcal{P}^*$ *such that* $T(x - \lambda u_0) \in \mathcal{P}$*, for all* $\lambda \ge 0$ *and* $x \in \partial U \cap (\Omega + \lambda u_0)$*, and one of the following conditions holds*:

(a) $Fx \nleq x - \lambda u_0$*, for all* $x \in \partial U$ *and for all* $\lambda \ge 0$.
(b) $Fx \in \mathcal{P}$, $\|Fx\| > \|x - \lambda u_0\|$*, for all* $x \in \partial U$*, for all* $\lambda \ge 0$*, and the cone* $\mathcal{P}$ *is normal with constant* $N = 1$.

Then the fixed point index $i_*(T + F, U \cap \Omega, \mathcal{P}) = 0$.

Remark 2.18. In condition (b), we may replace $\|Fx\| > \|x - \lambda u_0\|$ by $\|Fx\| > \|x\|$.

Proof. We show that conditions (a) or (b) imply that

$$Fx \neq (I-T)(x-\lambda u_0), \quad \text{for all } \lambda \geq 0 \text{ and } x \in \partial U \cap (\Omega + \lambda u_0).$$

On the contrary, assume the existence of $\lambda_0 \geq 0$ and $x_0 \in \partial U \cap (\Omega + \lambda_0 u_0)$ such that

$$Fx_0 = (I-T)(x_0 - \lambda_0 u_0).$$

Then $x_0 - \lambda_0 u_0 - Fx_0 = T(x_0 - \lambda_0 u_0) \in \mathcal{P}$. If condition (a) holds, then a contradiction is achieved. Otherwise, we deduce that

$$Fx_0 \leq x_0 - \lambda_0 u_0.$$

Since $\mathcal{P}$ is normal with constant $N = 1$, we deduce that

$$\|Fx_0\| \leq \|x_0 - \lambda_0 u_0\|,$$

contradicting condition (b) and ending the proof of Proposition 2.17. □

2.3. Fixed point theorems of cone compression and expansion type. In this section, two fixed point theorems of cone compression and expansion for an expansive operator perturbed by a k-set contraction are established. Assumption (a) is called the cone compression while (b) is the cone expansion.

Given two constants $0 < r < R$, define the open sets:

$$\mathcal{P}_r = \{x \in \mathcal{P} : \|x\| < r\} \quad \text{and} \quad \mathcal{P}_{r,R} = \{x \in \mathcal{P} : r < \|x\| < R\}.$$

Theorem 2.19. *Let E be a Banach space, $\mathcal{P} \subset E$ a normal cone with constant $N = 1$, and U_1 and U_2 two bounded open subsets of $\mathcal{P}$ such that $\overline{U_1} \subset U_2$ and $0 \in U_1 \cap \Omega$, where $\Omega \subset \mathcal{P}$. Assume that $T\colon \Omega \to E$ is an expansive mapping with constant $h > 1$, $F\colon \overline{U_2} \to E$ is a k-set contraction with $0 \leq k < h-1$ and $F(\overline{U_2}) \subset (I-T)(\Omega)$. Let $u_0 \in \mathcal{P}^*$ be such that $T(x - \lambda u_0) \in \mathcal{P}$, for all $\lambda \geq 0$ and $x \in \partial U_1 \cap \partial U_2 \cap (\Omega + \lambda u_0)$, and suppose that one of the following conditions is satisfied*:

(a) *$Fx \in \mathcal{P}$, $\|Fx\| > \|x - \lambda u_0\|$, for all $x \in \partial U_1$, for all $\lambda \in [0,1]$ and $\|Fx + T0\| \leq (h-1)\|x\|$, for all $x \in \partial U_2$.*

(b) *$\|Fx+T0\| \leq (h-1)\|x\|$, for all $x \in \partial U_1$ and $Fx \in \mathcal{P}$, $\|Fx\| > \|x-\lambda u_0\|$, for all $x \in \partial U_2$, for all $\lambda \geq 0$.*

Then $T + F$ has at least one fixed point in $(\overline{U_2} \setminus U_1) \cap \Omega$.

Proof. We only give the proof in case of the cone expansion. Without loss of generality, assume that $Tx + Fx \neq x$ on $\partial U_1 \cap \Omega$ and $Tx + Fx \neq x$ on $\partial U_2 \cap \Omega$, otherwise we are finished. By Propositions 2.12 and 2.17, we have

$$i_*(T+F, U_1 \cap \Omega, \mathcal{P}) = 1 \quad \text{and} \quad i_*(T+F, U_2 \cap \Omega, \mathcal{P}) = 0.$$

The additivity property of the index yields

$$i_*(T+F, (U_2 \setminus \overline{U_1}) \cap \Omega, \mathcal{P}) = -1.$$

By the existence property of the index, the sum $T+F$ has at least one fixed point in the closed set $(\overline{U_2}\setminus U_1)\cap\Omega$. □

Theorem 2.20. *Let E be a Banach space, $\mathcal{P}\subset E$ a cone, and $0\in\Omega\subset\mathcal{P}$. Let $\gamma,\beta>0$, $\gamma\neq\beta$, $r=\min(\gamma,\beta)$ and $R=\max(\gamma,\beta)$. Assume that $T\colon\Omega\to E$ is an expansive mapping with constant $h>1$ such that $\|T0\|<(h-1)\gamma$, $F\colon\overline{\mathcal{P}_R}\to E$ is a k-set contraction with $0\le k<h-1$, $F(\partial\mathcal{P}_\gamma\cap\Omega)\subset\mathcal{P}$ and $tF(\overline{\mathcal{P}_R})\subset(I-T)(\Omega)$, for all $t\in[0,1]$. Let $u_0\in\mathcal{P}^*$ be such that $T(x-\lambda u_0)\in\mathcal{P}$, for all $\lambda\ge 0$ and $x\in\partial\mathcal{P}_\beta\cap(\Omega+\lambda u_0)$, and suppose that the following conditions are satisfied*:

(a) *$Fx\not\geq x-Tx$, for all $x\in\partial\mathcal{P}_\gamma\cap\Omega$,*

(b) *$Fx\not\leq x-\lambda u_0$, for all $x\in\partial\mathcal{P}_\beta$, for all $\lambda\ge 0$.*

Then $T+F$ has a fixed point $x\in\mathcal{P}_{r,R}\cap\Omega$.

Remark 2.21. If $\beta<\gamma$, then conditions (a) and (b) represent a compression property of $T+F$ upon the conical shell $\mathcal{P}_{r,R}\cap\Omega$, while if $\beta>\gamma$, these conditions express an expansion property of the conical shell $\mathcal{P}_{r,R}\cap\Omega$.

