FIXED POINT RESULTS IN SET $P_{h,e}$ WITH APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, without assuming operators to be continuous or compact, by employing monotone iterative technique on ordered Banach space, we at first establish new fixed point theorems for some kinds of nonlinear mixed monotone operators in set $P_{h,e}$. Then, we study a new form of fractional two point boundary value problem depending on a certain constant and give the existence and uniqueness of solutions. We also show that the unique solution exists in set $P_{h,e}$ or P_h and can be uniformly approximated by constructing two iterative sequences for any initial values. At the end, a concrete example is given to illustrate our abstract results. The conclusions of this article enrich the fixed point theorems and provide new methods to deal with nonlinear differential equations.

1. Introduction

In the past decades, one can observe a remarkable amount of interest for the development of fixed point theory, since it has a huge number of applications in engineering, mechanics, the nuclear industry, fluid dynamics, biological chemistry technology and elasticity, etc. It is worthy to mention the recent papers [9],

[10], [15]–[17], [20], [23], [26], [27], [30], without assuming operators to be continuous, compact or having upper and lower solutions, the authors study fixed point theorems for the single operator, such as a generalized convex increasing operator, an $t-\alpha(t)$ mixed monotone model operator, a $t-\alpha(t,u,v)$ mixed monotone model operators, a mixed monotone e-concave-convex operator, a monotone τ-φ-convex operator and so on and obtain results on the existence of unique positive fixed points in P_h. On the other hand, the unique existence criterions of positive fixed point for some sum-type operators are also presented in P_h (note that P_h is a subset of a cone P), see [11], [13], [18], [19], [29]. As applications, authors establish some results for the local existence and uniqueness of positive solutions to boundary or initial value problems which including fractional differential equations, elastic beam equation, Lane–Emden–Fowler equation, second order impulsive differential equations and so on, see [1]–[3], [7], [12], [19], [21], [28] and the references therein.

Furthermore, in [25], the authors study the fixed point problems in a new set which is defined as $P_{h,e} = \{x \in E | x + e \in P_h\}$ with $\theta \leq e \leq h$, $e \in P$. It is obvious that $P_{h,e}$ is not a subset of P for some e and $P_h \subseteq P_{h,e}$. In this case, the $\varphi-(h,e)$ concave increasing operator $A \colon P_{h,e} \to E$ is introduced and satisfies

$$A(\lambda x + (\lambda - 1)e) \geq \varphi(\lambda)Ax + (\varphi(\lambda) - 1)e, \quad \text{for all } x \in P_{h,e}, \ \lambda \in (0,1),$$

with $\varphi(\lambda) > \lambda$. For such operator the authors obtain the uniqueness of fixed point in $P_{h,e}$. Also, by using the obtained fixed point theorem, the authors study the existence and uniqueness of solution for a class of fractional differential equation with integral boundary conditions, a fractional q-difference equation with three-point boundary conditions and a new coupled system of fractional equations, see [22], [24], [25].

Besides, in [14], Wardowski introduced a new type of (e,u)-concave-convex operator $A \colon C_{e,u} \times C_{e,u} \to C_{e,u}$ which satisfies

$$A(\lambda x + (\lambda - 1)u, \lambda^{-1}y + (\lambda^{-1} - 1)u) \geq L(x,y,\lambda)A(x,y) + (L(x,y,\lambda) - 1)u,$$

where $x, y \in C_{e,u}$, $\lambda \in (0,1)$, $L \colon C_{e,u} \times C_{e,u} \times (0,1) \to (0,+\infty)$ is a mapping and satisfies $L(x,y,\lambda) > \lambda$. And $C_{e,u} = \{x \in E : \alpha e \leq x + u \leq \beta e \text{ for some } \alpha, \beta > 0\}$ for $e \in P^+$, $u \in P$, $u \leq e$. According to the definition of the set $P_{h,e}$ and $C_{e,u}$, we know these two sets are essentially the same.

Inspired by the above literature, in this paper, we study the fixed point theorems for some kinds of mixed monotone operators in set $P_{h,e}$ with an order determined by a normal cone. We suppose that $T \colon P_{h,e} \times P_{h,e} \to E$ is a mixed monotone operator and satisfies

$$T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda}v + \left(\frac{1}{\lambda} - 1\right)e\right) \geq \varphi(\lambda)T(u,v) + (\varphi(\lambda) - 1)e,$$

with $\lambda \in (0,1)$, $u, v \in P_{h,e}$ and $\varphi(\lambda) > \lambda$. Or

$$T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda} v + \left(\frac{1}{\lambda} - 1\right)e\right) \geq \lambda^{\beta(\lambda)} T(u,v) + (\lambda^{\beta(\lambda)} - 1)e,$$

with $\lambda \in (0,1)$, $u, v \in P_{h,e}$ and $0 < \beta = \beta(\lambda) < 1$. Or

$$T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda} v + \left(\frac{1}{\lambda} - 1\right)e\right) \geq \lambda^{\beta(\lambda,u,v)} T(u,v) + (\lambda^{\beta(\lambda,u,v)} - 1)e,$$

with $\lambda \in (0,1)$, $u, v \in P_{h,e}$, and $0 < \beta = \beta(\lambda, u, v) < 1$. By using the cone theory and monotone iterative method, we present the unique existence results for the operator T. Then, we focus on studying the existence and uniqueness of solution for a certain type of nonlinear fractional differential equation:

(1.1)
$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t), u(t)) = a, & 0 < t < 1, \ n-1 < \alpha \leq n, \\ u^{(i)}(0) = 0, & 0 \leq i \leq n-2, \ n \in \mathbb{N}, \ n > 3, \\ D_{0+}^{\nu} u(t)|_{t=1} = 0, & 2 \leq \nu \leq n-2, \end{cases}$$

where D_{0+}^{α} is the Riemann–Liouvile fractional derivative. $a \in \mathbb{R}$ is a constant. When $a > 0$, by applying the fixed point theorem of mixed monotone operator on set $P_{h,e}$, we deduce the sufficient conditions which guarantee the existence and uniqueness of solution in $P_{h,e}$. When $a \leq 0$, we study the existence of a unique positive solution by applying the fixed point theorem of mixed monotone operator in set P_h. Also, for any given initial value points, two iterative sequences are constructed to approximate the unique solution. At last, an interesting concrete example is given to demonstrate the results.

From literature, no papers have considered the fixed point theorems of these kinds of mixed monotone operators in set $P_{h,e}$ and also no papers have concentrated on the equations (1.1) with $a \neq 0$. Therefore, our research is new and worthy. Straightly speaking, our work is different from those in [7], [9], [10], [14]–[17], [21], [27], [30].

(a) Our study is based on set $P_{h,e}$ to study the fixed point theorems, while papers [9], [10], [15]–[17], [27], [30] deal with the set P_h. Here we should point that if we set $e = \theta$, then $P_{h,e} = P_h$, which implies $P_h \subset P_{h,e}$. In this case, the corresponding fixed point theorems have been studied in [10], [16], [27]. So the operators in our study have a more general form.

(b) Comparing with results in [14], we do not require the mixed monotone operator defined by $A\colon C_{e,u} \times C_{e,u} \to C_{e,u}$. So our research improves the assumed condition for the operator.

(c) In case of the equation (1.1), the previous results are only focused with the case $a = 0$. For example, in [7], [21], the authors study the existence and uniqueness of positive solution by using fixed point theorems on set P_h. However, in this paper, we study the same results of problem (1.1) with $a \leq 0$ by applying the fixed point theorem on P_h. Besides, for the case $a > 0$, we can not deduce

the similar results by means of previously available methods in [7], [21], [27]. So by using the fixed point theorems on set $P_{h,e}$, we obtain the problem (1.1) has a unique solution in $P_{h,e}$.

In conclusion, our research not only extends and improves the previous fixed point theorems of mixed monotone operators, but also present some new ways to solve the nonlinear equation problems, especially for the differential system with a certain type like problem (1.1).

The outline of this paper is as follows. In next section, we recall some preliminary facts that we need in the sequel. In Section 3, cone theory and monotone iterative technique are used to discuss the unique existence of fixed point for some kinds of mixed monotone operators in set $P_{h,e}$. In Section 4, by using the fixed point theorems presented in this paper, we study the existence and uniqueness of solutions to a form of fractional two point boundary value problem dependent on a certain constant. Also, an example is given to illustrate the obtained main result.

2. Preliminaries

In the following, we first give some definitions and lemmas in ordered Banach space. For more details, we refer to [5], [6], [25], [27].

DEFINITION 2.1 [6]. Let E be a real Banach space. A non-empty closed convex set $P \subset E$ is called a cone if it satisfies the following conditions:

(a) If $x \in P$, $\lambda \geq 0$, then $\lambda x \in P$;
(b) If $x \in P$, and $-x \in P$, then $x = \theta$,

where θ denotes the zero element of E. The cone P includes a partial order in E given by

$$x, y \in E, \ x \leq y \ \Leftrightarrow \ y - x \in P,$$

in which we denote $x < y$ or $y > x$ if $x \neq y$.

DEFINITION 2.2 ([6]). A cone P is called normal if there exists a constant $N > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where N is called the normality constant of P.

DEFINITION 2.3 ([6]). Given $h > \theta$ (i.e. $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$, in which $\sim$ is the following equivalence relation: for $x, y \in E$, we write $x \sim y$ if there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$.

DEFINITION 2.4 ([25]). Given $h > \theta$ (i.e. $h \geq \theta$ and $h \neq \theta$), take an element $e \in P$, with $\theta \leq e \leq h$, and define $P_{h,e} = \{x \in E \mid x + e \in P_h\}$. Then we can see

that

$$P_{h,e} = \{x \in E \mid \text{there exist } \mu = \mu(h,e,x) > 0, \ \nu = \nu(h,e,x) > 0$$
$$\text{such that } \mu h \le x + e \le \nu h\}.$$

LEMMA 2.5 ([25]). *If $x \in P_{h,e}$, then $\lambda x + (\lambda - 1)e \in P_{h,e}$ for $\lambda > 0$.*

LEMMA 2.6 ([25]). *If $x, y \in P_{h,e}$, then there exist $0 < \mu < 1$, $\nu > 1$ such that*

$$\mu y + (\mu - 1)e \le x \le \nu y + (\nu - 1)e.$$

Further, we can choose a small $r \in (0,1)$ such that

$$ry + (r - 1)e \le x \le r^{-1}y + (r^{-1} - 1)e.$$

In the sequel, we present the basic concept of the fractional derivative, a useful lemma and the properties of Green function which can be used to study the differential equations (1.1).

DEFINITION 2.7 ([8]). Let $\alpha > 0$. Suppose that $u : (0, +\infty) \to \mathbb{R}$ is a continuous function. Then the Riemann–Liouvile fractional derivative of order α for the function u is defined as

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t - \tau)^{n-\alpha-1} u(\tau) \, d\tau,$$

where $n = [\alpha] + 1$ with $[\alpha]$ standing for the largest integer less than the number α. $\Gamma(\alpha)$ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} \, dt.$$

LEMMA 2.8 ([4]). *Let $g \in C[0,1]$ be given. Then the unique solution to problem*

$$\begin{cases} -D_{0+}^{\alpha} u(t) = g(t), & 0 < t < 1, \ n - 1 < \alpha \le n, \\ u^{(i)}(0) = 0, & 0 \le i \le n - 2, \ n \in \mathbb{N}, \ n > 3, \\ D_{0+}^{\nu} u(t)\big|_{t=1} = 0, & 2 \le \nu \le n - 2, \end{cases}$$

is

$$(2.1) \qquad u(t) = \int_0^1 G(t,s)g(s) \, ds, \quad t \in [0,1],$$

where

$$(2.2) \qquad G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-\nu-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\[3mm] \dfrac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases}$$

is the Green function for this problem.

