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ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS OF LIMIT PERIODIC DIFFERENCE SYSTEMS WITH COEFFICIENTS FROM COMMUTATIVE GROUPS

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Abstract. We study the behaviour of solutions of limit periodic difference systems over (infinite) fields with absolute values. The considered systems are described by the coefficient matrices that belong to commutative groups whose boundedness is not required. In particular, we are interested in special systems with solutions which vanish at infinity or which are not asymptotically almost periodic. We obtain a transparent condition on the matrix groups which ensures that the special systems form a dense subset in the space of all considered systems, i.e. that, in any neighbourhood of any considered limit periodic system, there exists a system which have non-asymptotically almost periodic or vanishing solutions. The presented results improve and extend known ones.

1. Introduction

The subject of the research presented in this paper is given by the theory of perturbations of difference systems in the form

$$x_{k+1} = A_k \cdot x_k, \tag{1.1}$$

where the coefficient matrices A_k are elements of a commutative group $\mathcal{X}$ for all considered k. In the center of our interest is the existence of non-asymptotically

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almost periodic solutions of systems (1.1). The analysis of non-almost periodic and non-asymptotically almost periodic solutions of general systems in the form (1.1) is usually based on special iterative constructions of sequences with required properties. In this sense, our approach is not an exception. A core of the used method is constructive as well as it is common in this research area. Nevertheless, to obtain our results, we apply an original construction of limit periodic sequences which differs from the constructions used before in the papers whose results are, among others, covered by ours. For such constructive proofs from this area, we can refer to strongly relevant articles [8], [9], [17]–[20], [37], and [39].

We point out that the motivation of our research is described in Section 3 below. At this place, we just mention that it comes mainly from papers [8], [9], [18]–[20] and we are interested in non-almost periodic solutions of systems (1.1) for which the sequence of the coefficient matrices A_k is limit periodic. Such limit periodic systems form the smallest class of the studied systems generalizing periodic systems which can possess at least one non-almost periodic solution.

Now, we proceed to a short literature overview, where we give a list of both books and papers that covers the background theory and the state of the research. We begin with monographs [5], [12], [24], [31], where the basic properties of limit periodic, almost periodic, and asymptotically almost periodic sequences and functions can be found. Next, we mention papers [1], [6], [7], [10], [16], [41]–[43] concerning the almost periodic solutions of almost periodic linear difference equations. Into the last list of references, we should add paper [15], which is the first one, where a construction was used to prove results about non-almost periodic solutions of homogeneous linear difference equations. Regarding the complex case of almost periodic systems of the treated form, we refer, e.g. to [3] and [21].

Our research is closely connected to the theory of the so-called transformable and weakly transformable groups (see [17], [39]). A special case of this theory can be found in [35] (see also [32], [33], [36]), where systems (1.1) are considered in the case, when $\mathcal{X}$ is the unitary group (or the orthogonal group). In [35], there is proved (among others) that, in any neighbourhood of an arbitrarily given almost periodic unitary system of the form (1.1), there exists an almost periodic unitary system whose fundamental matrix is not almost periodic. Note that the method from [35] cannot be used for commutative groups of coefficient matrices which are considered in this paper.

At the end of this references overview, we mention some papers, where the continuous case is treated. Of course, the corresponding differential systems are of the form

$$x'(t) = A(t) \cdot x(t). \tag{1.2}$$

Papers [22], [23], [34], [36] are devoted to the analysis of the almost periodic solutions of systems (1.2) with skew-Hermitian and skew-symmetric coefficient matrices A. The non-almost periodic solutions of skew-Hermitian and skew-symmetric differential systems (1.2) are studied in [38], [40] (see also [33]). These results about skew-Hermitian and skew-symmetric homogeneous linear differential systems corresponds to the results about unitary and orthogonal homogeneous linear difference systems recalled above. Last but not least, we refer to [25], [26], [29], [30], where the reader can find constructions of homogeneous linear differential systems with almost periodic coefficients.

The theory of almost periodic functions and sequences has applications in a lot of areas. For example, almost periodic patterns describe quasicrystals (see, e.g. [28]), the process of the transfer of information by neurons can be described via almost periodicity (see, e.g. [27]), the effect of almost periodicity is used to study population dynamics (see, e.g. [13], where the authors obtain sufficient conditions for the existence of almost periodic solutions of systems of difference equations), etc.

This paper is organized as follows. The upcoming section is devoted to definitions of limit periodic, almost periodic, and asymptotically almost periodic sequences. Then, in Section 3, we complete notations and describe the studied difference systems properly. Further, the basic motivation of our research is extensively described in Section 3, where the most relevant results from the above mentioned papers are explicitly formulated. In the final section, the main result is proved (see Theorem 4.1 below). At the end of Section 4, we also discuss the main result and formulate a consequence of the used method.

2. Limit periodicity, almost periodicity, and asymptotic almost periodicity in metric spaces

In this section, we mention the notion of limit periodic, almost periodic, and asymptotically almost periodic sequences. Let a metric space $(\mathcal{M}, \varrho)$ be given arbitrarily. We put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Definition 2.1. A sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{M}$ or $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{M}$ is called limit periodic if there exists a sequence of periodic sequences $\{\varphi_k^n\}_{k\in\mathbb{Z}} \subseteq \mathcal{M}$ or $\{\varphi_k^n\}_{k\in\mathbb{N}_0} \subseteq \mathcal{M}$, $n \in \mathbb{N}$, with the property that $\lim\limits_{n\to\infty} \varphi_k^n = \varphi_k$ uniformly with respect to $k \in \mathbb{Z}$ or $k \in \mathbb{N}_0$.

Remark 2.2. Note that the set of limit periodic sequences can be introduced in another equivalent way (see [4] and also [2]).

Definition 2.3. We say that a sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{M}$ is almost periodic if, for any $\varepsilon > 0$, there exists $p(\varepsilon) \in \mathbb{N}$ with the property that any set consisting of $p(\varepsilon)$ consecutive integers contains at least one integer l for which $\varrho(\varphi_{k+l}, \varphi_k) < \varepsilon$,

$k \in \mathbb{Z}$. A sequence $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{M}$ is called almost periodic if there exists an almost periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{M}$ with the property that $\varphi_k = \psi_k$, $k \in \mathbb{N}_0$.

Definition 2.4. A sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{M}$ or $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{M}$ is called asymptotically almost periodic if, for any $\varepsilon > 0$, there exist $p(\varepsilon), P(\varepsilon) \in \mathbb{N}$ with the property that any set consisting of $p(\varepsilon)$ consecutive positive integers contains at least one integer l for which $\varrho(\varphi_{k+l}, \varphi_k) < \varepsilon$, $k \geq P(\varepsilon)$, $k \in \mathbb{N}$.

The definitions mentioned above (as Definitions 2.3 and 2.4) are based on the so-called Bohr concept. The (almost periodicity and) asymptotic almost periodicity can be introduced in another equivalent way (the so-called Bochner concept) as follows.

Theorem 2.5. *Let a sequence $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{M}$ be given. The sequence $\{\varphi_k\}$ is asymptotically almost periodic if and only if any sequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}_0$ fulfilling $\lim_{n\to\infty} s_n = \infty$ has a subsequence $\{s_n^1\}_{n\in\mathbb{N}} \subseteq \{s_n\}$ with the property that, for any $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ for which*

$$\varrho(\varphi_{k+s_i^1}, \varphi_{k+s_j^1}) < \varepsilon, \quad i, j \geq N(\varepsilon),\ i, j \in \mathbb{N},\ k \in \mathbb{N}_0. \tag{2.1}$$

Proof. See [14, Part 2]. □

In addition, we also recall the following known theorem.

