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# SOME TWO-POINT PROBLEMS FOR SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS WITH ARGUMENT DEVIATIONS

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Abstract. In the paper we describe the classes of unique solvability of the Dirichlet and mixed two point boundary value problems for the second order linear integro-differential equation

$$u''(t) = p_0(t)u(t) + p_1(t)u(\tau_1(t)) + \int_a^b p(t,\, s)u(\tau(s))\, ds + q(t).$$

On the basis of the obtained and, in some sense, optimal results for the linear problems, by the a priori boundedness principle we prove the theorems of solvability and unique solvability for the second order nonlinear functional differential equations under the mentioned boundary conditions.

## 1. Statement of the main results

**1.1. Introduction.** In this paper we will consider the second order linear integro-differential equation

$$u''(t) = p_0(t)u(t) + p_1(t)u(\tau_1(t)) + \int_a^b p(t,s)u(\tau(s))\, ds + q(t) \tag{1.1}$$

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on the interval $I = [a, b]$, and nonlinear functional differential equation

$$u''(t) = F(u)(t), \tag{1.2}$$

under the Dirichlet two point boundary conditions

$$u(a) = c_1, \quad u(b) = c_2, \tag{1.3$_1$}$$

and mixed two point boundary conditions

$$u(a) = c_1, \quad u'(b) = c_2, \tag{1.3$_2$}$$

where $c_1, c_2 \in R$, $p \in L_\infty(I \times I, R)$, $p_0, p_1, q \in L_\infty(I, R)$, $\tau_1, \tau \colon I \to I$ are the measurable functions and $F \colon C'(I, R) \to L_\infty(I, R)$ is a continuous operator.

By a solution of problem (1.2), (1.3$_1$) ((1.2), (1.3$_2$)) we understand a function $u \in \widetilde{C}'(I, R)$, which satisfies equation (1.2) almost everywhere on $I$ and satisfies conditions (1.3$_1$) ((1.3$_2$)).

Ample interesting literature is devoted to the two-point boundary value problems for the integro-differential equations of special forms (see, e.g. [1], [10], [11], [13], [14] and the references therein). Our work is motivated by some original results for the functional differential equations with argument deviations (see [2]–[6], [12]) and the results of R.P. Agarwal [1], J. Morchalo [11] and B.G. Pachpatte [13], in which simple but quite general sufficient efficient conditions of solvability of BVP for nonlinear integro-differential equations are proved.

For example, in [1] Agarwal studied, $n$th order integro-differential equations which, for $n = 2$, have the form

$$u''(t) = f_0\bigg(t, u(t), u'(t), \int_a^b g(t, s, u(t), u'(s))\, ds\bigg). \tag{1.4}$$

Under the assumption that the functions $f$ and $g$ are continuous in all of their arguments, along with other results, it is proved by Schauder's fixed point theorem that problem (1.4), (1.3$_1$) is solvable if

$$\begin{aligned}\bigg| f_0\bigg(t, x(t), x'(t), \int_a^b g(t, s, x(t), x'(s))\, ds\bigg)\bigg| \\ \le L + \sum_{j=0}^{1} L_j |x^{(j)}(t)| + \sum_{j=0}^{1} L_j \int_a^b h_j(t, s) |x^{(j)}(s)|\, ds,\end{aligned}$$

on $[a, b] \times C'([a, b]; R)$, where the positive constants $L$, $L_j$, and the functions $\int_a^b h_j(t, s)\, ds$ satisfy a certain smallness condition.

Pachpatte in [13] studied nonlinear equations with the argument deviations of the form

$$u''(t) = f_0\bigg(t, u(t), u(\tau_1(t)), \int_a^b g(t, s, u(s), u(\tau(s)))\, ds\bigg), \tag{1.5}$$

where the right-hand side does not contain the first derivative of $u$. By using the Banach fixed point theorem, conditions for the unique solvability of problem (1.5), ($1.3_1$) in a specific set $B$ of asymptotically exponential functions, under the assumptions that the functions $f_0$ and $g$ satisfy the Lipschitz conditions and the inequality

$$\int_a^t \left| K(t,s) f_0\left(s, 0, 0, \int_a^b g(s,\xi,0,0)\, d\xi\right)\right| ds \leq P \exp{(Lt)}$$

holds on $[a,b]$, where the positive numbers $P, L$, satisfy to certain smallness conditions and $K(t,s)$ is the Green function of the problem $u''(t) = 0$, $u(a) = u(b) = 0$.

Morchalo in [11] and in his other studies imposes conditions on the partial derivatives of the functions involved in equation (1.5) which give some kinds of monotonicity of the functions $f_0$ and $g$.

In this paper we established theorems, which in some sense complete and generalize the results of the works cited above as well as certain other known results. We first describe some classes of unique solvability for linear problems (1.1), ($1.3_i$) $i \in \{1,2\}$ and find efficient sufficient conditions of unique solvability of the mentioned problem. The conditions we obtain take into account the effect of argument deviations and they are in some sense optimal (see Remark 1.5). On the basis of these results, by the a priori boundedness principle, we prove existence and uniqueness theorems for nonlinear problems (1.2), ($1.3_i$) when equation (1.2) is close in some sense to linear integro-differential equations. As corollaries of our main results, we obtain efficient sufficient conditions of solvability for the equation

$$u''(t) = f_0\left(t, u(t), u(\tau_1(t)), u'(\tau_2(t)), \int_a^b V(u)(t,s) u(\tau(s))\, ds\right), \tag{1.6}$$

under boundary conditions ($1.3_i$), where $f_0\colon I \times R^4 \to R$ is from the Carathéodory class, $V\colon C'(I,R) \to L_\infty(I \times I, R)$ is a continuous operator and $\tau, \tau_1, \tau_2\colon I \to I$ are measurable functions.

Our results allow also to obtain interesting efficient sufficient conditions of unique solvability for a large class of the two point BVP for $n$th order linear functional differential equations. As an example of such problems we consider here (see Corollary 1.9) $n$th order linear functional differential equation with argument deviation

$$u^{(n)}(t) = p_2(t) u(\tau(t)) + q(t), \tag{1.7}$$

under the boundary conditions

$$u(a) = c_1, \quad u^{(i-1)}(b) = c_2, \quad u^{(j)}(a) = c_{j+1} \quad (j = 2, \dots, n-1), \tag{$1.8_i$}$$

where $n \geq 3$, $c_1, \dots, c_n \in R$, $p_2 \in L_\infty(I,R)$ and $\tau\colon I \to I$ is a measurable function.