Proof. We only present the proof in case of the cone compression. It is analogous for the cone expansion. By Propositions 2.5 and 2.17, we have

$$i_*(T+F,\Omega\cap\mathcal{P}_R,\mathcal{P})=1\quad\text{and}\quad i_*(T+F,\Omega\cap\mathcal{P}_r,\mathcal{P})=0.$$

The additivity property of the fixed point index in Theorem 2.3 yields

$$i_*(T+F,\Omega\cap\mathcal{P}_{r,R},\mathcal{P})=1.$$

By the existence property, $T+F$ has at least one fixed point in the open set $\mathcal{P}_{r,R}\cap\Omega$, proving our claim. □

3. The limit case where F is an $(h-1)$-set contraction

3.1. Definition of a fixed point index. Suppose that $T\colon\Omega\to E$ is h-expansive and $F\colon\overline{U}\to E$ is an $(h-1)$-set contraction. Since $(I-T)^{-1}$ is $1/(h-1)$-Lipschtzian, then $(I-T)^{-1}F\colon\overline{U}\to\mathcal{P}$ is a 1-set contraction. Assume that

$$tF(\overline{U})\subset(I-T)(\Omega),\quad\text{for all } t\in[0,1] \tag{3.1}$$

$$0\notin\overline{(I-T-F)(\partial U\cap\Omega)}. \tag{3.2}$$

Then there exists $\gamma>0$ such that $\inf_{x\in\partial U\cap\Omega}\|x-Tx-Fx\|\ge\gamma$. Thus $0\notin(I-T-kF)(\partial U\cap\Omega)$, for all $k\in(1-\gamma/M,1)$, where $M=\gamma+\sup_{x\in\overline{U}}\|Fx\|$. In fact, for all $x\in\partial U\cap\Omega$, we have

$$\|0-(x-Tx-kFx)\|\ge\|x-Tx-Fx\|-(1-k)\|Fx\|\ge\gamma-(1-k)M>0.$$

In other words, $x\neq(I-T)^{-1}kFx$, for all $x\in\partial U$ and $k\in(1-\gamma/M,1)$. Clearly, $(I-T)^{-1}kF$ is a strict k-set contraction mapping. As a consequence, by Definition 2.1 and Lemma 1.3, the fixed point index $i_*(T+kF, U\cap\Omega,\mathcal{P})$ is well defined. Thus we set

Definition 3.1. For $k\in(1-\gamma/M,1)$ we have

$$i_*(T+F,U\cap\Omega,\mathcal{P}) = i_*(T+kF,U\cap\Omega,\mathcal{P}) = i((I-T)^{-1}kF,U,\mathcal{P}).$$

However we must check that $i_*(T+F,U\cap\Omega,\mathcal{P})$ does not depend on the parameter $k\in(1-\gamma/M,1)$. For this, let $G_j=k_jF\colon \overline{U}\to E$ be $k_j(h-1)$-set contractions with $k_j\in(1-\gamma/M,1)$ $(j=1,2)$. Then

$$\|G_jx-Fx\| = (1-k_j)\|Fx\| \le (1-k_j)M<\gamma, \quad \text{for all } x\in\partial U.$$

Define the convex deformation $H\colon[0,1]\times\overline{U}\to E$ by

$$H(t,x)=tG_1x+(1-t)G_2x.$$

The operator H is continuous, uniformly continuous in t for each x, and $H([0,1]\times\overline{U})\subset(I-T)(\Omega)$. In addition $H(t,\,\cdot\,)$ is a $\overline{k}(h-1)$-set contraction for each t, where $\overline{k}=\max(k_1,k_2)$ and $T+H(t,\,\cdot\,)$ has no fixed point on $\partial U\cap\Omega$. In fact, for all $x\in\partial U\cap\Omega$, we have

$$\begin{aligned}\|x-Tx-H(t,x)\| &= \|x-Tx-tG_1x-(1-t)G_2x\|\\ &\ge \|x-Tx-Fx\|-t\|Fx-G_1x\|-(1-t)\|Fx-G_2x\|\\ &> \gamma-t\gamma-(1-t)\gamma=0.\end{aligned}$$

From the invariance property by homotopy of the index in Theorem 2.3, we deduce that

$$i_*(T+G_1,U\cap\Omega,\mathcal{P}) = i_*(T+G_2,U\cap\Omega,\mathcal{P}),$$

which shows that the index $i_*(T+F,U\cap\Omega,\mathcal{P})$ does not depend on k.

The integer defined in Definition 3.1 will be called *the generalized fixed point index of the sum $T+F$ on $U\cap\Omega$ with respect to $\mathcal{P}$*. It satisfies some properties grouped in the following theorem.

Theorem 3.2.

(a) (Normalization) *If $U=\mathcal{P}_r=\mathcal{P}\cap\mathcal{B}_r$ is a conical shell and $Fx=z_0\in\mathcal{B}(-T0,(h-1)r)\cap\mathcal{P}$, for all $x\in\overline{\mathcal{P}_r}$, then $i_*(T+F,\mathcal{P}_r\cap\Omega,\mathcal{P})=1$.*

(b) (Additivity) *For any pair of disjoint open subsets U_1,U_2 in U such that $0\notin\overline{(I-T-F)((\overline{U}\backslash(U_1\cup U_2))\cap\Omega)}$, we have*

$$i_*(T+F,U\cap\Omega,\mathcal{P}) = i_*(T+F,U_1\cap\Omega,\mathcal{P})+i_*(T+F,U_2\cap\Omega,\mathcal{P}),$$

where $i_*(T+F,U_j\cap\Omega,X):=i_*(f|_{\overline{U_j}},U_j\cap\Omega,\mathcal{P})$, $j=1,2$.

(c) (Homotopy Invariance) *The fixed point index $i_*(T+H(t,\,\cdot\,),\,U\cap\Omega,\mathcal{P})$ does not depend on the parameter $t\in[0,1]$, where*

(i) *$H\colon[0,1]\times\overline{U}\to E$ is continuous and $H(t,x)$ is uniformly continuous in t with respect to $x\in\overline{U}$,*

(ii) $H(t,\cdot)\colon \overline{U}\to E$ *is an* $(h-1)$*-set contraction,*

(iii) $tH([0,1]\times\overline{U})\subset (I-T)(\Omega)$*, for all* $t\in[0,1]$*,*

(iv) $0\notin\overline{(I-T-H(t,\cdot))(\partial U\cap\Omega)}$*, for all* $t\in[0,1]$*,*

(d) (Solvability) *If* $i_*(T+F,U\cap\Omega,\mathcal{P})\neq 0$*, then* $0\in\overline{(I-T-F)(U\cap\Omega)}$*.*

Proof. (a) Since F is a constant mapping, it is a 0-set contraction (completely continuous), which implies that $(I-T)^{-1}F$ is a 0-set contraction. As in the proof of Theorem 2.3, part (a), $y_0=(I-T)^{-1}z_0\in\mathcal{P}_r$. By the normalization property in Lemma 1.3, we deduce that

$$i((I-T)^{-1}z_0,\mathcal{P}_r,\mathcal{P})=1.$$

Therefore $i_*(T+z_0,\mathcal{P}_r\cap\Omega,\mathcal{P})=1$, proving our claim.

