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A GLOBAL MULTIPLICITY RESULT FOR A VERY SINGULAR CRITICAL NONLOCAL EQUATION

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Abstract. In this article we show the global multiplicity result for the following nonlocal singular problem

$$(\mathrm{P}_\lambda) \qquad (-\Delta)^s u = u^{-q} + \lambda u^{2^*_s-1}, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > 2s$, $s \in (0,1)$, $\lambda > 0$, $q > 0$ satisfies $q(2s-1) < (2s+1)$ and $2^*_s = 2n/(n-2s)$. Employing the variational method, we show the existence of at least two distinct weak positive solutions for (P_λ) in X_0 when $\lambda \in (0,\Lambda)$ and no solution when $\lambda > \Lambda$, where $\Lambda > 0$ is appropriately chosen. We also prove a result of independent interest that any weak solution to (P_λ) is in $C^\alpha(\mathbb{R}^n)$ with $\alpha = \alpha(s,q) \in (0,1)$. The asymptotic behaviour of weak solutions reveals that this result is sharp.

1. Introduction

In this article we prove the existence, multiplicity and Hölder regularity of weak solutions to the following fractional critical and singular elliptic equation

$$(\mathrm{P}_\lambda) \qquad (-\Delta)^s u = u^{-q} + \lambda u^{2^*_s-1}, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > 2s$, $s \in (0,1)$, $\lambda > 0$, $q > 0$ satisfies $q(2s-1) < (2s+1)$ and $2^*_s = 2n/(n-2s)$. The fractional

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Laplace operator denoted by $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = 2C_s^n \quad \text{P.V.} \quad \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

where P.V. denotes the Cauchy principal value,

$$C_s^n = \pi^{-n/2}\, 2^{2s-1} s\, \frac{\Gamma[(n+2s)/2]}{\Gamma(1-s)},$$

and Γ is the Gamma function. The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance, see [3] for instance. The theory of fractional Laplacian and elliptic equations involving it as the principal part has been evolved immensely in recent years. There is a vast literature available on it, however without giving an exhaustive list we cite [7], [10], [14], [16], [19], [21], [22] for motivation to readers.

Nowadays, researchers are inspecting on various forms of singular nonlocal equations. We cite [11], [8], [9] as some contemporary woks related to it. The fractional elliptic equations with singular and critical nonlinearities was first studied by Barrios et al. in [5]. The authors considered the problem

$$(-\Delta)^s u = \lambda \frac{f(x)}{u^\gamma} + M u^p, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where $n > 2s$, $M \geq 0$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $1 < p < 2_s^* - 1$ and $f \in L^m(\Omega)$, $m \geq 1$ is a nonnegative function. Here, authors studied the existence of distributional solutions using the uniform estimates of $\{u_n\}$ which are solutions of the regularized problems with singular term $u^{-\gamma}$ replaced by $(u+1/n)^{-\gamma}$. Motivated by their results, Sreenadh and Mukherjee in [15] studied the singular problem

$$(-\Delta)^s u = \lambda a(x) u^{-q} + u^{2_s^*-1}, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where $\lambda > 0$, $0 < q \leq 1$ and $\theta \leq a(x) \in L^\infty(\Omega)$, for some $\theta > 0$. They showed that although the energy functional corresponding to this problem fails to be Fréchet differentiable, making use of its Gâteaux differentiability the Nehari manifold technique can still be benefitted to obtain existence of at least two solutions over a certain range of λ. The significance of q being less than 1 is the Gâteaux differentiability of the functional corresponding to the problem. Consider now the case $q > 1$. Let

$$X \stackrel{\text{def}}{=} \bigg\{ u \;\bigg|\; u \colon \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, \\ u|_\Omega \in L^2(\Omega) \text{ and } \frac{(u(x)-u(y))}{|x-y|^{n/2+s}} \in L^2(Q) \bigg\},$$

where $Q \overset{\text{def}}{=} \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$ endowed with the norm

$$\|u\|_X \overset{\text{def}}{=} \|u\|_{L^2(\Omega)} + [u]_X,$$

where

$$[u]_X = \left(\int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2}.$$

Let $J \colon X_0 \to \mathbb{R}$ be the functional defined by

$$J(u) \overset{\text{def}}{=} \frac{C_s^n}{2} \|u\|_{X_0}^2 - \frac{1}{1-q} \int_\Omega |u|^{1-q} \, dx - \frac{\lambda}{2_s^*} \int_\Omega |u|^{2_s^*} \, dx$$

for any u in the Hilbert space $X_0 \overset{\text{def}}{=} \{u \in X : u = 0 \text{ almost everywhere in } \mathbb{R}^n \setminus \Omega\}$ equipped with the inner product

$$\langle u, v \rangle \overset{\text{def}}{=} \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy.$$

J may not be defined on the whole space nor it is even continuous on $D(I) \equiv \{u \in X_0 : I(u) < \infty\}$ and the approach used for $q < 1$ can not be extended. Besides this, one has that the interior of $D(I) = \emptyset$ because of the singular term. But we notice that if we enforce the condition $q > 1$ satisfies $q(2s-1) < (2s+1)$, then we can prove that $D(I)$ is non empty and Gâteaux differentiable on a suitable convex cone of X_0.

The existence of weak solutions to (P_λ) when $\lambda \in (0, \Lambda)$ and no solution when $\lambda > \Lambda$ has been already obtained by Giacomoni et al. in [12]. But here the multiplicity of solutions has been achieved in $L^1_{\mathrm{loc}}(\Omega)$ only, by using non smooth critical point theory, so the questions of existence of solutions in the energy space and of Hölder regularity were still pending. This article is bringing answers to these two issues. For that, we followed the approach of [13] but we notify that the adversity and novelty of this article lies in extending Haitao's technique in a nonlocal framework. The regularity of weak solution of the purely singular problem

$$(-\Delta)^s u = u^{-q}, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega$$

plays a vital role in our study. This has been obtained by Adimurthi, Giacomoni and Santra in [1] in recent times. In the present paper we extend the Hölder regularity results proved in [1, Theorem 1.4] in our framework of weak solutions (see Definition 1.1 below) rather than the more restricted classical solutions framework (defined in [1]). It requires additional L^∞-estimates and the use of the weak comparison principle.

Our paper has been organized as follows. Section 2 contains some preliminary results used in the subsequent sections. Sections 3 and 4 contain the proof of existence of first and second weak solution to (P_λ) respectively (Theorem 1.2). The proof of the Hölder regularity result (Theorem 1.3) is given in Section 4 based on a priori estimates proved in Proposition 4.1. Now we state the main results proved in the paper. First we define the notion of weak solutions.

Definition 1.1. A function $u \in X_0$ is said to be a weak solution of (P_λ) if there exists a $m_K > 0$ such that $u > m_K$ in every compact subset K of Ω, and for all $\phi \in X_0$ it satisfies

$$C_s^n \int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega (u^{-q} + u^{2_s^*-1})\phi\,dx.$$

Given any $\phi \in C_0(\overline{\Omega})$ such that $\phi > 0$ in Ω we define

$$C_\phi(\Omega) \stackrel{\mathrm{def}}{=} \{u \in C_0(\overline{\Omega}) \mid \text{exists } c \geq 0 \text{ such that } |u(x)| \leq c\phi(x), \text{ for all } x \in \Omega\}$$

with the usual norm $\|u/\phi\|_{L^\infty(\Omega)}$ and the associated positive cone. We define the following open convex subset of $C_\phi(\Omega)$ as

$$C_\phi^+(\Omega) \stackrel{\mathrm{def}}{=} \left\{ u \in C_\phi(\Omega) \;\middle|\; \inf_{x\in\Omega} \frac{u(x)}{\phi(x)} > 0 \right\}.$$

In particular, C_ϕ^+ contains all those functions $u \in C_0(\Omega)$ with $k_1\phi \leq u \leq k_2\phi$ in Ω for some $k_1, k_2 > 0$. Let $\phi_{1,s}$ be the first positive normalized eigenfunction ($\|\phi_{1,s}\|_{L^\infty(\Omega)} = 1$) of $(-\Delta)^s$ in X_0. We recall that $\phi_{1,s} \in C^s(\mathbb{R}^N)$ and $\phi_{1,s} \in C_{\delta^s}^+(\Omega)$ where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ (see for instance Proposition 1.1 and Theorem 1.2 in [17]). We then define the barrier function ϕ_q as follows:

$$\phi_q \stackrel{\mathrm{def}}{=} \begin{cases} \phi_{1,s} & \text{if } 0 < q < 1, \\ \phi_{1,s}\left(\ln\left(\dfrac{2}{\phi_{1,s}}\right)\right)^{1/(q+1)} & \text{if } q = 1, \\ \phi_{1,s}^{2/(q+1)} & \text{if } q > 1. \end{cases} \tag{1.1}$$

We prove the following as the main results:

Theorem 1.2. *There exists $\Lambda > 0$ such that*

(a) (P_λ) *admits at least two solutions in $X_0 \cap C_{\phi_q}^+(\Omega)$ for every $\lambda \in (0, \Lambda)$;*
(b) (P_λ) *admits no solution for $\lambda > \Lambda$;*
(c) (P_Λ) *admits at least one positive solution $u_\Lambda \in X_0 \cap C_{\phi_q}^+(\Omega)$.*

Theorem 1.3. *Let $\lambda \in (0, \Lambda]$, $q > 0$ satisfy $q(2s-1) < (2s+1)$ and $u \in X_0$ be any positive weak solution of* (P_λ). *Then*

(a) $u \in C^s(\mathbb{R}^n)$ *when $0 < q < 1$;*
(b) $u \in C^{s-\varepsilon}(\mathbb{R}^n)$ *for any small enough $\varepsilon > 0$ when $q = 1$;*
(c) $u \in C^{2s/(q+1)}(\mathbb{R}^n)$ *when $q > 1$.*

Remark 1.4. Here, the Hölder regularity for the weak solutions of (P_λ) obtained is optimal because of the behavior of the solution near $\partial\Omega$ since we showed that any weak solution of (P_λ) lies in $C_{\phi_q}^+(\Omega)$.

Remark 1.5. It follows from Theorem 1.3 that the extremal solution (when $\lambda = \Lambda$), in case of critical growth nonlinearities is a classical solution which extends the results in [1] where in this regard only subcritical nonlinearities are considered.

2. Preliminaries

We start by some preliminary results. The energy functional corresponding to (P_λ) is given by $I_\lambda \colon X_0 \to \mathbb{R}$ defined as

$$I_\lambda(u) \stackrel{\text{def}}{=} \begin{cases} \dfrac{C_s^n \|u\|_{X_0}^2}{2} - \dfrac{1}{1-q}\displaystyle\int_\Omega |u|^{1-q}\,dx - \dfrac{1}{2_s^*}\int_\Omega |u|^{2_s^*}\,dx & \text{if } q \neq 1, \\ \dfrac{C_s^n \|u\|_{X_0}^2}{2} - \displaystyle\int_\Omega \ln|u|\,dx - \dfrac{1}{2_s^*}\int_\Omega |u|^{2_s^*}\,dx & \text{if } q = 1. \end{cases}$$

Let $q > 0$ satisfies $q(2s-1) < (2s+1)$. Then for any $\varphi \in X_0$ and $u \in C_{\phi_q}^+(\Omega)$, by Hardy's inequality (see [23, Lemma 3.2.6.1, p. 259]) we obtain

$$\int_\Omega u^{-q}\varphi \leq \bigg(\int_\Omega \frac{dx}{(\delta(x))^{2s(q-1)/(q+1)}}\bigg)^{1/2} \bigg(\frac{\varphi^2}{(\delta(x))^{2s}}\bigg)^{1/2} < K\|\varphi\| < +\infty \tag{2.1}$$

where $K > 0$ is a constant. If we define $D(I) = \{u \in X_0 : I_\lambda(u) < \infty\}$, then in virtue of (2.1) we get that $D(I) \neq \emptyset$. This gives an importance of the inequality $q(2s-1) < (2s+1)$. From the proof of [1, Theorem 1.2], we know that if $0 < q < 1$ and $u \in X_0$ satisfies $u \geq c\delta^s$, then I_λ is Gâteaux differentiable at u. In the proposition below, we show the same property of I_λ when $q \geq 1$ satisfies $q(2s-1) < (2s+1)$.