LEMMA 2.9 ([21]). *The function $G(t,s)$ satisfies the following conditions:*

(a) $G(t, s)$ *is a continuous function on the unit square* $[0, 1] \times [0, 1]$;
(b) $G(t, s) \geq 0$ *for each* $(t, s) \in [0, 1] \times [0, 1]$;
(c) $[1 - (1-s)^\nu](1-s)^{\alpha-\nu-1}t^{\alpha-1} \leq \Gamma(\alpha)G(t, s) \leq (1-s)^{\alpha-\nu-1}t^{\alpha-1}$, *for all* $t, s \in [0, 1]$.

3. Main results

In this section, we study the fixed point theorems for some kinds of mixed monotone operators on set $P_{h,e}$. We assume that E is a real Banach space with a partial order introduced by a normal cone P of E. Take $h \in E$, $h > \theta$, $e \in P$ with $\theta \leq e \leq h$, P_h and $P_{h,e}$ are given in the Definitions 2.3 and 2.4. The main abstract results are as follows.

THEOREM 3.1. *Let P be normal. Assume that $T \colon P_{h,e} \times P_{h,e} \to E$ is a mixed monotone operator and satisfies the following conditions:*

(L$_1$) *there exist $h \in E$, $e \in P$ with $\theta \leq e \leq h$, $h \neq \theta$ such that $T(h, h) \in P_{h,e}$;*
(L$_2$) *for any $u, v \in P_{h,e}$, and every $\lambda \in (0, 1)$, there exist $\varphi(\lambda) > \lambda$ such that*

$$(3.1) \qquad T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda}v + \left(\frac{1}{\lambda} - 1\right)e\right) \geq \varphi(\lambda)T(u, v) + (\varphi(\lambda) - 1)e.$$

Then:

(a) *there exist u_0, $v_0 \in P_{h,e}$ such that*

$$u_0 < v_0, \qquad u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0;$$

(b) *the operator T has a unique fixed point $x^* \in P_{h,e}$;*
(c) *for any initial values $x_0, y_0 \in P_{h,e}$ and the sequences $x_n = T(x_{n-1}, y_{n-1})$, $y_n = T(y_{n-1}, x_{n-1})$, for $n = 1, 2, \ldots$, we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.*

PROOF. *Step 1.* We prove the conclusion (a). By $h \in E$, $e \in P$ with $\theta \leq e \leq h$, $h \neq \theta$, we easily obtain $h \in P_{h,e}$. Considering the fact that $T(h, h) \in P_{h,e}$, by Lemma 2.6, we choose a small number $t_0 \in (0, 1)$ such that

$$(3.2) \qquad t_0 h + (t_0 - 1)e \leq T(h, h) \leq t_0^{-1}h + (t_0^{-1} - 1)e.$$

With the fact that $\varphi(t_0) > t_0$, we can choose a sufficiently large positive integer k such that

$$(3.3) \qquad \left(\frac{\varphi(t_0)}{t_0}\right)^k \geq \frac{1}{t_0}.$$

Let

$$(3.4) \qquad a_n = t_0^n h + (t_0^n - 1)e, \quad b_n = \frac{1}{t_0^n}h + \left(\frac{1}{t_0^n} - 1\right)e, \quad n = 1, 2, \ldots$$

By (3.4), we deduce that

$$(3.5) \quad a_n = t_0 a_{n-1} + (t_0 - 1)e, \quad b_n = \frac{1}{t_0} b_{n-1} + \left(\frac{1}{t_0} - 1\right)e, \quad n = 1, 2, \ldots$$

Now, set

$$(3.6) \quad u_0 := a_k, \qquad v_0 := b_k.$$

It is obvious that $u_0, v_0 \in P_{h,e}$ according to Lemma 2.5. Also, we get $u_0 < t_0^{2k} v_0 < v_0$. From (3.1)–(3.6), we deduce

$$
\begin{aligned}
T(u_0, v_0) &= T(a_k, b_k) = T\left(t_0 a_{k-1} + (t_0 - 1)e, \frac{1}{t_0} b_{k-1} + \left(\frac{1}{t_0} - 1\right)e\right) \\
&\geq \varphi(t_0) T(a_{k-1}, b_{k-1}) + (\varphi(t_0) - 1))e \\
&= \varphi(t_0) T\left(t_0 a_{k-2} + (t_0 - 1)e, \frac{1}{t_0} b_{k-2} + \left(\frac{1}{t_0} - 1\right)e\right) + (\varphi(t_0) - 1)e \\
&\geq \varphi(t_0)\left[\varphi(t_0) T(a_{k-2}, b_{k-2}) + (\varphi(t_0) - 1)e\right] + (\varphi(t_0) - 1))e \\
&= (\varphi(t_0))^2 T(a_{k-2}, b_{k-2}) + \left[(\varphi(t_0))^2 - 1\right]e \\
&\geq \ldots \geq (\varphi(t_0))^k T(a_0, b_0) + \left[(\varphi(t_0))^k - 1\right]e \\
&= (\varphi(t_0))^k T(h, h) + \left[(\varphi(t_0))^k - 1\right]e \\
&\geq t_0^{k-1}[t_0 h + (t_0 - 1)e] + \left(t_0^{k-1} - 1\right)e = t_0^k h + \left(t_0^k - 1\right)e = u_0.
\end{aligned}
$$

Condition (3.1) implies that

$$(3.7) \quad T(u, v) \leq \frac{1}{\varphi(\lambda)} T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda} v + \left(\frac{1}{\lambda} - 1\right)e\right) + \left(\frac{1}{\varphi(\lambda)} - 1\right)e.$$

By (3.2)–(3.7), we obtain that

$$
\begin{aligned}
T(v_0, u_0) &= T(b_k, a_k) = T\left(\frac{1}{t_0} b_{k-1} + \left(\frac{1}{t_0} - 1\right)e, t_0 a_{k-1} + (t_0 - 1)e\right) \\
&\leq \frac{1}{\varphi(t_0)} T\Bigg(b_{k-1} + (1 - t_0)e + (t_0 - 1)e, a_{k-1} \\
&\quad + \left(1 - \frac{1}{t_0}\right)e + \left(\frac{1}{t_0} - 1\right)e\Bigg) + \left(\frac{1}{\varphi(t_0)} - 1\right)e \\
&= \frac{1}{\varphi(t_0)} T(b_{k-1}, a_{k-1}) + \left(\frac{1}{\varphi(t_0)} - 1\right)e \\
&= \frac{1}{\varphi(t_0)} T\left(\frac{1}{t_0} b_{k-2} + \left(\frac{1}{t_0} - 1\right)e, t_0 a_{k-2} + (t_0 - 1)e\right) + \left(\frac{1}{\varphi(t_0)} - 1\right)e \\
&\leq \frac{1}{\varphi(t_0)}\left[\frac{1}{\varphi(t_0)} T(b_{k-2}, a_{k-2}) + \left(\frac{1}{\varphi(t_0)} - 1\right)e\right] + \left(\frac{1}{\varphi(t_0)} - 1\right)e \\
&= \left(\frac{1}{\varphi(t_0)}\right)^2 T(b_{k-2}, a_{k-2}) + \left[\left(\frac{1}{\varphi(t_0)}\right)^2 - 1\right]e
\end{aligned}
$$

$$\leq \ldots \leq \left(\frac{1}{\varphi(t_0)}\right)^k T(h,h) + \left[\left(\frac{1}{\varphi(t_0)}\right)^k - 1\right]e$$

$$\leq \left(\frac{1}{\varphi(t_0)}\right)^k \left[\frac{1}{t_0}h + \left(\frac{1}{t_0} - 1\right)e\right] + \left[\left(\frac{1}{\varphi(t_0)}\right)^k - 1\right]e$$

$$\leq \frac{1}{t_0^{k-1}}\left[\frac{1}{t_0}h + \left(\frac{1}{t_0} - 1\right)e\right] + \left(\frac{1}{t_0^{k-1}} - 1\right)e = \frac{1}{t_0^k}h + \left(\frac{1}{t_0^k} - 1\right)e = v_0.$$

It follows from mixed monotone property of operator T that

$$(3.8) \qquad u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0.$$

Step 2. We will prove the conclusion (b) hold. We at first show that x^* is a fixed point of T in $P_{h,e}$. Denote the sequences

$$(3.9) \qquad u_n = T(u_{n-1}, v_{n-1}), \quad v_n = T(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots$$

By (3.8) and (3.9), we get $u_0 \leq u_1 = T(u_0, v_0) \leq T(v_0, u_0) = v_1 \leq v_0$. From (3.9) and the mixed monotone property of T, we deduce that $u_n \leq v_n$. Hence, by induction, it is easy to get that

$$(3.10) \qquad u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0, \quad n = 1, 2, \ldots$$

Note that $u_0, v_0 \in P_{h,e}$ and $u_0 < v_0$, from Lemma 2.6 and (3.10), we can choose a constant $r_0 \in (0, 1)$ such that $u_0 \geq r_0 v_0 + (r_0 - 1)e$, thus

$$u_n \geq u_0 \geq r_0 v_0 + (r_0 - 1)e \geq r_0 v_n + (r_0 - 1)e, \quad n = 1, 2, \ldots$$

Define

$$(3.11) \qquad t_n = \sup\{t > 0 \mid u_n \geq t v_n + (t - 1)e\}, \quad n = 1, 2, \ldots$$

Thus we have

$$(3.12) \qquad t_n \in (0, 1), \quad u_n \geq t_n v_n + (t_n - 1)e, \quad n = 1, 2, \ldots$$

It follows from (3.10) and (3.12) that

$$u_{n+1} \geq u_n \geq t_n v_n + (t_n - 1)e \geq t_n v_{n+1} + (t_n - 1)e, \quad n = 0, 1, \ldots$$

By (3.11), we get $t_{n+1} \geq t_n$. Therefore, t_n is increasing with $t_n \subset (0, 1)$, i.e. $0 < t_0 \leq t_1 \leq \ldots \leq t_n \leq \ldots < 1$. Suppose $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. In fact, if $t^* \in (0, 1)$, we consider the following two cases:

Case 1. there exists an integer N such that $t_N = t^*$. As t_n is an increasing sequence, so we have $t_n = t^*$ as $n \geq N$. It follows from $u_0, v_0 \in P_{h,e}$ and (3.10) that $u_n, v_n \in P_{h,e}$. Also, (3.12) implies that

$$(3.13) \qquad v_n \leq \frac{1}{t_n}u_n + \left(\frac{1}{t_n} - 1\right)e, \quad t_n \in (0, 1).$$

Then, for $n \geq N$, by monotone property of T, (3.1), (3.9), (3.12) and (3.13), we have

$$
\begin{aligned}
u_{n+1} = T(u_n, v_n) &\geq T\left(t_n v_n + (t_n - 1)e, \frac{1}{t_n} u_n + \left(\frac{1}{t_n} - 1\right)e\right) \\
&= T\left(t^* v_n + (t^* - 1)e, \frac{1}{t^*} u_n + \left(\frac{1}{t^*} - 1\right)e\right) \\
&\geq \varphi(t^*) T(v_n, u_n) + (\varphi(t^*) - 1)e = \varphi(t^*) v_{n+1} + (\varphi(t^*) - 1)e.
\end{aligned}
$$

According to (3.11), we can easily obtain $t_{n+1} \geq \varphi(t^*)$ as $n \geq N$. So, $t^* = t_{n+1} \geq \varphi(t^*) > t^*$ as $n \to \infty$, which is a contradiction.