(b) Let

$$\gamma=\inf_{(\overline{U}\setminus(U_1\cup U_2))\cap\Omega}\|x-Tx-Fx\|>0.$$

Suppose that $G=kF\colon\overline{U}\to E$ is a $k(h-1)$-set contraction and

$$\|Gx-Fx\|<\gamma,\quad\text{for all } x\in\overline{U}\setminus(U_1\cup U_2)\cap\Omega. \tag{3.3}$$

From Definition 3.1, we have

$$\begin{aligned}i_*(T+F,U\cap\Omega,\mathcal{P})&=i_*(T+G,U\cap\Omega,\mathcal{P}),\\ i_*(T+F,U_j\cap\Omega,\mathcal{P})&=i_*(T+G,U_j\cap\Omega,\mathcal{P}),\quad j=1,2.\end{aligned}$$

Hence $T+G$ has no fixed point in $\overline{U}\setminus(U_1\cup U_2)\cap\Omega$. In fact, if the exists $x_0\in\overline{U}\setminus(U_1\cup U_2)\cap\Omega$ such that $x_0=Tx_0+Gx_0$, then

$$\gamma\le\|x_0-Tx_0-Fx_0\|=\|x_0-Tx_0-Gx_0+Gx_0-Fx_0\|=\|Gx_0-Fx_0\|,$$

which contradicts (3.3). The claim follows from Definition 3.1 and property (b) of the fixed point index in Theorem 2.3.

(c) By the property of the function H, there exist $\gamma>0$ and $N>0$ such that

$$\|x-Tx-H(t,x)\|\ge\gamma,\quad\text{for all } x\in\partial U\cap\Omega \text{ and } t\in[0,1],$$

as well as $\|H(t,x)\|\le N$, for all $x\in\overline{U}$ and $t\in[0,1]$. Let $K(t,x)=kH(t,x)$, where $k\in(1-\gamma/2N,1)$. Then for all $x\in\partial U\cap\Omega$ and $t\in[0,1]$, we have

$$\begin{aligned}\|x-Tx-K(t,x)\|&=\|x-Tx-H(t,x)\|+\|H(t,x)-K(t,x)\|\\ &\ge\gamma-(k-1)N>\gamma-\gamma/2>0.\end{aligned}$$

Obviously, $K(t,\cdot)\colon\overline{U}\to E$ is a $k(h-1)$-set contraction, where k does not depend on $t\in[0,1]$ and $K([0,1]\times\overline{U})\subset(I-T)(\Omega)$. Then our claim follows from Definition 3.1 and property (c) of the fixed point index in Theorem 2.3.

(d) Consider a sequence $(k_n)_n \subset (0,1)$ such that $k_n \to 1$, as $n \to \infty$ and define the function $G_n = k_n F$, $n = 1, 2, \ldots$ Then $G_n\colon \overline{U} \to E$ is a $k(h-1)$-set contraction. Since $\|Fx\| < \infty$, for all $x \in \overline{U}$, we obtain that

$$\|Fx - G_n x\| = \|Fx - k_n Fx\| = (1-k_n)\|F_n x\| \to 0, \quad \text{as } n \to +\infty.$$

Hence there exists $n_0 > 0$ such that, for every $n \geq n_0$,

$$\|Fx - G_n x\| < \gamma, \quad \text{where } 0 < \gamma < \inf_{x \in \partial U \cap \Omega} \|x - Tx - Fx\|.$$

By assumption and Definition 3.1,

$$i_*(T+F, U \cap \Omega, \cap P) = i_*(T + G_n, U \cap \Omega, \cap P) \neq 0.$$

Thus, property (d) in Theorem 2.3 guaranties that for all $n = 1, 2, \ldots$, the mapping $T + G_n$ has a fixed point x_n in $U \cap \Omega$. Consequently,

$$\begin{aligned}\|x_n - Tx_n - Fx_n\| &= \|x_n - Tx_n - G_n x_n + G_n x_n - Fx_n\| \\ &= \|G_n x_n - Fx_n\| \to 0, \quad \text{as } n \to +\infty.\end{aligned}$$

Then $x_n - Tx_n - Fx_n \to 0$, as $n \to +\infty$, that is $0 \in \overline{(I-T-F)(U \cap \Omega)}$. □

Remark 3.3. As for the additivity property in Theorem 3.2, we cannot replace the condition $0 \notin \overline{(I-T-F)((\overline{U} \backslash (U_1 \cup U_2)) \cap \Omega)}$ by the weaker one that $T+F$ has no fixed point on $(\overline{U} \backslash (U_1 \cup U_2)) \cap \Omega$. In fact, let us consider the Banach space c_0 of real sequences converging to zero with the sup-norm and the cone $\mathcal{P}$ of sequences (a_n) with only positive entries a_n. Let $r\colon \overline{\mathcal{P}}_5 \to \overline{\mathcal{P}}_1$ be the radial retraction, $s\colon \overline{\mathcal{P}}_1 \ni (a_1, a_2, \ldots) \mapsto (1, a_1, a_2, \ldots) \in \overline{\mathcal{P}}_1$ the well-known shift map, and let $\widehat{F} := s \circ r$. For $T = 2I$, $F = -\widehat{F}$, and $U = \Omega = \mathcal{P}_5$, $U_1 = \mathcal{P}_3 \setminus \overline{\mathcal{P}}_2$, $U_2 = \mathcal{P}_5 \setminus \overline{\mathcal{P}}_4$, we get

$$i_*(T+F, \mathcal{P}_5, \mathcal{P}) = 1 \neq 0 + 0 = i_*(T+F, U_1, \mathcal{P}) + i_*(T+F, U_2, \mathcal{P}).$$

Remark 3.4. Notice that a sufficient condition for (3.2) holds is:

$$\exists\, \delta > 0,\ \forall\, x \in \partial U \cup \Omega, \quad \|x - Tx - Fx\| \geq \delta.$$

3.2. Computation of the fixed point index. According to Theorem 2.3 and in a way similar to the one used to show Propositions 2.5–2.12, we can show the following results. The proofs are omitted.

Proposition 3.5. *Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$ *and* $F\colon \overline{\mathcal{P}_r} \to E$ *is a* $(h-1)$*-set contraction with* $F(\partial \mathcal{P}_r \cap \Omega) \subset \mathcal{P}$ *and* $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$, *for all* $t \in [0,1]$ *and* $0 \notin \overline{(I-T-F)(\partial \mathcal{P}_r \cap \Omega)}$. *If* $0 \in \Omega$, $\|T0\| < (h-1)r$, *and*

$$Fx \not\geq x - Tx, \quad \text{for all } x \in \partial \mathcal{P}_r \cap \Omega,$$

then the fixed point index $i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.6. *Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$ *and* $F\colon \overline{\mathcal{P}_r} \to E$ *is a* $(h-1)$*-set contraction with* $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$*, for all* $t \in [0,1]$ *and* $0 \notin \overline{(I-T-F)(\partial\mathcal{P}_r \cap \Omega)}$*. If* $0 \in \Omega$, $\|T0\| < (h-1)r$*, and*

$$Fx \neq \lambda(x - Tx) \quad \textit{for all } x \in \partial\mathcal{P}_r \cap \Omega \textit{ and } \lambda > 1,$$

then the fixed point index $i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.7. *Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$ *and* $F\colon \overline{\mathcal{P}_r} \to E$ *is an* $(h-1)$*-set contraction with* $tF(\overline{\mathcal{P}_r}) \subset (I-T)(\Omega)$*, for all* $t \in [0,1]$ *and* $0 \notin \overline{(I-T-F)(\partial\mathcal{P}_r \cap \Omega)}$*. If* $0 \in \Omega$, $\|T0\| < (h-1)r$*, and*