Throughout the paper, we use the following notations: $R$ is the set of all real numbers, $R_+ = [0,+\infty[$; $C(I;R)$ is the Banach space of the continuous functions $u\colon I \to R$ with the norm

$$\|u\|_C = \max\{|u(t)| : t \in I\};$$

$C'(I;R)$ is the Banach space of the functions $u\colon I \to R$ which are continuous together with their first derivatives with the norm

$$\|u\|_{C'} = \max\{|u(t)| + |u'(t)| : t \in I\};$$

$\widetilde{C}'(I;R)$ is the set of the functions $u\colon I \to R$ which are absolutely continuous together with their first derivatives; $L(I;R)$ is the Banach space of the Lebesgue integrable functions $p\colon I \to R$ with the norm

$$\|p\|_L = \int_a^b |p(s)|\,ds;$$

$L_\infty(I,R)$ is the space of the essentially bounded measurable functions $p\colon I \to R$ with the norm

$$||p||_\infty = \operatorname{ess\,sup}\{|p(t)| : t \in I\};$$

$L_\infty(I \times I, R)$ is the set of such functions $p\colon I \times I \to R$, that for any fixed $t \in I$, $p(t,\,\cdot\,) \in L(I,R)$ and

$$\int_a^b |p(\,\cdot\,,s)|\,ds \in L_\infty(I,R).$$

Let $i \in \{1,\,2\}$, then for arbitrary $p_0, p_1 \in L_\infty(I,\,R)$, $p \in L_\infty(I \times I,\,R)$, and measurable $\tau_1, \tau\colon I \to I$ we will use the notations:

$$\begin{aligned}\ell_i&(p_1,p,\tau_1,\tau)\\ &= \frac{\pi}{2^{i-1}(b-a)}\bigg(\int_a^b\bigg(|p_1(\xi)||\tau_1(\xi)-\xi| + \int_a^b |p(\xi,\,s)||\tau(s)-\xi|\,ds\bigg)\,d\xi\bigg)^{1/2},\end{aligned}$$

$$\ell_0(p_0,p_1,p)(t) = |p_0(t)| + |p_1(t)| + \int_a^b |p(t,s)|\,ds.$$

Definition 1.1. Let $i \in \{1,2\}$, $\sigma \in \{-1,1\}$ and $\tau_1, \tau\colon I \to I$ be measurable functions. We will say that the vector-function $(h_0,h_1,h)\colon I \to R^3$, where $h_0, h_1 \in L_\infty(I,R_+)$ and $h \in L_\infty(I \times I, R_+)$ belong to the set $D^{\sigma,i}_{\tau_1,\tau}$, if for an arbitrary vector-function $(p_0,p_1,p)\colon I \to R^3$ with measurable components such that

$$\begin{aligned}&0 \leq \sigma p_j(t) \leq h_j(t)\ (j=0,1) && \text{for } t \in I,\\ &0 \leq \sigma p(t,s) \leq h(t,s) && \text{for } (t,s) \in I^2,\end{aligned}\tag{1.9}$$

the homogeneous problem

$$v''(t) = p_0(t)v(t) + p_1(t)v(\tau_1(t)) + \int_a^b p(t,s)v(\tau(s))\,ds, \tag{1.10}$$

$$v(a) = 0, \quad v^{(i-1)}(b) = 0, \tag{1.11$_i$}$$

has no nontrivial solution.

**1.2. Linear problem.**

PROPOSITION 1.2. *Let* $i \in \{1,2\}$, $\sigma \in \{-1,1\}$,

$$h_0, h_1 \in L_\infty(I, R_+), \quad h \in L_\infty(I \times I, R_+), \tag{1.12}$$

*and, for almost all* $t \in I$, *the inequality*

$$\frac{1-\sigma}{2}\ell_0(h_0,h_1,h)(t) + \ell_i(h_1,h,\tau_1,\tau)\ell_0^{1/2}(h_0,h_1,h)(t) < \frac{\pi^2}{4^{i-1}(b-a)^2} \tag{1.13}$$

*holds. Then*

$$(h_0,h_1,h) \in D^{\sigma,i}_{\tau_1,\tau}. \tag{1.14}$$

THEOREM 1.3. *Let* $i \in \{1,2\}$, $\sigma \in \{-1,1\}$, *and*

$$\sigma p_0, \sigma p_1 \in L_\infty(I, R_+), \quad \sigma p \in L_\infty(I \times I, R_+).$$

*Moreover, let for almost all* $t \in I$ *the inequality*

$$\frac{1-\sigma}{2}\ell_0(p_0,p_1,p)(t) + \ell_i(p_1,p,\tau_1,\tau)\ell_0^{1/2}(p_0,p_1,p)(t) < \frac{\pi^2}{4^{i-1}(b-a)^2} \tag{1.15}$$

*holds. Then problem* (1.1), (1.3$_i$) *is uniquely solvable.*

REMARK 1.4. When $p_0$, $p_1$, $p$ are nonnegative functions, then $1-\sigma = 0$ and condition (1.15) in Theorem 1.3 becomes especially simple.

REMARK 1.5. The condition (1.15) is optimal in the sense that for the one term equation

$$v''(t) = p(t)v(t) \quad \text{for } t \in [0, \pi/2^{i-1}], \tag{1.16}$$

when $p(t) \le 0$, condition (1.15) transforms into the condition $|p(t)| < 1$, which is optimal in the sense that if $p \equiv -1$, then $\sin t$ is a nonzero solution of problem (1.16), (1.11$_i$).

Also, from condition (1.15) immediately follows the well known fact that if $p(t) \ge 0$, then problem (1.16), (1.11$_i$) has only the zero solution.