Theorem 2.6. *The uniform limit of almost periodic sequences is almost periodic.*

Proof. The statement of the theorem can be easily obtained using the proof of [11, Theorem 6.4]. □

Remark 2.7. From Theorem 2.6, we know that any limit periodic sequence is almost periodic. We add that any almost periodic sequence is asymptotically almost periodic (consider directly Definitions 2.3 and 2.4).

Remark 2.8. One can easily show that any asymptotically almost periodic sequence is bounded if $k \in \mathbb{N}_0$ (see Definition 2.4).

3. Considered difference systems over fields

Let F be an infinite field with an absolute value $|\cdot|$. Let $m \in \mathbb{N}$ be arbitrarily given. The set of all $m \times m$ matrices with elements from F will be denoted by $\mathrm{Mat}(F, m)$ and the set of all $m \times 1$ vectors with elements from F will be denoted by F^m. As usual, the identity matrix (in $\mathrm{Mat}(F, m)$) will be denoted by I. The absolute value on F gives the norms $\|\cdot\|$ on F^m, $\mathrm{Mat}(F, m)$ as the sums of the absolute values of their elements. The absolute value on F and the norms on F^m, $\mathrm{Mat}(F, m)$ imply the metrics denoted by ϱ, where ε-neighbourhoods will be denoted using the symbol $\mathcal{O}_\varepsilon^\varrho$.

Now, we are ready to introduce the studied homogeneous linear difference systems over the field F. Let a matrix group $\mathcal{X} \subset \mathrm{Mat}(F, m)$ be given. We consider systems in the form

$$x_{k+1} = A_k \cdot x_k, \tag{3.1}$$

where $k \in \mathbb{N}_0$ or $k \in \mathbb{Z}$ and $A_k \in \mathcal{X}$ for all k. As $\mathcal{LP}(\mathcal{X})$, we will denote the set of all systems (3.1) with the property that the sequence $\{A_k\}$ is limit periodic. Note that we identify the sequence $\{A_k\}$ with the system (3.1) which is determined by $\{A_k\}$. Using this convention, for $\{A_k\}_{k\in\mathbb{N}_0}, \{B_k\}_{k\in\mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$ or $\{A_k\}_{k\in\mathbb{Z}}, \{B_k\}_{k\in\mathbb{Z}} \in \mathcal{LP}(\mathcal{X})$, we define the metric

$$\sigma(\{A_k\}, \{B_k\}) := \sup_k \|A_k - B_k\|$$

in $\mathcal{LP}(\mathcal{X})$. As $\mathcal{O}^\sigma_\varepsilon(\{A_k\})$, we will denote the ε-neighbourhood of $\{A_k\}$ in $\mathcal{LP}(\mathcal{X})$.

In the theorems below, we collect the most relevant results which give the basic motivation for the current research and which are completed (in a certain sense) or improved by the presented results in the next section. We repeat that this motivation comes from papers [8], [9], [18]–[20]. The results deal with two cases, when the considered group $\mathcal{X}$ is commutative or bounded. We begin with two results about bounded groups.

THEOREM 3.1. *Let (F, ϱ) be separable. Let $\mathcal{X}$ be bounded and have the property that there exists $\xi > 0$ such that, for any $\delta > 0$, there exists $l \in \mathbb{N}$ with the property that, for any $u \in F^m$, $\|u\| \geq 1$, there exist matrices $M_1, \dots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}^\varrho_\delta(I), \quad i \in \{1, \dots, l\}, \qquad \|M_l \cdot \ldots \cdot M_1 \cdot u - u\| > \xi.$$

Then, for any $\varepsilon > 0$ and any $\{A_k\}_{k\in\mathbb{Z}} \in \mathcal{LP}(\mathcal{X})$, there exists $\{B_k\}_{k\in\mathbb{Z}} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\})$ with the property that the system $x_{k+1} = B_k \cdot x_k$, $k \in \mathbb{Z}$, does not possess any non-trivial asymptotically almost periodic solution.

PROOF. See [19, Theorem 7]. □

THEOREM 3.2. *Let $\mathcal{X}$ be bounded and have the property that there exists $\xi > 0$ such that, for any $\delta > 0$, there exist matrices $M_1, \dots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}^\varrho_\delta(I), \quad i \in \{1, \dots, l\}, \qquad \|M_l \cdot \ldots \cdot M_1 - I\| > \xi.$$

Then, for any $\varepsilon > 0$ and any $\{A_k\}_{k\in\mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$, there exists $\{B_k\}_{k\in\mathbb{N}_0} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\})$ with the property that the fundamental matrix of $x_{k+1} = B_k \cdot x_k$, $k \in \mathbb{N}_0$, is not asymptotically almost periodic.

PROOF. See [9, Theorem 4.9]. □

Now, we mention the strongest known results about commutative groups which correspond with Theorems 3.1 and 3.2. For the reader's convenience, we mention these results in full.

Theorem 3.3. *Let $\mathcal{X}$ be commutative and have the property that there exists $\xi > 0$ such that, for any $\delta > 0$, there exists $l \in \mathbb{N}$ with the property that, for any $u \in F^m$, $\|u\| \geq 1$, there exist matrices $M_1, \dots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}^{\varrho}_{\delta}(I), \quad i \in \{1, \dots, l\}, \qquad \|M_l \cdot \ldots \cdot M_1 \cdot u - u\| > \xi.$$

Then, for any $\varepsilon > 0$, $\{A_k\}_{k \in \mathbb{Z}} \in \mathcal{LP}(\mathcal{X})$, and for any sequence $\{u_n\}_{n \in \mathbb{N}}$ of non-zero vectors $u_n \in F^m$, there exists $\{B_k\}_{k \in \mathbb{Z}} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ with the property that the solution of $x_{k+1} = B_k \cdot x_k$, $k \in \mathbb{Z}$, $x_0 = u_n$ is not almost periodic for all $n \in \mathbb{N}$.

Proof. See [8, Theorem 5.1]. □

Remark 3.4. Note that Theorem 3.3 improves the main result of [18].

Theorem 3.5. *Let the unit ball $\{u \in F^m;\ \|u\| \leq 1\}$ be compact. Let $\mathcal{X}$ be commutative and have the property that there exists $\xi > 0$ such that, for any $\delta > 0$, there exist matrices $M_1, \dots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}^{\varrho}_{\delta}(I), \quad i \in \{1, \dots, l\}, \qquad \|M_l \cdot \ldots \cdot M_1 - I\| > \xi.$$

Then, for any $\varepsilon > 0$ and any $\{A_k\}_{k \in \mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$, there exists $\{B_k\}_{k \in \mathbb{N}_0} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ with the property that the fundamental matrix $\{X_k\}_{k \in \mathbb{N}_0}$ of system $x_{k+1} = B_k \cdot x_k$, $k \in \mathbb{N}_0$, is not asymptotically almost periodic or this system has at least one non-trivial solution $\{x_k\}_{k \in \mathbb{N}_0}$ for which $\liminf\limits_{k \to \infty} \|x_k\| = 0$.

Proof. See [20, Theorem 10]. □

The aim of this paper is to improve Theorems 3.3 and 3.5 concerning non-asymptotically almost periodic solutions and solutions vanishing at infinity. More precisely, we generalize Theorem 3.3 with regard to the statement of Theorem 3.5. To obtain such a generalization, we use an iterative construction of limit periodic sequences which is different from the constructions applied in [8], [20]. Note that the construction from [8] is not applicable for solutions vanishing at infinity and that the construction from [20] does not cover any case of at least two initial conditions.

4. Results

Now, we prove the main result. We point out that $\mathcal{X}$ does not need to be bounded (cf. Theorems 3.1 and 3.2).