$$\|Fx\| \leq \|x - Tx\| \quad \textit{for all } x \in \partial\mathcal{P}_r \cap \Omega,$$

then the fixed point index $i_*(T+F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.8. *Let* U *be a bounded open subset of* $\mathcal{P}$ *such that* $0 \in U$*. Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$ *and* $F\colon \overline{U} \to E$ *is an* $(h-1)$*-set contraction with* $F(\overline{U}) \subset (I-T)(\Omega)$ *and* $0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}$*. If*

$$Fx \neq (I-T)(\lambda x), \quad \textit{for all } x \in \partial U \cap \Omega \textit{ and } \lambda > 1,$$

then the fixed point index $i_*(T+F, U \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.9. *Let* U *be a bounded open subset of* $\mathcal{P}$ *such that* $0 \in U \cap \Omega$*. Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$ *and* $F\colon \overline{U} \to E$ *is an* $(h-1)$*-set contraction with* $F(\overline{U}) \subset (I-T)(\Omega)$ *and* $0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}$*. If*

$$\|Fx + T0\| \leq (h-1)\|x\| \quad \textit{for all } x \in \partial U \cap \Omega, \tag{3.4}$$

then the fixed point index $i_*(T+F, U \cap \Omega, \mathcal{P}) = 1$.

Proposition 3.10. *Let* U *be a bounded open subset of* $\mathcal{P}$*. Assume that* $T\colon \Omega \subset \mathcal{P} \to E$ *is an expansive mapping with constant* $h > 1$ *such that* $F\colon \overline{U} \to E$ *is an* $(h-1)$*-set contraction with* $tF(\overline{U}) \subset (I-T)(\Omega)$*, for all* $t \in [0,1]$ *and* $0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}$*. If there exists* $u_0 \in \mathcal{P}^*$ *such that*

$$\gamma Fx \neq (I-T)(x - \lambda u_0), \tag{3.5}$$

for all $\lambda \geq 0$, $x \in \partial U \cap (\Omega + \lambda u_0)$ *and* $\gamma \in (0,1)$*, then the fixed point index* $i_*(T+F, U \cap \Omega, \mathcal{P}) = 0$.

Proof. The mapping $(I-T)^{-1}F\colon \overline{U} \to \mathcal{P}$ is a 1-set contraction. Suppose that $i_*(T+F, U \cap \Omega, \mathcal{P}) \neq 0$. Then, from Definition 3.1, for $k \in (k_0, 1)$ with some $0 < k_0 < 1$, we can see that

$$i((I-T)^{-1}kF, U, \mathcal{P}) \neq 0.$$

For each $k \in (k_0, 1)$ and $r > 0$, define the homotopy:

$$H(t, x) = (I - T)^{-1} kFx + tru_0, \quad \text{for } x \in \overline{U} \text{ and } t \in [0, 1].$$

The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t, \cdot\,)$ is a strict k-set contraction for each t and

$$H\big([0, 1] \times \overline{U}\big) = (I - T)^{-1} kF(U) + tru_0 \subset \mathcal{P}.$$

We check that $H(t, x) \neq x$, for all $(t, x) \in [0, 1] \times \partial U$. If $H(t_0, x_0) = x_0$ for some $(t_0, x_0) \in [0, 1] \times \partial U$, then

$$x_0 - t_0 ru_0 = (I - T)^{-1} kFx_0,$$

and so $x_0 - t_0 ru_0 \in \Omega$. Hence

$$(I - T)(x_0 - t_0 ru_0) = kFx_0,$$

for $x_0 \in \partial U \cap (\Omega + t_0 ru_0)$, contradicting assumption (3.5). By the property (c) of the index in Lemma 1.3, for $k \in (k_0, 1)$, we deduce that

$$i\big((I - T)^{-1} kF + ru_0, U \cap \Omega, \mathcal{P}\big) = i\big((I - T)^{-1} kF, U, \mathcal{P}\big) \neq 0.$$

By property (f) of the index in Lemma 1.3, for each $k \in (k_0, 1)$ and $r > 0$, there exists $x_r \in U$ such that

$$x_r - (I - T)^{-1} kFx_r = ru_0. \tag{3.6}$$

Letting $r \to +\infty$ in (3.6), the left-hand side of (3.6) is bounded while the right-hand side is not, which is a contradiction. Therefore

$$i_*(T + F, U \cap \Omega, \mathcal{P}) = 0,$$

which completes the proof. □

Proposition 3.11. *Let U be a bounded open subset of $\mathcal{P}$. Assume that $T\colon \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$ such that $F\colon \overline{U} \to E$ is an $(h - 1)$-set contraction with $tF(\overline{U}) \subset (I - T)(\Omega)$, for all $t \in [0, 1]$ and $0 \notin \overline{(I - T - F)(\partial U \cap \Omega)}$. Suppose that there exists $u_0 \in \mathcal{P}^*$ such that $T(x - \lambda u_0) \in \mathcal{P}$, for all $\lambda \geq 0$ and $x \in \partial U \cap (\Omega + \lambda u_0)$, and one of the following conditions is satisfied*:

(a) *$\gamma Fx \not\leq x - \lambda u_0$, for all $x \in \partial U$, $\lambda \geq 0$, and $\gamma \in (0, 1)$.*
(b) *$Fx \in \mathcal{P}$, $\gamma\|Fx\| > \|x - \lambda u_0\|$, for all $x \in \partial U$, $\lambda \geq 0$, $\gamma \in (0, 1)$, and the cone $\mathcal{P}$ is normal with constant $N = 1$.*

Then the fixed point index $i_*(T + F, U \cap \Omega, \mathcal{P}) = 0$.

The proof is similar to that of Proposition 2.17.

Proposition 3.12. *Let U be a bounded open subset of $\mathcal{P}$. Assume that $T\colon \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, and $F\colon \overline{U} \to E$ is an $(h-1)$-set contraction with $F(\partial U) \subset \mathcal{P}$ and $tF(\overline{U}) \subset (I-T)(\Omega)$, for all $t \in [0,1]$ and $0 \notin \overline{(I-T-F)(\partial U \cap \Omega)}$. Suppose further that there exists $u_0 \in \mathcal{P}^*$ such that*

$$c_0 Fx \not\leq x - T(x - \lambda u_0), \quad \text{for all } \lambda \geq 0,\ x \in \partial U \cap (\Omega + \lambda u_0) \text{ and } c_0 \in (0,1). \tag{3.7}$$

Then the fixed point index $i_*(T+F, U \cap \Omega, \mathcal{P}) = 0$.