When $p_0 \equiv p_1 \equiv 0$, i.e. when equation (1.1) is of the form

$$u''(t) = \int_a^b p(t,s)u(\tau(s))\,ds + q(t), \tag{1.17}$$

from Theorem 1.3, it follows

Corollary 1.6. *Let $i \in \{1,2\}$, $\sigma \in \{-1,1\}$, and $\sigma p \in L_\infty(I \times I, R_+)$. Moreover, let for almost all $t \in I$ the inequality*

$$\frac{1-\sigma}{2}\int_a^b p(t,s)\,ds + \ell_i(0,p,t,\tau)\bigg(\int_a^b p(t,s)\,ds\bigg)^{1/2} < \frac{\pi^2}{4^{i-1}(b-a)^2} \tag{1.18}$$

*holds. Then problem* (1.17), $(1.3_i)$ *is uniquely solvable.*

Remark 1.7. If in equation (1.17) the coefficient $p$ is nonnegative then condition (1.18) transforms into the condition

$$\int_a^b\int_a^b p(\xi,s)|\tau(s)-\xi|\,ds\,d\xi\int_a^b p(t,s)\,ds < \frac{\pi^2}{4^{i-1}(b-a)^2}.$$

Also for equation (1.7), when $n=2$, from Theorem 1.3 we have:

Corollary 1.8. *Let $i \in \{1,2\}$, $n=2$, and the function $p_2 \in L_\infty(I,R_+)$ be such that the condition*

$$\int_a^b p_2(s)|\tau(s)-s|\,ds < \frac{\pi^2}{4^{i-1}(b-a)^2}$$

*holds. Then problem* (1.7), $(1.3_i)$ *is uniquely solvable.*

Some interesting results for higher order functional differential equations also follow from our main theorem:

Corollary 1.9. *Let $i \in \{1,2\}$, and the function $p_2 \in L_\infty(I,R_+)$ be such that for almost all $t \in I$ the condition*

$$\int_a^b\int_a^t p_2(s)|\tau(s)-t|\,ds\,dt\int_a^b p_2(s)\,ds \le \frac{\pi^2[(n-3)!]^2}{4^{i-1}(b-a)^{2(n-2)}}$$

*holds. Then problem* (1.7), $(1.8_i)$ *is uniquely solvable.*

Remark 1.10. If in Corollaries 1.6 and 1.8 we assume that $\sigma p_j = h_j$ and $\sigma p = h$, then we get the conditions which guarantee inclusion (1.14).

**1.3. Nonlinear problem.** Now we shall consider results on the solvability and the unique solvability of nonlinear problems (1.2), $(1.3_i)$ $i \in \{1,2\}$. Firstly we will introduce some definitions.

Definition 1.11. We say that $F \in K(C', L_\infty)$, if $F\colon C'(I,R) \to L_\infty(I,R)$ is a continuous operator and for an arbitrary $r > 0$

$$\sup\big\{|F(x)(t)| : \|x\|_{C'} \le r,\ x \in C'(I,R)\big\} \in L_\infty(I,R_+).$$

Definition 1.12. Let $i \in \{1,2\}$, $\sigma \in \{-1,1\}$, $\tau_1,\tau\colon I \to I$ be measurable functions and the operators $V_j\colon C'(I,R) \to L_\infty(I,R)$ $(j=0,1)$, $V\colon C'(I,R) \to L_\infty(I \times I, R)$ be continuous. Then we say that

$$(V_0,V_1,V) \in E(h_0,h_1,h,D^{\sigma,i}_{\tau_1,\tau}),$$

if $(h_0, h_1, h) \in D^{\sigma,i}_{\tau_1,\tau}$ and for an arbitrary $x \in C'(I,R)$ the conditions

$$
\begin{aligned}
&0 \le \sigma V_j(x)(t) \le h_j(t) \ (j = 0,1) && \text{for } t \in I, \\
&0 \le \sigma V(x)(t,s) \le h(t,s) && \text{for } (t,s) \in I^2,
\end{aligned} \tag{1.19}
$$

hold.

Throughout the paper it is assumed that

$$
L(x,u)(t) = V_0(x)(t)u(t) + V_1(x)(t)u(\tau_1(t)) + \int_a^b V(x)(t,s)u(\tau(s))\,ds, \tag{1.20}
$$

and the function sgn is defined by the equality

$$
\operatorname{sgn} x = \begin{cases} 1 & \text{for } x \ge 0, \\ -1 & \text{for } x < 0. \end{cases}
$$

Theorem 1.13. *Let $i \in \{1,2\}$, the numbers $\sigma \in \{-1,1\}$, $r > 0$ and the operators $(V_0, V_1, V) \in E(h_0, h_1, h, D^{\sigma,i}_{\tau_1,\tau})$, $F \in K(C', L_\infty)$, be such that the condition*

$$
|F(x)(t) - L(x,x)(t)| \le \eta(t, \|x\|_{C'}) \quad \textit{for } t \in I,\ \|x\|_{C'} \ge r, \tag{1.21}
$$

*holds, where the function $\eta\colon I \times R_+ \to R_+$ is summable in the first argument, nondecreasing in the second one and satisfies the condition*

$$
\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \eta(s,\rho)\,ds = 0. \tag{1.22}
$$

*Then problem* (1.2), (1.3$_i$) *has at least one solution.*

Theorem 1.14. *Let $i \in \{1,2\}$, $\sigma \in \{-1,1\}$, $r > 0$, $F \in K(C', L_\infty)$, $(V_0, V_1, V) \in E(h_0, h_1, h, D^{\sigma,i}_{\tau_1,\tau})$, the operator $\widetilde{V}_0\colon C'(I,R) \to L_\infty(I,R)$ be continuous and almost everywhere on $I$ the conditions*

$$
0 \le \sigma[F(x)(t) - L(x,x)(t)] \operatorname{sgn} x(t) \le \big|\widetilde{V}_0(x)(t)x(t)\big| + \eta(t, \|x\|_{C'}) \tag{1.23}
$$

*for $\|x\|_{C'} \ge r$, and*

$$
0 \le \sigma \widetilde{V}_0(x)(t) \le h_0(t) - \sigma V_0(x)(t) \quad \textit{for } x \in C'(I,R), \tag{1.24}
$$

*hold true, where the function $\eta\colon I \times R_+ \to R_+$ is summable in the first argument, nondecreasing in the second one and satisfies the condition* (1.22). *Then problem* (1.2), (1.3$_i$) *has at least one solution.*