Case 2. $t_n < t^*$ for all integers n. By monotone property of T, (3.1), (3.9), (3.12) and (3.13), we get

$$
\begin{aligned}
u_{n+1} = T(u_n, v_n) &\geq T\left(t_n v_n + (t_n - 1)e, \frac{1}{t_n} u_n + \left(\frac{1}{t_n} - 1\right)e\right) \\
&= T\left(\frac{t_n}{t^*}(t^* v_n + (t^* - 1)e) + \left(\frac{t_n}{t^*} - 1\right)e, \frac{t^*}{t_n}\left(\frac{1}{t^*} u_n + \left(\frac{1}{t^*} - 1\right)e\right)\right. \\
&\quad \left. + \left(\frac{t^*}{t_n} - 1\right)e\right) \\
&\geq \varphi\left(\frac{t_n}{t^*}\right) T\left(t^* v_n + (t^* - 1)e, \frac{1}{t^*} u_n + \left(\frac{1}{t^*} - 1\right)e\right) + \left(\varphi\left(\frac{t_n}{t^*}\right) - 1\right)e \\
&\geq \varphi\left(\frac{t_n}{t^*}\right)[\varphi(t^*) T(v_n, u_n) + (\varphi(t^*) - 1)e] + \left(\varphi\left(\frac{t_n}{t^*}\right) - 1\right)e \\
&= \varphi\left(\frac{t_n}{t^*}\right)\varphi(t^*) v_{n+1} + \left(\varphi\left(\frac{t_n}{t^*}\right)\varphi(t^*) - 1\right)e.
\end{aligned}
$$

By application of (3.11) again, we know that $t_{n+1} \geq \varphi(t_n/t^*)\varphi(t^*) > (t_n/t^*)\varphi(t^*)$ for all integers n. Taking $n \to \infty$, we obtain $t^* \geq \varphi(t^*) > t^*$, which is impossible. Hence $t^* = 1$. That is $\lim_{n\to\infty} t_n = 1$. In what follows, we prove u_n, v_n are two Cauchy sequences. For any natural number p, it follows from (3.10) and (3.12) that, for $n = 0, 1, \ldots$,

$$
\begin{aligned}
\theta \leq u_{n+p} - u_n &\leq v_n - u_n \leq v_n - t_n v_n - (t_n - 1)e \\
&= (1 - t_n)v_n + (1 - t_n)e \leq (1 - t_n)v_0 + (1 - t_n)e.
\end{aligned}
$$

Similarly, for $n = 0, 1, \ldots$,

$$
\theta \leq v_n - v_{n+p} \leq v_n - u_n \leq (1 - t_n)v_0 + (1 - t_n)e.
$$

From the normality of cone P, we have

$$
\|u_{n+p} - u_n\| \leq M(1 - t_n)\|v_0 + e\| \to 0, \quad \|v_n - v_{n+p}\| \leq \overline{M}(1 - t_n)\|v_0 + e\| \to 0,
$$

as $n \to \infty$, where M, $\overline{M}$ are the normality constants, which implies $\{u_n\}$, $\{v_n\}$ are Cauchy sequences. Since E is complete, there exist u^*, $v^* \in [u_0, v_0]$ satisfy

$$u_n \to u^*, \quad v_n \to v^* \quad \text{as } n \to \infty.$$

By (3.10), for all $n \in \mathbb{N}$, we have $u_n \leq u^* \leq v^* \leq v_n$ with $u^*, v^* \in P_{h,e}$. Clearly,

$$\theta \leq v^* - u^* \leq v_n - u_n \leq v_n - t_n v_n - (t_n - 1)e = (1 - t_n)(v_0 + e).$$

By using the normality of cone P again, we have

$$\|v^* - u^*\| \leq \overline{N}(1 - t_n)\|v_0 + e\| \to 0 \quad (n \to \infty),$$

where $\overline{N}$ is the normality constant. Thus $u^* = v^*$. Set $x^* := u^* = v^*$. Then we get $u_{n+1} = T(u_n, v_n) \leq T(x^*, x^*) \leq T(v_n, u_n) = v_{n+1}$. Letting $n \to \infty$, we have $x^* = T(x^*, x^*)$. That is x^* is a fixed point of T in $P_{h,e}$.

Further, let us illustrate the uniqueness of the fixed point. In fact, suppose T has another fixed point $\overline{x} \in P_{h,e}$. According to Lemma 2.6, there exists $s_0 \in (0, 1)$ such that

$$s_0 \overline{x} + (s_0 - 1)e \leq x^* \leq \frac{1}{s_0} \overline{x} + \left(\frac{1}{s_0} - 1\right)e.$$

Set

$$(3.14) \qquad \overline{t} = \sup\left\{t > 0 \,\middle|\, t\overline{x} + (t - 1)e \leq x^* \leq \frac{1}{t}\overline{x} + \left(\frac{1}{t} - 1\right)e\right\}.$$

Then we can easily obtain that $0 < \overline{t} \leq 1$ and

$$(3.15) \qquad \overline{t}\overline{x} + (\overline{t} - 1)e \leq x^* \leq \frac{1}{\overline{t}}\overline{x} + \left(\frac{1}{\overline{t}} - 1\right)e.$$

Next we prove $\overline{t} = 1$. Suppose that $0 < \overline{t} < 1$, it follows from (3.1), (3.15) and the mixed monotone property of T that

$$(3.16) \qquad x^* = T(x^*, x^*) \geq T\left(\overline{t}\overline{x} + (\overline{t} - 1)e, \frac{1}{\overline{t}}\overline{x} + \left(\frac{1}{\overline{t}} - 1\right)e\right)$$

$$\geq \varphi(\overline{t})\, T(\overline{x}, \overline{x}) + (\varphi(\overline{t}) - 1)e = \varphi(\overline{t})\overline{x} + (\varphi(\overline{t}) - 1)e.$$

Also, by (3.7) and (3.15), we get

$$(3.17) \qquad x^* = T(x^*, x^*) \leq T\left(\frac{1}{\overline{t}}\overline{x} + \left(\frac{1}{\overline{t}} - 1\right)e, \overline{t}\overline{x} + (\overline{t} - 1)e\right)$$

$$\leq \frac{1}{\varphi(\overline{t})}\, T(\overline{x}, \overline{x}) + \left(\frac{1}{\varphi(\overline{t})} - 1\right)e = \frac{1}{\varphi(\overline{t})}\overline{x} + \left(\frac{1}{\varphi(\overline{t})} - 1\right)e.$$

It follows from (3.14)–(3.17) that $\overline{t} \geq \varphi(\overline{t}) > \overline{t}$, which is a contradiction. So $\overline{t} = 1$, which gives $x^* = \overline{x}$. Hence, the fixed point of T is unique in $P_{h,e}$.

Step 3. Defining successively the sequences

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots$$

for any initial points $x_0, y_0 \in P_{h,e}$, we are to prove that $x_n \to x^*$, $y_n \to x^*$ as $n \to \infty$. Since $x^*, x_0, y_0 \in P_{h,e}$, there exist $q_0, q_1 \in (0,1)$ such that

$$q_0 x^* + (q_0 - 1)e \le x_0 \le \frac{1}{q_0} x^* + \left(\frac{1}{q_0} - 1 \right)e,$$

$$q_1 x^* + (q_1 - 1)e \le y_0 \le \frac{1}{q_1} x^* + \left(\frac{1}{q_1} - 1 \right)e.$$

Let $q^* = \min\{q_0, q_1\}$, then $q^* \in (0,1)$, and

$$(3.18) \qquad q^* x^* + (q^* - 1)e \le x_0, \quad y_0 \le \frac{1}{q^*} x^* + \left(\frac{1}{q^*} - 1 \right)e.$$

Choosing a sufficiently large positive integer m such that

$$\left(\frac{\varphi(q^*)}{q^*} \right)^m \ge \frac{1}{q^*}.$$

Put

$$(3.19) \qquad u_0' = q^{*m} x^* + (q^{*m} - 1)e, \qquad v_0' = \frac{1}{q^{*m}} x^* + \left(\frac{1}{q^{*m}} - 1 \right)e.$$

Obviously, it follows from Lemma 2.6, (3.18) and (3.19) that

$$(3.20) \qquad u_0', v_0' \in P_{h,e} \quad \text{and} \quad u_0' \le x_0, \quad y_0 \le v_0'.$$

Set

$$(3.21) \qquad u_n' = T(u_{n-1}', v_{n-1}'), \quad v_n' = T(v_{n-1}', u_{n-1}'), \quad n = 1, 2, \dots$$

Similar to the previous argument, it follows that there exist $y^* \in P_{h,e}$ such that $T(y^*, y^*) = y^*$, and

$$\lim_{n \to \infty} u_n' = \lim_{n \to \infty} v_n' = y^*.$$

By the uniqueness of fixed point of operator T in $P_{h,e}$, we have $x^* = y^*$. From (3.20) and (3.21), we induce that $u_n' \le x_n$, $y_n \le v_n'$, for $n = 1, 2, \dots$ Thus, we have

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = x^*. \qquad \square$$

REMARK 3.2. If we assume that $T \colon P_{h,e} \times P_{h,e} \to P_{h,e}$, then $T(h,h) \in P_{h,e}$ is automatically satisfied. So we can get the following conclusion.