Proof. The mapping $(I-T)^{-1}F\colon \overline{U} \to \mathcal{P}$ is a 1-set contraction. By contradiction, suppose that $i_*(T+F, U\cap\Omega, \mathcal{P}) \neq 0$. From Definition 3.1, for $k \in (k_0, 1)$ with $c_0 \leq k_0 < 1$, we have

$$i\big((I-T)^{-1}kF, U, \mathcal{P}\big) \neq 0.$$

For each $k \in (k_0, 1)$ and $r > 0$, consider the homotopic deformation

$$H(t,x) = (I-T)^{-1}kF + tru_0, \quad \text{for } x \in \overline{U} \text{ and } t \in [0,1].$$

The operator H is continuous and uniformly continuous in t, for each x. Moreover, $H(t,\,\cdot\,)$ is a strict k-set contraction, for each t and $H([0,1] \times \overline{U}) \subset \mathcal{P}$. We prove that $H(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U$. If $H(t_0, x_0) = x_0$ for some $(t_0, x_0) \in [0,1] \times \partial U$, then

$$x_0 - t_0 r u_0 = (I-T)^{-1} kFx_0,$$

and so $x_0 - t_0 r u_0 \in \Omega$. Hence

$$x_0 - t_0 r u_0 - T(x_0 - t_0 r u_0) = kFx_0,$$

for $x_0 \in \partial U \cap (\Omega + t_0 r u_0)$, which implies that

$$x_0 - T(x_0 - t_0 r u_0) \geq kFx_0 \geq c_0 Fx_0,$$

contradicting assumption (3.7). As a consequence, by property (c) of the index in Lemma 1.3, for $k \in (k_0, 1)$, we get

$$i((I-T)^{-1}kF + ru_0, U \cap \Omega, \mathcal{P}) = i((I-T)^{-1}kF, U, \mathcal{P}) \neq 0.$$

By the existence property of the index in Lemma 1.3, for each $k \in (k_0, 1)$ and $r > 0$, there exists $x_r \in U$ such that

$$x_r - (I-T)^{-1}kFx_r = ru_0. \tag{3.8}$$

Letting $r \to +\infty$ in (3.8), the left side of (3.8) is bounded, but the right side is not, leading to a contradiction. Therefore

$$i_*(T+F, U \cap \Omega, \mathcal{P}) = 0. \qquad \square$$

4. Fixed point theorems

In this section, we present fixed point theorems for some special mappings, including 1-set contractions. The first two follow from Corollary 2.14 and Corollary 2.15, respectively.

Corollary 4.1. *Assume that* $T\colon \Omega\subset\mathcal{P}\to E$ *is an expansive mapping with constant* $h>1$ *and* $F\colon \overline{\mathcal{P}_r}\to E$ *is compact. Assume that* $F(\overline{\mathcal{P}_r})\subset (I-T)(\Omega)$ *and* $\|Fx+T0\|<(h-1)r$, *for all* $x\in\overline{\mathcal{P}_r}$. *Then the sum operator* $T+F$ *has at least one solution in* $\mathcal{P}_r\cap\Omega$.

Corollary 4.2. *Assume that* $T\colon \Omega\subset\mathcal{P}\to E$ *is an expansive mapping with constant* $h>2$ *and* $F\colon \overline{\mathcal{P}_r}\to E$ *is a* 1*-set contraction such that* $F(\overline{\mathcal{P}_r})\subset (I-T)(\Omega)$ *and* $\|Fx\|\le\|x\|$, *for all* $x\in\overline{\mathcal{P}_r}$ *with* $r>\|T0\|/(h-2)$. *Then* $T+F$ *has at least one fixed point in* $\mathcal{P}_r\cap\Omega$.

Recall that an operator L is said to be semi-closed if the identity perturbation $I-L$ is a closed operator.

Corollary 4.3. *Let* Ω *be a closed subset of* $\mathcal{P}$. *Assume that* $T\colon\Omega\to E$ *is a* 2*-expansive mapping and* $F\colon\overline{\mathcal{P}_r}\to E$ *is a* 1*-set contraction with* $tF(\overline{\mathcal{P}_r})\subset (I-T)(\Omega)$, *for all* $t\in[0,1]$. *Assume further that* $0\in\Omega$, $\|T0\|<r$, *and* $T+F$ *is semi-closed and satisfies*

$$\|Fx\|+\|Tx\|<\|x\|, \quad \text{for all } x\in\partial\mathcal{P}_r\cap\Omega.$$

Then $T+F$ *has a fixed point in* $\mathcal{P}_r\cap\Omega$.

Proof. Assume that $0\notin\overline{(I-T-F)(\partial\mathcal{P}_r\cap\Omega)}$, otherwise we are finished. Since F and T satisfy the assumptions of Proposition 3.7, then

$$0\in\overline{(I-T-F)(\mathcal{P}_r\cap\Omega)}.$$

So there exists a sequence $(x_n)_n$ in $\mathcal{P}_r\cap\Omega$ such that $x_n-Tx_n-Fx_n\to 0$ as $n\to+\infty$. Since $I-T-F$ is closed, then $0\in(I-T-F)(\overline{\mathcal{P}_r}\cap\Omega)$. Hence there exists $x\in\overline{\mathcal{P}_r}\cap\Omega$ such that $x=Tx+Fx$. Because $0\notin(I-T-F)(\partial\mathcal{P}_r\cap\Omega)$, we obtain $x\in\mathcal{P}_r\cap\Omega$. □

According to Proposition 3.5, the following corollary is proven in the same way.

Corollary 4.4. *Let* Ω *be a closed subset of* $\mathcal{P}$. *Assume that* $T\colon\Omega\to E$ *is a* 2*-expansive mapping and* $F\colon\overline{\mathcal{P}_r}\to E$ *is a* 1*-set contraction with* $F(\partial\mathcal{P}_r\cap\Omega)\subset\mathcal{P}$ *and* $tF(\overline{\mathcal{P}_r})\subset(I-T)(\Omega)$, *for all* $t\in[0,1]$. *Assume further that* $0\in\Omega$, $\|T0\|<r$, *and* $T+F$ *is semi-closed and satisfies*

$$Fx+Tx<x, \quad \text{for all } x\in\partial\mathcal{P}_r\cap\Omega.$$

Then $T+F$ *has a fixed point in in* $\mathcal{P}_r\cap\Omega$.

5. Applications

5.1. Example 1. Consider the nonlinear integral equation:

$$x(t) = x^3(t) + p(t)x(t) - \int_a^b K(t,s,x(s))\,ds, \quad a < t < b, \tag{5.1}$$

where $p\colon [a,b] \to \mathbb{R}_+$ is continuous and $K\colon [a,b]\times[a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ is continuous together with its first partial derivatives. We let

($\mathcal{H}_0$) $1 < p_1 \colon = \min_{a\le t\le b} p(t) \le p_2 =\colon \max_{a\le t\le b} p(t)$.

($\mathcal{H}_1$) There exists $R > 0$ such that

$$\int_a^b K(t,s,u)\,ds < R^3 + (p(t)-1)R, \quad \text{for all } (t,u) \in [a,b]\times[0,R].$$

Our main existence result for equation (5.1) is

Theorem 5.1. *Under assumptions* ($\mathcal{H}_0$) *and* ($\mathcal{H}_1$), *the integral equation* (5.1) *has at least one bounded solution* $x \in C([a,b])$ *such that* $0 \le x(t) \le R$, $a \le t \le b$.

