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THE CONTINUITY OF ADDITIVE AND CONVEX FUNCTIONS WHICH ARE UPPER BOUNDED ON NON-FLAT CONTINUA IN $\mathbb{R}^n$

Taras Banakh — Eliza Jabłońska — Wojciech Jabłoński

Abstract. We prove that for a continuum $K \subset \mathbb{R}^n$ the sum K^{+n} of n copies of K has non-empty interior in $\mathbb{R}^n$ if and only if K is not flat in the sense that the affine hull of K coincides with $\mathbb{R}^n$. Moreover, if K is locally connected and each non-empty open subset of K is not flat, then for any (analytic) non-meager subset $A \subset K$ the sum A^{+n} of n copies of A is not meager in $\mathbb{R}^n$ (and then the sum A^{+2n} of $2n$ copies of the analytic set A has non-empty interior in $\mathbb{R}^n$ and the set $(A-A)^{+n}$ is a neighbourhood of zero in $\mathbb{R}^n$). This implies that a mid-convex function $f\colon D \to \mathbb{R}$ defined on an open convex subset $D \subset \mathbb{R}^n$ is continuous if it is upper bounded on some non-flat continuum in D or on a non-meager analytic subset of a locally connected nowhere flat subset of D.

1. Introduction

Let X be a linear topological space over the field of real numbers. A function $f\colon X \to \mathbb{R}$ is called *additive* if $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

A function $f\colon D \to \mathbb{R}$ defined on a convex subset $D \subset X$ is called *mid-convex* if $f((x+y)/2) \le (f(x)+f(y))/2$ for all $x, y \in D$.

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Many classical results concerning additive or mid-convex functions state that the boundedness of such functions on "sufficiently large" sets implies their continuity. That is why Ger and Kuczma [8] introduced the following three families of sets in X:

- the family $\mathcal{A}(X)$ of all subsets $T \subset X$ such that any mid-convex function $f\colon D \to \mathbb{R}$ defined on a convex open subset $D \subset X$ containing T is continuous if $\sup f(T) < \infty$;
- the family $\mathcal{B}(X)$ of all subsets $T \subset X$ such that any additive function $f\colon X \to \mathbb{R}$ with $\sup f(T) < \infty$ is continuous;
- the family $\mathcal{C}(X)$ of all subsets $T \subset X$ such that any additive function $f\colon X \to \mathbb{R}$ is continuous if the set $f(T)$ in bounded in $\mathbb{R}$.

It is clear that

$$\mathcal{A}(X) \subset \mathcal{B}(X) \subset \mathcal{C}(X). \tag{1.1}$$

By the example of Erdős [4] (discussed in [8]) the classes $\mathcal{B}(X)$ and $\mathcal{C}(X)$ are not equal even if $X = \mathbb{R}^n$, $n \in \mathbb{N}$. On the other hand, Ger and Kominek [7] proved that $\mathcal{A}(X) = \mathcal{B}(X)$ for any Baire topological vector space X. In particular, $\mathcal{A}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^n)$ for every $n \in \mathbb{N}$ (cf. [15]).

There are lots of papers devoted to the problem of recognizing sets in the families $\mathcal{A}(X)$, $\mathcal{B}(X)$ or $\mathcal{C}(X)$, see e.g. [1], [5], [17], [19], [20]. The classical results concerning mid-convex functions (namely, Bernstein–Doetsch Theorem [2] and its generalizations, see e.g. [23]) imply that a subset T with non-empty interior in a topological vector space X belongs to the family $\mathcal{A}(X)$. By (the proofs of) Lemma 9.2.1 and Theorem 9.2.5 in [14], a subset T of a topological vector space X belongs to a family $\mathcal{K} \in \{\mathcal{A}(X), \mathcal{B}(X), \mathcal{C}(X)\}$ if and only if for some $n \in \mathbb{N}$ its n-fold sum

$$T^{+n} = \underbrace{T + \ldots + T}_{n \text{ times}}$$

belongs to the family $\mathcal{K}$. Combining these two facts, we obtain the following well-known folklore result.

Theorem 1.1. *A subset T of a topological vector space X belongs to the family $\mathcal{A}(X)$ if for some $n \in \mathbb{N}$ its n-fold sum T^{+n} has non-empty interior in X.*

Theorem 1.1 has been used many times to show that various "thin" sets actually belong to the class $\mathcal{A}(X)$, $\mathcal{B}(X)$ or $\mathcal{C}(X)$. In this respect let us mention the following result of Ger [6].