COROLLARY 3.3. *Let P be a normal cone. The mixed monotone operator $T \colon P_{h,e} \times P_{h,e} \to P_{h,e}$ satisfies (L_2), Then the conclusions (a)–(c) in Theorem 3.1 hold true.*

THEOREM 3.4. *Let P be a normal cone of a real Banach space. Assume that $T \colon P_{h,e} \times P_{h,e} \to E$ is a mixed monotone operator, satisfies (L_1) and*

(L_3) *for any $u, v \in P_{h,e}$, all $\lambda \in (0,1)$, there exists $0 < \beta = \beta(\lambda) < 1$ such that*

$$(3.22) \quad T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda}v + \left(\frac{1}{\lambda} - 1\right)e\right) \geq \lambda^{\beta(\lambda)}T(u, v) + (\lambda^{\beta(\lambda)} - 1)e.$$

Then the conclusions (a)–(c) *in Theorem* 3.1 *still hold true.*

PROOF. Taking into account that $\lambda \in (0,1)$, $0 < \beta = \beta(\lambda) < 1$, we immediately conclude that $\lambda^{\beta(\lambda)} > \lambda$. So, if we set $\varphi(\lambda) = \lambda^{\beta(\lambda)}$ in Theorem 3.1, we easily obtain that results (a)–(c) hold. Here we omit the proof. $\square$

COROLLARY 3.5. *Let P be a normal cone of a real Banach space. The mixed monotone operator $T\colon P_{h,e} \times P_{h,e} \to P_{h,e}$ satisfies* (L_3). *Then the conclusions* (a)–(c) *in Theorem* 3.1 *hold true.*

THEOREM 3.6. *Let P be a normal cone of a real Banach space E. Assume that $T\colon P_{h,e} \times P_{h,e} \to E$ is a mixed monotone operator, satisfies* (L_1) *and* (L_4) *for all $u, v \in P_{h,e}$, $\lambda \in (0,1)$, there exists $0 < \beta = \beta(\lambda, u, v) < 1$ such that*

$$(3.23) \quad T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda}v + \left(\frac{1}{\lambda} - 1\right)e\right) \geq \lambda^{\beta(\lambda,u,v)}T(u, v) + (\lambda^{\beta(\lambda,u,v)} - 1)e.$$

where $\beta(\lambda, u, v)$ may be multitudinous for the fixed λ, u, v. Let $\underline{\beta}(\lambda, u, v) = \inf \beta(\lambda, u, v)$. Suppose that

(a) *$\underline{\beta}(\lambda, u, v)$ is nondecreasing in $u \in P_{h,e}$ for fixed $\lambda \in (0,1)$, $v \in P_{h,e}$;*
(b) *$\underline{\beta}(\lambda, u, v)$ is nonincreasing in $v \in P_{h,e}$ for fixed $\lambda \in (0,1)$, $u \in P_{h,e}$.*

Then the conclusions (a)–(c) *in Theorem* 3.1 *still hold true.*

PROOF. Firstly, we will prove the conclusion (a), i.e. there exist $u_0, v_0 \in P_{h,e}$ such that

$$u_0 < v_0, \qquad u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0.$$

According to $\beta(\lambda, u, v)$ is a bounded set for the fixed λ, u, v, so there exists $0 < \underline{\beta}(\lambda, u, v) = \inf \beta(\lambda, u, v) < 1$. Combining with (3.23), for any $\lambda \in (0,1)$, $u, v \in P_{h,e}$, we have

$$(3.24) \quad T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda}v + \left(\frac{1}{\lambda} - 1\right)e\right) \geq \lambda^{\underline{\beta}(\lambda,u,v)}T(u, v) + (\lambda^{\underline{\beta}(\lambda,u,v)} - 1)e.$$

Formula (3.24) can be written as

$$(3.25) \quad T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda}v + \left(\frac{1}{\lambda} - 1\right)e\right)$$
$$\geq \lambda(1 + \eta(\lambda, u, v))T(u, v) + [\lambda(1 + \eta(\lambda, u, v)) - 1]e.$$

if we set $\eta(\lambda, u, v) = \lambda^{\underline{\beta}(\lambda,u,v)-1} - 1$, for all $\lambda \in (0,1)$, $u, v \in P_{h,e}$. Obviously, $\eta(\lambda, u, v) > 0$. Let

$$(3.26) \quad \phi(\lambda, u, v) = \lambda(1 + \eta(\lambda, u, v)).$$

We easily see that $\phi(\lambda, u, v)$ is nonincreasing in u for fixed λ, v, nondecreasing in v for fixed λ, u, and

$$(3.27) \qquad \phi(\lambda, u, v) > \lambda, \quad \text{for all } \lambda \in (0,1), \ u, v \in P_{h,e}.$$

From (3.25) and (3.26), we have

$$(3.28) \quad T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda} v + \left(\frac{1}{\lambda} - 1\right)e\right)$$
$$\geq \phi(\lambda, u, v) T(u,v) + [\phi(\lambda, u, v) - 1]e.$$

Together with the fact that $T(h, h) \in P_{h,e}$, by Lemma 2.6, we choose a small number $t_0 \in (0, 1)$ such that

$$(3.29) \ \ t_0 h + (t_0 - 1)e \leq T(h, h) \leq \frac{1}{t_0^{1 - \underline{\beta}(t_0, t_0^{-1}h, t_0 h)}} h + \left(\frac{1}{t_0^{1 - \underline{\beta}(t_0, t_0^{-1}h, t_0 h)}} - 1\right)e.$$

Since $\phi(t_0, h, h) > t_0$, we can select some positive integer k such that

$$(3.30) \qquad \left(\frac{\phi(t_0, h, h)}{t_0}\right)^k \geq \frac{1}{t_0}.$$

Let (3.4)–(3.6) hold, then we have $u_0, v_0 \in P_{h,e}$, and $u_0 < t_0^{2k} v_0 < v_0$. From (3.4)–(3.6), (3.27)–(3.30), we have

$$T(u_0, v_0) = T(a_k, b_k) = T\left(t_0 a_{k-1} + (t_0 - 1)e, \frac{1}{t_0} b_{k-1} + \left(\frac{1}{t_0} - 1\right)e\right)$$
$$\geq \phi(t_0, a_{k-1}, b_{k-1}) T(a_{k-1}, b_{k-1}) + [\phi(t_0, a_{k-1}, b_{k-1}) - 1]e$$
$$= \phi(t_0, a_{k-1}, b_{k-1}) T\left(t_0 a_{k-2} + (t_0 - 1)e, \frac{1}{t_0} b_{k-2} + \left(\frac{1}{t_0} - 1\right)e\right)$$
$$+ [\phi(t_0, a_{k-1}, b_{k-1}) - 1]e$$
$$\geq \phi(t_0, a_{k-1}, b_{k-1}) \phi(t_0, a_{k-2}, b_{k-2}) T(a_{k-2}, b_{k-2})$$
$$+ [\phi(t_0, a_{k-1}, b_{k-1}) \phi(t_0, a_{k-2}, b_{k-2}) - 1]e$$
$$\geq \ldots \geq \phi(t_0, a_{k-1}, b_{k-1}) \ldots \phi(t_0, a_1, b_1) T(a_1, b_1)$$
$$+ [\phi(t_0, a_{k-1}, b_{k-1}) \ldots \phi(t_0, a_1, b_1) - 1]e$$
$$\geq (\phi(t_0, h, h))^{k-1} T(a_1, b_1) + [(\phi(t_0, h, h))^{k-1} - 1]e$$
$$= (\phi(t_0, h, h))^{k-1} T\left(t_0 a_0 + (t_0 - 1)e, \frac{1}{t_0} b_0 + \left(\frac{1}{t_0} - 1\right)e\right)$$
$$+ [(\phi(t_0, h, h))^{k-1} - 1]e$$
$$\geq (\phi(t_0, h, h))^k T(h, h) + [(\phi(t_0, h, h))^k - 1]e$$
$$\geq t_0^{k-1}[t_0 h + (t_0 - 1)e] + (t_0^{k-1} - 1)e = t_0^k h + (t_0^k - 1)e = u_0.$$

The (3.28) implies that

$$(3.31) \quad T(u,v) \le \frac{1}{\phi(\lambda, u, v)} T\left(\lambda u + (\lambda - 1)e, \frac{1}{\lambda}v + \left(\frac{1}{\lambda} - 1\right)e\right)$$
$$+ \left(\frac{1}{\phi(\lambda, u, v)} - 1\right)e.$$

By (3.4)–(3.6), (3.27), (3.29), (3.30), (3.31) and conditions (a), (b), we obtain that

$$T(v_0, u_0) = T(b_k, a_k) \le \frac{1}{\phi(t_0, b_k, a_k)} T\left(t_0 b_k + (t_0 - 1)e, \frac{1}{t_0}a_k + \left(\frac{1}{t_0} - 1\right)e\right)$$
$$+ \left(\frac{1}{\phi(t_0, b_k, a_k)} - 1\right)e$$
$$\le \frac{1}{\phi(t_0, b_k, a_k)} T(b_{k-1}, a_{k-1}) + \left(\frac{1}{\phi(t_0, b_k, a_k)} - 1\right)e$$
$$\le \frac{1}{\phi(t_0, b_k, a_k)} \frac{1}{\phi(t_0, b_{k-1}, a_{k-1})} T\left(b_{k-2}, a_{k-2}\right)$$
$$+ \left[\frac{1}{\phi(t_0, b_k, a_k)} \frac{1}{\phi(t_0, b_{k-1}, a_{k-1})} - 1\right]e$$
$$\le \frac{1}{\phi(t_0, b_k, a_k)} \cdot \ldots \cdot \frac{1}{\phi(t_0, b_2, a_2)} T(b_1, a_1)$$
$$+ \left[\frac{1}{\phi(t_0, b_k, a_k)} \cdot \ldots \cdot \frac{1}{\phi(t_0, b_2, a_2)} - 1\right]e$$
$$\le \frac{1}{(\phi(t_0, b_k, a_k))^{k-1}} T(b_1, a_1) + \left[\frac{1}{(\phi(t_0, b_k, a_k))^{k-1}} - 1\right]e$$
$$\le \frac{1}{(\phi(t_0, b_k, a_k))^{k-1}} \frac{T(h, h)}{\phi(t_0, b_1, a_1)} + \left[\frac{1}{(\phi(t_0, b_k, a_k))^{k-1}\phi(t_0, b_1, a_1)} - 1\right]e$$
$$\le \frac{1}{(\phi(t_0, b_k, a_k))^{k-1}} \frac{1}{t_0^{\beta(t_0, t_0^{-1}h, t_0 h)}}$$
$$\times \left[\frac{1}{t_0^{1-\beta(t_0, t_0^{-1}h, t_0 h)}} h + \left(\frac{1}{t_0^{1-\beta(t_0, t_0^{-1}h, t_0 h)}} - 1\right)e\right]$$
$$+ \left[\frac{1}{(\phi(t_0, b_k, a_k))^{k-1}t_0^{\beta(t_0, t_0^{-1}h, t_0 h)}} - 1\right]e$$
$$\le \frac{1}{t_0^{k-1}} \frac{1}{t_0} h + \left(\frac{1}{t_0^{k-1}} \frac{1}{t_0} - 1\right)e = \frac{1}{t_0^k} h + \left(\frac{1}{t_0^k} - 1\right)e = v_0.$$

It follows from mixed monotone property of operator T that

$$u_0 \le T(u_0, v_0) \le T(v_0, u_0) \le v_0.$$

Secondly, we will show x^* is a fixed point of T. Denote the sequences in (3.9), similar to the step two in the proof of Theorem 3.1, we can easily obtain

$$u_0 \le u_1 \le \ldots \le u_n \le \ldots \le v_n \le \ldots \le v_1 \le v_0, \quad n = 1, 2, \ldots$$

Let $t_n = \sup\{t > 0 \mid u_n \geq tv_n + (t-1)e\}$, $n = 1, 2, \ldots$ Then $u_n \geq t_n v_n + (t_n - 1)e$. We can easily obtain that t_n is increasing with $t_n \subset (0, 1)$, and there exists t^* such that $\lim\limits_{n \to \infty} t_n = t^*$. Next we prove $t^* = 1$. In fact, if $t^* \in (0, 1)$, consider the two cases:

Case 1. There exists an integer N such that $t_N = t^*$. In this case, we have $t_n = t^*$ and $u_n \geq t^* v_n + (t^* - 1)e$ as $n \geq N$. So

$$
\begin{aligned}
u_{n+1} = T(u_n, v_n) &\geq T\left(t^* v_n + (t^* - 1)e, \frac{1}{t^*} u_n + \left(\frac{1}{t^*} - 1\right)e\right) \\
&\geq (t^*)^{\underline{\beta}(t^*, v_n, u_n)} T(v_n, u_n) + [(t^*)^{\underline{\beta}(t^*, v_n, u_n)} - 1]e \\
&= (t^*)^{\underline{\beta}(t^*, v_n, u_n)} v_{n+1} + [(t^*)^{\underline{\beta}(t^*, v_n, u_n)} - 1]e, \quad n \geq N.
\end{aligned}
$$

Hence, by the definition of t_n we get $t_{n+1} \geq (t^*)^{\underline{\beta}(t^*, v_n, u_n)} > t^*, n \geq N$. This is a contradiction.