Proof. Consider the Banach space $E = C([a,b],\mathbb{R})$ normed by $\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$, the positive cone $\mathcal{P} = \{x \in E : x(t) \ge 0\}$, and the conical shell $\mathcal{P}_R := \mathcal{P} \cap \mathcal{B}_R$. Define the operators $T, F\colon \overline{\mathcal{P}}_R \to E$ by

$$(Tx)(t) = x^3(t) + p(t)x(t) \quad \text{and} \quad (Fx)(t) = -\int_a^b K(t,s,x(s))\,ds$$

respectively, for $t \in [a,b]$. Then the integral equation (5.1) is equivalent to the operational equation $x = Tx + Fx$. We check that all assumptions of Propositon 2.8 are satisfied.

(a) Obviously, T and F map $\overline{\mathcal{P}}_R$ into E. Moreover

$$\|Tx - Ty\|_\infty \ge p_1\|x-y\|_\infty, \quad \text{for all } x,y \in \overline{\mathcal{P}}_R,$$

that is $T\colon \overline{\mathcal{P}}_R \to \mathcal{P}$ is expansive with constant $h = p_1 > 1$.

(b) If $x \in \overline{\mathcal{P}}_R$, then $\|x\|_\infty \le R$ and ($\mathcal{H}_1$) guarantees that

$$\|Fx\|_\infty \le R^3 + (p_2 - 1)R, \tag{5.2}$$

which implies that $F(\overline{\mathcal{P}}_R)$ is uniformly bounded. For $x \in \overline{\mathcal{P}}_R$, differentiating $(Fx)(t)$ with respect to t yields

$$(Fx)'(t) = -\int_a^b \frac{\partial K(t,s,x(s))}{\partial t}\,ds.$$

Hence

$$\exists\, N > 0, \quad \|(Fx)'\|_\infty \le N. \tag{5.3}$$

Estimates (5.2)–(5.3) imply that $F(\overline{\mathcal{P}_R})$ is an equicontinuous subset of E. Appealing to the Arzela–Ascoli compactness criterion, we can show that F maps bounded sets of $\mathcal{P}$ into relatively compact sets. In view of the sup-norm and the continuity of function K, it is easily checked that F is continuous. Therefore, $F\colon \overline{\mathcal{P}}_R \to E$ is completely continuous, i.e., is a 0-set contraction.

(c) Checking (2.3). Assume that there exist $x_0 \in \partial\mathcal{P}_R$ and $\lambda_0 \geq 1$ such that

$$Fx_0 = \lambda_0(x_0 - Tx_0).$$

Let $t_0 \in [a,b]$ with $x_0(t_0) = R$. From $(\mathcal{H}_1)$, we have

$$-(Fx_0)(t_0) = \int_a^b K(t_0, s, x_0(s))\, ds < R^3 + (p(t_0) - 1)R. \tag{5.4}$$

In addition

$$\begin{aligned} -(Fx_0)(t_0) &= -\lambda_0(x_0(t_0) - (Tx_0)(t_0)) \\ &= \lambda_0\big(R^3 + (p(t_0) - 1)R\big) \geq R^3 + \big(p(t_0) - 1\big)R, \end{aligned}$$

contradicting (5.8). Hence (2.3) holds.

(d) It remains to check that $\mu F(\overline{\mathcal{P}}_R) \subset (I - T)(\overline{\mathcal{P}}_R)$, for every $\mu \in [0,1]$. For this, let $z \in \overline{\mathcal{P}}_R$ and $\mu \in [0,1]$ be fixed and define the nonlinear operator on $\overline{\mathcal{P}}_R$ by $Ax = Tx + \mu Fz$. As T is, $A\colon \overline{\mathcal{P}}_R \to E$ is p_1-expansive mapping.

In order to use Proposition 1.11, we check that $\overline{\mathcal{P}}_R \subset A(\overline{\mathcal{P}}_R)$. Indeed, for given $v \in \overline{\mathcal{P}}_R$, define the operational equation set on $\overline{\mathcal{P}}_R$:

$$Au = v. \tag{5.5}$$

On the one hand, $v(t) - \mu Fz(t) \geq 0$, for all $t \in [a,b]$ and Assumption $(\mathcal{H}_1)$ guarantees that

$$\|v - \mu Fz\| \leq \|v\| + \mu\|Fz\| \leq R + R^3 + (p_2 - 1)R = R^3 + p_2 R.$$

Hence $v - \mu Fz \in \overline{\mathcal{B}}\big(0, R^3 + p_2 R\big) \cap \mathcal{P}$.

On the other hand, from the expression of operator T, we can easily check that $T\colon \overline{\mathcal{P}}_R \to T(\overline{\mathcal{P}}_R)$ is a bijection, where $T(\overline{\mathcal{P}}_R)$ is a conical shell centered at origin with radius R^3+p_2R. Then there exists some $u \in \overline{\mathcal{P}}_R$ such that (5.5) holds. By Proposition 1.11, A has a unique fixed point $w \in \overline{\mathcal{P}}_R$, that is $(I-T)w = \mu Fz$. Since z and μ are arbitrary, we conclude that $\mu F(\overline{\mathcal{P}}_R) \subset (I - T)(\overline{\mathcal{P}}_R)$, for every $\mu \in [0,1]$. Therefore Proposition 2.8 applies with $\Omega = \overline{\mathcal{P}_R}$ and gives the desired conclusion. □

5.2. Example 2. Let $(X, \|\cdot\|)$ be a Banach space ordered by a cone $\mathcal{K}$. Consider the nonlinear integral equation:

$$x(t) = a(t)x(t) - b(t)e^{mt}x_0 - \int_0^{+\infty} G(t,s)g(s)f(s,x(s))\, ds, \quad t \geq 0, \tag{5.6}$$

where $x_0 \in \mathcal{K}$ is such that $\|x_0\| = 1$, $a, b\colon [0,+\infty) \to [0,+\infty)$ are bounded functions, and m is a real number. $f \in C([0,+\infty) \times \mathcal{K}, \mathcal{K})$, $g \in C([0,+\infty), \mathbb{R}^+)$ and the kernel $G\colon [0,+\infty) \times [0,+\infty) \to \mathbb{R}^+$ is defined by:

$$G(t,s) = \frac{1}{2m}\begin{cases} e^{-ms}(e^{mt} - e^{-mt}) & \text{if } 0 \le t \le s < \infty, \\ e^{-mt}(e^{ms} - e^{-ms}) & \text{if } 0 \le s \le t < \infty. \end{cases}$$

Suppose that the following conditions hold:

($\mathcal{H}_1$) There exist $p, q \in L^\infty([0,+\infty), \mathbb{R}^+)$ such that

$$\|f(t,x)\| \le p(t) + q(t)\|x\|, \quad \text{for all } (t,x) \in [0,+\infty) \times \mathcal{K},$$

where, for some $\theta \ge m$, the coefficients p and q satisfy:

$$\int_0^{+\infty} e^{\theta t} g(t) q(t)\, dt < \infty \quad \text{and} \quad \int_0^{+\infty} g(t)p(t)\, dt < \infty.$$

($\mathcal{H}_2$) For all $r > 0$ and all subinterval $[a,b] \subset [0,+\infty)$, the nonlinearity $f(\,\cdot\,,\,\cdot\,)$ is uniformly continuous on $[a,b] \times \mathcal{B}_X(0,r)$ and there exists a nonnegative function $l \in L^1([0,+\infty))$ with

$$\int_0^{+\infty} e^{(\theta-m)t} g(t) l(t)\, dt = 2m(a_0 - 1),$$
$$\alpha(f(t,B)) \le l(t)\alpha(B), \quad t \in [0,+\infty),$$

for every bounded subset $B \subset X$.