Theorem 1.2 (Ger). *Let $I \subset \mathbb{R}$ be a nontrivial interval, $n \geq 2$ and let $\varphi\colon I \to \mathbb{R}^n$ be a C^1-smooth function defining in $\mathbb{R}^n$ a curve which does not lie entirely in an $(n-1)$ dimensional affine hyperplane. Let $Z \subset I$ be a set of positive Lebesgue measure. If one of the conditions is fulfilled:*

(a) *the Lebesgue measure of* $Z \cap (a,b)$ *is positive for every nontrivial interval* $(a,b) \subset I$,

(b) *the determinant*

$$\begin{vmatrix} \varphi_1'(x_1) & \dots & \varphi_1'(x_n) \\ \vdots & \ddots & \vdots \\ \varphi_n'(x_1) & \dots & \varphi_n'(x_n) \end{vmatrix}$$

is non-zero for almost every $(x_1,\dots,x_n)$ *in* I^n,

then the image $\varphi(Z)$ *belongs to the class* $\mathcal{A}(\mathbb{R}^n)$.

In this paper we prove a topological counterpart of Ger's Theorem 1.2.

A subset A of a topological vector space X is defined to be

- *flat* if the affine hull of A is nowhere dense in X;
- *nowhere flat* if each non-empty relatively open subset $U \subset A$ is not flat in X.

By a *continuum* we understand a connected compact metrizable space.

The main aim of the paper is to prove the following result.

Theorem 1.3. *Let* $n \in \mathbb{N}$. *For any non-flat continuum* $K \subset \mathbb{R}^n$ *its* n*-fold sum* K^{+n} *has non-empty interior in* $\mathbb{R}^n$ *and hence* K *belongs to the class* $\mathcal{A}(\mathbb{R}^n)$. *Moreover, if* K *is locally connected and nowhere flat in* $\mathbb{R}^n$, *then for any non-meager analytic subspace* A *of* K *the* $2n$*-fold sum* A^{+2n} *has non-empty interior in* $\mathbb{R}^n$, *which implies that* $A \in \mathcal{A}(\mathbb{R}^n)$.

The first part of Theorem 1.3 will follow from Corollary 2.5 and the second one from Corollary 3.2.

Remark 1.4. The first part of Theorem 1.3 answers a problem posed by the last author in [10].

2. Algebraic sum of n continua in $\mathbb{R}^n$

In the proof of Theorem 1.3 we shall apply a non-trivial result of Holsztyński [9] and Lifanov [18] on the dimension properties of products of continua. Let us recall that a closed subset S of a topological space X is called a *partition* between subsets A and B of X if there exist two sets U and W open in $X \setminus S$ such that $A \subset U$, $B \subset W$, $U \cap W = \emptyset$ and $X \setminus S = U \cup W$.

The following proposition can be derived from results of Holsztyński [9] and Lifanov [18] and is discussed by Engelking in [3, 1.8.K].

Proposition 2.1 (Holsztyński, Lifanov). *Let* $K_1,\dots,K_n$ *be continua and* $K := \prod_{i=1}^{n} K_i$ *be their product. For every positive integer* $i \le n$ *let* a_i^-, a_i^+ *be two* *distinct points in* K_i *and let* S_i *be a partition between the sets*

$$A_i^- := \left\{(x_k)_{k=1}^n \in K : x_i = a_i^-\right\} \quad \textit{and} \quad A_i^+ := \left\{(x_k)_{k=1}^n \in K : x_i = a_i^+\right\}$$

in K. *Then the intersection* $\bigcap_{i=1}^n S_i$ *is not empty.*

The principal ingredient in the proof of Theorem 1.3 is the following result, which can have an independent value.

Theorem 2.2. *Let* $K_1, \dots, K_n$ *be continua in* $\mathbb{R}^n$ *containing the origin of* $\mathbb{R}^n$. *Assume that each continuum* K_i *contains a point* e_i *such that the vectors* $e_1, \dots, e_n$ *are linearly independent. Then the algebraic sum* $K := K_1 + \dots + K_n$ *has non-empty interior in* $\mathbb{R}^n$ *and the Lebesgue measure of* K *is not smaller than the volume of the parallelotope* $P := [0,1] \cdot e_1 + \dots + [0,1] \cdot e_n$.

Proof. After a suitable linear transformation of $\mathbb{R}^n$, we can assume that $e_1, \dots, e_n$ coincide with the standard basis of $\mathbb{R}^n$, i.e. $e_1 = (1,0,\dots,0)$, $e_2 = (0,1,0,\dots,0)$, $\dots$, $e_n = (0,\dots,0,1)$. In this case we should prove that the sum $K = K_1 + \dots + K_n$ has non-empty interior in $\mathbb{R}^n$ and the Lebesgue measure $\lambda(K)$ of K is not smaller than the volume $\lambda(P) = 1$ of the cube $P = [0,1]^n$.