THEOREM 4.1. *Let $\mathcal{X}$ be commutative and have the property that there exists $\xi > 0$ such that, for any $\delta > 0$, there exists $l \in \mathbb{N}$ with the property that, for any $u \in F^m$, $\|u\| \geq 1$, there exist matrices $M_1, \dots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}^{\varrho}_{\delta}(I), \quad i \in \{1, \dots, l\}, \qquad \|M_l \cdot \ldots \cdot M_1 \cdot u - u\| > \xi. \tag{4.1}$$

Then, for any $\varepsilon > 0$, $\{A_k\}_{k \in \mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$, and for any sequence $\{v_n\}_{n \in \mathbb{N}}$ of non-zero vectors $v_n \in F^m$, there exists $\{B_k\}_{k \in \mathbb{N}_0} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ with the property that the solution $\{x^n_k\}_{k \in \mathbb{N}_0}$ of $x_{k+1} = B_k \cdot x_k$, $k \in \mathbb{N}_0$, $x_0 = v_n$ is not asymptotically almost periodic or $\liminf\limits_{k \to \infty} \|x^n_k\| = 0$ for all $n \in \mathbb{N}$.

PROOF. For the reader's convenience, the presented proof is divided into 5 parts denoted as Parts I–V. In Part I, we show that it suffices to consider only $1 < \|v_n\| < 2$, $n \in \mathbb{N}$, without loss of generality. In Part II, we introduce auxiliary values $\delta_i > 0$ and $l_i \in \mathbb{N}$ for $i \in \mathbb{N}$. Then, in Part III, we present a construction which gives the resulting sequence $\{B_k\}_{k \in \mathbb{N}_0}$. The construction is an iterative process. For simplicity, the steps of this process are denoted as Steps i.j for the j part of the i step. In Part IV, we show that $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ for the obtained sequence $\{B_k\}_{k \in \mathbb{N}_0}$. In the final Part V, we prove that $\{B_k\}_{k \in \mathbb{N}_0}$ has the required properties from the statement of the theorem.

PART I. From the statement of the theorem (see (4.1)), one can easily see that, for any $\vartheta > 0$, there exist elements $f_1, f_2 \in F$ with the property that $|f_1| \in (1 - \vartheta, 1)$, $|f_2| \in (1, 1 + \vartheta)$ (consider properties of the absolute value of an inverse element). Hence, for any non-zero vector $u \in F^m$, there exists $\langle u \rangle \in F$ for which

$$1 < \left\| \frac{u}{\langle u \rangle} \right\| = \left\| \langle u \rangle^{-1} \cdot u \right\| < 2. \tag{4.2}$$

Consequently, to prove Theorem 4.1, it suffices to consider only solutions given by the Cauchy problems $x_{k+1} = B_k \cdot x_k$, $k \in \mathbb{N}_0$, $x_0 = u$, where $1 < \|u\| < 2$. Therefore, we can assume that $\{v_n;\, n \in \mathbb{N}\} \subseteq \{u \in F^m;\, 1 < \|u\| < 2\}$.

PART II. Let $\varepsilon > 0$ and $\{A_k\}_{k \in \mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$ be arbitrarily given. We know (see, e.g. Remarks 2.7 and 2.8) that there exists $K > 0$ for which

$$\sup_{k \in \mathbb{N}_0} \|A_k\| < K. \tag{4.3}$$

We put

$$\delta_i := \frac{\varepsilon}{2^i K}, \quad i \in \mathbb{N}, \tag{4.4}$$

and the corresponding $l \in \mathbb{N}$ will be denoted by l_i. Without loss of generality, we can assume that $\xi \in (0, 1)$ and that $l_{i+1} \geq l_i \geq 2$, $i \in \mathbb{N}$.

PART III. We find the resulting sequence $\{B_k\}_{k \in \mathbb{N}_0}$ by the following construction using the below described sequences of C^i_k for $k \in \mathbb{N}_0$.

Step 1.1. At first, we consider the solution $\{x_k^{(1,1,1)}\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_1.$$

For δ_1 and for v_1, there exist matrices $M_1^1, \dots, M_{l_1}^1 \in \mathcal{X}$ satisfying

$$(4.5) \qquad M_j^1 \in \mathcal{O}_{\delta_1}^{\varrho}(I), \quad j \in \{1, \dots, l_1\}, \qquad \|M_1^1 \cdot \ldots \cdot M_{l_1}^1 \cdot v_1 - v_1\| > \xi.$$

We put $r_1 := 2l_1$. We define the periodic sequence $\{C_k^1\}_{k\in\mathbb{N}_0}$ with the period $p(1,1,1) := 2l_1$ by the choice

$$\begin{aligned} C_{2j-1}^1 &= M_j^1 && \text{for all } j \in \{1, \dots, l_1\}, \\ C_j^1 &= I && \text{for the other } j \in \{0, \dots, 2l_1 - 1\} \end{aligned}$$

if $\|x_{2l_1}^{(1,1,1)} - v_1\| < \xi/4$; and by $\{C_k^1\}_{k\in\mathbb{N}_0} \equiv \{I\}_{k\in\mathbb{N}_0}$ otherwise. We put $B_k^1 = A_k \cdot C_k^1$ for $k \in \mathbb{N}_0$.

Step 2.1. Next, we consider the solution $\{x_k^{(2,1,1)}\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = B_k^1 \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_1.$$

We put $r_2 := 64 \cdot l_1 \cdot l_2$ and $P(2,1,1) := 4p(1,1,1)$. For δ_2 and for $x_{P(2,1,1)+r_2}^{(2,1,1)}$, there exist matrices

$$(4.6) \qquad M_1^{(2,1,1)}, \dots, M_{l_2}^{(2,1,1)} \in \mathcal{O}_{\delta_2}^{\varrho}(I)$$

satisfying (see (4.2))

$$\left\| M_1^{(2,1,1)} \cdot \ldots \cdot M_{l_2}^{(2,1,1)} \cdot \frac{x_{P(2,1,1)+r_2}^{(2,1,1)}}{\left\langle x_{P(2,1,1)+r_2}^{(2,1,1)} \right\rangle} - \frac{x_{P(2,1,1)+r_2}^{(2,1,1)}}{\left\langle x_{P(2,1,1)+r_2}^{(2,1,1)} \right\rangle} \right\| > \frac{\xi}{2}.$$

We define periodic sequence $\{C_k^{(2,1,1)}\}_{k\in\mathbb{N}_0}$ with period $p(2,1,1) := P(2,1,1) \cdot r_2$ in the following way. If

$$\left\| x_{P(2,1,1)}^{(2,1,1)} \right\| > \frac{1}{2}, \quad \left\| x_{P(2,1,1)+r_2}^{(2,1,1)} - x_{P(2,1,1)}^{(2,1,1)} \right\| \le \frac{\xi}{8} \cdot \left\| x_{P(2,1,1)}^{(2,1,1)} \right\|,$$

then

$$\begin{aligned} C_{P(2,1,1)+4(j-1)+2}^{(2,1,1)} &= M_j^{(2,1,1)} && \text{for all } j \in \{1, \dots, l_2\}, \\ C_j^{(2,1,1)} &= I && \text{for the other } j \in \{0, \dots, p(2,1,1) - 1\}. \end{aligned}$$

In the other cases, we put

$$C_0^{(2,1,1)} = \ldots = C_{p(2,1,1)-1}^{(2,1,1)} = I.$$

We put $B_k^{(2,1,1)} = B_k^1 \cdot C_k^{(2,1,1)}$, $k \in \mathbb{N}_0$.