Proposition 2.1. *If $M = \{u \in X_0 : u_1 \leq u \leq u_2\}$ where $u_1 \in C_{\phi_q}^+(\Omega)$ and $u_2 \in X_0$, then I_λ is Gâteaux differentiable at u in the direction $(v-u)$ where $v, u \in M$.*

Proof. We need to show that

$$\begin{aligned} \lim_{t \to 0} \frac{I_\lambda(u + t(v-u)) - I_\lambda(u)}{t} &= C_s^n \int_Q \frac{(v(x)-v(y))((v-u)(x)-(v-u)(y))}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad - \int_\Omega u^{-q}(v-u)\,dx - \lambda\int_\Omega u^{2_s^*-1}(v-u)\,dx. \end{aligned}$$

It is enough to show this for the singular term; for the rest two terms, the proof is standard. For any $t \in (0,1)$, $u + t(v-u) \in M$ since M is convex. Consider

$$F(u) = \frac{1}{1-q}\int_\Omega u^{1-q}\,dx.$$

Using Mean Value Theorem we get

$$\begin{aligned}\frac{F(u+t(v-u))-F(u)}{t} &= \frac{1}{t(1-q)}\int_\Omega \big((u+t(v-u))^{1-q}-u^{1-q}\big)(x)\,dx\\ &= \int_\Omega (u+t\theta(v-u))^{-q}(x)(v-u)(x)\,dx\end{aligned}$$

for some $\theta \in (0,1)$. Since $(u+t\theta(v-u)) \in M$ and (2.1), we have

$$\int_\Omega (u+t\theta(v-u))^{-q}(v-u)\,dx \le \int_\Omega u_1^{-q}(v-u)\,dx < +\infty.$$

So, using the Lebesgue Dominated Convergence theorem, we pass to the limit $t \to 0$ and get

$$\lim_{t\to 0}\frac{F(u+t(v-u))-F(u)}{t} = \int_\Omega u^{-q}(v-u)\,dx. \qquad \square$$

Let $L(u) := (-\Delta)^s u - u^{-q}$. Then L forms a monotone operator. So we have the following comparison principle:

Lemma 2.2. *Let $u_1, u_2 \in X_0 \cap C^+_{\phi_q}(\Omega)$ are weak solutions to*

$$L(u_1) = g_1 \quad \text{in } \Omega, \qquad L(u_2) = g_2 \quad \text{in } \Omega,$$

with $g_1, g_2 \in L^2(\Omega)$ such that $g_1 \le g_2$ almost everywhere in Ω. Then $u_1 \le u_2$ almost everywhere in Ω. Moreover, if $g \in L^\infty(\Omega)$, then the problem

$$L(u) = g \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

has a unique solution in X_0.

3. Existence result

Let us define $\Lambda := \sup\{\lambda > 0 : (\mathrm{P}_\lambda) \text{ has a weak solution }\}$. Also let $\underline{w} \in C_0(\overline{\Omega})$ solves the purely singular problem

$$(-\Delta)^s \underline{w} = \underline{w}^{-q}, \quad \underline{w} > 0 \quad \text{in } \Omega, \quad \underline{w} = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.$$

Then [1, Theorems 1.2 and 1.4] give us that $\underline{w}$ is unique, $\underline{w} \in X_0 \cap C^+_{\phi_q}(\Omega)$ and $\underline{w} \in C^{s_q}(\mathbb{R}^n)$, where

$$s_q \overset{\text{def}}{=} \begin{cases} s & \text{if } q < 1,\\ s-\varepsilon \text{ for any } \varepsilon > 0 & \text{if } q = 1,\\ \dfrac{2s}{q+1} & \text{if } 1 < q \text{ and } q(2s-1) < 2s+1.\end{cases}$$

For the sake of clarity we basically focus on the case $1 \le q$ and $q(2s-1) < (2s+1)$. Indeed, when $q \in (0,1)$, the case follows easily along the same line. In this context, the next result is an important lemma for Λ.

Lemma 3.1. *It holds $0 < \Lambda < +\infty$.*

Proof. First we prove that $\Lambda < +\infty$. Using $\phi_{1,s}$ as the test function in (P_λ) we get

$$\begin{aligned}(3.1)\qquad \int_\Omega (u^{-q}\phi_{1,s} + \lambda u^{2_s^*-1}\phi_{1,s})\,dx &= \int_{\mathbb{R}^n} \phi_{1,s}(-\Delta)^s u\,dx \\ &= \int_{\mathbb{R}^n} u(-\Delta)^s \phi_{1,s}\,dx = \lambda_{1,s}\int_\Omega u\phi_{1,s}\,dx.\end{aligned}$$

If we choose a $\lambda > 0$ which satisfies $t^{-q} + \lambda t^{2_s^*-1} > 2\lambda_{1,s}t$ for all $t > 0$, then we get a contradiction to (3.1). Therefore $\Lambda < +\infty$. Now to prove $\Lambda > 0$ we need sub- and supersolution for (P_λ). It is easy to see that $\underline{u_\lambda} = \underline{w}$ forms a subsolution of (P_λ) and $\overline{u_\lambda} = \underline{u_\lambda} + Mz$ for $\lambda > 0$ small enough and for a $M = M(\lambda) > 0$ forms a supersolution of (P_λ), where $0 < z \in X_0$ solves $(-\Delta)^s z = 1$ in Ω. Now we define the closed convex subset M_λ of X_0 as

$$M_\lambda := \{u \in X_0 : \underline{u_\lambda} \le u \le \overline{u_\lambda}\}.$$

Consider the iterative scheme ($k \ge 1$):

$$(\mathrm{P}_{\lambda,k})\qquad (P_{\lambda,k}) \begin{cases} (-\Delta)^s u_k - u_k^{-q} = \lambda u_{k-1}^{2_s^*-1},\ u_k > 0 & \text{in } \Omega, \\ u_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

with $u_0 = \underline{u_\lambda}$. The existence of $\{u_k\}$ in $X_0 \cap M_\lambda \cap C^+_{\phi_q}(\Omega)$ can be proved by considering the approximated problem corresponding to $(\mathrm{P}_{\lambda,k})$, for instance we refer [6, Proposition 2.3]. From Lemma 2.2, it follows that $\{u_k\}$ is increasing and $u_k \in M_\lambda$ for all k. Let $\lim\limits_{k\uparrow\infty} u_k = u_\lambda$. Then testing $(\mathrm{P}_{\lambda,k})$ by u_k we get

$$\|u_k\|^2 \le 2\int_\Omega \overline{u_\lambda}^2\,dx + \lambda\int_\Omega \overline{u_\lambda}^{2_s^*}\,dx + \int_\Omega \overline{u_\lambda}\,\underline{u_\lambda}^{-q} \le K_\lambda$$

where $K_\lambda > 0$ is a constant depending on λ. So, up to a subsequence, $u_k \rightharpoonup u_\lambda$ in X_0. Finally, using the Lebesgue dominated convergence Theorem, we pass through the limit in $(\mathrm{P}_{\lambda,k})$ to obtain u_λ solves (P_λ) weakly and obviously, $u_\lambda \in M_\lambda$. This proves that $\Lambda > 0$. □

In the next result, we prove the existence of a weak solution for (P_λ) whenever $\lambda \in (0, \Lambda)$. In the proof, we use a minimization on a conical shell argument similar as in [2, Lemma 4.1] and in [6, Proposition 3.5]. But here we take advantage of the existence of a strict positive subsolution to control the singular nonlinearity.

Proposition 3.2. *For each $\lambda \in (0, \Lambda)$, (P_λ) admits a weak solution w in $C^+_{\phi_q}(\Omega)$.*

Proof. The proof goes along the line of Perron's method adapted over a nonlocal framework (see [13, Lemma 2.2]). Let $\lambda \in (0, \Lambda)$ and $\lambda' \in (\lambda, \Lambda)$. Then it is easy to see that $u_{\lambda'}$, a weak solution of $(\mathrm{P}_{\lambda'})$, forms a supersolution for (P_λ). Such a λ' exists because of the definition of Λ and Lemma 3.1. Let $\underline{u_\lambda}$ be the same function as defined in Lemma 3.1 and consider the closed convex subset W_λ of X_0 as $W_\lambda = \{u \in X_0 : \underline{u_\lambda} \leq u \leq u_{\lambda'}\}$. Then, for each $u \in W_\lambda$, because of fractional Sobolev embedding I_λ satisfies

$$I_\lambda(u) \geq \begin{cases} \dfrac{C_s^n\|u\|^2}{2} - \dfrac{C}{2_s^*}\|u\|^{2_s^*} & \text{if } q > 1, \\ \dfrac{C_s^n\|u\|^2}{2} - \dfrac{C}{2_s^*}\|u\|^{2_s^*} - C(\lambda') & \text{if } q \leq 1, \end{cases}$$

where $C(\lambda')$ is a positive constant depending solely on λ'. Then I_λ is bounded from below and coercive over W_λ. If $\{u_k\} \subset W_\lambda$ be such that $u_k \rightharpoonup u_0$ in X_0 as $k \to \infty$, then since for each k, $u_k \geq \underline{u_\lambda}$ for $q > 1$ and $u_k \leq u_{\lambda'}$ for $q \in (0,1]$, $\int_\Omega u_k^{1-q}\,dx \leq \int_\Omega \underline{u_\lambda}^{1-q}\,dx$, we can use Lebesgue dominated convergence theorem to get that

$$\int_\Omega u_k^{1-q}\,dx \to \int_\Omega u_0^{1-q}\,dx \quad \text{as}\, k \to \infty.$$

Hence from weak lower semicontinuity of norms, it follows that I_λ is weakly lower semicontinuous over W_λ. Moreover, W_λ is weakly sequentially closed subset of X_0. Therefore there exists a $w \in W_\lambda$ such that

$$\inf_{u \in W_\lambda} I_\lambda(u) = I_\lambda(w). \tag{3.2}$$

Claim. w is a weak solution of (P_λ).