Case 2. $t_n < t^*$ for all integers n. In this case, we have

$$
\begin{aligned}
u_{n+1} = T(u_n, v_n) &\geq T\left(t_n v_n + (t_n - 1)e, \frac{1}{t_n} u_n + \left(\frac{1}{t_n} - 1\right)e\right) \\
&= T\left(\frac{t_n}{t^*}(t^* v_n + (t^* - 1)e) + \left(\frac{t_n}{t^*} - 1\right)e, \frac{t^*}{t_n}\left(\frac{1}{t^*} u_n + \left(\frac{1}{t^*} - 1\right)e\right)\right. \\
&\qquad \left. + \left(\frac{t^*}{t_n} - 1\right)e\right) \\
&\geq \left(\frac{t_n}{t^*}\right)^{\underline{\beta}(t_n/t^*, t^* v_n + (t^* - 1)e, u_n/t^* + (1/t^* - 1)e)} \\
&\qquad \times T\left(t^* v_n + (t^* - 1)e, \frac{1}{t^*} u_n + \left(\frac{1}{t^*} - 1\right)e\right) \\
&\qquad + \left[\left(\frac{t_n}{t^*}\right)^{\underline{\beta}(t_n/t^*, t^* v_n + (t^* - 1)e, u_n/t^* + (1/t^* - 1)e)} - 1\right]e \\
&\geq \left(\frac{t_n}{t^*}\right)^{\underline{\beta}(t_n/t^*, t^* v_n + (t^* - 1)e, u_n/t^* + (1/t^* - 1)e)} (t^*)^{\underline{\beta}(t^*, v_n, u_n)} T(v_n, u_n) \\
&\qquad + \left[\left(\frac{t_n}{t^*}\right)^{\underline{\beta}(t_n/t^*, t^* v_n + (t^* - 1)e, u_n/t^* + (1/t^* - 1)e)} (t^*)^{\underline{\beta}(t^*, v_n, u_n)} - 1\right]e \\
&\geq \frac{t_n}{t^*} \times t^* \left[\frac{(t^*)^{\underline{\beta}(t^*, v_n, u_n)}}{t^*}\right] v_{n+1} + \left[\frac{t_n}{t^*} \times t^* \left(\frac{(t^*)^{\underline{\beta}(t^*, v_n, u_n)}}{t^*}\right) - 1\right]e \\
&= t_n (t^*)^{\underline{\beta}(t^*, v_n, u_n) - 1} v_{n+1} + \left[t_n (t^*)^{\underline{\beta}(t^*, v_n, u_n) - 1} - 1\right]e.
\end{aligned}
$$

Applying the definition of t_n again and the monotone property of $\underline{\beta}$, we have

$$
t_{n+1} \geq t_n (t^*)^{\underline{\beta}(t^*, v_n, u_n) - 1} \geq t_n (t^*)^{\underline{\beta}(t^*, v_0, u_0) - 1}.
$$

Taking $n \to \infty$, we obtain $t^* \geq t^*(t^*)^{\underline{\beta}(t^*, v_0, u_0)-1} > t^*$, which is impossible. Hence $t^* = 1$. That is $\lim\limits_{n\to\infty} t_n = 1$. Furthermore, according to proof of Theorem 3.1, $\{u_n\}$, $\{v_n\}$ are Cauchy sequences, and $u_{n+1} = T(u_n, v_n) \leq T(x^*, x^*) \leq T(v_n, u_n) = v_{n+1}$. Letting $n \to \infty$, we have $x^* = T(x^*, x^*)$. So x^* is the fixed point of operator T.

Next, we show the uniqueness of fixed point. Suppose $\overline{x} \in P_{h,e}$ is another fixed point of operator T. Let

$$\overline{t} = \sup\left\{t > 0 \ \Big| \ t\overline{x} + (t-1)e \leq x^* \leq \frac{1}{t}\overline{x} + \left(\frac{1}{t}-1\right)e\right\}.$$

Obviously, $0 < \overline{t} \leq 1$ and

$$\overline{t}\overline{x} + (\overline{t}-1)e \leq x^* \leq \frac{1}{\overline{t}}\overline{x} + \left(\frac{1}{\overline{t}}-1\right)e.$$

When $0 < \overline{t} < 1$, then

$$(3.32) \qquad x^* = T(x^*, x^*) \geq T\left[\overline{t}\overline{x} + (\overline{t}-1)e, \frac{1}{\overline{t}}\overline{x} + \left(\frac{1}{\overline{t}}-1\right)e\right]$$

$$\geq \phi(\overline{t}, \overline{x}, \overline{x})T(\overline{x}, \overline{x}) + [\phi(\overline{t}, \overline{x}, \overline{x}) - 1]e$$

$$= \phi(\overline{t}, \overline{x}, \overline{x})\overline{x} + [\phi(\overline{t}, \overline{x}, \overline{x}) - 1]e.$$

Also we get

$$(3.33) \qquad x^* = T(x^*, x^*) \leq T\left(\frac{1}{\overline{t}}\overline{x} + \left(\frac{1}{\overline{t}}-1\right)e, \overline{t}\overline{x} + (\overline{t}-1)e\right)$$

$$\leq \frac{1}{\phi(\overline{t}, (1/\overline{t})\overline{x} + (1/\overline{t}-1)e, \overline{t}\overline{x} + (\overline{t}-1)e)} T(\overline{x}, \overline{x})$$

$$+ \left(\frac{1}{\phi(\overline{t}, (1/\overline{t})\overline{x} + (1/\overline{t}-1)e, \overline{t}\overline{x} + (\overline{t}-1)e)} - 1\right)e$$

$$= \frac{1}{\phi(\overline{t}, (1/\overline{t})\overline{x} + (1/\overline{t}-1)e, \overline{t}\overline{x} + (\overline{t}-1)e)} \overline{x}$$

$$+ \left(\frac{1}{\phi(\overline{t}, (1/\overline{t})\overline{x} + (1/\overline{t}-1)e, \overline{t}\overline{x} + (\overline{t}-1)e)} - 1\right)e.$$

By the monotone property of ϕ, we have

$$(3.34) \qquad \phi\left(\overline{t}, \frac{1}{\overline{t}}\overline{x} + \left(\frac{1}{\overline{t}}-1\right)e, \overline{t}\overline{x} + (\overline{t}-1)e\right) \leq \phi(\overline{t}, \overline{x}, \overline{x}).$$

Hence, by (3.32)–(3.34), we deduce that

$$\phi\left(\overline{t}, \frac{1}{\overline{t}}\overline{x} + \left(\frac{1}{\overline{t}}-1\right)e, \overline{t}\overline{x} + (\overline{t}-1)e\right)\overline{x}$$

$$+ \left[\phi\left(\overline{t}, \frac{1}{\overline{t}}\overline{x} + \left(\frac{1}{\overline{t}}-1\right)e, \overline{t}\overline{x} + (\overline{t}-1)e\right) - 1\right]e$$

$$\leq x^* \leq \frac{1}{\phi\big(\overline{t}, (1/\overline{t})\overline{x} + (1/\overline{t} - 1)e, \overline{t}\,\overline{x} + (\overline{t} - 1)e\big)}\,\overline{x}$$
$$+ \left(\frac{1}{\phi\big(\overline{t}, (1/\overline{t})\overline{x} + (1/\overline{t} - 1)e, \overline{t}\,\overline{x} + (\overline{t} - 1)e\big)} - 1\right)e.$$

It follows from the definition of $\overline{t}$ that

$$\overline{t} \geq \phi\left(\overline{t}, \frac{1}{\overline{t}}\,\overline{x} + \left(\frac{1}{\overline{t}} - 1\right)e, \overline{t}\,\overline{x} + (\overline{t} - 1)e\right) > \overline{t}.$$

This is a contradiction. So $\overline{t} = 1$, which implies $x^* = \overline{x}$. As a result, the fixed point of T is unique in $P_{h,e}$.

Lastly, similar to the proof of step three in Theorem 3.1, we can obtain for any initial values x_0, $y_0 \in P_{h,e}$, constructing successively the sequences

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$. $\qquad\square$

COROLLARY 3.7. *Let P be a normal cone of a real Banach space. The mixed monotone operator $T\colon P_{h,e} \times P_{h,e} \to P_{h,e}$ satisfies (L_4). Let $\underline{\beta}(\lambda, u, v) = \inf \beta(\lambda, u, v)$. Suppose that $\underline{\beta}(\lambda, u, v)$ is nondecreasing in $u \in P_{h,e}$ for fixed $\lambda \in (0,1)$, $v \in P_{h,e}$ and nonincreasing in $v \in P_{h,e}$ for fixed $\lambda \in (0,1)$, $u \in P_{h,e}$. Then the conclusions (a)–(c) in Theorem 3.1 hold.*

REMARK 3.8. In the Theorem 3.1, Theorem 3.4 and Theorem 3.6, suppose the operator $T\colon P_{h,e} \times P_{h,e} \to E$ is a mixed monotone operator, and satisfies the conditions (L_1) and (L_2) or (L_3) or (L_4), then the operator $T\colon P_{h,e} \times P_{h,e} \to P_{h,e}$.

PROOF. We will take the following case as an example, and other cases are similar to this proof, whose proof is omitted here.