On the basis of Theorem 1.14 we can prove the next existence and the uniqueness theorem

Theorem 1.15. *Let $i \in \{1,2\}$, $\sigma \in \{-1,1\}$, $F \in K(C', L_\infty)$, $(V_0, V_1, V) \in E(h_0, h_1, h, D^{\sigma,i}_{\tau_1,\tau})$, operator $\widetilde{V}_0\colon C'(I,R) \to L_\infty(I,R)$ be continuous and the conditions* (1.24),

$$
0 \le \sigma[F(x)(t) - F(y)(t) - L(z,z)(t)] \operatorname{sgn} z(t) \le \big|\widetilde{V}_0(z)(t)z(t)\big| \tag{1.25}
$$

*for* $t \in I$, $x, y \in C'(I,R)$, *hold if* $z = x - y$. *Then problem* (1.2), (1.3$_i$) *is uniquely solvable.*

From Theorem 1.13 the next corollary follows.

Corollary 1.16. *Let* $i \in \{1,2\}$, $\sigma \in \{-1,1\}$, *the Caratheodory's class functions* $p_0, p_1 \colon I \times R^3 \to R$ *and continuous operator* $V \colon C'(I,R) \to L_\infty(I \times I, R)$ *be such, that almost everywhere on* $I$ *the conditions*:

$$\begin{aligned} &0 \le \sigma p_j(t,\widetilde{x}) \le h_j(t) && \text{for } \widetilde{x} \equiv (x_1,x_2,x_3) \in R^3, \\ &0 \le \sigma V(y)(t,s) \le h(t,s) && \text{for } y \in C'(I,R), \end{aligned} \tag{1.26}$$

*and*

$$|f_0(t,\widetilde{x},x_0) - p_0(t,\widetilde{x})x_1 - p_1(t,\widetilde{x})x_2 - x_0| \le q\bigg(t, \sum_{j=0}^{2} |x_j|\bigg) \tag{1.27}$$

*for* $(\widetilde{x}, x_0) \in R^4$, *are satisfied, where* $(h_0,h_1,h) \in D^{\sigma,i}_{\tau_1,\tau}$, *the function* $q \colon I \times R_+ \to R_+$ *is summable in the first argument, nondecreasing in the second one and equality*

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b q(s,\rho)\,ds = 0 \tag{1.28}$$

*holds. Then problem* (1.6), (1.3$_i$) *has at least one solution.*

In the same way we can get the corollaries concerning the solvability and the unique solvability of problem (1.6), (1.3$_i$), from the Theorems 1.14 and 1.15.

As an example let us consider the integro-differential equation

$$\begin{aligned} u''(t) = {}& \frac{|u'(\tau_2(t))|}{1+|u'(\tau_2(t))|}\, u(\tau_1(t)) \\ &+ \int_a^t u(\tau(s)) \sin^2(u'(t)u(s))\,ds + [u'(t)u(t)]^\alpha + 1, \end{aligned} \tag{1.29}$$

where $\alpha \in [0,1/2)$ and $\tau_2, \tau_1, \tau \colon I \to I$ are measureble functions. Then if $h_1 \equiv 0$, and $h_0 \equiv h \equiv 1$, we conclude from Corollary 1.16 and by Remark 1.7 that problems (1.29), (1.3$_i$) $i \in \{1,2\}$ are solvable if

$$\int_a^b \bigg( |\tau_1(\xi) - \xi| + \int_a^\xi |\tau(s) - \xi|\,ds \bigg) d\xi < \frac{\pi^2}{4^{i-1}(b-a)^2(1+b-a)}.$$

The solvability of problems (1.29), (1.3$_i$) does not follow from the previously known results.

## 2. Auxiliary propositions

First we will introduce here the well known inequalities (see Theorems 256 and 257 in [7]).

Lemma 2.1. *Let* $i \in \{1,2\}$, $z' \in L^2(I,R)$ *and* $z(a)=0$, $z^{(i-1)}(b)=0$. *Then*

$$\int_a^b z^2(s)\,ds \le \frac{4^{i-1}(b-a)^2}{\pi^2}\int_a^b z'^2(s)\,ds. \tag{2.1}$$

On the other hand if $v'' \in L_\infty(I,R)$ and $v(a)=0$, $v^{(i-1)}(b)=0$, then by the integration by parts and Schwarz inequality we get the estimate

$$\left(\int_a^b v'^2(s)\,ds\right)^2 \le \int_a^b (v''(s))^2\,ds \int_a^b v^2(s)\,ds,$$

from which by (2.1) (with $z=v$) immediately follows that

$$\int_a^b v'^2(s)\,ds \le \frac{4^{i-1}(b-a)^2}{\pi^2}\int_a^b (v''(s))^2\,ds. \tag{2.2$_i$}$$

Lemma 2.2. *Let* $i \in \{1,2\}$ *and all the conditions of Proposition* 1.2, *and conditions* (1.9) *hold. Then problem* (1.10), (1.11$_i$) *has only the trivial solution.*

Proof. Assume, that problem (1.10), (1.11$_i$) has a nontrivial solution $v$. Due to (1.11$_i$) it is clear that $v' \not\equiv Const$, and then there exist $t_*, t^* \in I$ such that $t_* < t^*$ and $v'(t^*) - v'(t_*) \neq 0$. Therefore from (1.10) follows that

$$0 < |v'(t^*) - v'(t_*)| \le \int_a^b \left| \int_a^b p(\xi,s)v(\tau(s))\,ds + p_1(\xi)v(\tau_1(\xi)) + p_0(\xi)v(\xi)\right| d\xi.$$

Introducing the notation $\delta = \int_a^b \delta_0(\xi)d\xi$, where

$$\delta_0(\xi) = \sigma\left(\int_a^b p(\xi,s)v^2(\tau(s))\,ds + p_1(\xi)v^2(\tau_1(\xi)) + p_0(\xi)v^2(\xi)\right),$$

due to the last inequality and (1.9), it is clear that

$$\delta > 0. \tag{2.3}$$

From (1.10) by (1.9), the Schwarz and the Cauchy–Schwarz inequalities we have

$$\int_a^b (v''(\xi))^2\,d\xi \le \int_a^b \ell_0(h_0,h_1,h)(\xi)\delta_0(\xi)\,d\xi. \tag{2.4}$$