Let $\varphi \in X_0$ and $\varepsilon > 0$; we define

$$v_\varepsilon = \min\{u_{\lambda'}, \max\{\underline{u_\lambda}, w + \varepsilon\varphi\}\} = w + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon$$

where $\varphi^\varepsilon = \max\{0, w+\varepsilon\varphi-u_{\lambda'}\}$ and $\varphi_\varepsilon = \max\{0, \underline{u_\lambda}-w-\varepsilon\varphi\}$. By construction $v_\varepsilon \in W_\lambda$ and $\varphi^\varepsilon, \varphi_\varepsilon \in X_0 \cap L^\infty(\Omega)$. Since $w + t(v_\varepsilon - w) \in W_\lambda$ for each $t \in (0,1)$, using (3.2) and Proposition 2.1 we get that

$$\begin{aligned} 0 &\leq \lim_{t \to 0^+} \frac{I_\lambda(w + t(v_\varepsilon - w)) - I_\lambda(w)}{t} \\ &= \int_Q (v_\varepsilon - w)(-\Delta)^s w\,dx - \int_\Omega w^{-q}(v_\varepsilon - w)\,dx - \int_\Omega w^{2_s^*-1}(v_\varepsilon - w)\,dx. \end{aligned}$$

This gives

$$\int_{\mathbb{R}^n} \varphi(-\Delta)^s w\,dx - \int_\Omega (w^{-q} + \lambda w^{2^*s-1})\varphi\,dx \geq \frac{1}{\varepsilon}(E^\varepsilon - E_\varepsilon), \tag{3.3}$$

where

$$\begin{aligned} E^\varepsilon &= \int_{\mathbb{R}^n} \varphi^\varepsilon(-\Delta)^s w\,dx - \int_\Omega (w^{-q} + \lambda w^{2_s^*-1})\varphi^\varepsilon\ dx \\ &= \int_{\mathbb{R}^n} \varphi^\varepsilon(-\Delta)^s(w - u_{\lambda'})\,dx + \int_{\mathbb{R}^n} \varphi^\varepsilon(-\Delta)^s u_{\lambda'}\,dx - \int_\Omega (w^{-q} + \lambda w^{2_s^*-1})\varphi^\varepsilon\ dx \end{aligned}$$

and

$$
\begin{aligned}
E_\varepsilon &= \int_{\mathbb{R}^n} \varphi_\varepsilon (-\Delta)^s w \, dx - \int_\Omega (w^{-q} + \lambda w^{2_s^*-1}) \varphi_\varepsilon \; dx \\
&= \int_{\mathbb{R}^n} \varphi_\varepsilon (-\Delta)^s (w - \underline{u_\lambda}) \, dx + \int_{\mathbb{R}^n} \varphi_\varepsilon (-\Delta)^s \underline{u_\lambda} \, dx - \int_\Omega (w^{-q} + \lambda w^{2_s^*-1}) \varphi_\varepsilon \; dx.
\end{aligned}
$$

We define

$$
\Omega^\varepsilon = \{x \in \Omega : (w + \varepsilon\varphi)(x) \geq u_{\lambda'} > w(x)\}
$$

so that $\mathcal{L}(\Omega^\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and also

$$
\mathcal{C}\Omega^\varepsilon := \Omega \setminus \Omega_\varepsilon \subset \{x \in \Omega : (w + \varepsilon\varphi)(x) < u_{\lambda'}(x)\},
$$

which implies that $\mathcal{L}(\Omega^\varepsilon \times \mathcal{C}\Omega^\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Now we consider the term

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \varphi^\varepsilon (-\Delta)^s (w - u_{\lambda'}) \, dx \\
&= \int_Q \frac{((w - u_{\lambda'})(x) - (w - u_{\lambda'})(y))(\varphi^\varepsilon(x) - \varphi^\varepsilon(y)}{)} |x-y|^{n+2s} \, dx \, dy \\
&= \int_{\Omega^\varepsilon} \int_{\Omega^\varepsilon} \frac{|(w - u_{\lambda'})(x) - (w - u_{\lambda'})(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \\
&\quad + \varepsilon \int_{\Omega^\varepsilon} \int_{\Omega^\varepsilon} \frac{((w - u_{\lambda'})(x) - (w - u_{\lambda'})(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dx \, dy \\
&\quad + 2 \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega^\varepsilon} \frac{(w - u_{\lambda'})^2(x)}{|x-y|^{n+2s}} \, dx \, dy + 2\varepsilon \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega^\varepsilon} \frac{(w - u_{\lambda'})(x)\varphi(x)}{|x-y|^{n+2s}} \, dx \, dy \\
&\quad - 2 \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega^\varepsilon} \frac{(w - u_{\lambda'})(x)(w - u_{\lambda'})(y)}{|x-y|^{n+2s}} \, dx \, dy \\
&\quad + 2\varepsilon \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega^\varepsilon} \frac{(w - u_{\lambda'})(y)\varphi(x)}{|x-y|^{n+2s}} \, dx \, dy \\
&\quad + 2 \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega} \frac{(w - u_{\lambda'})^2(x)}{|x-y|^{n+2s}} \, dx \, dy + 2\varepsilon \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega} \frac{(w - u_{\lambda'})(x)\varphi(x)}{|x-y|^{n+2s}} \, dx \, dy \\
&\geq \varepsilon \int_{\Omega^\varepsilon} \int_{\Omega^\varepsilon} \frac{((w - u_{\lambda'})(x) - (w - u_{\lambda'})(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dx \, dy \\
&\quad + 2\varepsilon \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega^\varepsilon} \frac{(w - u_{\lambda'})(x)\varphi(x)}{|x-y|^{n+2s}} \, dx \, dy - 2\varepsilon^2 \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega^\varepsilon} \frac{\varphi(x)\varphi(y)}{|x-y|^{n+2s}} \, dx \, dy \\
&\quad + 2\varepsilon \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega^\varepsilon} \frac{(w - u_{\lambda'})(y)\varphi(x)}{|x-y|^{n+2s}} \, dx \, dy + 2\varepsilon \int_{\Omega^\varepsilon} \int_{\mathcal{C}\Omega} \frac{(w - u_{\lambda'})(x)\varphi(x)}{|x-y|^{n+2s}} \, dx \, dy,
\end{aligned}
$$

where, in order to obtain the last inequality, we use the fact that if $(x, y) \in \Omega^\varepsilon \times \mathcal{C}\Omega^\varepsilon$, then

$$
(w - u_{\lambda'})(x)(w - u_{\lambda'})(y) \leq \varepsilon^2 \varphi(x)\varphi(y).
$$

Therefore we get

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^n} \varphi^\varepsilon(-\Delta)^s(w-u_{\lambda'})\,dx \geq o(1) \quad \text{as } \varepsilon \to 0^+.$$

Moreover, using the fact that $u_{\lambda'}$ is a supersolution of (P_λ), the other terms of $E^\varepsilon/\varepsilon$ can be estimated as

$$\begin{aligned}
\frac{1}{\varepsilon}&\int_{\mathbb{R}^n} \varphi^\varepsilon(-\Delta)^s u_{\lambda'}\,dx - \frac{1}{\varepsilon}\int_\Omega \big(w^{-q}+\lambda w^{2^*s-1}\big)\varphi^\varepsilon\,dx \\
&\geq \frac{1}{\varepsilon}\int_{\Omega^\varepsilon}\big(u_{\lambda'}^{-q}-w^{-q}\big)\varphi^\varepsilon\,dx + \frac{1}{\varepsilon}\int_{\Omega^\varepsilon}\big(u_{\lambda'}^{2^*_s-1}-w^{2^*_s-1}\big)\varphi^\varepsilon\,dx \\
&\geq -\int_{\Omega^\varepsilon}\big|u_{\lambda'}^{-q}-w^{-q}\big|\,|\varphi|\,dx = o(1)
\end{aligned}$$

as $\varepsilon \to 0^+$. Altogether we get $E^\varepsilon/\varepsilon \geq o(1)$ as $\varepsilon \to 0^+$ and similarly we obtain $E_\varepsilon/\varepsilon \leq o(1)$ as $\varepsilon \to 0^+$. Hence (3.3) gives that for all $\varphi \in X_0$

$$\int_{\mathbb{R}^n} \varphi(-\Delta)^s w\,dx - \int_\Omega \big(w^{-q}+\lambda w^{2^*_s-1}\big)\varphi\,dx \geq o(1) \quad \text{as } \varepsilon \to 0^+$$

but since φ was arbitrary, this implies that w is a weak solution of (P_λ). $\square$

We now prove a special property of w, the weak solution of (P_λ) obtained in Proposition 3.2 following the proof in [6, Proposition 3.5]. We also refer [2, Proposition 5.2] where similar ideas were already used.

Lemma 3.3. *Let $\lambda \in (0,\Lambda)$ and w denotes the weak solution of* (P_λ) *obtained in Proposition* 3.2. *Then w forms a local minimum of the functional I_λ.*

Proof. We argue by contradiction, so suppose w is not a local minimum of I_λ. Then there exists a sequence $\{u_k\} \subset X_0$ satisfying

$$\|u_k - w\| \to 0 \quad \text{as } k \to \infty \quad \text{and} \quad I_\lambda(u_k) < I_\lambda(w). \tag{3.4}$$

We define $\underline{u} = \underline{u_\lambda}$ and $\overline{u} = u_{\lambda'}$ as sub- and supersolution of (P_λ) as defined in the proof of Proposition 3.2. Also we define

$$v_k = \max\{\underline{u}, \min\{u_k, \underline{u}\}\} = \begin{cases} \underline{u} & \text{if } u_k < \underline{u}, \\ u_k & \text{if } \underline{u} \leq u_k \leq \overline{u}, \\ \overline{u} & \text{if } u_k > \overline{u}, \end{cases}$$

and $\underline{w_k} = (u_k - \underline{u})^-$, $\overline{w_k} = (u_k - \overline{u})^+$. Correspondingly, we define the sets

$$\underline{S_k} = \mathrm{Supp}(\underline{w_k}) \quad \text{and} \quad \overline{S_k} = \mathrm{Supp}(\overline{w_k}).$$

Then $u_k = v_k - \underline{w_k} + \overline{w_k}$ and $v_k \in W_\lambda$ where W_λ has been defined in Proposition 3.2. The main idea of the proof is to establish that the measures of $\overline{S_k}$ and

$\underline{S_k}$ tend to 0 as $k \to \infty$ which together with $v_k \in W_\lambda$ force $I_\lambda(u_k)$ to be beyond $I_\lambda(w)$. First, we have that

$$\begin{aligned}\int_\Omega (u_k^+)^{1-q}\,dx &= \int_{\underline{S_k}} (u_k^+)^{1-q}\,dx + \int_{\overline{S_k}} (u_k^+)^{1-q}\,dx + \int_{\underline{u}\le v_k \le \overline{u}} (v_k)^{1-q}\,dx \\ &= \int_{\underline{S_k}} \big((u_k^+)^{1-q} - \underline{u}^{1-q}\big)\,dx + \int_{\overline{S_k}} \big((u_k^+)^{1-q} - \overline{u}^{1-q}\big)\,dx + \int_\Omega (v_k)^{1-q}\,dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega (u_k^+)^{2_s^*}\,dx &= \int_{\underline{S_k}} (u_k^+)^{2_s^*}\,dx + \int_{\overline{S_k}} (u_k^+)^{2_s^*}\,dx + \int_{\underline{u}\le v_k \le \overline{u}} (v_k)^{2_s^*}\,dx \\ &= \int_{\underline{S_k}} \big((u_k^+)^{2_s^*} - \underline{u}^{2_s^*}\big)\,dx + \int_{\overline{S_k}} \big((u_k^+)^{2_s^*} - \overline{u}^{2_s^*}\big)\,dx + \int_\Omega (v_k)^{2_s^*}\,dx.\end{aligned}$$