The operator $T\colon P_{h,e} \times P_{h,e} \to E$ satisfies the conditions (L_1) and (L_2). Since $T(h,h) \in P_{h,e}$, then $T(h,h) + e \in P_h$, there exists $t_0 \in (0,1)$ such that

$$t_0 h \leq T(h,h) + e \leq t_0^{-1} h.$$

By $h \in E$, $e \in P$ with $\theta \leq e \leq h$, $h \neq \theta$, we easily obtain $h \in P_{h,e}$. For any $(x,y) \in P_{h,e} \times P_{h,e}$, by Lemma 2.6, there exist $\mu_1, \mu_2 \in (0,1)$ such that

$$\mu_1 h + (\mu_1 - 1)e \leq x \leq \frac{1}{\mu_1}\,h + \left(\frac{1}{\mu_1} - 1\right)e,$$
$$\mu_2 h + (\mu_2 - 1)e \leq y \leq \frac{1}{\mu_2}\,h + \left(\frac{1}{\mu_2} - 1\right)e.$$

Let $\mu = \min\{\mu_1, \mu_2\}$, then $\mu \in (0,1)$. From (3.1), (3.7) and the mixed monotone property of operator T, we obtain

$$T(x,y) \geq T\left(\mu_1 h + (\mu_1 - 1)e, \frac{1}{\mu_2}h + \left(\frac{1}{\mu_2} - 1\right)e\right)$$

$$\geq T\left(\mu h + (\mu - 1)e, \frac{1}{\mu}h + \left(\frac{1}{\mu} - 1\right)e\right)$$

$$\geq \varphi(\mu)T(h,h) + (\varphi(\mu) - 1)e = \varphi(\mu)(T(h,h) + e) - e \geq \varphi(\mu)t_0 h - e,$$

$$T(x,y) \leq \left(\frac{1}{\mu_1}h + \left(\frac{1}{\mu_1} - 1\right)e, \mu_2 h + (\mu_2 - 1)e\right)$$

$$\leq \left(\frac{1}{\mu}h + \left(\frac{1}{\mu} - 1\right)e, \mu h + (\mu - 1)e\right)$$

$$\leq \frac{1}{\varphi(\mu)}T(h,h) + \left(\frac{1}{\varphi(\mu)} - 1\right)e = \frac{1}{\varphi(\mu)}(T(h,h) + e) - e \leq \frac{1}{\varphi(\mu)t_0}h - e.$$

Hence $\varphi(\mu)t_0 h \leq T(x,y) + e \leq h/(\varphi(\mu)t_0)$, which means $T(x,y) + e \in P_h$. Then $T(x,y) \in P_{h,e}$. So we have $T: P_{h,e} \times P_{h,e} \to P_{h,e}$. $\qquad\square$

REMARK 3.9. If we set $e = \theta$, then $P_{h,e} = P_h$, which implies $P_h \subset P_{h,e}$. In this case, we can deduce the following Corollaries 3.10–3.13 as special cases of the Theorems 3.1–3.6 and Corollaries 3.3–3.7. Here we should point that these special cases of our main results not only show the fixed point of mixed monotone operator T is unique, but also guarantee the fixed point is positive.

COROLLARY 3.10 (see Theorem 2.1 of [27]). *Suppose that P is a normal cone of E, and the mixed monotone operator $T: P \times P \to P$ satisfies the following conditions*:

(A_1) *there exists $h \in P$ with $h \neq \theta$ such that $T(h,h) \in P_h$*;
(A_2) *for any $u, v \in P$ and $\lambda \in (0,1)$, there exists $\varphi(\lambda) \in (\lambda, 1]$ such that $T(\lambda u, \lambda^{-1}v) \geq \varphi(\lambda)A(u,v)$*.

Then:

(a) *$T: P_h \times P_h \to P_h$*;
(b) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0$, $u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0$*;
(c) *the operator T has a unique fixed point x^* in P_h*;
(d) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

COROLLARY 3.11 (see Theorem 2.1 of [16]). *Let P be a normal cone of a real Banach space E, $h > \theta$. $T: P_h \times P_h \to P_h$ is a mixed monotone operator.*

Assume that for all $0 < \lambda < 1$ and $u, v \in P_h$, there exists $0 < \beta(\lambda) < 1$ such that

$$T\left(\lambda u, \frac{1}{\lambda} v\right) \geq \lambda^{\beta(\lambda)} T(u, v).$$

Then T has a exactly one fixed point x^* in P_h. Moreover, for any initial point $x_0, y_0 \in P_h$, constructing successively the sequences $x_n = T(x_{n-1}, y_{n-1})$, $y_n = T(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

COROLLARY 3.12 (see Theorem 3.2 of [16]). *Let P be a normal cone of a real Banach space E, $h > \theta$. $T: P_h \times P_h \to P_h$ is a mixed monotone operator. Assume that, for all $0 < \lambda < 1$ and $u, v \in P_h$, there exists $0 < \beta(\lambda, u, v) < 1$ such that*

$$T\left(\lambda u, \frac{1}{\lambda} v\right) \geq t^{\beta(\lambda, u, v)} T(u, v).$$

Let $\underline{\beta}(\lambda, u, v) = \inf \beta(\lambda, u, v)$. Suppose $\underline{\beta}(\lambda, u, v)$ is nondecreasing in u and nonincreasing in v. Then there exists a unique fixed point $x^* \in P_h$ such that $T(x^*, x^*) = x^*$. Moreover, for any initial point $x_0, y_0 \in P_h$, constructing successively the sequences $x_n = T(x_{n-1}, y_{n-1})$, $y_n = T(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

COROLLARY 3.13 (see Theorem 2.1 of [10]). *Let P be a normal, solid cone of E. Suppose that $T: P_h \times P_h \to P_h$ is a mixed monotone operator and there exists a constant $\beta \in [0, 1)$, such that*

$$T(\lambda x, \lambda^{-1} y) \geq \lambda^{\beta} T(x, y), \quad \text{for all } x, y \in P_h, \ \lambda \in (0, 1),$$

then the operator T has a unique fixed point $x^* \in P_h$. Moreover, for any $(x_0, y_0) \in P_h \times P_h$, $x_n = T(x_{n-1}, y_{n-1})$, $y_n = T(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, satisfy $x_n \to x^*$, $y_n \to x^*$ as $n \to \infty$.

4. Application

In the sequel, we take into account the existence and uniqueness of solutions for the problem (1.1). We will work in the Banach space $E = C[0, 1] = \{x: [0, 1] \to \mathbb{R} \text{ is continuous}\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$. Notice that this space can be equipped with a partial order given by

$$x, y \in C[0, 1], \ x \leq y \ \Leftrightarrow x(t) \leq y(t) \text{ for all } t \in [0, 1].$$

Set $P = \{x \in C[0, 1] \mid x(t) \geq 0, t \in [0, 1]\}$, the standard cone. It is clear that P is a normal cone in E and the normality constant is 1. Define the set

$$P_h = \{x \in P \mid \exists \zeta, \eta > 0 : \zeta h(t) \leq x(t) \leq \eta h(t), \ t \in [0, 1]\},$$
$$P_{h,e} = \{x \in E \mid x + e \in P_h\}.$$

Here we choose

$$(4.1) \quad e(t) = a \int_0^1 G(t,s)\, ds = \frac{a}{(\alpha-\nu)\Gamma(\alpha)}\, t^{\alpha-1} - \frac{a}{\alpha\Gamma(\alpha)}\, t^\alpha, \quad \text{for all } t \in [0,1],$$

where $G(t,s)$ is given in (2.2). It is easy to obtain $e(t) \geq 0$ by the nonnegative of $G(t,s)$ and $a > 0$, which means $e \in P$. And $e(t) \equiv 0$ by $a = 0$, which means $e = \theta$. Further, set

$$(4.2) \quad h(t) = Mt^{\alpha-1}, \quad \text{for all } t \in [0,1]$$

with $M \geq a/[(\alpha-\nu)\Gamma(\alpha)]$. It follows from (4.1) and (4.2) that

$$(4.3) \quad e(t) = \frac{a}{(\alpha-\nu)\Gamma(\alpha)}\, t^{\alpha-1} - \frac{a}{\alpha\Gamma(\alpha)}\, t^\alpha \leq \frac{a}{(\alpha-\nu)\Gamma(\alpha)}\, t^{\alpha-1} \leq Mt^{\alpha-1} = h(t).$$

So, $0 \leq e(t) \leq h(t)$ implies $\theta \leq e \leq h$. Then we can see that $h \in P_{h,e}$ and

$$P_{h,e} = \{x \in E \mid \text{there exist } \mu_1 = \mu_1(h,e,x) > 0,\ \mu_2 = \mu_2(h,e,x) > 0$$
$$\text{such that } \mu_1 h \leq x + e \leq \mu_2 h\}.$$

THEOREM 4.1. *Assume the following conditions*:

(H$_1$) *the constant $a > 0$, the function $f\colon [0,1] \times [-e^*, +\infty) \times [-e^*, +\infty) \to (-\infty, +\infty)$ is continuous, where $e^* = \max\{e(t) : t \in [0,1]\}$;*

(H$_2$) *$f(t,x,y)$ is increasing in $x \in [-e^*, +\infty)$ for fixed $t \in [0,1]$ and $y \in [-e^*, +\infty)$, decreasing in $y \in [-e^*, +\infty)$ for fixed $t \in [0,1]$ and $x \in [-e^*, +\infty)$;*

(H$_3$) *for any $\lambda \in (0,1)$, $t \in [0,1]$, $x,y \in (-\infty, +\infty)$, $z \in [0, e^*]$ there exists $\varphi(\lambda) > \lambda$ such that*

$$f(t, \lambda x + (\lambda-1)z, \lambda^{-1}y + (\lambda^{-1}-1)z) \geq \varphi(\lambda) f(t,x,y);$$

(H$_4$) *$f(t,0,M) \geq 0$ with $f(t,0,M) \not\equiv 0$ for $t \in [0,1]$ and $M \geq a/[(\alpha-\nu)\Gamma(\alpha)] > 0$. Then:*

  (a) *there exists $u_0, v_0 \in P_{h,e}$ such that $u_0 < v_0$ and*

$$\begin{cases} u_0(t) \leq \displaystyle\int_0^1 G(t,s)f(s, u_0(s), v_0(s)) - e(t), & t \in [0,1], \\[2mm] v_0(t) \geq \displaystyle\int_0^1 G(t,s)f(s, v_0(s), u_0(s)) - e(t), & t \in [0,1], \end{cases}$$

  where $e(t)$, $h(t)$ are given in (4.1) and (4.2);

  (b) *the problem (1.1) has a unique solution u^* in $P_{h,e}$;*

  (c) *for any $x_0, y_0 \in P_{h,e}$, constructing successively the sequences*

$$\begin{cases} x_n(t) = \displaystyle\int_0^1 G(t,s)f(s, x_{n-1}(s), y_{n-1}(s))\, ds - e(t), & n = 1,2,\ldots, \\[2mm] y_n(t) = \displaystyle\int_0^1 G(t,s)f(s, y_{n-1}(s), x_{n-1}(s))\, ds - e(t), & n = 1,2,\ldots, \end{cases}$$

we have both $x_n(t)$ and $y_n(t)$ that converge uniformly to $u^*(t)$ for all $t \in [0,1]$.

PROOF. From Lemma 2.8 and (4.1), the problem (1.1) has an integral formulation given by

$$
\begin{aligned}
u(t) &= \int_0^1 G(t,s)f(s,u(s),u(s))\,ds - a\int_0^1 G(t,s)\,ds \\
&= \int_0^1 G(t,s)f(s,u(s),u(s))\,ds - e(t) \\
&= \int_0^1 G(t,s)f(s,u(s),u(s))\,ds - \frac{a}{(\alpha-\nu)\Gamma(\alpha)}\,t^{\alpha-1} + \frac{a}{\alpha\Gamma(\alpha)}\,t^\alpha.
\end{aligned}
$$

We define the operator $T\colon P_{h,e} \times P_{h,e} \to E$ by

$$
T(u,v)(t) = \int_0^1 G(t,s)f(s,u(s),v(s))\,ds - e(t), \quad t \in [0,1].
$$

It is easy to prove that u is the solution of problem (1.1) if and only if it is a fixed point of the operator T. Next, we divide the proof into three steps.