Step 2.2. We consider the solution $\{x_k^{(2,1,2)}\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = B_k^{(2,1,1)} \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_1.$$

We put $P(2,1,2) := 8\,p(2,1,1)$. For δ_2 and for vector $x^{(2,1,2)}_{P(2,1,2)+r_2-r_1}$, there exist matrices

$$M_1^{(2,1,2)}, \dots, M_{l_2}^{(2,1,2)} \in \mathcal{O}^{\varrho}_{\delta_2}(I) \tag{4.7}$$

such that

$$\left\| M_1^{(2,1,2)} \cdot \ldots \cdot M_{l_2}^{(2,1,2)} \cdot \frac{x^{(2,1,2)}_{P(2,1,2)+r_2-r_1}}{\left\langle x^{(2,1,2)}_{P(2,1,2)+r_2-r_1} \right\rangle} - \frac{x^{(2,1,2)}_{P(2,1,2)+r_2-r_1}}{\left\langle x^{(2,1,2)}_{P(2,1,2)+r_2-r_1} \right\rangle} \right\| > \frac{\xi}{2}.$$

We define the periodic sequence $\{C_k^{(2,1,2)}\}_{k\in\mathbb{N}_0}$ with the period $p(2,1,2) := P(2,1,2)\cdot(r_2 - r_1)$ as

$$\begin{aligned} C^{(2,1,2)}_{P(2,1,2)+8(j-1)+4} &= M_j^{(2,1,2)} && \text{for all } j \in \{1,\dots,l_2\}, \\ C_j^{(2,1,2)} &= I && \text{for the other } j \in \{0,\dots,p(2,1,2)-1\} \end{aligned}$$

if

$$\left\| x^{(2,1,2)}_{P(2,1,2)} \right\| > \frac{1}{2}, \quad \left\| x^{(2,1,2)}_{P(2,1,2)+r_2-r_1} - x^{(2,1,2)}_{P(2,1,2)} \right\| \le \frac{\xi}{8} \cdot \left\| x^{(2,1,2)}_{P(2,1,2)} \right\|;$$

and as

$$C_0^{(2,1,2)} = \ldots = C^{(2,1,2)}_{p(2,1,2)-1} = I$$

in the other cases. We put $B_k^{(2,1,2)} = B_k^{(2,1,1)} \cdot C_k^{(2,1,2)}$, $k \in \mathbb{N}_0$.

Step 2.3. We consider the solution $\{x_k^{(2,2,1)}\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = B_k^{(2,1,2)} \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_2.$$

We put $P(2,2,1) := 16\,p(2,1,2)$. For δ_2 and vector $x^{(2,2,1)}_{P(2,2,1)+r_2}$, there exist matrices

$$M_1^{(2,2,1)}, \dots, M_{l_2}^{(2,2,1)} \in \mathcal{O}^{\varrho}_{\delta_2}(I) \tag{4.8}$$

with the property that

$$\left\| M_1^{(2,2,1)} \cdot \ldots \cdot M_{l_2}^{(2,2,1)} \cdot \frac{x^{(2,2,1)}_{P(2,2,1)+r_2}}{\left\langle x^{(2,2,1)}_{P(2,2,1)+r_2} \right\rangle} - \frac{x^{(2,2,1)}_{P(2,2,1)+r_2}}{\left\langle x^{(2,2,1)}_{P(2,2,1)+r_2} \right\rangle} \right\| > \frac{\xi}{2}.$$

We define the periodic auxiliary matrix sequence $\{C_k^{(2,2,1)}\}_{k\in\mathbb{N}_0}$ with the period $p(2,2,1) := P(2,2,1)\cdot r_2$ as

$$\begin{aligned} C^{(2,2,1)}_{P(2,2,1)+16(j-1)+8} &= M_j^{(2,2,1)} && \text{for all } j \in \{1,\dots,l_2\}, \\ C_j^{(2,2,1)} &= I && \text{for the other } j \in \{0,\dots,p(2,2,1)-1\} \end{aligned}$$

if

$$\left\| x^{(2,2,1)}_{P(2,2,1)} \right\| > \frac{1}{2}, \quad \left\| x^{(2,2,1)}_{P(2,2,1)+r_2} - x^{(2,2,1)}_{P(2,2,1)} \right\| \le \frac{\xi}{8} \cdot \left\| x^{(2,2,1)}_{P(2,2,1)} \right\|;$$

and as

$$C_0^{(2,2,1)} = \ldots = C^{(2,2,1)}_{p(2,2,1)-1} = I$$

otherwise. We put $B_k^{(2,2,1)} = B_k^{(2,1,2)} \cdot C_k^{(2,2,1)}$, $k \in \mathbb{N}_0$.

Step 2.4. We consider the solution $\{x_k^{(2,2,2)}\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = B_k^{(2,2,1)} \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_2.$$

We put $P(2,2,2) := 32\,p(2,2,1)$. For δ_2 and for vector $x^{(2,2,2)}_{P(2,2,2)+r_2-r_1}$, we consider arbitrary matrices

$$M_1^{(2,2,2)}, \dots, M_{l_2}^{(2,2,2)} \in \mathcal{O}^{\varrho}_{\delta_2}(I) \tag{4.9}$$

satisfying

$$\left\| M_1^{(2,2,2)} \cdot \ldots \cdot M_{l_2}^{(2,2,2)} \cdot \frac{x^{(2,2,2)}_{P(2,2,2)+r_2-r_1}}{\left\langle x^{(2,2,2)}_{P(2,2,2)+r_2-r_1} \right\rangle} - \frac{x^{(2,2,2)}_{P(2,2,2)+r_2-r_1}}{\left\langle x^{(2,2,2)}_{P(2,2,2)+r_2-r_1} \right\rangle} \right\| > \frac{\xi}{2}.$$

Now we define the periodic sequence $\{C_k^{(2,2,2)}\}_{k\in\mathbb{N}_0}$ with the period $p(2,2,2) := P(2,2,2) \cdot (r_2 - r_1)$ by

$$\begin{aligned} C^{(2,2,2)}_{P(2,2,2)+32(j-1)+16} &= M_j^{(2,2,2)} && \text{for all } j \in \{1,\dots,l_2\},\\ C_j^{(2,2,2)} &= I && \text{for the other } j \in \{0,\dots,p(2,2,2)-1\} \end{aligned}$$

if

$$\left\| x^{(2,2,2)}_{P(2,2,2)} \right\| > \frac{1}{2}, \quad \left\| x^{(2,2,2)}_{P(2,2,2)+r_2-r_1} - x^{(2,2,2)}_{P(2,2,2)} \right\| \le \frac{\xi}{8} \cdot \left\| x^{(2,2,2)}_{P(2,2,2)} \right\|.$$

In the other cases, we define

$$C_0^{(2,2,2)} = \ldots = C^{(2,2,2)}_{p(2,2,2)-1} = I.$$

We denote $B_k^2 = B_k^{(2,2,1)} \cdot C_k^{(2,2,2)}$ and $C_k^2 = C_k^{(2,1,1)} \cdot C_k^{(2,1,2)} \cdot C_k^{(2,2,1)} \cdot C_k^{(2,2,2)}$ for $k \in \mathbb{N}_0$.

We continue the construction in the same manner. Before the i step for arbitrary integer $i \ge 3$, we construct the sequence

$$\{B_k^{i-1}\}_{k\in\mathbb{N}_0} \equiv \{A_k \cdot C_k^1 \cdot \ldots \cdot C_k^{i-1}\}_{k\in\mathbb{N}_0},$$

where the sequence $\{C_k^1 \cdot \ldots \cdot C_k^{i-1}\}_{k\in\mathbb{N}_0}$ has period $p(i-1,i-1,i-1)$.