Then we can express $I_\lambda(u_k)$ as

$$\begin{aligned}I_\lambda(u_k) = {}& I_\lambda(v_k) + \frac{J_0}{2} \\ & - \frac{1}{1-q}\left(\int_{\underline{S_k}} \big((u_k^+)^{1-q} - \underline{u}^{1-q}\big)\,dx + \int_{\overline{S_k}} \big((u_k^+)^{1-q} - \overline{u}^{1-q}\big)\,dx\right) \\ & - \frac{\lambda}{2_s^*}\left(\int_{\underline{S_k}} \big((u_k^+)^{2_s^*} - \underline{u}^{2_s^*}\big)\,dx + \int_{\overline{S_k}} \big((u_k^+)^{2_s^*} - \overline{u}^{2_s^*}\big)\,dx\right)\end{aligned} \tag{3.5}$$

where $J_0 = C_s^n\big(\|u_k\|^2 - \|v_k\|^2\big)$. While denoting $S_k = \{x \in \Omega : \underline{u} \le v_k \le \overline{u}\}$ and $h_k(x,y) = (u_k(x) - u_k(y))^2 - (v_k(x) - v_k(y))^2$, we get

$$\begin{aligned}J_0 = {}& \int_{\underline{S_k}}\int_{\underline{S_k}} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy + \int_{\overline{S_k}}\int_{\overline{S_k}} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy \\ & + 2\int_{\underline{S_k}}\int_{\overline{S_k}} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy + 2\int_{\underline{S_k}}\int_{S_k} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy \\ & + 2\int_{\overline{S_k}}\int_{S_k} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned}$$

Since $u_k = \overline{w_k} + \overline{u}$ and $v_k = \overline{u}$ in $\overline{S_k}$ and $u_k = \underline{u} - \underline{w_k}$ and $v_k = \underline{u}$ in $\underline{S_k}$ we get that

$$\begin{aligned}\int_{\underline{S_k}}\int_{\underline{S_k}} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy = {}& \int_{\underline{S_k}}\int_{\underline{S_k}} \frac{(\underline{w_k}(x) - \underline{w_k}(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\ & - 2\int_{\underline{S_k}}\int_{\underline{S_k}} \frac{(\underline{w_k}(x) - \underline{w_k}(y))(\underline{u}(x) - \underline{u}(y))}{|x-y|^{n+2s}}\,dx\,dy\end{aligned}$$

$$\begin{aligned}\int_{\overline{S_k}}\int_{\overline{S_k}} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy = {}& \int_{\overline{S_k}}\int_{\overline{S_k}} \frac{(\overline{w_k}(x) - \overline{w_k}(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\ & + 2\int_{\overline{S_k}}\int_{\overline{S_k}} \frac{(\overline{w_k}(x) - \overline{w_k}(y))(\overline{u}(x) - \overline{u}(y))}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned}$$

Also similarly we obtain

$$\begin{aligned}
\int_{\underline{S_k}}\int_{\overline{S_k}} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy = {}& \int_{\underline{S_k}}\int_{\overline{S_k}} \frac{(\underline{w_k}(x)+\overline{w_k}(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\
& - 2\int_{\underline{S_k}}\int_{\overline{S_k}} \frac{(\underline{w_k}(x)+\overline{w_k}(y))(\underline{u}(x)-\overline{u}(y))}{|x-y|^{n+2s}}\,dx\,dy,
\end{aligned}$$

$$\begin{aligned}
\int_{\underline{S_k}}\int_{S_k} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy = {}& \int_{\underline{S_k}}\int_{S_k} \frac{\underline{w_k}^2(x)}{|x-y|^{n+2s}}\,dx\,dy \\
& - 2\int_{\underline{S_k}}\int_{S_k} \frac{\underline{w_k}(x)(\underline{u}(x)-u_k(y))}{|x-y|^{n+2s}}\,dx\,dy, \\
\int_{\overline{S_k}}\int_{S_k} \frac{h_k(x,y)}{|x-y|^{n+2s}}\,dx\,dy = {}& \int_{\overline{S_k}}\int_{S_k} \frac{\overline{w_k}^2(x)}{|x-y|^{n+2s}}\,dx\,dy \\
& + 2\int_{\overline{S_k}}\int_{S_k} \frac{\overline{w_k}(x)(\overline{u}(x)-u_k(y))}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

Since $\mathcal{C}\underline{S_k} = \overline{S_k}\cup S_k$, $\mathcal{C}\overline{S_k} = \underline{S_k}\cup S_k$ and

$$\begin{aligned}
\|\underline{w_k}\|^2 &= \int_{\underline{S_k}}\int_{\underline{S_k}} \frac{(\underline{w_k}(x)-\underline{w_k}(y))^2}{|x-y|^{n+2s}}\,dx\,dy + 2\int_{\underline{S_k}}\int_{\mathcal{C}S_k} \frac{\underline{w_k}^2(x)}{|x-y|^{n+2s}}\,dx\,dy \\
\|\overline{w_k}\|^2 &= \int_{\overline{S_k}}\int_{\overline{S_k}} \frac{(\overline{w_k}(x)-\overline{w_k}(y))^2}{|x-y|^{n+2s}}\,dx\,dy + 2\int_{\overline{S_k}}\int_{\mathcal{C}S_k} \frac{\overline{w_k}^2(x)}{|x-y|^{n+2s}}\,dx\,dy,
\end{aligned}$$

using all above estimates, we can express J_0 as

$$\begin{aligned}
J_0 = {}& C^n_s(\|\underline{w_k}\|^2 + \|\overline{w_k}\|^2) + 2\bigg(\int_{\underline{S_k}}\int_{\overline{S_k}} \frac{(\underline{w_k}(x)+\overline{w_k}(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\
& - \int_{\underline{S_k}}\int_{\overline{S_k}} \frac{\underline{w_k}^2(x)}{|x-y|^{n+2s}} - \int_{\overline{S_k}}\int_{\underline{S_k}} \frac{\overline{w_k}^2(x)}{|x-y|^{n+2s}}\,dx\,dy\bigg) \\
& - 2\int_{\underline{S_k}}\int_{\underline{S_k}} \frac{(\underline{w_k}(x)-\underline{w_k}(y))(\underline{u}(x)-\underline{u}(y))}{|x-y|^{n+2s}}\,dx\,dy \\
& + 2\int_{\overline{S_k}}\int_{\overline{S_k}} \frac{(\overline{w_k}(x)-\overline{w_k}(y))(\overline{u}(x)-\overline{u}(y))}{|x-y|^{n+2s}}\,dx\,dy \\
& - 4\int_{\underline{S_k}}\int_{\overline{S_k}} \frac{(\underline{w_k}(x)+\overline{w_k}(y))(\underline{u}(x)-\overline{u}(y))}{|x-y|^{n+2s}}\,dx\,dy \\
& - 4\int_{\underline{S_k}}\int_{S_k} \frac{\underline{w_k}(x)(\underline{u}(x)-u_k(y))}{|x-y|^{n+2s}}\,dx\,dy \\
& + 4\int_{\overline{S_k}}\int_{S_k} \frac{\overline{w_k}(x)(\overline{u}(x)-u_k(y))}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

Now we notice that, if $(x,y) \in \underline{S_k} \times S_k$, then $(\underline{u}(x) - u_k(y)) \le (\underline{u}(x) - \underline{u}(y))$; if $(x,y) \in \overline{S_k} \times S_k$, then $(\overline{u}(x) - u_k(y)) \ge (\overline{u}(x) - \overline{u}(y))$ and

$$\int_{\underline{S_k}} \int_{\overline{S_k}} \frac{(\underline{w_k}(x) + \overline{w_k}(y))^2}{|x-y|^{n+2s}}\, dx\, dy - \int_{\underline{S_k}} \int_{\overline{S_k}} \frac{\underline{w_k}^2(x)}{|x-y|^{n+2s}}\, dx\, dy \\ - \int_{\overline{S_k}} \int_{\underline{S_k}} \frac{\overline{w_k}^2(x)}{|x-y|^{n+2s}}\, dx\, dy = 2 \int_{\overline{S_k}} \int_{\underline{S_k}} \frac{\overline{w_k}(x)\underline{w_k}(y)}{|x-y|^{n+2s}}\, dx\, dy.$$

Also, using change of variables, we have

$$\int_{\underline{S_k}} \int_{\overline{S_k}} \frac{(\underline{w_k}(x) + \overline{w_k}(y))(\underline{u}(x) - \overline{u}(y))}{|x-y|^{n+2s}}\, dx\, dy \\ = \int_{\underline{S_k}} \int_{\overline{S_k}} \frac{\underline{w_k}(x)(\underline{u}(x) - \overline{u}(y))}{|x-y|^{n+2s}}\, dx\, dy - \int_{\overline{S_k}} \int_{\underline{S_k}} \frac{\overline{w_k}(x)(\overline{u}(x) - \underline{u}(y))}{|x-y|^{n+2s}}\, dx\, dy.$$

Therefore altogether we obtain

$$\begin{aligned} J_0 &\ge C_s^n(\|\underline{w_k}\|^2 + \|\overline{w_k}\|^2) + 4 \int_{\overline{S_k}} \int_{\underline{S_k}} \frac{\overline{w_k}(x)\underline{w_k}(y)}{|x-y|^{n+2s}}\, dx\, dy + 2 \int_{\mathbb{R}^n} \overline{w_k}(-\Delta)^s \overline{u}\, dx \\ &\quad - 2 \int_{\mathbb{R}^n} \underline{w_k}(-\Delta)^s \underline{u}\, dx - 4 \int_{\overline{S_k}} \int_{\underline{S_k}} \frac{\overline{w_k}(x)(\overline{u}(x) - \underline{u}(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad + 4 \int_{\underline{S_k}} \int_{\overline{S_k}} \frac{\underline{w_k}(x)(\underline{u}(x) - \overline{u}(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad - 4 \int_{\underline{S_k}} \int_{\overline{S_k}} \frac{(\underline{w_k}(x) + \overline{w_k}(y))(\underline{u}(x) - \overline{u}(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &\ge C_s^n\left(\|\underline{w_k}\|^2 + \|\overline{w_k}\|^2\right) + 2 \int_{\mathbb{R}^n} \overline{w_k}(-\Delta)^s \overline{u}\, dx - 2 \int_{\mathbb{R}^n} \underline{w_k}(-\Delta)^s \underline{u}\, dx \end{aligned}$$

where we used the fact that if $(x,y) \in \overline{S_k} \times \underline{S_k}$, then $\overline{w_k}(x)\underline{w_k}(y) \ge 0$. Now, recalling that $\underline{u}$ and $\overline{u}$ forms sub- and supersolution of (P_λ), respectively, inserting the above inequality in (3.5) we obtain

$$\begin{aligned} I_\lambda(u_k) &\ge I_\lambda(v_k) + \frac{C_s^n\|\underline{w_k}\|^2}{2} + \frac{C_s^n\|\overline{w_k}\|^2}{2} \\ &\quad + \int_{\overline{S_k}} \left(\frac{\overline{u}^{1-q} - (\overline{u} + \overline{w_k})^{1-q}}{1-q} + \overline{u}^{-q}\overline{w_k} \right) dx \\ &\quad + \int_{\underline{S_k}} \left(\frac{\underline{u}^{1-q} - (\underline{u} - \underline{w_k})^{1-q}}{1-q} - \underline{u}^{-q}\underline{w_k} \right) dx \\ &\quad + \lambda \int_{\overline{S_k}} \left(\frac{\overline{u}^{2_s^*} - (\overline{u} + \overline{w_k})^{2_s^*}}{2_s^*} + \overline{u}^{2_s^*-1}\overline{w_k} \right) dx \\ &\quad + \lambda \int_{\underline{S_k}} \left(\frac{\underline{u}^{2_s^*} - (\underline{u} - \underline{w_k})^{2_s^*}}{2_s^*} - \underline{u}^{2_s^*-1}\underline{w_k} \right) dx. \end{aligned}$$