Step 1. We show operator $T\colon P_{h,e} \times P_{h,e} \to E$ satisfies the condition (L_2). For any $u,v \in P_{h,e}$, $\lambda \in (0,1)$, from (H_3), we obtain

$$
\begin{aligned}
T(\lambda u &+ (\lambda-1)e, \lambda^{-1}v + (\lambda^{-1}-1)e)(t) \\
&= \int_0^1 G(t,s)f(s, \lambda u(s) + (\lambda-1)e(s), \lambda^{-1}v(s) + (\lambda^{-1}-1)e(s))\,ds - e(t) \\
&\geq \varphi(\lambda)\int_0^1 G(t,s)f(s,u(s),v(s))\,ds - e(t) \\
&= \varphi(\lambda)\left[\int_0^1 G(t,s)f(s,u(s),v(s))\,ds - e(t)\right] + (\varphi(\lambda)-1)e(t) \\
&= \varphi(\lambda)T(u,v)(t) + (\varphi(\lambda)-1)e(t).
\end{aligned}
$$

Hence, for all $u,v \in P_{h,e}$, we have

$$
T(\lambda u + (\lambda-1)e, \lambda^{-1}v + (\lambda^{-1}-1)e) \geq \varphi(\lambda)T(u,v) + (\varphi(\lambda)-1)e.
$$

Step 2. We prove operator $T\colon P_{h,e} \times P_{h,e} \to E$ is a mixed monotone operator. For $u \in P_{h,e}$, we have $u+e \in P_h$, so there exist $\zeta > 0$ such that $u(t)+e(t) \geq \zeta h(t)$, $t \in [0,1]$, by (H_1), we get

$$
u(t) \geq \zeta h(t) - e(t) \geq -e(t) \geq -e^*.
$$

for all $u_i, v_i \in P_{h,e}$ $(i=1,2)$ with $u_1 \geq u_2$, $v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, and $u_i(t), v_i(t) \in [-e^*, +\infty)$, for all $t \in [0,1]$. According to the

nonnegative property of $G(t,s)$ and (H$_2$), we obtain

$$T(u_1,v_1)(t) = \int_0^1 G(t,s)f(s,u_1(s),v_1(s))\,ds - e(t)$$

$$\geq \int_0^1 G(t,s)f(s,u_2(s),v_2(s))\,ds - e(t) = T(u_2,v_2)(t),$$

which means $T(u_1,v_1) \geq T(u_2,v_2)$, that is, T is a mixed monotone operator.

Step 3. We prove that $T(h,h) \in P_{h,e}$. By the definition of $P_{h,e}$, we only need to prove $T(h,h) + e \in P_h$. On the one hand, from (H$_2$), Lemma 2.9, (4.2), for any $t \in [0,1]$, we have

$$T(h,h)(t) + e(t) = \int_0^1 G(t,s)f(s,h(s),h(s))\,ds$$

$$= \int_0^1 G(t,s)f(s,Ms^{\alpha-1},Ms^{\alpha-1})\,ds$$

$$\leq \int_0^1 \frac{(1-s)^{\alpha-\nu-1}t^{\alpha-1}}{\Gamma(\alpha)}f(s,M,0)\,ds$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}f(s,M,0)\,ds \cdot t^{\alpha-1}$$

$$= \frac{1}{M\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}f(s,M,0)\,ds \cdot h(t).$$

On the other hand, we also obtain

$$T(h,h)(t) + e(t) = \int_0^1 G(t,s)f\big(s,Ms^{\alpha-1},Ms^{\alpha-1}\big)\,ds$$

$$\geq \int_0^1 \frac{[1-(1-s)^\nu](1-s)^{\alpha-\nu-1}t^{\alpha-1}}{\Gamma(\alpha)}f(s,0,M)\,ds$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^1 [1-(1-s)^\nu](1-s)^{\alpha-\nu-1}f(s,0,M)\,ds \cdot t^{\alpha-1}$$

$$= \frac{1}{M\Gamma(\alpha)}\int_0^1 [1-(1-s)^\nu](1-s)^{\alpha-\nu-1}f(s,0,M)\,ds \cdot h(t).$$

Let

$$l_1 := \frac{1}{M\Gamma(\alpha)}\int_0^1 [1-(1-s)^\nu](1-s)^{\alpha-\nu-1}f(s,0,M)\,ds,$$

$$l_2 := \frac{1}{M\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}f(s,M,0)\,ds.$$

So we have $l_1 h(t) \leq T(h,h)(t) + e(t) \leq l_2 h(t)$, $t \in [0,1]$. It follows from (H$_2$) and (H$_4$) that

$$\int_0^1 (1-s)^{\alpha-\nu-1}f(s,M,0)\,ds \geq \int_0^1 [1-(1-s)^\nu](1-s)^{\alpha-\nu-1}f(s,0,M)\,ds > 0,$$

which implies $l_2 \geq l_1 > 0$. Hence $T(h,h) + e \in P_h$ holds, that is $T(h,h) \in P_{h,e}$.

As a result, Theorem 3.1 (a) implies that there exist $u_0, v_0 \in P_{h,e}$ such that

$$u_0(t) \leq \int_0^1 G(t,s) f(s, u_0(s), v_0(s)) \, ds - e(t), \quad t \in [0,1],$$

$$v_0(t) \geq \int_0^1 G(t,s) f(s, u_0(s), v_0(s)) \, ds - e(t), \quad t \in [0,1].$$

Theorem 3.1 (b) means that the operator T has a unique fixed point u^* in $P_{h,e}$. Moreover, by conclusion (c) in Theorem 3.1, for any $x_0, y_0 \in P_{h,e}$, the sequences

$$x_n(t) = \int_0^1 G(t,s) f(s, x_{n-1}(s), y_{n-1}(s)) \, ds - e(t), \quad n = 1, 2, \ldots,$$

$$y_n(t) = \int_0^1 G(t,s) f(s, y_{n-1}(s), x_{n-1}(s)) \, ds - e(t), \quad n = 1, 2, \ldots,$$

and $x_n \to u^*$, $y_n \to u^*$ as $n \to \infty$, where $G(t,s)$ and $e(t)$ are given in (2.2) and (4.1). $\square$

From Theorem 3.4 and Theorem 3.6 and similar to the proofs of Theorem 4.1, we can deduce the following Corollary 4.2.

COROLLARY 4.2. *If we replace condition* (H_3) *in Theorem 4.1 with the following condition* (H_5) *or* (H_6) *or* (H_7):

(H_5) *for any* $\lambda \in (0,1)$, $t \in [0,1]$, $x, y \in (-\infty, +\infty)$, $z \in [0, e^*]$ *there exists* $\beta \in (0,1)$ *such that*

$$f(t, \lambda x + (\lambda - 1)z, \lambda^{-1} y + (\lambda^{-1} - 1)z) \geq \lambda^\beta f(t, x, y);$$

(H_6) *for any* $\lambda \in (0,1)$, $t \in [0,1]$, $x, y \in (-\infty, +\infty)$, $z \in [0, e^*]$ *there exists* $\beta(\lambda) \in (0,1)$ *such that*

$$f(t, \lambda x + (\lambda - 1)z, \lambda^{-1} y + (\lambda^{-1} - 1)z) \geq \lambda^{\beta(\lambda)} f(t, x, y);$$

(H_7) *for any* $\lambda \in (0,1)$, $t \in [0,1]$, $x, y \in (-\infty, +\infty)$, $z \in [0, e^*]$ *there exists* $\beta(\lambda, x, y) \in (0,1)$ *such that*

$$f(t, \lambda x + (\lambda - 1)z, \lambda^{-1} y + (\lambda^{-1} - 1)z) \geq \lambda^{\beta(\lambda, x, y)} f(t, x, y).$$

Then the results (a)–(c) *in Theorem 4.1 still hold.*

Next, by using Corollary 3.10 we can easily obtain the problem (1.1) has a unique positive solution.

THEOREM 4.3. *Suppose that*:

(H_1') *the constant* $a \leq 0$, *the function* $f \colon [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *is continuous with* $f(t, 0, 1) \not\equiv a$;

(H_2') $f(t, x, y)$ *is increasing in* $x \in [0, +\infty)$ *for fixed* $t \in [0,1]$ *and* $y \in [0, +\infty)$, *decreasing in* $y \in [0, +\infty)$ *for fixed* $t \in [0,1]$ *and* $x \in [0, +\infty)$;

(H$'_3$) *for any* $\lambda \in (0,1)$, $t \in [0,1]$, $x,y \in [0,+\infty)$, *there exists* $\varphi(\lambda) \in (\lambda,1]$ *such that*

$$f(t, \lambda x, \lambda^{-1} y) \geq \varphi(\lambda) f(t, x, y);$$

Then:

(a) *there exists* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 < u_0 < v_0$ *and*

$$\begin{cases} u_0(t) \leq \displaystyle\int_0^1 G(t,s)[f(s, u_0(s), v_0(s)) - a] \, ds, & t \in [0,1], \\[2mm] v_0(t) \geq \displaystyle\int_0^1 G(t,s)[f(s, v_0(s), u_0(s)) - a] \, ds, & t \in [0,1], \end{cases}$$

where $h(t) = t^{\alpha-1}$, $G(t,s)$ *is given in* (2.2);

(b) *the problem* (1.1) *has a unique positive solution* u^* *in* P_h;

(c) *for any* x_0, $y_0 \in P_h$, *constructing successively the sequences*

$$\begin{cases} x_n(t) = \displaystyle\int_0^1 G(t,s)[f(s, x_{n-1}(s), y_{n-1}(s)) - a] \, ds, & n = 1, 2, \ldots, \\[2mm] y_n(t) = \displaystyle\int_0^1 G(t,s)[f(s, y_{n-1}(s), x_{n-1}(s)) - a] \, ds, & n = 1, 2, \ldots, \end{cases}$$

we have $\|x_n - u^*\| \to 0$ *and* $\|y_n - u^*\| \to 0$ *as* $n \to \infty$.

PROOF. It is well known that u is a solution of problem (1.1) if and only if

$$u(t) = \int_0^1 G(t,s)[f(s, u(s), u(s)) - a] \, ds,$$

where the Green function $G(t,s)$ is given in (2.2), constant $a \leq 0$. Define an operator $T \colon P \times P \to E$ by

$$T(u,v)(t) = \int_0^1 G(t,s)[f(s, u(s), v(s)) - a] \, ds, \quad \text{for all } u, v \in P.$$

Obviously, u is the solution of problem (1.1) if and only if $u = T(u,u)$. From (H$'_1$), we know that $T \colon P \times P \to P$. In the sequel we prove the T satisfies all the assumptions of Corollary 3.10.

Firstly, we show that T is a mixed monotone operator. For any $u_i, v_i \in P_h$, $i = 1, 2$ with $u_1 \geq u_2$, $v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, $t \in [0,1]$. Since $G(t,s)$ is nonnegative and in view of (H$'_2$), we obtain

$$T(u_1, v_1)(t) = \int_0^1 G(t,s)[f(s, u_1(s), v_1(s)) - a] \, ds$$

$$\geq \int_0^1 G(t,s)[f(s, u_2(s), v_2(s)) - a] \, ds = T(u_2, v_2)(t),$$

which means $T(u_1, v_1) \geq T(u_2, v_2)$, that is, T is a mixed monotone operator.