Step i.1. We consider the solution $\{x_k^{(i,1,1)}\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = B_k^{i-1} \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_1.$$

We put

$$r_i := l_1 \cdot \ldots \cdot l_i \cdot 2^{1+\sum_{n=1}^{i} n^2}, \tag{4.10}$$

$$P(i,1,1) := p(i-1,i-1,i-1) \cdot 2^{1+\sum_{n=1}^{i-1} n^2}. \tag{4.11}$$

For δ_i and vector $x^{(i,1,1)}_{P(i,1,1)+r_i}$, there exist matrices

$$M_1^{(i,1,1)}, \dots, M_{l_i}^{(i,1,1)} \in \mathcal{O}^{\varrho}_{\delta_i}(I) \tag{4.12}$$

with the property that

$$\left\| M_1^{(i,1,1)} \cdot \ldots \cdot M_{l_i}^{(i,1,1)} \cdot \frac{x^{(i,1,1)}_{P(i,1,1)+r_i}}{\left\langle x^{(i,1,1)}_{P(i,1,1)+r_i} \right\rangle} - \frac{x^{(i,1,1)}_{P(i,1,1)+r_i}}{\left\langle x^{(i,1,1)}_{P(i,1,1)+r_i} \right\rangle} \right\| > \frac{\xi}{2}.$$

We define the periodic sequence $\{C_k^{(i,1,1)}\}_{k\in\mathbb{N}_0}$ with the period $p(i,1,1) := P(i,1,1)\cdot r_i$ (see (4.10) and (4.11)) by the values

$$C^{(i,1,1)}_{P(i,1,1)+(j-1)2^{1+\sum_{n=1}^{i-1} n^2}+2^{\sum_{n=1}^{i-1} n^2}} = M_j^{(i,1,1)} \quad \text{for all } j\in\{1,\ldots,l_i\},$$

$$C_j^{(i,1,1)} = I \quad \text{for the other } j\in\{0,\ldots,p(i,1,1)-1\}$$

if

$$\left\| x^{(i,1,1)}_{P(i,1,1)} \right\| > \frac{1}{2^{i-1}}, \qquad \left\| x^{(i,1,1)}_{P(i,1,1)+r_i} - x^{(i,1,1)}_{P(i,1,1)} \right\| \le \frac{\xi}{8}\cdot \left\| x^{(i,1,1)}_{P(i,1,1)} \right\|;$$

and by the values

$$C_0^{(i,1,1)} = \ldots = C^{(i,1,1)}_{p(i,1,1)-1} = I$$

in the other cases. We introduce $B_k^{(i,1,1)} = B_k^{i-1}\cdot C_k^{(i,1,1)}$, $k\in\mathbb{N}_0$.

We continue in the same manner in the i step of the construction.

Step i.i. We consider the solution $\{x_k^{(i,1,i)}\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = B_k^{(i,1,i-1)}\cdot x_k, \quad k\in\mathbb{N}_0, \qquad x_0 = v_1.$$

We put

$$P(i,1,i) := p(i,1,i-1)\cdot 2^{i+\sum_{n=1}^{i-1} n^2}.$$

For δ_i and for $x^{(i,1,i)}_{P(i,1,i)+r_i-r_{i-1}}$, there exist matrices

$$M_1^{(i,1,i)},\ldots,M_{l_i}^{(i,1,i)} \in \mathcal{O}^{\varrho}_{\delta_i}(I) \tag{4.13}$$

satisfying

$$\left\| M_1^{(i,1,i)} \cdot \ldots \cdot M_{l_i}^{(i,1,i)} \cdot \frac{x^{(i,1,i)}_{P(i,1,i)+r_i-r_{i-1}}}{\left\langle x^{(i,1,i)}_{P(i,1,i)+r_i-r_{i-1}} \right\rangle} - \frac{x^{(i,1,i)}_{P(i,1,i)+r_i-r_{i-1}}}{\left\langle x^{(i,1,i)}_{P(i,1,i)+r_i-r_{i-1}} \right\rangle} \right\| > \frac{\xi}{2}.$$

We define the periodic sequence $\{C_k^{(i,1,i)}\}_{k\in\mathbb{N}_0}$ with the period

$$p(i,1,i) := P(i,1,i)\cdot (r_i - r_{i-1})$$

by the values

$$C^{(i,1,i)}_{P(i,1,i)+(j-1)2^{i+\sum_{n=1}^{i-1} n^2}+2^{i-1+\sum_{n=1}^{i-1} n^2}} = M_j^{(i,1,i)} \quad \text{for all } j\in\{1,\ldots,l_i\},$$

$$C_j^{(i,1,i)} = I \quad \text{for the other } j\in\{0,\ldots,p(i,1,i)-1\}$$

if

$$\left\| x^{(i,1,i)}_{P(i,1,i)} \right\| > \frac{1}{2^{i-1}}, \qquad \left\| x^{(i,1,i)}_{P(i,1,i)+r_i-r_{i-1}} - x^{(i,1,i)}_{P(i,1,i)} \right\| \le \frac{\xi}{8}\cdot \left\| x^{(i,1,i)}_{P(i,1,i)} \right\|;$$

and we put

$$C_0^{(i,1,i)} = \ldots = C_{p(i,1,i)-1}^{(i,1,i)} = I$$

in the other cases. We define $B_k^{(i,1,i)} = B_k^{(i,1,i-1)} \cdot C_k^{(i,1,i)}$, $k \in \mathbb{N}_0$.

We continue our construction.

Step i.i(i − 1) + 1. We consider the solution $\{x_k^{(i,i,1)}\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = B_k^{(i,i-1,i)} \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_i.$$

We put

$$P(i,i,1) := p(i,i-1,i) \cdot 2^{i(i-1)+1+\sum\limits_{n=1}^{i-1} n^2}.$$

For δ_i and $x_{P(i,i,1)+r_i}^{(i,i,1)}$, there exist matrices

$$M_1^{(i,i,1)}, \ldots, M_{l_i}^{(i,i,1)} \in \mathcal{O}_{\delta_i}^{\varrho}(I) \tag{4.14}$$

with the property that

$$\left\| M_1^{(i,i,1)} \cdot \ldots \cdot M_{l_i}^{(i,i,1)} \cdot \frac{x_{P(i,i,1)+r_i}^{(i,i,1)}}{\left\langle x_{P(i,i,1)+r_i}^{(i,i,1)} \right\rangle} - \frac{x_{P(i,i,1)+r_i}^{(i,i,1)}}{\left\langle x_{P(i,i,1)+r_i}^{(i,i,1)} \right\rangle} \right\| > \frac{\xi}{2}.$$

We define the periodic sequence $\{C_k^{(i,i,1)}\}_{k\in\mathbb{N}_0}$ with the period

$$p(i,i,1) := P(i,i,1) \cdot r_i$$

by the values

$$C^{(i,i,1)}_{P(i,i,1)+(j-1)2^{i(i-1)+1+\sum\limits_{n=1}^{i-1} n^2}+2^{i(i-1)+\sum\limits_{n=1}^{i-1} n^2}} = M_j^{(i,i,1)} \quad \text{for all } j \in \{1,\ldots,l_i\},$$

$$C_j^{(i,i,1)} = I \quad \text{for the other } j \in \{0,\ldots,p(i,i,1)-1\}$$

if

$$\left\| x_{P(i,i,1)}^{(i,i,1)} \right\| > \frac{1}{2^{i-1}}, \quad \left\| x_{P(i,i,1)+r_i}^{(i,i,1)} - x_{P(i,i,1)}^{(i,i,1)} \right\| \le \frac{\xi}{8} \cdot \left\| x_{P(i,i,1)}^{(i,i,1)} \right\|;$$

and by the constant values

$$C_0^{(i,i,1)} = \ldots = C_{p(i,i,1)-1}^{(i,i,1)} = I$$

in other cases. We define $B_k^{(i,i,1)} = B_k^{(i,i-1,i)} \cdot C_k^{(i,i,1)}$, $k \in \mathbb{N}_0$.