Now, from Mean Value Theorem it follows that there exists $\theta \in (0,1)$ (where θ may change its value for different function below) such that

$$
\begin{aligned}
(3.6) \qquad I_\lambda(u_k) \geq{} & I_\lambda(v_k) + \frac{C_s^n\|\underline{w_k}\|^2}{2} + \frac{C_s^n\|\overline{w_k}\|^2}{2} \\
& - \int_{\overline{S_k}} ((\overline{u}+\theta\overline{w_k})^{-q} - \overline{u}^{-q})\overline{w_k}\,dx \\
& - \int_{\underline{S_k}} (\underline{u}^{-q} - (\underline{u}+\theta\underline{w_k})^{-q})\underline{w_k}\,dx \\
& - \lambda\int_{\overline{S_k}} \big((\overline{u}+\theta\overline{w_k})^{2_s^*-1} - \overline{u}^{2_s^*-1}\big)\overline{w_k}\,dx \\
& - \lambda\int_{\underline{S_k}} \big(\underline{u}^{2_s^*-1} - (\underline{u}+\theta\underline{w_k})^{2_s^*-1}\big)\underline{w_k}\,dx \\
\geq{} & I_\lambda(v_k) + \frac{C_s^n\|\underline{w_k}\|^2}{2} + \lambda\int_{\overline{S_k}} \big((\overline{u}+\theta\overline{w_k})^{2_s^*-1} - \overline{u}^{2_s^*-1}\big)\overline{w_k}\,dx \\
& - \lambda\int_{\underline{S_k}} \big(\underline{u}^{2_s^*-1} - (\underline{u}+\theta\underline{w_k})^{2_s^*-1}\big)\underline{w_k}\,dx.
\end{aligned}
$$

Now, since $2_s^* > 2$, there exists constant $C > 0$ such that (3.6) reduces to

$$
\begin{aligned}
(3.7) \quad I_\lambda(u_k) \geq{} & I_\lambda(v_k) + \frac{C_s^n\|\underline{w_k}\|^2}{2} + \frac{C_s^n\|\overline{w_k}\|^2}{2} \\
& - C\int_{\underline{S_k}} (\underline{u}^{2_s^*-2} - \underline{w_k}^{2_s^*-2})\underline{w_k}^2\,dx \\
& - C\int_{\overline{S_k}} (\overline{u}^{2_s^*-2} - \overline{w_k}^{2_s^*-2})\overline{w_k}^2\,dx \\
\geq{} & I_\lambda(v_k) + \frac{C_s^n\|\underline{w_k}\|^2}{2} + \frac{C_s^n\|\overline{w_k}\|^2}{2} \\
& - C\bigg(\int_{\underline{S_k}} |\underline{u}|^{2_s^*}\bigg)^{(2_s^*-2)/2_s^*}\|\underline{w_k}\|^2 - C\bigg(\int_{\overline{S_k}} |\overline{u}|^{2_s^*}\bigg)^{(2_s^*-2)/2_s^*}\|\overline{w_k}\|^2.
\end{aligned}
$$

Claim. $\lim_{k\to\infty} |\overline{S_k}| = 0$ and $\lim_{k\to\infty} |\underline{S_k}| = 0$.

Let $\alpha > 0$ and define

$$
\begin{aligned}
A_k &= \{x\in\Omega : u_k \geq \overline{u} \text{ and } \overline{u} > w+\alpha\}, & \widehat{A}_k &= \{x\in\Omega : u_k \leq \underline{u} \text{ and } \underline{u} < w-\alpha\}, \\
B_k &= \{x\in\Omega : u_k \geq \overline{u} \text{ and } \overline{u} \leq w+\alpha\}, & \widehat{B}_k &= \{x\in\Omega : u_k \leq \underline{u} \text{ and } \underline{u} \geq w-\alpha\}.
\end{aligned}
$$

Since

$$
0 = \mathcal{L}(\{x\in\Omega : \overline{u} < w\}) = \mathcal{L}\bigg(\bigcap_{j=1}^{\infty}\{x\in\Omega : \overline{u} < w+1/j\}\bigg)
$$

so there exists $j_0 \geq 1$ large enough and $\alpha < 1/j_0$ such that $\mathcal{L}(\{x \in \Omega : \overline{u} < w+\alpha\}) \leq \varepsilon/2$. This implies that $\mathcal{L}(B_k) \leq \varepsilon/2$ and similarly, we obtain

$\mathcal{L}(\widehat{B}_k) \leq \varepsilon/2$. From (3.4) we already have $|u_k - w|_2 \to 0$ as $k \to \infty$. So, for $k \geq k_0$ large enough, we get that

$$\frac{\alpha^2 \varepsilon}{2} \geq \int_\Omega |u_k - w|^2 \, dx \geq \int_{A_k} |u_k - w|^2 \, dx \geq \alpha^2 \mathcal{L}(A_k)$$

which implies that $\mathcal{L}(A_k) \leq \varepsilon/2$ for $k \geq k_0$. Similarly, we obtain $\mathcal{L}(\widehat{A}_k) \leq \varepsilon/2$ for $k \geq k_0$. Now, since $\overline{S_k} \subset A_k \cap B_k$ and $\underline{S_k} \subset \widehat{A}_k \cap \widehat{B}_k$, we get that $\mathcal{L}(\overline{S_k}) \leq \varepsilon$ and $\mathcal{L}(\underline{S_k}) \leq \varepsilon$ for $k \geq k_0$. This proves the claim. Thus

$$\left(\int_{\overline{S_k}} |\overline{u}|^{2_s^*} \right)^{(2_s^*-2)/2_s^*} \leq o(1) \quad \text{and} \quad \left(\int_{\underline{S_k}} |\underline{u}|^{2_s^*} \right)^{(2_s^*-2)/2_s^*} \leq o(1)$$

which imposing in (3.7) gives that, for large enough k,

$$I_\lambda(u_k) \geq I_\lambda(v_k) \geq I_\lambda(w),$$

which is a contradiction to (3.4). Therefore w must be a local minimum of I_λ over X_0. □

THEOREM 3.4. *There exists a positive weak solution to* (P_Λ).

PROOF. Let $\lambda_m \uparrow \Lambda$ as $m \to \infty$ and $\{u_{\lambda_m}\}$ be a sequence of positive weak solutions to (P_{λ_m}), such that u_{λ_m} forms the local minimum of I_{λ_m} as seen in Lemma 3.3. Since we consider the minimal solutions, we get $u_m \leq u_{m+1}$ for each m. Then it is easy to see that $I_{\lambda_m} < 0$ in the case $0 < q < 1$, whereas there exists a constant K independent of m such that $I_{\lambda_m} \leq K$ for all m when $q > 1$ but $q(2s-1) < (2s+1)$. This implies that $\{u_{\lambda_m}\}$ is uniformly bounded in X_0. Therefore, up to a subsequence, there exists $u_\Lambda \in X_0$ such that $u_{\lambda_m} \rightharpoonup u_\Lambda$ weakly and pointwise almost everywhere in X_0 as $m \to \infty$. Also, by construction $u_{\lambda_m} \geq \underline{u_{\lambda_1}}$ as defined in Lemma 3.1. Therefore, u_Λ is a positive weak solution of (P_Λ). □

4. Multiplicity result

We have already obtained the first solution for (P_λ) in the previous section when $\lambda \in (0, \Lambda)$ in X_0-topology. We fix $\lambda \in (0, \Lambda)$ and let w denotes the first weak solution of (P_λ) obtained in Proposition 3.2. In this section, we prove the existence of second solution of (P_λ) using the machinery of Mountain Pass Lemma and with the help of Ekeland variational principle. Precisely, we extend the approach used in [13] to the non local setting and for $q \geq 1$. This can be done by using the asymptotic boundary behavior of $w \in C^+_{\phi_q}(\Omega)$ and the Hardy's inequality to control the singular nonlinearity in the cone T defined as:

$$T \stackrel{\text{def}}{=} \{x \in X_0 : u \geq w \text{ almost everywhere in } \Omega\}$$

and, since w forms a local minimizer of I_λ, we get that $I_\lambda(u) \geq I_\lambda(w)$ whenever $\|u - w\| \leq \sigma_0$, for some constant $\sigma_0 > 0$. Then one of the following cases holds:

(ZA) (Zero Altitude) $\inf\{I_\lambda(u) \mid u \in T, \|u-w\| = \sigma\} = I_\lambda(w)$ for all σ in $(0,\sigma_0)$.

(MP) (Mountain Pass) There exists a $\sigma_1 \in (0,\sigma_0)$ such that $\inf\{I_\lambda(u) \mid u \in T, \|u-w\| = \sigma_1\} > I_\lambda(w)$.

Before investigating the two distinguished cases (ZA) and (MP), we prove the following regularity result for weak solutions to (P_λ):

Proposition 4.1. *Any weak solution to* (P_λ) *for* $\lambda \in (0,\Lambda]$ *belongs to* $L^\infty(\Omega) \cap C^+_{\phi_q}(\Omega)$.

Proof. Let $u \in X_0$ denotes a weak solution to (P_λ). We know that $\underline{u_\lambda} \in X_0 \cap C^+_{\phi_q}(\Omega)$ (defined in Lemma 3.1) forms a subsolution to (P_λ) satisfying $(-\Delta)^s \underline{u_\lambda} = \underline{u_\lambda}^{-q}$ in Ω. We first have:

Claim. $\underline{u_\lambda} \leq u$ almost everywhere in Ω.