Now, note that for $\delta$ we have the following representation

$$\begin{aligned}\delta = \sigma &\int_a^b v(\xi)\left[\int_a^b p(\xi,s)v(\tau(s))\,ds + p_1(\xi)v(\tau_1(\xi)) + p_0(\xi)v(\xi)\right] d\xi \\ &+ \int_a^b |p_1(\xi)|v(\tau_1(\xi))\left(\int_\xi^{\tau_1(\xi)} v'(\eta)\,d\eta\right) d\xi \\ &+ \int_a^b \int_a^b |p(\xi,s)|v(\tau(s))\left(\int_\xi^{\tau(s)} v'(\eta)\,d\eta\right) ds\,d\xi.\end{aligned} \tag{2.5}$$

In view of (1.10), (2.2$_i$) and (2.4), by the integration by parts and in view of the boundary conditions (1.11$_i$) we obtain

$$\sigma\int_a^b v(\xi)\left(\int_a^b p(\xi,s)v(\tau(s))\,ds + p_1(\xi)v(\tau_1(\xi)) + p_0(\xi)v(\xi)\right) d\xi \tag{2.6}$$

$$
\begin{aligned}
&= \sigma \int_a^b v(\xi) v''(\xi)\, d\xi \\
&= -\sigma \int_a^b v'^2(\xi)\, d\xi \le \frac{1-\sigma}{2} \frac{4^{i-1}(b-a)^2}{\pi^2} \int_a^b (v''(\xi))^2\, d\xi \\
&\le \frac{1-\sigma}{2} \frac{4^{i-1}(b-a)^2}{\pi^2} \int_a^b \ell_0(h_0, h_1, h)(\xi) \delta_0(\xi)\, d\xi.
\end{aligned}
$$

Also by the use of the Schwarz inequality, the Cauchy–Schwarz inequality and inequalities $(2.2_i)$, (2.4) we get that

$$
\begin{aligned}
(2.7)\quad &\int_a^b |p_1(\xi)| v(\tau_1(\xi)) \bigg( \int_\xi^{\tau_1(\xi)} v'(\eta) d\eta \bigg)\, d\xi \\
&\quad + \int_a^b \int_a^b |p(\xi,s)| v(\tau(s)) \bigg( \int_\xi^{\tau(s)} v'(\eta) d\eta \bigg)\, ds\, d\xi \\
&\le \bigg( \int_a^b |p_1(\xi) v(\tau_1(\xi))|\, |\tau_1(\xi) - \xi|^{1/2}\, d\xi \\
&\quad + \int_a^b \int_a^b |p(\xi,s) v(\tau(s))| |\tau(s) - \xi|^{1/2}\, ds\, d\xi \bigg) \bigg( \int_a^b v'^2(\eta)\, d\eta \bigg)^{1/2} \\
&\le \bigg[ \bigg( \int_a^b |p_1(\xi)| v^2(\tau_1(\xi))|\, d\xi \int_a^b |p_1(\xi)|\, |\tau_1(\xi) - \xi|\, d\xi \bigg)^{1/2} \\
&\quad + \bigg( \int_a^b \int_a^b |p(\xi,s)| v^2(\tau(s))\, ds\, d\xi \int_a^b \int_a^b |p(\xi,s)| |\tau(s) - \xi|\, ds\, d\xi \bigg)^{1/2} \bigg] \\
&\quad \times \frac{2^{i-1}(b-a)}{\pi} \bigg( \int_a^b (v''(\eta))^2\, d\eta \bigg)^{1/2} \\
&\le \bigg( \int_a^b |p_1(\xi)| v^2(\tau_1(\xi))|\, d\xi + \int_a^b \int_a^b |p(\xi,s)| v^2(\tau(s))\, ds\, d\xi \bigg)^{1/2} \\
&\quad \times \bigg( \int_a^b |p_1(\xi)|\, |\tau_1(\xi) - \xi|\, d\xi + \int_a^b \int_a^b |p(\xi,\, s)| |\tau(s) - \xi|\, ds\, d\xi \bigg)^{1/2} \\
&\quad \times \frac{2^{i-1}(b-a)}{\pi} \bigg( \int_a^b \ell_0(p_0, p_1, p)(\xi) \delta_0(\xi)\, d\xi \bigg)^{1/2} \\
&\le \frac{4^{i-1}(b-a)^2}{\pi^2} \bigg( \delta \int_a^b \ell_0(p_0, p_1, p)(\xi) \delta_0(\xi)\, d\xi \bigg)^{1/2} \ell_i(p_1, p, \tau_1, \tau).
\end{aligned}
$$

Therefore from (2.3) and (2.5), by estimates (2.6), (2.7) and inequalities (1.9), we get

$$
\begin{aligned}
(2.8)\quad 0 < \delta \le \frac{4^{i-1}(b-a)^2}{\pi^2} \bigg[ &\frac{1-\sigma}{2} \int_a^b \ell_0(h_0, h_1, h)(\xi), \delta_0(\xi)\, d\xi \\
&+ \bigg( \delta \int_a^b \ell_0(h_0, h_1, h)(\xi) \delta_0(\xi)\, d\xi \bigg)^{1/2} \ell_i(h_1, h, \tau_1, \tau) \bigg].
\end{aligned}
$$

Let now $M_j = \|h_j\|_\infty$, $M_3 = \big\| \int_a^b h(t,s)\,ds \big\|_\infty$, and $M = M_0 + M_1 + M_3$. Consequently, due to condition (1.13), either

$$\ell_0(h_0,h_1,h)(t) < M \quad \text{almost every on } I, \quad \text{and}$$

$$\frac{1-\sigma}{2} M + \ell_i(h_1,h,\tau_1,\tau) M^{1/2} = \frac{\pi^2}{4^{i-1}(b-a)^2}, \tag{2.9}$$

or

$$\frac{1-\sigma}{2} M + \ell_i(h_1,h,\tau_1,\tau) M^{1/2} < \frac{\pi^2}{4^{i-1}(b-a)^2}. \tag{2.10}$$