We continue the construction in the same way.

Step i.i². We consider the solution $\{x_k^{(i,i,i)}\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = B_k^{(i,i,i-1)} \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_i$$

and we put

$$P(i,i,i) := p(i,i,i-1) \cdot 2^{\sum\limits_{n=1}^{i} n^2}.$$

For δ_i and for vector $x_{P(i,i,i)+r_i-r_{i-1}}^{(i,i,i)}$, there exist matrices

$$M_1^{(i,i,i)}, \ldots, M_{l_i}^{(i,i,i)} \in \mathcal{O}_{\delta_i}^{\varrho}(I) \tag{4.15}$$

such that

$$\left\| M_1^{(i,i,i)} \cdot \ldots \cdot M_{l_i}^{(i,i,i)} \cdot \frac{x_{P(i,i,i)+r_i-r_{i-1}}^{(i,i,i)}}{\left\langle x_{P(i,i,i)+r_i-r_{i-1}}^{(i,i,i)} \right\rangle} - \frac{x_{P(i,i,i)+r_i-r_{i-1}}^{(i,i,i)}}{\left\langle x_{P(i,i,i)+r_i-r_{i-1}}^{(i,i,i)} \right\rangle} \right\| > \frac{\xi}{2}.$$

We define the periodic sequence $\{C_k^{(i,i,i)}\}_{k\in\mathbb{N}_0}$ with the period

$$p(i,i,i) := P(i,i,i) \cdot (r_i - r_{i-1})$$

by the values

$$C^{(i,i,i)}_{P(i,i,i)+(j-1)2^{\sum_{n=1}^{i} n^2}+2^{-1+\sum_{n=1}^{i} n^2}} = M_j^{(i,i,i)} \quad \text{for all } j \in \{1,\ldots,l_i\},$$

$$C_j^{(i,i,i)} = I \quad \text{for the other } j \in \{0,\ldots,p(i,i,i)-1\}$$

if

$$\left\| x_{P(i,i,i)}^{(i,i,i)} \right\| > \frac{1}{2^{i-1}}, \qquad \left\| x_{P(i,i,i)+r_i-r_{i-1}}^{(i,i,i)} - x_{P(i,i,i)}^{(i,i,i)} \right\| \le \frac{\xi}{8} \cdot \left\| x_{P(i,i,i)}^{(i,i,i)} \right\|;$$

and by the constant values

$$C_0^{(i,i,i)} = \ldots = C_{p(i,i,i)-1}^{(i,i,i)} = I$$

in the other cases. We introduce $B_k^i = B_k^{(i,i,i)} = B_k^{(i,i,i-1)} \cdot C_k^{(i,i,i)}$ and

$$C_k^i = C_k^{(i,1,1)} \cdot \ldots \cdot C_k^{(i,1,i)} \cdot C_k^{(i,2,1)} \cdot \ldots \cdot C_k^{(i,2,i)} \cdot \ldots \cdot C_k^{(i,i,1)} \cdot \ldots \cdot C_k^{(i,i,i)}$$

for all $k \in \mathbb{N}_0$.

Analogously, we continue this construction for all $i + \bar{i}$, $\bar{i} \in \mathbb{N}$. We define the resulting sequence of B_k by the formula

$$B_k = A_k \cdot C_k^1 \cdot \ldots \cdot C_k^i \cdot \ldots, \quad k \in \mathbb{N}_0. \tag{4.16}$$

This definition is correct. Indeed, based on the construction, for any $k \in \mathbb{N}_0$, there exists $t(k) \in \mathbb{N}$ such that

$$B_k = A_k \cdot C_k^{t(k)}. \tag{4.17}$$

PART IV. Now, we show that the resulting sequence $\{B_k\}$ is limit periodic. The limit periodicity of $\{A_k\}$ implies the existence of periodic sequences $\{D_k^n\}_{k\in\mathbb{N}_0} \subseteq \mathcal{X}$ for $n \in \mathbb{N}$ with the property that

$$\|A_k - D_k^n\| < \frac{1}{2^n}, \quad k \in \mathbb{N}_0,\ n \in \mathbb{N}. \tag{4.18}$$

We have (see (4.3), (4.16), and (4.18))

$$
\begin{aligned}
(4.19)\quad \|B_k - D_k^n \cdot C_k^1 \cdot \ldots \cdot C_k^i\| &= \|A_k \cdot C_k^1 \cdot \ldots \cdot C_k^i \cdot \ldots - D_k^n \cdot C_k^1 \cdot \ldots \cdot C_k^i\| \\
&\le \|A_k \cdot C_k^1 \cdot \ldots \cdot C_k^i \cdot \ldots - A_k \cdot C_k^1 \cdot \ldots \cdot C_k^i\| \\
&\quad + \|A_k \cdot C_k^1 \cdot \ldots \cdot C_k^i - D_k^n \cdot C_k^1 \cdot \ldots \cdot C_k^i\| \\
&\le \|A_k\| \cdot \|C_k^1 \cdot \ldots \cdot C_k^i (C_k^{i+1} \cdot C_k^{i+2} \cdot \ldots - I)\| \\
&\quad + \|A_k - D_k^n\| \cdot \|C_k^1 \cdot \ldots \cdot C_k^i\| \\
&\le K \cdot \|C_k^1 \cdot \ldots \cdot C_k^i\| \cdot \|C_k^{i+1} \cdot C_k^{i+2} \cdot \ldots - I\| + \frac{1}{2^n} \cdot \|C_k^1 \cdot \ldots \cdot C_k^i\|
\end{aligned}
$$

for all $k \in \mathbb{N}_0$ and $n, i \in \mathbb{N}$. Considering (4.5)–(4.9), ..., (4.12), ..., (4.13), ..., (4.14), ..., (4.15), we see that

$$(4.20)\qquad C_k^i \in \mathcal{O}^{\varrho}_{\delta_i}(I), \quad k \in \mathbb{N}_0,\ i \in \mathbb{N}.$$

Since (see (4.16) and (4.17))

$$(4.21)\qquad C_k^1 \cdot \ldots \cdot C_k^i \cdot \ldots = C_k^{t(k)}, \quad k \in \mathbb{N}_0,$$

there exists $L > 0$ with the property that (see (4.20))

$$\|C_k^1 \cdot \ldots \cdot C_k^i\| \le L, \quad k \in \mathbb{N}_0,\ i \in \mathbb{N},$$

which implies (see (4.19))

$$(4.22)\qquad \|B_k - D_k^n \cdot C_k^1 \cdot \ldots \cdot C_k^i\| \le KL \cdot \|C_k^{i+1} \cdot C_k^{i+2} \cdot \ldots - I\| + \frac{L}{2^n}$$

for $k \in \mathbb{N}_0$, $n, i \in \mathbb{N}$. Applying (4.20) and (4.21), where $t(k) \le k$ for all $k \in \mathbb{N}_0$ (consider the construction), and putting $i = n$, from (4.22), we obtain (see (4.4))

$$\|B_k - D_k^n \cdot C_k^1 \cdot \ldots \cdot C_k^n\| \le KL \cdot \delta_{n+1} + \frac{L}{2^n} = \frac{\varepsilon L}{2^{n+1}} + \frac{L}{2^n}, \quad k \in \mathbb{N}_0,\ n \in \mathbb{N}.$$

Therefore, for $n \to \infty$, the sequence $\{B_k\}$ is the uniform limit of periodic sequences $\{D_k^n \cdot C_k^1 \cdot \ldots \cdot C_k^n\}_{k \in \mathbb{N}_0}$. In particular, $\{B_k\}$ is limit periodic. We also have (see (4.4), (4.17), and (4.20))

$$(4.23)\quad \|B_k - A_k\| = \|A_k \cdot C_k^{t(k)} - A_k\| \le \|A_k\| \cdot \|C_k^{t(k)} - I\| \le K \cdot \delta_1 = \frac{\varepsilon}{2}, \quad k \in \mathbb{N}_0.$$

Together with the limit periodicity of $\{B_k\}$, (4.23) gives $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$.