Let us prove the above claim. Suppose it is not true. First it is easy to see that for any $v \in X_0$ it holds

$$(v(x)-v(y))(v^+(x)-v^-(y)) \geq |v^+(x)-v^+(y)|^2, \quad \text{for any } x,y \in \mathbb{R}^n.$$

Therefore, using $(\underline{u_\lambda}-u)^+$ as the test function in $(-\Delta)^s(\underline{u_\lambda}-u) \leq \underline{u_\lambda}^{-q} - u^{-q}$ in Ω we get

$$\begin{aligned}
0 &\leq C^n_s \int_Q \frac{|(\underline{u_\lambda}-u)^+(x) - (\underline{u_\lambda}-u)^+(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\leq C^n_s \int_Q \frac{((\underline{u_\lambda}-u)^+(x) - (\underline{u_\lambda}-u)^+(y))((\underline{u_\lambda}-u)(x) - (\underline{u_\lambda}-u)(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\leq \int_\Omega (\underline{u_\lambda}^{-q} - u^{-q})(\underline{u_\lambda}-u)^+\,dx \leq 0.
\end{aligned}$$

Hence it must be that $\mathrm{meas}\{x \in \Omega : \underline{u_\lambda}(x) \geq u(x)\} = 0$ which gives a contradiction. Therefore $\underline{u_\lambda} \leq u$ almost everywhere in Ω. Let us now prove that $u \in L^\infty(\Omega)$. We follow the approach in [4, Proposition 2.2]. By virtue of the above claim and Hardy's inequality, we know that

$$\int_\Omega u^{-q}\phi\,dx < \infty \quad \text{for any } \phi \in X_0.$$

We aim to show that $(u-1)^+$ belongs to $L^\infty(\Omega)$ which will imply that $u \in L^\infty(\Omega)$. If $f(t) = (t-1)^+$ for $t \in \mathbb{R}$ and $\psi(t) \in C^\infty(\mathbb{R})$ is a convex and increasing function such that $\psi'(t) \leq 1$ when $t \in [0,1]$ and $\psi'(t) = 1$ when $t \geq 1$, then we can define

$$\psi_\varepsilon(t) = \varepsilon\psi(t/\varepsilon)$$

so that $\psi_\varepsilon \to f$ uniformly as $\varepsilon \to 0$. Also, since ψ_ε is smooth, by regularity results and the uniform convergence of ψ_ε to f we get that

$$(-\Delta)^s\psi_\varepsilon(u) \to (-\Delta)^s(u-1)^+ \quad \text{as } \varepsilon \to 0.$$

Moreover, because ψ_ε is convex and differentiable, we know that

$$(-\Delta)^s\psi_\varepsilon(u) \leq \psi'_\varepsilon(u)(-\Delta)^s u \leq \chi_{\{u>1\}}(-\Delta)^s u$$

where $\chi_{\{u>1\}}$ denotes the characteristic function over the set $\{x \in \Omega : u(x) > 1\}$. Then, passing on the limits $\varepsilon \to 0$ in above equation, we obtain

$$(-\Delta)^s(u-1)^+ \leq \chi_{\{u>1\}}(-\Delta)^s u \leq \chi_{\{u>1\}}(u^{-q}+\lambda u^{2_s^*-1}) \leq C(1+((u-1)^+)^{2_s^*-1})$$

for some constant $C > 0$. Therefore using [4, Proposition 2.2], we conclude that $(u-1)^+ \in L^\infty(\Omega)$. Finally, we show that $u \in C^+_{\phi_q}(\Omega)$. Let z_λ be the unique solution (refer to [1, Theorem 1.1] with $\delta = q$ and $\beta = 0$) to

$$(-\Delta)^s z_\lambda = {z_\lambda}^{-q} + \lambda c, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

with $c = \|u\|_\infty^{2_s^*-1}$. Similarly we can prove that $u \leq z_\lambda$. Therefore ,using local regularity results in [17, Propositions 2.2 and 2.3] derived from [20, Propositions 2.8 and 2.9], $u \in C^+_{\phi_q}(\Omega)$. □

Lemma 4.2. *Let* (ZA) *holds; then for any* $\sigma \in (0, \sigma_0)$ *there exists a solution* $v \in C^+_\phi(\Omega)$ *to* (P$_\lambda$) *such that* $0 < w < v$ *in* Ω *and* $\|v - w\| = \sigma$.

Proof. We follow the proof of Lemma 2.6 of [13] in a nonlocal framework. We fix $\sigma \in (0, \sigma_0)$ and $r > 0$ such that $\sigma - r > 0$ and $\sigma + r < \sigma_0$. Let us define the set

$$W = \{u \in T \mid 0 < \sigma - r \leq \|u - w\| \leq \sigma + r\}$$

which is closed in X_0 and, by (ZA), $\inf\limits_{u\in W} I_\lambda(u) = I_\lambda(w)$. So, using Ekeland variational principle, for any minimizing sequence $\{u_k\} \subset X_0$ satisfying $\|u_k\| = \sigma$ and $I_\lambda(u_k) \leq I_\lambda(w) + 1/k$, we get another sequence $\{v_k\} \subset W$ such that

$$\begin{cases} I_\lambda(v_k) \leq I_\lambda(u_k) \leq I_\lambda(w) + \dfrac{1}{k}, \ \|u_k - v_k\| \leq \dfrac{1}{k}, & \\ I_\lambda(v_k) \leq I_\lambda(z) + \dfrac{1}{k}\|z - v_k\| & \text{for all } z \in W. \end{cases} \tag{4.1}$$

We can choose $\varepsilon > 0$ small enough so that $v_k + \varepsilon(z - v_k) \in W$ for $z \in T$. So from (4.1) we obtain

$$\frac{I_\lambda(v_k + \varepsilon(z - v_k)) - I_\lambda(v_k)}{\varepsilon} \geq -\frac{1}{k}\,\|z - v_k\|.$$

Letting $\varepsilon \to 0^+$ and using the fact that $v_k \geq w$ for each k, for $z \in T$ we get

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta)^s v_k(z - v_k) - \int_\Omega v_k^{-q}(z - v_k)\,dx \\ - \lambda \int_\Omega v_k^{2_s^*-1}(z - v_k)\,dx \geq -\frac{1}{k}\|z - v_k\|. \end{aligned} \tag{4.2}$$

Now, since $\{v_k\}$ forms a bounded sequence in X_0, we get that there exists a v in X_0 such that, up to a subsequence, $v_k \rightharpoonup v$ weakly in X_0 and pointwise almost everywhere in Ω as $k \to \infty$. Since $v_k \geq w$ for each k, we get $v \geq w$

almost everywhere in Ω. In what follows, we will prove that v is a weak solution of (P_λ). For $\phi \in X_0$ and $\varepsilon > 0$, we set $\phi_{k,\varepsilon} = (v_k + \varepsilon\phi - w)^- \in X_0$ which implies that $(v_k + \varepsilon\phi + \phi_{k,\varepsilon}) \in T$. Putting $z = v_k + \varepsilon\phi + \phi_{k,\varepsilon}$ in (4.2) we get

$$\begin{aligned}(4.3)\quad & C_s^n \int_Q \frac{(v_k(x)-v_k(y))((\varepsilon\phi+\phi_{k,\varepsilon})(x)-(\varepsilon\phi+\phi_{k,\varepsilon})(y))}{|x-y|^{n+2s}}\,dx\,dy \\ & - \int_\Omega v_k^{-q}(\varepsilon\phi+\phi_{k,\varepsilon})\,dx - \lambda\int_\Omega v_k^{2_s^*-1}(\varepsilon\phi+\phi_{k,\varepsilon})\,dx \geq \frac{-1}{k}\|(\varepsilon\phi+\phi_{k,\varepsilon})\|.\end{aligned}$$

We define the sets

$$\Omega_{k,\varepsilon} = \operatorname{Supp}\phi_{k,\varepsilon}, \quad \Omega_\varepsilon = \operatorname{Supp}\phi_\varepsilon \quad \text{and} \quad \Omega_0 = \{x \in \Omega : v(x) = w(x)\}.$$

Then we get that $\mathcal{L}(\Omega_\varepsilon \setminus \Omega_0) \to 0$ as $\varepsilon \to 0$ and $\mathcal{L}(\Omega_{k,\varepsilon} \setminus \Omega_\varepsilon) + \mathcal{L}(\Omega_\varepsilon \setminus \Omega_{k,\varepsilon}) \to 0$ as $k \to \infty$. Also, since $|\phi_{k,\varepsilon}| \leq w + \varepsilon|\phi|$, using Lebesgue Dominated Convergence Theorem we get $\phi_{k,\varepsilon} \to \phi_\varepsilon = (v+\varepsilon\phi-w)^-$ in $L^m(\Omega)$ for all $m \in [1, 2_s^*]$. Moreover $\phi_{k,\varepsilon} \rightharpoonup \phi_\varepsilon$ weakly in X_0 and pointwise almost everywhere in Ω as $k \to \infty$. Now we estimate the following integral

$$\begin{aligned}(4.4)\quad & \int_Q \frac{(v_k(x)-v_k(y))(\phi_{k,\varepsilon}(x)-\phi_{k,\varepsilon}(y))}{|x-y|^{n+2s}}\,dx\,dy \\ &= \int_Q \frac{(v_k(x)-v_k(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad + \int_Q \frac{(v_k(x)-v_k(y))((\phi_{k,\varepsilon}-\phi_\varepsilon)(x)-(\phi_{k,\varepsilon}-\phi_\varepsilon)(y))}{|x-y|^{n+2s}}\,dx\,dy := I_1 + I_2.\end{aligned}$$

We show that $I_2 \leq o_k(1)$ for which we split the integrals and estimate them separately. Let $H_k = \Omega_{k,\varepsilon} \cap \Omega_\varepsilon$ and $G_k = \Omega_{k,\varepsilon} \setminus \Omega_\varepsilon \cup \Omega_\varepsilon \setminus \Omega_{k,\varepsilon}$. Then

$$\begin{aligned}(4.5)\quad & \int_\Omega\int_{\mathcal{C}\Omega} \frac{(v_k(x)-v_k(y))((\phi_{k,\varepsilon}-\phi_\varepsilon)(x)-(\phi_{k,\varepsilon}-\phi_\varepsilon)(y))}{|x-y|^{n+2s}} \\ &\leq \int_{H_k}\int_{\mathcal{C}\Omega} \frac{v(x)(v-v_k)(x)}{|x-y|^{n+2s}} + \int_{G_k}\int_{\mathcal{C}\Omega} \frac{v_k(x)(\phi_{k,\varepsilon}-\phi_\varepsilon)(x)}{|x-y|^{n+2s}} \\ &\leq \int_{H_k}\int_{\mathcal{C}\Omega} \frac{v(x)(v-v_k)(x)}{|x-y|^{n+2s}} + \int_{G_k}\int_{\mathcal{C}\Omega} \frac{v_k(x)\phi_{k,\varepsilon}(x)}{|x-y|^{n+2s}} \\ &= \int_{H_k}\int_{\mathcal{C}\Omega} \frac{v(x)(v-v_k)(x)}{|x-y|^{n+2s}} + o_k(1)\end{aligned}$$

using the fact that $\mathcal{L}(\Omega_{k,\varepsilon} \setminus \Omega_\varepsilon) + \mathcal{L}(\Omega_\varepsilon \setminus \Omega_{k,\varepsilon}) \to 0$ as $k \to \infty$ and Lebesgue Dominated Convergence Theorem. Similarly

$$\begin{aligned}(4.6)\quad & \int_\Omega\int_\Omega \frac{(v_k(x)-v_k(y))((\phi_{k,\varepsilon}-\phi_\varepsilon)(x)-(\phi_{k,\varepsilon}-\phi_\varepsilon)(y))}{|x-y|^{n+2s}} \\ &\leq \int_{H_k}\int_{H_k} \frac{(v(x)-v(y))((v-v_k)(x)-(v-v_k)(y))}{|x-y|^{n+2s}} \\ &\quad + 2\int_{H_k}\int_{G_k} \frac{(v_k(x)-v_k(y))((\phi_{k,\varepsilon}-\phi_\varepsilon)(x)-(\phi_{k,\varepsilon}-\phi_\varepsilon)(y))}{|x-y|^{n+2s}}\end{aligned}$$

$$
\begin{aligned}
&+\int_{G_k}\int_{G_k}\frac{(v_k(x)-v_k(y))((\phi_{k,\varepsilon}-\phi_\varepsilon)(x)-(\phi_{k,\varepsilon}-\phi_\varepsilon)(y))}{|x-y|^{n+2s}}\\
&\le \int_{H_k}\int_{H_k}\frac{(v(x)-v(y))((v-v_k)(x)-(v-v_k)(y))}{|x-y|^{n+2s}}+o_k(1)
\end{aligned}
$$

using again the Lebesgue Dominated Convergence Theorem with the fact that $v_k - v \to 0$ and $\phi_{k,\varepsilon}-\phi_\varepsilon \to 0$ pointwise as $k\to\infty$. Combining (4.5) and (4.6) we obtain that

$$
I_2 \le \int_{H_k}\int_{H_k\cup\mathcal{C}\Omega}\frac{(v(x)-v(y))((v-v_k)(x)-(v-v_k)(y))}{|x-y|^{n+2s}}+o_k(1)=o_k(1).
$$