Secondly, we prove the operator T satisfies the condition (H_3'). In fact, for any $\lambda \in (0,1)$, $t \in [0,1]$, $x,y \in [0,+\infty)$, by (H_3') and $a \leq 0$, we have

$$T(\lambda u, \lambda^{-1}v)(t) = \int_0^1 G(t,s)[f(s,\lambda u(s),\lambda^{-1}v(s)) - a]\,ds$$

$$\geq \int_0^1 G(t,s)[\varphi(\lambda)f(s,u(s),v(s)) - a]\,ds$$

$$\geq \varphi(\lambda)\int_0^1 G(t,s)[f(s,u(s),v(s)) - a]\,ds = \varphi(\lambda)T(u,v)(t),$$

which implies $T(\lambda u, \lambda^{-1}v) \geq \varphi(\lambda)T(u,v)$ for $u,v \in P, \lambda \in (0,1)$.

Thirdly, we show that $T(h,h) \in P_h$. According to (H_1'), (H_2') and Lemma 2.9, we have

$$T(h,h)(t) = \int_0^1 G(t,s)[f(s,h(s),h(s)) - a]\,ds$$

$$= \int_0^1 G(t,s)[f(s,s^{\alpha-1},s^{\alpha-1}) - a]\,ds$$

$$\geq \int_0^1 \frac{[1-(1-s)^\nu](1-s)^{\alpha-\nu-1}t^{\alpha-1}}{\Gamma(\alpha)}[f(s,0,1) - a]\,ds$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^1 [1-(1-s)^\nu](1-s)^{\alpha-\nu-1}[f(s,0,1) - a]\,ds \cdot h(t),$$

and

$$T(h,h)(t) = \int_0^1 G(t,s)[f(s,s^{\alpha-1},s^{\alpha-1}) - a]\,ds$$

$$\leq \int_0^1 \frac{(1-s)^{\alpha-\nu-1}t^{\alpha-1}}{\Gamma(\alpha)}[f(s,1,0) - a]\,ds$$

$$= \int_0^1 \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\nu-1}[f(s,1,0) - a]\,ds \cdot h(t).$$

Set

$$l_3 := \frac{1}{\Gamma(\alpha)}\int_0^1 [1-(1-s)^\nu](1-s)^{\alpha-\nu-1}[f(s,0,1) - a]\,ds,$$

$$l_4 := \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}[f(s,1,0) - a]\,ds.$$

Then we have

$$l_3 h(t) \leq T(h,h)(t) \leq l_4 h(t).$$

It follows from $f(t,1,0) \geq f(t,0,1) \not\equiv a$ that

$$\int_0^1 [f(s,1,0) - a]\,ds \geq \int_0^1 [f(s,0,1) - a]\,ds > 0,$$

which means $l_3 > 0$, $l_4 > 0$. Hence $T(h,h) \in P_h$.

Lastly, an application of Corollary 3.10 implies the conclusions (a)–(c) in Theorem 4.3 hold.   □

REMARK 4.4. For the problem (1.1), by the use of Theorem 3.1, we can prove the existence and uniqueness of nontrivial solution $x^* \in P_{h,e}$, see Theorem 4.1. However, we can not guarantee that this solution is positive unless the constant $a \leq 0$. When $a \leq 0$, by applying the Corollary 3.10, we can obtain the problem (1.1) has a unique positive solution $x^* \in P_h$, see Theorem 4.3. Thereinto, When $a = 0$, we have $e = \theta$. In this case, the set $P_{h,e} = P_h$, which implies the theorem 4.3 is a special case of theorem 4.1. Moreover, if we set $a = 0$, $\varphi(\lambda) = \lambda^\beta$ in Theorem 4.3, the corresponding result has been obtained in [21]. Therefore, our study for the problem (1.1) is more general. Besides, we should point that we can not deduce the similar results for the problem (1.1) with $a > 0$ by means of previously available methods in [27], [21]. So our study of the fixed point theorems in set $P_{h,e}$ is significant.

REMARK 4.5. If the Green function satisfies some properties similar to Lemma 2.9, then our main results (Theorem 4.1, Corollary 4.2, Theorem 4.3) can be applied to many fractional differential equations boundary value problems depending on certain constant.

In what follows, we give an concrete example to illustrate our main result.

EXAMPLE 4.6. Consider the following problem:

$$
(4.4) \quad
\begin{cases}
D_{0^+}^{7/2} u(t) + \left(\dfrac{14}{3}\, u(t) + \dfrac{1}{\Gamma(7/2)} \right)^{1/3} \left(\dfrac{1}{2} - \dfrac{2}{7}\, t \right)^{1/3} t^{5/6} \\[3mm]
\quad + \left[\left(\dfrac{14}{3}\, u(t) + \dfrac{1}{\Gamma(7/2)} \right) \left(\dfrac{1}{2} - \dfrac{2}{7} t \right) t^{5/2} + u(t) + \dfrac{3}{14\Gamma(7/2)} \right]^{-1/3} = 1 \\[3mm]
\hspace{8cm} \text{for } 0 < t < 1; \\[3mm]
u(0) = u'(0) = u''(0) = 0; \\[2mm]
D_{0^+}^{3/2} u(t)\big|_{t=1} = 0,
\end{cases}
$$

where $a = 1$, $\alpha = 7/2$, $\nu = 3/2$, $n = 4$. This example can be written in the form of (1.1) with the function f defined by

$$
\begin{aligned}
f(t, x, y) &= \left(\frac{14}{3}\, x + \frac{1}{\Gamma(7/2)} \right)^{1/3} \left(\frac{1}{2} - \frac{2}{7} t \right)^{1/3} t^{5/6} \\[2mm]
&\quad + \left[\left(\frac{14}{3}\, y + \frac{1}{\Gamma(7/2)} \right) \left(\frac{1}{2} - \frac{2}{7} t \right) t^{5/2} + y + \frac{3}{14\Gamma(7/2)} \right]^{-1/3} \\[2mm]
&= \left(\frac{e(t)}{e^*}\, x + e(t) \right)^{1/3} + \left[\left(\frac{e(t)}{e^*} + 1 \right) y + e(t) + e^* \right]^{-1/3},
\end{aligned}
$$

where

$$e(t) = \frac{a}{(\alpha - \beta)\Gamma(\alpha)} t^{\alpha-1} - \frac{a}{\alpha\Gamma(\alpha)} t^{\alpha} = \frac{1}{2\Gamma(7/2)} t^{5/2} - \frac{2}{7\Gamma(7/2)} t^{7/2},$$

$$e^*(t) = \max\{e(t) : t \in [0, 1]\} = \frac{3}{14\Gamma(7/2)}.$$

It is obvious that $f : [0, 1] \times (-\infty, +\infty) \times (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous and $f(t, x, y)$ is increasing with respect to the variable x, decreasing with respect to the variable y.

Take $h(t) = Mt^{5/2}$, $t \in [0, 1]$ with $M = a/(\alpha - \nu)\Gamma(\alpha) = 1/2\Gamma(7/2)$. Then

$$e(t) = \frac{t^{5/2}}{2\Gamma(7/2)} - \frac{2t^{7/2}}{7\Gamma(7/2)} \leq \frac{t^{5/2}}{2\Gamma(7/2)} = Mt^{5/2} = h(t),$$

$$f(t, 0, M) = \left(e(t)\right)^{1/3} + \left[\frac{10}{3\Gamma(7/2)}\left(\frac{1}{2} - \frac{2}{7}t\right)t^{5/2} + \frac{5}{7\Gamma(7/2)}\right]^{-1/3}$$

$$= \left(\frac{t^{5/2}}{\Gamma(7/2)}\left(\frac{1}{2} - \frac{2t}{7}\right)\right)^{1/3} + \left(\left(\frac{1}{2} - \frac{2t}{7}\right)t^{5/2}\left(\frac{14M}{3} + \frac{1}{\Gamma(7/2)}\right)\right)^{-1/3} \geq 0$$

with $f(t, 0, M) \not\equiv 0$, for all $t \in [0, 1]$.

Besides, for $\lambda \in (0, 1)$, $x, y \in (-\infty, +\infty)$, $z \in [0, e^*]$, we have

$$f(t, \lambda x + (\lambda - 1)z, \lambda^{-1}y + (\lambda^{-1} - 1)z) = \left[\frac{e(t)}{e^*}(\lambda x + (\lambda - 1)z) + e(t)\right]^{1/3}$$

$$+ \left[\left(\frac{e(t)}{e^*} + 1\right)(\lambda^{-1}y + (\lambda^{-1} - 1)z) + e(t) + e^*\right]^{-1/3}$$

$$= \lambda^{1/3}\left[\frac{e(t)}{e^*}(x + \left(1 - \frac{1}{\lambda}\right)z) + \frac{e(t)}{\lambda}\right]^{1/3}$$

$$+ \lambda^{1/3}\left[\left(\frac{e(t)}{e^*} + 1\right)(y + (1 - \lambda)z) + \lambda(e(t) + e^*)\right]^{-1/3}$$

$$= \lambda^{1/3}\left[\frac{e(t)}{e^*}x + \left(1 - \frac{1}{\lambda}\right)\frac{e(t)}{e^*}z + \frac{e(t)}{\lambda}\right]^{1/3}$$

$$+ \lambda^{1/3}\left[\left(\frac{e(t)}{e^*} + 1\right)y + (1 - \lambda)\left(\frac{e(t)}{e^*} + 1\right)z + \lambda(e(t) + e^{\star})\right]^{-1/3}$$

$$\geq \lambda^{1/3}\left\{\left[\frac{e(t)}{e^*}x + \left(1 - \frac{1}{\lambda}\right)\frac{e(t)}{e^*}e^* + \frac{e(t)}{\lambda}\right]^{1/3}\right.$$

$$\left. + \left[\left(\frac{e(t)}{e^*} + 1\right)y + (1 - \lambda)\left(\frac{e(t)}{e^*} + 1\right)e^* + \lambda(e(t) + e^*)\right]^{-1/3}\right\}$$

$$= \lambda^{1/3}\left[\left(\frac{e(t)}{e^*}x + e(t)\right)^{1/3} + \left(\left(\frac{e(t)}{e^*} + 1\right)y + e(t) + e^*\right)^{-1/3}\right]$$

$$= \lambda^{1/3}f(t, x, y),$$

for $t \in [0, 1]$, where $\varphi(\lambda) = \lambda^{1/3} > \lambda$. Hence all the conditions of Theorem 4.1 are satisfied. An application of Theorem 4.1 implies that the problem (4.4) has a unique solution u^* in $P_{h,e}$. Taking any initial value $x_0, y_0 \in P_{h,e}$, and constructing the sequences:

$$x_n(t) = \int_0^1 G(t, s) f(s, x_{n-1}(s), y_{n-1}(s)) \, ds$$
$$- \frac{1}{\Gamma(7/2)} \left(\frac{1}{2} t^{5/2} - \frac{2}{7} t^{7/2} \right), \quad n = 1, 2, \ldots,$$

$$y_n(t) = \int_0^1 G(t, s) f(s, y_{n-1}(s), x_{n-1}(s)) \, ds$$
$$- \frac{1}{\Gamma(7/2)} \left(\frac{1}{2} t^{5/2} - \frac{2}{7} t^{7/2} \right), \quad n = 1, 2, \ldots,$$

we have $x_n, y_n \to u^*$ as $n \to \infty$, where

$$G(t, s) = \begin{cases} \dfrac{t^{5/2}(1-s)^2 - (t-s)^{5/2}}{\Gamma(7/2)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{5/2}(1-s)^2}{\Gamma(7/2)}, & 0 \le t \le s \le 1. \end{cases}$$

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