Part V. Let $z \in \mathbb{N}$ be given arbitrarily. Now, we consider the solution $\{x_k^z\}_{k \in \mathbb{N}_0}$ of the Cauchy problem

$$(4.24)\qquad x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_z.$$

To prove the statement of the theorem, we assume that there exists $b \ge z$, $b \in \mathbb{N}$, with the property that

$$(4.25)\qquad \|x_k^z\| \ge \frac{1}{2^{b-1}}, \quad k \in \mathbb{N}_0.$$

We have to show that $\{x_k^z\}$ is not asymptotically almost periodic.

Let $i > b$, $i \in \mathbb{N}$, be arbitrary. Since (see (4.25))

$$\left\|x^z_{P(i,z,1)}\right\| = \left\|x^{(i,z,1)}_{P(i,z,1)}\right\| > \frac{1}{2^b} \geq \frac{1}{2^{i-1}}, \tag{4.26}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left\|x^z_{P(i,z,i)}\right\| = \left\|x^{(i,z,i)}_{P(i,z,i)}\right\| > \frac{1}{2^b} \geq \frac{1}{2^{i-1}}, \tag{4.27}$$

we have the following possibilities. If

$$\left\|x^{(i,z,1)}_{P(i,z,1)+r_i} - x^{(i,z,1)}_{P(i,z,1)}\right\| > \frac{\xi}{8} \cdot \left\|x^{(i,z,1)}_{P(i,z,1)}\right\|,$$

then (see (4.26))

$$\left\|x^z_{P(i,z,1)+r_i} - x^z_{P(i,z,1)}\right\| = \left\|x^{(i,z,1)}_{P(i,z,1)+r_i} - x^{(i,z,1)}_{P(i,z,1)}\right\| > \frac{\xi}{8} \cdot \frac{1}{2^b}. \tag{4.28}$$

If

$$\left\|x^{(i,z,1)}_{P(i,z,1)+r_i} - x^{(i,z,1)}_{P(i,z,1)}\right\| \leq \frac{\xi}{8} \cdot \left\|x^{(i,z,1)}_{P(i,z,1)}\right\|,$$

then (see (4.2) and (4.26))

$$\begin{aligned}
&\left\|x^z_{P(i,z,1)+r_i} - x^z_{P(i,z,1)}\right\| \\
&= \left\|M_1^{(i,z,1)} \cdot \ldots \cdot M_{l_i}^{(i,z,1)} \cdot x^{(i,z,1)}_{P(i,z,1)+r_i} - x^{(i,z,1)}_{P(i,z,1)}\right\| \\
&\geq \left\|M_1^{(i,z,1)} \cdot \ldots \cdot M_{l_i}^{(i,z,1)} \cdot x^{(i,z,1)}_{P(i,z,1)+r_i} - x^{(i,z,1)}_{P(i,z,1)+r_i}\right\| \\
&\quad - \left\|x^{(i,z,1)}_{P(i,z,1)+r_i} - x^{(i,z,1)}_{P(i,z,1)}\right\| \\
&= \left|\left\langle x^{(i,z,1)}_{P(i,z,1)+r_i}\right\rangle\right| \\
&\quad \times \left\|M_1^{(i,z,1)} \cdot \ldots \cdot M_{l_i}^{(i,z,1)} \cdot \frac{x^{(i,z,1)}_{P(i,z,1)+r_i}}{\left\langle x^{(i,z,1)}_{P(i,z,1)+r_i}\right\rangle} - \frac{x^{(i,z,1)}_{P(i,z,1)+r_i}}{\left\langle x^{(i,z,1)}_{P(i,z,1)+r_i}\right\rangle}\right\| \\
&\quad - \left\|x^{(i,z,1)}_{P(i,z,1)+r_i} - x^{(i,z,1)}_{P(i,z,1)}\right\| \\
&\geq \frac{\left\|x^{(i,z,1)}_{P(i,z,1)+r_i}\right\|}{2} \cdot \frac{\xi}{2} - \frac{\xi}{8} \cdot \left\|x^{(i,z,1)}_{P(i,z,1)}\right\| > \frac{\xi}{16} \cdot \left\|x^{(i,z,1)}_{P(i,z,1)}\right\| > \frac{\xi}{16} \cdot \frac{1}{2^b},
\end{aligned}$$

i.e.

$$\left\|x^z_{P(i,z,1)+r_i} - x^z_{P(i,z,1)}\right\| > \frac{\xi}{16} \cdot \frac{1}{2^b}. \tag{4.29}$$

Note that, in the estimations above, we also use the fact that

$$\left|\left\langle x^{(i,z,1)}_{P(i,z,1)+r_i}\right\rangle\right| \leq \left\|x^{(i,z,1)}_{P(i,z,1)+r_i}\right\| \leq 2\left|\left\langle x^{(i,z,1)}_{P(i,z,1)+r_i}\right\rangle\right|.$$

We continue these estimations. Finally, we get the following observation. If

$$\left\|x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}} - x^{(i,z,i)}_{P(i,z,i)}\right\| > \frac{\xi}{8} \cdot \left\|x^{(i,z,i)}_{P(i,z,i)}\right\|,$$

then (see (4.27))

$$\left\|x^z_{P(i,z,i)+r_i-r_{i-1}} - x^z_{P(i,z,i)}\right\| = \left\|x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}} - x^{(i,z,i)}_{P(i,z,i)}\right\| > \frac{\xi}{8}\cdot\frac{1}{2^b}. \tag{4.30}$$

If

$$\left\|x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}} - x^{(i,z,i)}_{P(i,z,i)}\right\| \le \frac{\xi}{8}\cdot\left\|x^{(i,z,i)}_{P(i,z,i)}\right\|,$$

then (see (4.27))

$$\begin{aligned}
&\left\|x^z_{P(i,z,i)+r_i-r_{i-1}} - x^z_{P(i,z,i)}\right\| \\
&= \left\|M_1^{(i,z,i)}\cdot\ldots\cdot M_{l_i}^{(i,z,i)}\cdot x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}} - x^{(i,z,i)}_{P(i,z,i)}\right\| \\
&\ge \left\|M_1^{(i,z,i)}\cdot\ldots\cdot M_{l_i}^{(i,z,i)}\cdot x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}} - x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}}\right\| \\
&\quad - \left\|x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}} - x^{(i,z,i)}_{P(i,z,i)}\right\| \\
&= \left|\left\langle x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}}\right\rangle\right| \\
&\quad\times\left\|M_1^{(i,z,i)}\cdot\ldots\cdot M_{l_i}^{(i,z,i)}\cdot\frac{x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}}}{\left\langle x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}}\right\rangle} - \frac{x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}}}{\left\langle x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}}\right\rangle}\right\| \\
&\quad - \left\|x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}} - x^{(i,z,i)}_{P(i,z,i)}\right\| \\
&\ge \frac{\left\|x^{(i,z,i)}_{P(i,z,i)+r_i-r_{i-1}}\right\|}{2}\cdot\frac{\xi}{2} - \frac{\xi}{8}\cdot\left\|x^{(i,z,i)}_{P(i,z,i)}\right\| > \frac{\xi}{16}\cdot\left\|x^{(i,z,i)}_{P(i,z,i)}\right\| > \frac{\xi}{16}\cdot\frac{1}{2^b},
\end{aligned}$$

i.e.