Remark 5.2. (a) According to [15, Lemma 2.3], G is the kernel to the second-order differential operator $-x'' + m^2 x = 0$, $0 < t < +\infty$ subject to homogeneous Dirichlet conditions.

(b) Let

$$M_1 := \int_0^{+\infty} e^{-ms} G(s,s)\, g(s)\, p(s)\, ds,$$
$$M_2 := \int_0^{+\infty} e^{(\theta-m)s}\, G(s,s)\, g(s)\, q(s)\, ds.$$

By ($\mathcal{H}_1$), these integrals are convergent. Indeed

$$M_1 = \frac{1}{2m}\int_0^{+\infty} \big(e^{-ms} - e^{-3ms}\big)\, g(s)\, p(s)\, ds < \frac{1}{2m}\int_0^{+\infty} g(s)p(s)\, ds < +\infty,$$
$$M_2 < \frac{1}{2m}\int_0^{+\infty} e^{\theta s} g(s)\, q(s)\, ds < +\infty.$$

We prove an existence result for equation (5.6).

Theorem 5.3. *Further to ($\mathcal{H}_1$)–($\mathcal{H}_2$), assume that*

($\mathcal{H}_3$) $a_0 := \inf_{t \in [0,+\infty)} a(t) > 1 + M_2$.

Then the integral equation (5.6) *has a sequence of approximate solutions. If further X is reflexive and, for each $t \in J$, the mapping*

$$(\mathcal{H}_4)\ \ t \mapsto \int_0^{+\infty} G(t,s)g(s)f(s,x(s))\,ds$$

is weakly sequentially continuous, then (5.6) *has at least one positive solution $x \in C(J,\mathcal{K})$.*

PROOF. Consider the Banach space

$$E = \Big\{x \in C([0,+\infty),X) : \lim_{t\to+\infty} e^{-\theta t}\|x(t)\| \text{ exists}\Big\}$$

equipped with the weighted Bielecki sup-norm $\|x\|_\theta = \sup_{t\in[0,+\infty)} \big(e^{-\theta t}\|x(t)\|\big)$. Let the positive cone $\mathcal{P} = \{x \in E : x(t) \geq 0,\ \text{for all } t \geq 0\}$ and the conical shell $\mathcal{P}_R := \{x \in \mathcal{P} : \|x\|_\theta < R\}$, where $R \geq (M_1 + \|b\|_\infty)/(a_0 - M_2 - 1)$. Define the operators on E:

$$(Tx)(t) = a(t)x(t) - x_0 b(t)e^{mt}x_0,$$
$$(Fx)(t) = -\int_0^{+\infty} G(t,s)g(s)f(s,x(s))\,ds.$$

Then the integral equation (5.6) is equivalent to the operational equation $x = Tx + Fx$.

We check that all assumptions of Proposition 3.9 are fulfilled. Obviously, T and F map $\overline{\mathcal{P}}_R$ into E. From $(\mathcal{H}_3)$, we have

$$\|Tx - Ty\|_\theta \geq a_0\|x-y\|_\theta, \quad \text{for all } x,y \in E,$$

that is T is expansive mapping with constant $a_0 > 1$. If $x \in \overline{\mathcal{P}}_R$, then

$$\|x\|_\theta \leq R \quad \text{and} \quad \|Fx\|_\theta \leq M_1 + M_2 R,$$

which implies that $F(\overline{\mathcal{P}}_R)$ is uniformly bounded. According to ([15, Lemmas 3.2 and 3.5]), under assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, F is a k-set-contraction on $\overline{\mathcal{P}}_R$ with

$$k = \frac{1}{2m}\int_0^{+\infty} e^{(\theta-m)s}g(s)\,l(s)\,ds.$$

In addition, by assumption $(\mathcal{H}_1)$, for all $x \in \partial\mathcal{P}_R$ and $t \in [0,+\infty)$, we have

$$\begin{aligned} e^{-\theta t}\|Fx(t) + T0(t)\| &= e^{-\theta t}\bigg\|\int_0^{+\infty} G(t,s)g(s)f(s,x(s))\,ds + b(t)e^{mt}\,x_0\bigg\| \\ &\leq \int_0^{+\infty} e^{-ms}G(s,s)g(s)[p(s) + q(s)e^{\theta s}\|x\|_\theta]\,ds + \|b\|_\infty \\ &\leq M_1 + M_2R + \|b\|_\infty \leq (a_0-1)R. \end{aligned}$$

Passing to the supremum over t guarantees that

$$\|Fx + T0\|_\theta \leq (a_0-1)\|x\|_\theta, \quad \text{for all } x \in \partial\mathcal{P}_R.$$

It remains to show that $F(\overline{\mathcal{P}}_R) \subset (I-T)(\overline{\mathcal{P}}_R)$. Arguing as in the first example, it is sufficient to prove that

$$y \in F(\overline{\mathcal{P}}_R) \quad \text{implies} \quad y + T(\overline{\mathcal{P}}_R) \supset \overline{\mathcal{P}}_R. \tag{5.7}$$

For any $x, u \in \overline{\mathcal{P}}_R$, define

$$v(t) = \frac{1}{a(t)}\bigg(u(t) + b(t)e^{mt}x_0 + \int_0^{+\infty} G(t,s)g(s)f(s,x(s))\,ds\bigg), \tag{5.8}$$

for $t \geq 0$. Hence $v(t) \geq 0$, $t \geq 0$ and

$$\begin{aligned} e^{-\theta t}\|v(t)\| &= \frac{1}{a(t)}\bigg(e^{-\theta t}u(t) + b(t)e^{(m-\theta)t}x_0 \\ &\qquad + e^{-\theta t}\int_0^{+\infty} G(t,s)g(s)f(s,x(s))\,ds\bigg) \\ &\leq \frac{1}{a_0}\big(\|u\|_\theta + \|b\|_\infty\|x_0\| + M_1 + M_2R\big) \\ &\leq \frac{1}{a_0}\big((M_2+1)R + M_1 + \|b\|_\infty\big) \leq R. \end{aligned}$$

Thus $v \in \overline{\mathcal{P}}_R$. Consequently, from (5.8) we have

$$u(t) = -\int_0^{+\infty} G(t,s)g(s)f(s,x(s))\,ds + a(t)v(t) - b(t)e^{mt}x_0 = (Fx)(t) + (Tv)(t),$$

for $t \in [0,+\infty)$, proving that $Fx + T(\overline{\mathcal{P}}_R) \supset \overline{\mathcal{P}}_R$.