Therefore, using this in (4.4), we obtain

$$
\begin{aligned}
&\int_Q \frac{(v_k(x)-v_k(y))(\phi_{k,\varepsilon}(x)-\phi_{k,\varepsilon}(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&\qquad\le \int_Q \frac{(v_k(x)-v_k(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))}{|x-y|^{n+2s}}\,dx\,dy+o_k(1).
\end{aligned}
$$

Moreover, we have that $|v_k^{-q}(\varepsilon\phi+\phi_{k,\varepsilon})| \le w^{-q}(w+2\varepsilon\phi)\in L^1(\Omega)$ using the Hardy's inequality. Thus using Lebesgue Dominated Convergence Theorem and passing on the limits $k\to\infty$ in (4.3) we get

$$
\begin{aligned}
0 \le C_s^n &\int_Q \frac{(v_k(x)-v_k(y))((\varepsilon\phi+\phi_\varepsilon)(x)-(\varepsilon\phi+\phi_\varepsilon)(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&-\int_\Omega (v^{-q}+\lambda v^{2_s^*-1})(\varepsilon\phi+\phi_\varepsilon)\,dx.
\end{aligned}
$$

Using the fact that w is a weak solution of (P_λ) and $v\ge w$, the above inequality implies that

$$
\begin{aligned}
&C_s^n\int_Q \frac{(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy-\int_\Omega v^{-q}\phi\,dx-\lambda\int_\Omega v^{2_s^*-1}\phi\,dx\\
&\quad\ge -\frac{1}{\varepsilon}\bigg(C_s^n\int_Q \frac{(v(x)-v(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&\qquad -\int_\Omega v^{-q}\phi_\varepsilon\,dx-\lambda\int_\Omega v^{2_s^*-1}\phi_\varepsilon\,dx\bigg)\\
&\quad\ge \frac{1}{\varepsilon}\bigg(C_s^n\int_Q \frac{((w-v)(x)-(w-v)(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&\qquad +\int_\Omega (v^{-q}-w^{-q})\phi_\varepsilon\,dx\bigg)\\
&\quad\ge C_s^n\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon}\frac{((v-w)(x)-(v-w)(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy
\end{aligned}
$$

$$+ 2C_s^n \int_{\Omega_\varepsilon} \int_{\{w \le v + \varepsilon\phi\}} \frac{((v-w)(x) - (v-w)(y))\phi(x)}{|x-y|^{n+2s}} \, dx \, dy$$
$$+ 2C_s^n \int_{\Omega_\varepsilon} \int_{\mathcal{C}\Omega} \frac{(v-w)(x)\phi(x)}{|x-y|^{n+2s}} \, dx \, dy + \int_{\Omega_\varepsilon} (v^{-q} - w^{-q})\phi \, dx = o(1)$$

as $\varepsilon \to 0^+$, using the fact that $|\Omega_\varepsilon \setminus \Omega_0| \to 0$ as $\varepsilon \to 0^+$. From this, we get that

$$C_s^n \int_Q \frac{(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}} \, dx \, dy - \int_\Omega v^{-q}\phi \, dx - \lambda \int_\Omega v^{2_s^*-1}\phi \, dx = 0$$

for all $\phi \in X_0$.

Claim. The sequence $v_k \to v$ strongly in X_0 as $k \to \infty$.

From the Brezis–Lieb Lemma we have

$$\|v_k\|^2 - \|v_k - v\|^2 = \|v\|^2 + o(1),$$
$$\int_\Omega |v_k|^{2_s^*} \, dx - \int_\Omega |v_k - v|^{2_s^*} \, dx = \int_\Omega |v|^{2_s^*} \, dx + o(1).$$

Since $v_k, v \ge w$ almost everywhere in Ω, we get

$$\int_\Omega |v_k|^{1-q} \, dx - \int_\Omega |v|^{1-q} \, dx = \int_\Omega (v_k + \theta v)^{-q}(v_k - v) \, dx, \quad \text{for } , \theta \in [0,1].$$

We know that $(v_k + \theta v)^{-q}(v_k - v) \to 0$ pointwise almost everywhere in Ω and $v_k, v \ge w \in C_\phi^+(\Omega)$. Therefor,e for any $E \subset \Omega$, we have

$$\int_\Omega (v_k + \theta v)^{-q}(v_k - v) \, dx \le C\|\delta^{(1-q)s/(1+q)}(x)\|_{L^2(E)} \|v_k - v\|, \tag{4.7}$$

using Hardy's inequality. Since $q(2s-1) < (2s+1)$, for any $\varepsilon > 0$, there exists a $\rho > 0$ such that $\left\|\delta^{(1-q)s/(1+q)}(x)\right\|_{L^2(E)} < \varepsilon$ whenever $\mathcal{L}(E) < \rho$. Hence from (4.7) and Vitali's convergence theorem we obtain

$$\int_\Omega (v_k + \theta v)^{-q}(v_k - v) \, dx \to 0 \quad \text{as } k \to \infty$$

that is

$$\int_\Omega |v_k|^{1-q} \, dx \to \int_\Omega |v|^{1-q} \, dx \quad \text{as } k \to \infty.$$

Taking v as the testing function in (4.2), we deduce

$$C_s^n \|v_k - v\|^2 \le \lambda \|v_k - v\|_{L^{2_s^*}(\Omega)}^{2_s^*} + o_k(1). \tag{4.8}$$

In the other hand, taking $z = 2v_k$ in (4.2), we infer

$$C_s^n \|v_k\|^2 - \int_\Omega v_k^{1-q} \, dx - \lambda \|v_k\|_{L^{2_s^*}(\Omega)}^{2_s^*} \ge o_k(1). \tag{4.9}$$

Since v is a weak solution, we have that

$$C_s^n \|v\|^2 - \int_\Omega v^{1-q} \, dx - \lambda \|v\|_{L^{2_s^*}(\Omega)}^{2_s^*} = 0. \tag{4.10}$$

From (4.9) and (4.10),

$$C_s^n\|v_k - v\|^2 \geq \lambda\|v_k - v\|^{2_s^*}_{L^{2_s^*}(\Omega)} + o_k(1). \tag{4.11}$$

From (4.8) and (4.11), we have that

$$C_s^n\|v_k - v\|^2 = \lambda\|v_k - v\|^{2_s^*}_{L^{2_s^*}(\Omega)} + o_k(1). \tag{4.12}$$

Without loss of generality, we can assume that $I_\lambda(w) \leq I_\lambda(v)$. Then, we easily get

$$I_\lambda(v_k) - I_\lambda(v) \leq I_\lambda(w) - I_\lambda(v) + o_k(1) \leq o_k(1)$$

from which it follows that

$$\frac{C_s^n}{2}\|v_k - v\|^2 - \frac{\lambda}{2_s^*}\|v_k - v\|^{2_s^*}_{L^{2_s^*}(\Omega)} \leq o_k(1). \tag{4.13}$$

From (4.12) and (4.13), we infer that $v_k \to v$ strongly in X_0. This proves the claim. Since $v_k \in W$ we conclude that $v \in W$ and $v \not\equiv w$. Next we prove that $w < v$ in Ω. For that, we first observe that from Proposition 4.1 $w, v \in L^\infty(\Omega) \cap C^+_{\phi_q}(\Omega)$. Now suppose that there exists $x_0 \in \Omega$ such that $v(x_0) = w(x_0)$. Then, since $v \geq w$, $v, w \in C(\mathbb{R}^n)$ and $v \not\equiv w$, we get

$$\begin{aligned} 0 &> C_s^n \int_{\mathbb{R}^n} \frac{(v-w)(x_0) - (v-w)(y)}{|x_0 - y|^{n+2s}} \\ &= v^{-q}(x_0) + \lambda v^{2_s^*-1}(x_0) - (w^{-q}(x_0) + \lambda w^{2_s^*-1}(x_0)) = 0 \end{aligned}$$

from which we get a contradiction. Therefore $v > w$ in Ω. $\square$

We define

$$S_s = \inf_{u \in X_0\setminus\{0\}} \frac{\displaystyle\int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\left(\displaystyle\int_\Omega |u|^{2_s^*}\,dx\right)^{2/2_s^*}}$$

as the best constant for the embedding $X_0 \hookrightarrow L^{2_s^*}(\Omega)$. Consider the family of minimizers $\{U_\varepsilon\}$ of S_s (refer [18]) defined as

$$U_\varepsilon(x) = \varepsilon^{-(n-2s)/2} u^*\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^n$$

where $u^*(x) = \overline{u}\big(x/S_s^{1/2s}\big)$, $\overline{u}(x) = \widetilde{u}(x)/|u|_{2_s^*}$ and $\widetilde{u}(x) = \alpha\big(\beta^2 + |x|^2\big)^{-(n-2s)/2}$ with $\alpha \in \mathbb{R}\setminus\{0\}$ and $\beta > 0$ are fixed constants. Then, for each $\varepsilon > 0$, U_ε satisfies

$$(-\Delta)^s u = |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^n.$$

Let $\nu > 0$ be such that $B_{4\nu} \subset \Omega$ and let $\zeta \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \zeta \leq 1$ in $\mathbb{R}^n$, $\zeta \equiv 0$ in $\mathbb{R}^n\setminus B_{2\nu}$ and $\zeta \equiv 1$ in B_ν. For each $\varepsilon > 0$ and $x \in \mathbb{R}^n$, we define $\Phi_\varepsilon(x) := \zeta(x)U_\varepsilon(x)$. From [12, Lemma 4.13], we have the following lemma.

Lemma 4.3. $\sup\{I_\lambda(u + t\Phi_\varepsilon) : t \geq 0\} < I_\lambda(u) + s(C_s^n S_s)^{n/2s}/n\lambda^{(n-2s)/2s}$, *for any sufficiently small* $\varepsilon > 0$.

Now we prove the existence of second solution if (MP) holds.

Lemma 4.4. *Let* (MP) *holds; then there exists a* $v \in X_0 \cap C_\phi^+(\Omega)$, *verifying* $w < v$ *in* Ω, *which solves* (P_λ) *weakly.*

Proof. From Lemma 4.3, it follows that there exists $\varepsilon > 0$ and $R_0 \geq 1$ such that

(a) $I_\lambda(w + RU_\varepsilon) < I_\lambda(w)$ for $\varepsilon \in (0, \varepsilon_0)$ and $R \geq R_0$.
(b) $I_\lambda(w + tR_0U_\varepsilon) < I_\lambda(w) + s(C_s^n S_s)^{n/2s}/n\lambda^{(n-2s)/2s}$ for $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, 1]$.