Let now condition (2.9) ((2.10)) be satisfied; then due to (2.8) we get

$$\delta < \frac{4^{i-1}(b-a)^2}{\pi^2} \left( \frac{1-\sigma}{2} M + \ell_i(h_1,h,\tau_1,\tau) M^{1/2} \right) \delta = \delta$$

$$\left( \delta \le \frac{4^{i-1}(b-a)^2}{\pi^2} \left( \frac{1-\sigma}{2} M + \ell_i(h_1,h,\tau_1,\tau) M^{1/2} \right) \delta < \delta \right).$$

Thus in both cases we get that $\delta < \delta$. The obtained contradiction shows that $v$ is the trivial solution of problem (1.10), $(1.11_i)$. □

Lemma 2.3. *Let $i \in \{1,2\}$, $\sigma \in \{-1,1\}$, $\tau_1, \tau \colon I \to I$ be measurable functions and $(V_0,V_1,V) \in E(h_0,h_1,h,D^{\sigma,i}_{\tau_1,\tau})$. Then there exists such positive number $\rho_0$, that for an arbitrary $x \in C'(I,R)$, any solution $u$ of the equation*

$$u''(t) = L(x,u)(t) + q(t), \tag{2.11}$$

*under boundary conditions $(1.3_i)$, where the operator $L$ is defined by the equality (1.20), admits to the estimate*

$$\|u\|_{C'} \le \rho_0(|c_1| + |c_2| + ||q||_L). \tag{2.12}$$

To prove this lemma, we need Lemma 2.4 below which follows from Lemma 1.1 of [8].

Lemma 2.4. *Let $y, y_k \in L(I,R)$, $v_0, v_{0k} \in L_\infty(I,R)$ $(k = 1,2,\dots)$,*

$$\lim_{k\to+\infty} ||v_{0k} - v_0||_\infty = 0, \quad \lim\sup_{k\to+\infty} ||y_k||_L < +\infty,$$

*and*

$$\lim_{k\to+\infty} \int_a^t y_k(s)\,ds = \int_a^t y(s)\,ds \quad \text{uniformly on } I.$$

*Then*

$$\lim_{k\to+\infty} \int_a^t y_k(s) v_{0k}(s)\,ds = \int_a^t y(s) v_0(s)\,ds \quad \text{for } t \in I.$$

Proof of Lemma 2.3. Assume that the lemma is not true. Then, for an arbitrary natural $k$, there exist operators

$$(V_{0k}, V_{1k}, V_k) \in E(h_0,h_1,h,D^{\sigma,i}_{\tau_1,\tau}), \tag{2.13}$$

and functions $x_k \in C'(I,R)$, $q_k \in L_\infty(I,R)$ such that the problem

$$\begin{aligned} u_k''(t) &= V_{0k}(x_k)(t)u_k(t) + V_{1k}(x_k)(t)u_k(\tau_1(t)) \\ &\qquad + \int_a^b V_k(x_k)(t,s)u_k(\tau(s))\,ds + q_k(t), \\ u_k(a) &= c_1, \quad u_k^{(i-1)}(b) = c_2, \end{aligned}$$

has such a solution $u_k$ that $\|u_k\|_{C'} \geq k(|c_1| + |c_2| + \|q_k\|_L)$. If we suppose that $v_k(t) = u_k(t)/\|u_k\|_{C'}$, $q_{0k}(t) = q_k(t)/\|u_k\|_{C'}$, $c_{0k} = (|c_1|+|c_2|)/\|u_k\|_{C'}$ then

$$\|v_k\|_{C'} = 1, \quad \|q_{0k}\|_L \leq \frac{1}{k}, \quad |v_k(a)| + |v_k^{(i-1)}(b)| = c_{0k} \leq \frac{1}{k}, \tag{2.14}$$

and almost everywhere on $I$ the equality

$$\begin{aligned} v_k''(t) &= V_{0k}(x_k)(t)v_k(t) + V_{1k}(x_k)(t)v_k(\tau_1(t)) \\ &\qquad + \int_a^b V_k(x_k)(t,s)v_k(\tau(s))\,ds + q_{0k}(t) \end{aligned} \tag{2.15}$$

holds. Therefore according to conditions (1.19) and (2.14) we have

$$|v_k''(t)| \leq h_0(t) + h_1(t) + \int_a^b h(t,s)\,ds + |q_{0k}(t)| \quad \text{for } t \in I. \tag{2.16}$$

In view of inequalities (2.14) and (2.16), the sequences $(v_k)_{k=1}^{+\infty}$ and $(v_k')_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous on $I$. By the Arzela–Ascoli lemma, without loss of generality it can be assumed that these sequences are uniformly convergent on $I$. Suppose $v(t) = \lim\limits_{k\to+\infty} v_k(t)$ for $t \in I$ and $v \in C'(I,R)$. Also due to (2.14), conditions $(1.11_i)$ hold and

$$\lim_{k\to+\infty} ||v_k - v||_{C'} = 0, \quad ||v||_{C'} = 1. \tag{2.17}$$

Set $P_{jk}(t) = \int_a^t V_{jk}(x_k)(s)\,ds$ $(j = 0,1)$, $P_k(t,s) = \int_a^s V_k(x_k)(t,\xi)\,d\xi$, then from (1.19) we get

$$\begin{aligned} P_{jk}(a) = 0, &\qquad 0 \leq \sigma(P_{jk}(t_2) - P_{jk}(t_1)) \leq \int_{t_1}^{t_2} h_j(s)\,ds, \\ P_k(t,\,a) = 0, &\quad 0 \leq \sigma(P_k(t,s_2) - P_k(t,s_1)) \leq \int_{s_1}^{s_2} h(t,s)\,ds, \end{aligned} \tag{2.18}$$

for $a \leq t_1 \leq t_2 \leq b$, $a \leq s_1 \leq s_2 \leq b$, $t \in I$, and then the sequences $(P_{jk}(t))_{k=1}^{+\infty}$ and for an arbitrary fixed $t_0 \in I$ sequence $(P_k(t_0,s))_{k=1}^{+\infty}$, are uniformly bounded and equicontinuous on $I$. Then, by the Arzela–Ascoli lemma, without loss of generality it can be assumed that these sequences uniformly converge. Therefore if we denote the limits of these sequences by $P_j(t)$ and $P(t_0,s)$ we get

$$\lim_{k\to+\infty} P_{jk}(t) = P_j(t), \qquad \lim_{k\to+\infty} P_k(t_0,s) = P(t_0,s), \tag{2.19}$$

uniformly on $I$ and then, from (2.18), it follows that

$$\begin{aligned} 0 \le \sigma(P_j(t_2) - P_j(t_1)) &\le \int_{t_1}^{t_2} h_j(s)\,ds, \\ 0 \le \sigma(P(t_0,s_2) - P(t_0,s_1)) &\le \int_{s_1}^{s_2} h(t_0,s)\,ds. \end{aligned} \tag{2.20}$$