Without loss of generality, assume that $0 \notin \overline{(I-T-F)(\partial\mathcal{P}_R)}$, otherwise we are finished. By Proposition 3.9 with $U = \mathcal{P}_R$ and $\Omega = \overline{\mathcal{P}}_R$, we get

$$i_*(T+F, \mathcal{P}_R, \mathcal{P}) = 1.$$

By the existence property of the index, $0 \in \overline{(I-T-F)(\mathcal{P}_R)}$, i.e. there exists a sequence $(x_n)_n$ in $\mathcal{P}_R$ such that $x_n - Tx_n - Fx_n \to 0$, as $n \to +\infty$. Since the sequence $(x_n)_n$ is bounded in E, then so is $(x_n(t))_n$ in X for each t. X being reflexive, there exists a subsequence still denoted $(x_n(t))_n$ which converges weakly in X. By [8, Theorem 5], the sequence $(x_n)_n$ also converges weakly in $C([0,+\infty), X)$ to some limit x. Assumption $(\mathcal{H}_3)$ implies that $x - Tx - Fx = 0$, that is the mapping $T + F$ has at least one positive fixed point in $\overline{\mathcal{P}}_R$, which is a solution of the integral equation (5.6). □

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References

[1] H. Amann, *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Funct. Anal. **11** (1972), no. 4, 346–384.

[2] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), no. 4, 620–709.

[3] R. Avery and D.R. Anderson, *Fixed point theorem of cone expansion and compression of functional type*, Special issue in honour of Professor Allan Peterson on the occasion of his 60th birthday, Part I, J. Difference Equ. Appl. **8** (2002), no. 11, 1073–1083.

[4] R.I. Avery, D.R. Anderson and R.J. Krueger, *An extension of the fixed point theorem of cone expansion and compression of functional type*, Comm. Appl. Nonlinear Anal. **13** (2006), no. 1, 15–26.

[5] J. Banas and Z. Knap, *Measures of weak noncompactness and nonlinear integral equations of convolution type*, J. Math. Anal. Appl. **146** (1990), no. 2, 353–362.

[6] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker, Inc., New York, 1980.

[7] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Translated from the Romanian. Editura Academiei Republicii Socialiste Romania, Bucharest; Noordhoff International Publishing, Leiden, 1976.

[8] W.M. Bogdanowicz, *Representations of linear continuous functionals on the space $C(X,Y)$ of continuous functions from compact X into locally convex Y*, Proc. Japan Acad. **42** (1966), 1122–1127.

[9] F.E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968), Amer. Math. Soc., Providence, R.I., 1976, pp. 1–308.

[10] T.A. Burton, *A fixed-point theorem of Krasnosel'skiĭ*, Appl. Math. Lett. **11** (1998), no. 1, 85–88.

[11] Y.Q. Cheen, K.S. Ha and Y.J. Cho, *The fixed point index for accretive mappings with k-set contraction perturbations in cones*, J. Korean Math. Soc. **34** (1997), no. 1, 237–245.

[12] G. Darbo, *Punti uniti in transformationi a condominio non-compactto*, Rend. Sem. Mat. Univ. Padova **24** (1955), 84–92.

[13] K. Deimling, *Nonlinear Functional Analysis*, Springer–Verlag, Berlin, Heidelberg, 1985.

[14] S. Djebali, *Fixed point theory for* 1*-set contractions: a survey*, Applied Mathematics in Tunisia, Proceedings of the International Conference on Advances in Applied Mathematics (ICAAM), Hammamet, Tunisia, December 2013; Springer Proceedings in Mathematics and Statistics, Vol. 131, Springer–Birkhäuser, 2015, pp. 53–100.

[15] S. Djebali, F. Madjidi and K. Mebarki, *Existence results for singular boundary value problems on unbounded Domains in Banach spaces*, Mediterr. J. Math. **11** (2014), 45–74.

[16] S. Djebali and K. Mebarki, *Fixed point index on translates of cones and applications*, Nonlinear Stud. **21** (2014), no. 4, 579–589.

[17] G. Feltrin, *A note on a fixed point theorem on topological cylinders*, Ann. Mat. Pura Appl. **196** (2017), 1441–1458.

[18] M. Feng, X. Zhang and W. Ge, *Positive fixed point of strict set contraction operators on ordered Banach spaces and applications*, Abstr. Appl. Anal. (2010), Art. ID 439137, 13 pp.

[19] D. Guo, Y.I. Cho and J. Zhu, *Partial Ordering Methods in Nonlinear Problems*, Shangdon Science and Technology Publishing Press, Shangdon, 1985.

[20] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5, Academic Press, Boston, Mass, USA, 1988.

[21] L. Guozhen, *The fixed point index and the fixed point theorems of* 1*-set contraction mappings*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1163–1170.

[22] M.A. Krasnosel'skiĭ, *Two remarks on the method of successive approximations*, Uspekhi Mat. Nauk **10** (1955), 123–127.

[23] M.A. Krasnosel'skiĭ, *Fixed points of cone-compressing or cone-extending operators*, Soviet Math. Dokl. **1** (1960), 1285–1288.

[24] M.A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, The Netherlands, 1964.

[25] M.K. Kwong, *On Krasnoselskiĭ's cone fixed point theorem*, Fixed Point Theory Appl. (2008), Art. ID 164537, 18 pp.

[26] M.K. Kwong, *The topological nature of Krasnoselskiĭ's cone fixed point theorem*, Nonlinear Anal. **69** (2008), 891–897.

[27] textscR.D. Nussbaum, *The fixed point index and asymptotic fixed point theorems for* k*-set-contractions*, Bull. Amer. Math. Soc. **75** (1969), 490–495.

[28] R.D. Nussbaum, *The fixed point index and fixed point theorems. Topological methods for ordinary differential equations* (Montecatini Terme, 1991), Lecture Notes in Math., vol. 1537, Springer, Berlin, 1993, 143–205

[29] D. O'Regan, *Fixed-point theory for the sum of two operators*, Appl. Math. Lett. **9** (1996), no. 1, 1–8.

[30] D. O'Regan and R. Precup, *Compression-expansion fixed point theorem in two norms and applications*, J. Math. Anal. Appl. **309** (2005), no. 2, 383–391.

[31] W.V. Petryshyn, *Fixed point theorems for various classes of* 1*-set-contractive and* 1*-ball-contractive mappings in Banach spaces*, Trans. Amer. Math. Soc. **182** (1973), 323–3582.

[32] B.N. Sadovskiĭ, *On a fixed point principle*, Funkc. Anal. Prilož. **1** (1967), no. 2, 74–76. (in Russian)

[33] D.R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1980.

[34] F. Wang, *Fixed-point theorems for the sum of two operators under* ω*-condensing*, Fixed Point Theory Appl. **102** (2013), 13 pp.

[35] T. Xiang and R. Yuan, *A class of expansive-type Krasnosel'skiĭ fixed point theorems*, Nonlinear Anal. **71** (2009), no. 7–8, 3229–3239.

[36] T. Xiang, S.G. Georgiev, *Noncompact-type Krasnosel'skiĭ fixed-point theorems and their applications*, Math. Methods Appl. Sci. **39** (2016), no. 4, 833–863.

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