We define the complete metric space

$$\Gamma := \{\eta \in C([0,1], T) : \eta(0) = w,\ \|\eta(1) - w\| > \sigma_1,\ I_\lambda(\eta(1)) < I_\lambda(w)\}$$

with metric defined as $d(\eta', \eta) = \max\limits_{t \in [0,1]} \{\|\eta'(t) - \eta(t)\|\}$ for all $\eta, \eta' \in \Gamma$. From (a) above, we get that $\eta(t) = w + tR_0U_\varepsilon \in \Gamma$ for large enough $R_0 > 0$. This gives that $\Gamma \neq \emptyset$. Let $\gamma_0 = \inf\limits_{\eta \in \Gamma} \max\limits_{t \in [0,1]} I_\lambda(\eta(t))$. In virtue of (b) above and condition (MP), we get

$$I_\lambda(w) < \gamma_0 \leq I_\lambda(w) + \frac{s(C_s^n S_s)^{n/2s}}{n\lambda^{(n-2s)/2s}}.$$

Now let $\Psi(\eta) = \max\limits_{t \in [0,1]} I_\lambda(\eta(t))$ for $\eta \in \Gamma$. Then, using Ekeland's variational principle, we get a sequence $\{\eta_k\} \subset \Gamma$ such that

$$\Psi(\eta_k) < \gamma_0 + \frac{1}{k} \quad \text{and} \quad \Psi(\eta_k) < \Psi(\eta) + \frac{1}{k}\|\Psi(\eta) - \eta(\eta_k)\|_\Gamma, \quad \text{for all}\ \eta \in \Gamma. \tag{4.14}$$

We define

$$\Lambda_k = \Big\{t \in [0,1] : I_\lambda(\eta_k(t)) = \max_{x \in [0,1]} I_\lambda(\eta_k(x))\Big\}.$$

Claim. There exists a $t_k \in \Lambda_k$ such that, if $v_k = \eta_k(t_k)$ and $z \in T$, then

$$\int_{\mathbb{R}^n} (-\Delta)^s v_k (z - v_k) - \int_\Omega \big(v_k^{-q} + \lambda v_k^{2_s^*-1}\big)(z - v_k)\, dx \geq -\frac{1}{k} \max\{1, \|z - v_k\|\}.$$

We prove it by contradiction, so assume that for every $t \in \Lambda_k$ there exists $z_t \in T$ such that

$$\begin{aligned} &\int_{\mathbb{R}^n} (-\Delta)^s \eta_k(t) \bigg(\frac{z_t - \eta_k(t)}{\max\{1, \|z_t - \eta_k(t)\|\}}\bigg)\, dx \\ &\quad - \int_\Omega ((\eta_k(t))^{-q} + \lambda\big(\eta_k(t)\big)^{2_s^*-1}) \bigg(\frac{z_t - \eta_k(t)}{\max\{1, \|z_t - \eta_k(t)\|\}}\bigg)\, dx < -\frac{1}{k}. \end{aligned} \tag{4.15}$$

Since I_λ is locally Lipschitz in T, z_t can be chosen to be locally constant on Λ_t. Therefore for each $t \in \Lambda_k$ there exists a neighbourhood N_t of t in $(0,1)$ such that, for each $r \in N_t \cap \Gamma_k$, (4.15) holds, that is

$$\begin{aligned}(4.16)\quad &\int_{\mathbb{R}^n} (-\Delta)^s \eta_k(r) \left(\frac{z_t - \eta_k(r)}{\max\{1, \|z_t - \eta_k(r)\|\}} \right) dx \\ &\quad - \int_{\Omega} \left((\eta_k(r))^{-q} + \lambda (\eta_k(r))^{2_s^*-1} \right) \left(\frac{z_t - \eta_k(r)}{\max\{1, \|z_t - \eta_k(r)\|\}} \right) dx < -\frac{1}{k}.\end{aligned}$$

It is possible to choose a finite set $\{r_1, \ldots, r_m\} \subset \Lambda_k$ such that $\Lambda_k \subset \bigcup_{i=1}^{m} J_{r_i}$. For notational convenience, we set $z_i = z_{r_i}$ and denote $\{\kappa_1, \ldots, \kappa_m\}$ as the partition of unity associated with covering $\{J_{r_1}, \ldots, J_{r_m}\}$ of Λ_k. Now, if we define $z(r) = \sum_{i=1}^{m} \kappa_i(r) z_i$ for $r \in [0,1]$, then $z(r) \in T$ for each $r \in [0,1]$. Therefore, from (4.16) we deduce that, for all $r \in [0,1]$,

$$\begin{aligned}&\int_{\mathbb{R}^n} (-\Delta)^s \eta_k(r) \left(\frac{z(r) - \eta_k(r)}{\max\{1, \|z(r) - \eta_k(r)\|\}} \right) dx \\ &\quad - \int_{\Omega} \left((\eta_k(r))^{-q} + \lambda (\eta_k(r))^{2_s^*-1} \right) \left(\frac{z(r) - \eta_k(r)}{\max\{1, \|z(r) - \eta_k(r)\|\}} \right) dx < -\frac{1}{k}.\end{aligned}$$

Let $h\colon [0,1] \to [0,1]$ be a continuous function such that $h(t) = 1$ in a neighbourhood of Λ_k and $h(0) = h(1) = 0$. Also we set $\mu_k(t) = \max\{1, \|z(t) - \eta_k(t)\|\}$ and

$$\eta(t) = \eta_k(t) + \frac{h(t)\varepsilon}{\mu_k(t)} (z(t) - \eta_k(t)).$$

For $\varepsilon \in (0,1)$, $\eta(t) \in T$ for all $t \in [0,1]$. Hence (4.14) gives us that

$$(4.17)\qquad \max_{t \in [0,1]} I_\lambda(\eta_k(t)) \le \max_{t \in [0,1]} I_\lambda(\eta(t)) + \frac{\varepsilon}{k} \max_{t \in [0,1]} \left(h(t) \frac{\|z(t) - \eta_k(t)\|}{\mu_k(t)} \right).$$

If $t_{k,\varepsilon} \in [0,1]$ denotes the value such that $I_\lambda(\eta(t_{k,\varepsilon})) = \max_{t \in [0,1]} I_\lambda(\eta(t))$, then we can assume that $t_{k,\varepsilon_j} \to t_k$ for some $t_k \in [0,1]$, where ε_j is a sequence such that $\varepsilon_j \to 0$. Using the continuity of η, we deduce that

$$\eta(t_{k,\varepsilon_j}) \to \eta_k(t_k) \quad \text{as } \varepsilon_j \to 0.$$

Hence from (4.17) we obtain that $\max_{t \in [0,1]} I_\lambda(\eta_k(t)) \le \max_{t \in [0,1]} I_\lambda(\eta_k(t_k))$ which implies $I_\lambda(\eta_k(t_k)) = \max_{t \in [0,1]} I_\lambda(\eta_k(t))$. So $t_k \in \Gamma_k$ and $h(t_{k,\varepsilon_j}) = 1$ for $j > 0$ large enough, by definition. If we set $v_k = \eta_k(t_k)$, $v_{k,j} = \eta_k(t_{k,\varepsilon_j})$ and $\mu_{k,j} = \max\{1, \|z(t_{k,\varepsilon_j}) - v_{k,j}\|\}$, then, for large enough j, we obtain

$$(4.18)\qquad I_\lambda(v_{k,j}) \le I_\lambda(v_k) \le I_\lambda \left(v_{k,j} + \frac{\varepsilon_j}{\mu_{k,j}} (z(t_{k,\varepsilon_j}) - v_{k,j}) \right) + \frac{\varepsilon_j}{k}.$$

It is easy to see that $\mu_{k,j} \to \theta_k := \max\{1, \|z(t_k) - v_k\|\}$ and $\|v_k - v_{k,j}\| \to 0$ as $j \to \infty$. Let $p_j = v_{k,j} - v_k$ and

$$k_j = p_j + \varepsilon_j \left(\frac{z(t_{k,j}) - v_{k,j}}{\mu_{k,j}} - \frac{z(t_k) - v_k}{\theta_k} \right) = p_j + o(1).$$

Then, from (4.18), we obtain

$$\frac{1}{\varepsilon_j}\bigg(I_\lambda\bigg(v_k+\varepsilon_j\bigg(\frac{z(t_k)-v_k}{\theta_k}\bigg)+k_j\bigg)+I_\lambda(v_k+p_j)\bigg)\geq -\frac{1}{k} \quad \text{as } j\to\infty.$$

But since $v_k+\varepsilon_j((z(t_k)-v_k)/\theta_k)\geq w$, using the fact that $z(t_k)\in T$, from Proposition 2.1 and the above inequality we get

$$\int_{\mathbb{R}^n}(-\Delta)^s v_k\bigg(\frac{z(t_k)-v_k}{\theta_k}\bigg)\,dx-\int_\Omega\big(v_k^{-q}+\lambda v_k^{2_s^*-1}\big)\bigg(\frac{z(t_k)-v_k}{\theta_k}\bigg)\,dx\geq -\frac{1}{k}.$$

This is a contradiction to (4.15). Thus, the claim holds. So there exists a sequence $\{v_k\}$ satisfying

$$\text{(4.19)}\qquad \begin{cases} \displaystyle\int_{\mathbb{R}^n}(-\Delta)^s v_k(z-v_k)-\int_\Omega(v_k^{-q}+\lambda v_k^{2_s^*-1})(z-v_k)\,dx \\ \qquad\qquad\qquad\qquad\qquad \displaystyle\geq -\frac{c}{k}(1+\|z\|) & \text{for all } z\in T, \\ I_\lambda(v_k)\to\gamma_0 & \text{as } k\to\infty, \end{cases}$$

where $c>0$ is some constant. Setting $z=2v_k$ in (4.14) and using (4.19) we get

$$\gamma_0+o(1)\geq \frac{sC_s^n}{n}\|v_k\|^2-\frac{2_s^*-1+q}{2_s^*(1-q)}\int_\Omega|v_k|^{1-q}\,dx-\frac{c}{2_s^*k}(1+2\|v_k\|).$$

Now this implies that $\{v_k\}$ must be bounded in X_0, thus up to a subsequence, $v_k\rightharpoonup v$ weakly in X_0 as $k\to\infty$. Using similar ideas as in (ZA) case, it can be shown that v is a weak solution of (P_λ). Then the remaining part of the proof is similar as in [12, Proposition 4.12] (see also [13, Lemma 2.7] in the local setting) and consists of proving the strong convergence of the sequence $\{v_k\}$ to v. To this aim we use that the energy $I_\lambda(v_k)$ is strictly below the first critical level $I_\lambda(w)+s(C_s^nS_s)^{n/2s}/n\lambda^{(n-2s)/2s}$ which implies

$$\text{(4.20)}\qquad \frac{C_s^n}{2}\|v_k-v\|^2-\frac{\lambda}{2_s^*}\|v_k-v\|_{L^{2_s^*}(\Omega)}^{2_s^*}<\frac{s(C_s^nS_s)^{n/2s}}{n\lambda^{(n-2s)/2s}}.$$

Now (4.12), (4.20) and the fact that $S_s\|v_k-v\|^2_{L^{2_s^*}(\Omega)}\leq\|v_k-v\|^2$ force $\|v_k-v\|\to 0$ as $k\to\infty$. Thus, we infer that $I_\lambda(v)=\gamma_0$ and $v\not\equiv w$ and the proof of $w<v$ in Ω can be performed as in the proof of Lemma 4.2. □

Proof of Theorem 1.2. The proof follows from Lemmas 4.2 and4.4, Proposition 3.4 along with Proposition 3.2. □

Proof of Theorem 1.3. The proof follows directly from Proposition 4.1 and [1, Theorem 1.2] with $\delta=q$ and $\beta=0$. To see that the regularity result falls into the scope of [1, Theorem 1.2], note that u is a classical solution as defined in [1, Definition 1]. □

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