Consequently the functions $P_j$ and $P(t_0,\,\cdot\,)$ are absolutely continuous and there exist the functions $p_j$, $p(t_0,\,\cdot\,) \in L(I,R)$ such that

$$P_j(t) = \int_a^t p_j(s)\,ds, \qquad P(t_0,s) = \int_a^s p(t_0,\xi)\,d\xi,$$

and

$$0 \le \sigma p_j(t) \le h(t), \quad 0 \le \sigma p(t_0,s) \le h(t_0,s) \quad \text{for } t,s \in I. \tag{2.21}$$

Then, due to (2.17), (2.19) and (2.21), by Lemma 2.4 with $y_k(s) = V_k(x_k)(t,s)$, $y(s) = p(t_0,s)$ and $v_{0k}(t) = v_k(\tau(t))$, $v_0(t) = v(\tau(t))$, we get

$$\lim_{k\to+\infty} \int_a^b V_k(x_k)(t,s)v_k(\tau(s))\,ds = \int_a^b p(t,s)v(\tau(s))\,ds \tag{2.22}$$

for $t \in I$. Analogously, due to (2.17), (2.19) and (2.21), from Lemma 2.4, we get that on $I$ the following equalities hold

$$\begin{aligned} \lim_{k\to+\infty} \int_a^t V_{0k}(x_k)(s)v_k(s)\,ds &= \int_a^t p_0(s)v(s)\,ds, \\ \lim_{k\to+\infty} \int_a^t V_{1k}(x_k)(s)v_k(\tau_1(s))\,ds &= \int_a^t p_1(s)v(\tau_1(s))\,ds. \end{aligned} \tag{2.23}$$

Therefore according to definition of the set $E(h_0,h_1,h,D^{\sigma,i}_{\tau_1,\tau})$ and conditions (2.13), (2.14), the functions

$$g_k(t) = \int_a^b V_k(x_k)(t,s)v_k(\tau(s))\,ds$$

are measurable and the inequality

$$|g_k(t)| \le \int_a^b h(t,s)\,ds$$

holds. Thus due to (2.22) the Lebesgue's bounded convergence theorem implies that the function

$$g(t) = \int_a^b p(t,s)v(\tau(s))\,ds$$

is integrable and the equality

$$\lim_{k\to+\infty} \int_a^t \int_a^b \big[V_k(x_k)(\xi,s)v_k(\tau(s)) - p(\xi,s)v(\tau(s))\big]\,ds\,d\xi = 0 \tag{2.24}$$

holds on $I$. If we integrate equation (2.15) from $a$ to $t$ and pass to the limit as $k \to +\infty$, then due to conditions (2.14), (2.17), (2.23) and (2.24) we find that $v$ is a solution of problem (1.10), ($1.11_i$). Let

$$p_0(t) + p_1(t) + \int_a^b p(t,s)\, ds \equiv 0,$$

then $v'' \equiv 0$ and conditions ($1.11_i$) yield $v \equiv 0$. If

$$p_0(t) + p_1(t) + \int_a^b p(t,s)\, ds \not\equiv 0,$$

then conditions (2.21) and the inclusion $(h_0, h_1, h) \in D^{\sigma,i}_{I,\tau}$ implies that $v \equiv 0$. Consequently in both of cases we get the contradiction with (2.17), which proves our lemma. □

Remark 2.5. The meaning of Lemma 2.3 is that the operator $L$ is consistent (see Definition 1 in paper [9]) with boundary conditions ($1.3_i$) if

$$(V_0, V_1, V) \in E(h_0, h_1, h, D^{\sigma,i}_{\tau_1,\tau}).$$

Now, for an arbitrary $x \in C'(I,R)$, consider the linear problem

$$v''(t) = L(x,v)(t), \quad v(a) = 0, \quad v^{(i-1)}(b) = 0, \tag{2.25}$$

where the operator $L$ is defined by (1.20) and the lemma below which is the modifications of Theorem 1 of paper [9].

Lemma 2.6. *Let $i \in \{1,2\}$, problem* (2.25) *has only the trivial solution for arbitrary $x \in C'(I,R)$, and there exist a positive number $\rho_1$ such, that for any $\lambda \in (0,1)$ every solution of the problem*

$$\begin{gathered} u''(t) = L(u,u)(t) + \lambda[F(u)(t) - L(u,u)(t)], \\ u(a) = \lambda c_1, \quad u^{(i-1)}(b) = \lambda c_2, \end{gathered} \tag{2.26}$$

*satisfies the estimate*

$$\|u(t)\|_{C'} \le \rho_1. \tag{2.27}$$

*Then problem* (1.2), ($1.3_i$) *has at least one solution.*

## 3. Proof of main results

Proof of Proposition 1.2. Follows from Lemma 2.2 and Definition 1.1. □

Proof of Theorem 1.3. In view of the well known fact that linear problem (1.1), ($1.3_i$) has the Fredholm property, the proof immediately follows from Proposition 1.2, with $h(t,s) \equiv \sigma p(t,s)$, $h_j(t) \equiv \sigma p_j(t)$ $(j = 0,1)$. □

PROOF OF COROLLARY 1.9. By the integration by parts, we can rewrite the homogeneous problem corresponding to the problem (1.7), ($1.8_i$) as (1.10), ($1.11_i$) with $p_0 \equiv p_1 \equiv 0$,

$$p(t,s) = \begin{cases} \dfrac{(t-s)^{n-3}}{(n-3)!}\, p_2(s) & \text{for } t \geq s, \\ 0 & \text{for } t < s. \end{cases}$$

Therefore

$$\int_a^b |p(t,s)|\, ds \leq \frac{(b-a)^{n-3}}{(n-3)!} \int_a^t |p_2(s)|\, ds,$$

and from Corollary 1.3 our corollary immediately follows. □

PROOF OF THEOREM 1.13. Let $\rho_0$ be a number defined in Lemma 2.3, then due to condition (1.22) there exists the constant $\rho_1 > r$ such, that

$$\rho_0 \left( b - a + |c_1| + |c_2| + \int_a^b \eta(s,\rho)\, ds \right) < \rho \quad \text{for } \rho \geq \rho_1. \tag{3.1}$$

Let also $\lambda \in (0,1)$ be an arbitrary fixed number, $u$ be a solution of problem (2.26), and assume that $\|u\|_{C'} \geq \rho_1$.