$$\left\|x^z_{P(i,z,i)+r_i-r_{i-1}} - x^z_{P(i,z,i)}\right\| > \frac{\xi}{16}\cdot\frac{1}{2^b}. \tag{4.31}$$

In all these cases, we obtain the existence of numbers $q^i_j \in \mathbb{N}$ for $j \in \{1,\ldots,i\}$ with the property that (see (4.28) together with (4.29), ..., (4.30) with (4.31))

$$\begin{gathered}
\left\|x^z_{q^i_1+r_i} - x^z_{q^i_1}\right\| > \frac{\xi}{16}\cdot\frac{1}{2^b}, \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\left\|x^z_{q^i_i+r_i-r_{i-1}} - x^z_{q^i_i}\right\| > \frac{\xi}{16}\cdot\frac{1}{2^b},
\end{gathered}$$

i.e.

$$\left\|x^z_{p^i_1+r_i} - x^z_{p^i_1}\right\| > \frac{\xi}{16}\cdot\frac{1}{2^b}, \tag{4.32}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left\|x^z_{p^i_i+r_i} - x^z_{p^i_i+r_{i-1}}\right\| > \frac{\xi}{16}\cdot\frac{1}{2^b}, \tag{4.33}$$

where $p^i_1 = q^i_1, \ldots, p^i_i = q^i_i - r_{i-1}$. We repeat that these inequalities follow from the construction for any sufficiently large $i \in \mathbb{N}$.

We apply Theorem 2.5, where we put $s_1 := 0$, $s_{i+1} := r_i$, $i \in \mathbb{N}$. Of course (see (4.10)), $\lim\limits_{i\to\infty} s_i = \infty$. Evidently, for any large $i \in \mathbb{N}$ and all considered integers j, we also have $p^i_j \in \mathbb{N}$. Thus, (4.32), ..., (4.33) imply a contradiction with (2.1) for $\varepsilon = 2^{-b-4}\,\xi$; i.e. the solution of the Cauchy problem (4.24) is not asymptotically almost periodic. □

Remark 4.2. We point out that Theorem 4.1 improves the main results of papers [8], [18], [20] (see also Remark 3.4). Hence, concerning other references, remarks, and examples, we can refer to those articles (and also to [9], [19]). We explicitly recall only the fact that any non-trivial solution $\{x_k\}_{k\in\mathbb{N}_0}$ of any homogeneous linear almost periodic difference system cannot be almost periodic if

$$\liminf_{k\to\infty} \|x_k\| = 0.$$

We refer to [39, Lemma 3.10].

To illustrate the fact that our main result is new even in the real and complex case, we mention the example below. In addition, in this example, we show how our result improves results of [8], [20] (see Theorems 3.3 and 3.5).

Example 4.3. Let us consider the real case (i.e. let $F = \mathbb{R}$) with the usual absolute value. Let $m = 4$ and let the considered group $\mathcal{X}$ be the set of all matrices in the following form

$$\begin{pmatrix} a\cos\alpha & -a\sin\alpha & 0 & 0 \\ a\sin\alpha & a\cos\alpha & 0 & 0 \\ 0 & 0 & b\cos\beta & -b\sin\beta \\ 0 & 0 & b\sin\beta & b\cos\beta \end{pmatrix},$$

where $a, b \neq 0$, $\alpha, \beta \in \mathbb{R}$. Based on Theorem 4.1, we know that, for any countable set of non-zero vectors $u \in \mathbb{R}^4$, in any neighbourhood of any system $\{A_k\}_{k\in\mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$, there exists a system $\{B_k\}_{k\in\mathbb{N}_0}$ for which the solution $\{x_k\}_{k\in\mathbb{N}_0}$ of the Cauchy problem

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = u$$

is not asymptotically almost periodic or $\liminf_{k\to\infty} \|x_k\| = 0$. For comparison, Theorem 3.3 guarantees that the solutions are not almost periodic. Of course, a sequence can be asymptotically almost periodic and, at the same time, non-almost periodic. Using Theorem 3.5, it is possible to guarantee only the existence of one solution $\{x_k\}_{k\in\mathbb{N}_0}$ of $x_{k+1} = B_k \cdot x_k$ which is not asymptotically almost periodic or $\liminf_{k\to\infty} \|x_k\| = 0$.

Example 4.4. In fact, from the proof of Theorem 4.1, it follows that the statement of Theorem 4.1 is true also for concrete vectors $u \in F^m$ satisfying $\|u\| \geq 1$ for which there exist matrices $M_1, \dots, M_l \in \mathcal{X}$ such that (4.1) is valid.

Hence, similarly as in Example 4.3, for the set $\mathcal{X}$ of all matrices

$$\begin{pmatrix} a\cos\alpha & -a\sin\alpha & 0 & 0 & 0 \\ a\sin\alpha & a\cos\alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b\cos\beta & -b\sin\beta \\ 0 & 0 & 0 & b\sin\beta & b\cos\beta \end{pmatrix},$$

where $a, b \neq 0$, $\alpha, \beta \in \mathbb{R}$, we obtain the same conclusion as in Example 4.3 for any countable set of vectors $u = (u_1, u_2, u_3, u_4, u_5)^T \in \mathbb{R}^5$ with the property that $|u_1| + |u_2| + |u_4| + |u_5| > 0$.

In addition, using the proof of Theorem 4.1, it is possible to prove the following result (cf. Theorem 3.5).

Theorem 4.5. *Let the unit ball $\{u \in F^m;\ \|u\| \leq 1\}$ be compact. Let $\mathcal{X}$ be commutative and have the property that there exists $\xi > 0$ such that, for any $\delta > 0$ and for any $u \in F^m$, $\|u\| \geq 1$, there exist matrices $M_1, \dots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}^{\varrho}_{\delta}(I), \quad i \in \{1, \dots, l\}, \qquad \|M_l \cdot \ldots \cdot M_1 \cdot u - u\| > \xi. \tag{4.34}$$

Then, for any $\varepsilon > 0$, $\{A_k\}_{k\in\mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$, and for any sequence $\{v_n\}_{n\in\mathbb{N}}$ of non-zero vectors $v_n \in F^m$, there exists $\{B_k\}_{k\in\mathbb{N}_0} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ with the property that the solution $\{x^n_k\}_{k\in\mathbb{N}_0}$ of

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{N}_0, \qquad x_0 = v_n$$

is not asymptotically almost periodic or $\liminf_{k\to\infty} \|x^n_k\| = 0$ for all $n \in \mathbb{N}$.

Proof. It suffices to show that the number l of matrices $M_1, \dots, M_l$ in (4.34) can be taken as the same for the given $\delta > 0$ and for all $u \in F^m$, $1 \leq \|u\| \leq 2$ (consider the beginning of the proof of Theorem 4.1). For any $v \in \{u \in F^m;\ 1 \leq \|u\| \leq 2\}$, there exist matrices $M^v_1, \dots, M^v_{l(v)} \in \mathcal{X}$ such that

$$M^v_j \in \mathcal{O}^{\varrho}_{\delta}(I), \quad j \in \{1, \dots, l(v)\}, \qquad \|M^v_{l(v)} \cdot \ldots \cdot M^v_1 \cdot v - v\| > \xi.$$

Thus, there exists an open set $U \subseteq F^m$ such that $v \in U$ and

$$\|M^v_{l(v)} \cdot \ldots \cdot M^v_1 \cdot u - u\| > \frac{\xi}{2}, \quad u \in U.$$

From the set of all such sets U, one can extract a finite set $\{U_1, \dots, U_i\}$ which covers the considered set $\{u \in F^m;\ 1 \leq \|u\| \leq 2\}$. Let us consider l as the maximum of the used $l(v)$. Thus, this theorem follows from the proof of Theorem 4.1, where ξ is replaced by $\xi/2$. □

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