Now note that all the assumptions of Lemma 2.3 are fulfilled; due to condition (1.21) we have the estimate

$$\begin{aligned} \|u\|_{C'} &\leq \rho_0 \left( |c_1| + |c_2| + \lambda \int_a^b |F(u)(s) - L(u,u)(s)|\, ds \right) \\ &< \rho_0 \left( |c_1| + |c_2| + \int_a^b \eta(s, \|u\|_{C'})\, ds \right), \end{aligned}$$

which, in view of our assumption, contradicts the inequality (3.1). Therefore our assumption is invalid, inequality (2.27) holds and then, from Lemma 2.6, the validity of our Theorem follows. □

PROOF OF THEOREM 1.14. Let $\rho_0$ be a number defined in Lemma 2.3, then due to condition (1.22) there exists constant $\rho_1 > r$, such that inequality (3.1) holds. Let also $\lambda \in (0,1)$ be an arbitrary fixed number, $u$ be a solution of problem (2.26) and assume that $\|u\|_{C'} \geq \rho_1$. Then, on account of condition (1.24) and nonnegativity of the function $\eta$, function $u$ is a solution of the equation

$$u''(t) = L(u,u)(t) + \lambda \nu(t) \widetilde{V}_0(u)(t) u(t) + \eta_1(t, \|u\|_{C'}),$$

where

$$\eta_1(t, \|u\|_{C'}) = \lambda \sigma \nu(t) (\eta(t, \|u\|_{C'}) + 1) \operatorname{sgn} u(t)$$

and

$$\nu(t) = \frac{\sigma[F(u)(t) - L(u,u)(t)] \operatorname{sgn} u(t)}{|\widetilde{V}_0(u)(t) u(t)| + \eta(t, \|u\|_{C'}) + 1}.$$

Moreover, due to condition (1.23), the estimates

$$0 \le \nu(t) \le 1, \tag{3.2}$$

and

$$|\eta_1(t, \|u\|_{C'})| \le 1 + \eta(t, \|u\|_{C'}), \tag{3.3}$$

are true on $I$. Now note, that according to conditions (1.24) and (3.2), the estimate

$$0 \le \sigma(V_0(u)(t) + \lambda\nu(t)\widetilde{V}_0(u)(t)) \le h_0(t)$$

holds on $I$. Consequently, the inclusion

$$(V_0 + \lambda\nu\widetilde{V}_0, V_1, V) \in E(h_0, h_1, h, D^{\sigma,i}_{\tau_1,\tau}) \tag{3.4}$$

is valid and then, from Lemma 2.3, due to inequality (3.3), we get the estimate

$$\|u\|_{C'} \le \rho_0\bigg(|c_1| + |c_2| + \int_a^b [\eta(s, \|u\|_{C'}) + 1]\, ds\bigg),$$

which contradicts with inequality (3.1). Therefore our assumption is invalid and inequality (2.27) holds, from which due to Lemma 2.6 validity of our theorem follows. □

Proof of Theorem 1.15. From conditions (1.25) it follows that all the assumptions of Theorem 1.14 hold. Assume that $v_1$, $v_2$, are the solutions of problem (1.2), ($1.3_i$), and let $v = v_1 - v_2$. Consequently, conditions ($1.11_i$) hold and due to condition (1.25) we have

$$v''(t) = L(v, v)(t) \quad \text{for } t \in I_0,$$

if $I_0 = \{t \in I : v(t) = 0\}$. On the other hand condition (1.24) yields that $v$ is a solution of the equation

$$v''(t) = L(v, v)(t) + \mu(t)\widetilde{V}_0(v)(t)v(t) \quad \text{for } t \in I \setminus I_0,$$

where

$$\mu(t) = \frac{\sigma(F(v_1)(t) - F(v_2)(t) - L(v, v)(t)) \operatorname{sgn} v(t)}{|\widetilde{V}_0(v)(t)v(t)|} \quad \text{for } t \in I \setminus I_0.$$

Let now

$$\nu(t) = \begin{cases} \mu(t) & \text{for } t \in I \setminus I_0, \\ 0 & \text{for } t \in I_0. \end{cases}$$

Then, according to (1.25), the estimate (3.2) holds on $I$ and

$$v''(t) = L(v, v)(t) + \nu(t)\widetilde{V}_0(v)(t)v(t) \quad \text{for } t \in I. \tag{3.5}$$

Moreover, analogously as in the proof of Theorem 1.14, conditions (1.24) and (3.2) yield the inclusion (3.4). Therefore all the assumptions of Lemma 2.3, with $c_1 = c_2 = 0$, $q \equiv 0$ are fulfilled and then equality $v \equiv 0$ holds, which proves our theorem. □

Proof of Corollary 1.16. If we assume that

$$V_j(x)(t) = p_j(t, x(t), x(\tau_1(t)), x'(\tau_2(t))),$$

$$F(x)(t) = f_0\bigg(t,\, x(t),\, x(\tau_1(t)),\, x'(\tau_2(t)),\, \int_a^b V(x)(t,s)x(\tau(s))\,ds\bigg),$$

and

$$\eta(t, \|x\|_{C'}) = q(t, (2+\delta)\|x\|_{C'}),$$

where

$$\delta = \bigg\| \int_a^b h(t,s)\,ds \bigg\|_\infty,$$

then in view of conditions (1.26)–(1.28) it is clear, that all the assumptions of Theorem 1.13 hold, from which the validity of our corollary immediately follows. □

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