On the space $\mathbb{R}^n$ we consider the sup-norm $\|x\| = \max_{1\le i\le n} |x_i|$. Let

$$\delta := \max\left\{\|x\| : x \in \bigcup_{i=1}^n K_i\right\}.$$

Choose numbers $s, l \in \mathbb{N}$ such that $l > s + (n-1)\delta$. Moreover, if $\lambda(K) < 1$, then we can replace s and l by larger numbers and additionally assume that $[2s/(2l+1)]^n > \lambda(K)$.

For every positive integer $i \le n$, consider the finite set $Z_i = \{k \cdot e_i : k \in \mathbb{Z}, |k| \le l\}$ in $\mathbb{R}^n$ and observe that the sum $\widetilde{K}_i := K_i + Z_i$ is a continuum containing the set Z_i (as K_i contains zero). Let $Z := Z_1 + \dots + Z_n \subset \mathbb{Z}^n \subset \mathbb{R}^n$ and observe that

$$\begin{aligned} K + Z &= (K_1 + \dots + K_n) + (Z_1 + \dots + Z_n) \\ &= (K_1 + Z_1) + \dots + (K_n + Z_n) = \widetilde{K}_1 + \dots + \widetilde{K}_n. \end{aligned}$$

Claim 2.3. $[-s,s]^n \subset K + Z$.

Proof. To derive a contradiction, suppose that $[-s,s]^n \not\subset K + Z$ and fix a point $(y_i)_{i=1}^n \in [-s,s]^n \setminus (K+Z)$. For every positive integer $k \le n$ denote by $\mathrm{pr}_k \colon \mathbb{R}^n \to \mathbb{R}$, $\mathrm{pr}_k \colon (x_i)_{i=1}^n \mapsto x_k$, the coordinate projection. Also let

$$\begin{aligned} Y_k &:= \{x \in \mathbb{R}^n : \mathrm{pr}_k(x) = y_k\}, \\ Y_k^- &:= \{x \in \mathbb{R}^n : \mathrm{pr}_k(x) < y_k\}, \quad Y_k^+ := \{x \in \mathbb{R}^n : \mathrm{pr}_k(x) > y_k\}. \end{aligned}$$

Consider the continuous map

$$\Sigma\colon \prod_{i=1}^{n} \widetilde{K}_i \to K+Z, \qquad \Sigma\colon (x_i)_{i=1}^n \mapsto \sum_{i=1}^{n} x_i.$$

For every positive integer $k \le n$ let

$$p_k\colon \prod_{i=1}^{n} \widetilde{K}_i \to \widetilde{K}_k, \qquad p_k\colon (x_i)_{i=1}^n \mapsto x_k,$$

be the coordinate projection. The sets $F_k^- := p_k^{-1}(-l\cdot e_k)$ and $F_k^+ := p_k^{-1}(l\cdot e_k)$ will be called the negative and positive k-faces of the "cube" $\widetilde{K} := \prod_{i=1}^{n} \widetilde{K}_i$.

We claim that $\Sigma(F_k^-) \subset Y_k^-$ and $\Sigma(F_k^+) \subset Y_k^+$. Indeed, for any $\widetilde{x} = (\widetilde{x}_i)_{i=1}^n$ in $F_k^+ \subset \widetilde{K}$ we can find sequences $(x_i)_{i=1}^n \in \prod_{i=1}^{n} K_i$ and $(z_i)_{i=1}^n \in \prod_{i=1}^{n} Z_i$ such that $(\widetilde{x}_i)_{i=1}^n = (x_i+z_i)_{i=1}^n$. Taking into account that $\mathrm{pr}_k(z_i) = 0$ for all $i \ne k$, we conclude that

$$\begin{aligned}
\mathrm{pr}_k \circ \Sigma(\widetilde{x}) &= \sum_{i=1}^{n} \mathrm{pr}_k(\widetilde{x}_i) = \mathrm{pr}_k(\widetilde{x}_k) + \sum_{i\ne k} \mathrm{pr}_k(\widetilde{x}_i)\\
&= \mathrm{pr}_k(l\cdot e_k) + \sum_{i\ne k} \mathrm{pr}_k(x_i+z_i)\\
&= l + \sum_{i\ne k} \mathrm{pr}_k(x_i) \ge l - \sum_{i\ne k} \|x_i\| \ge l-(n-1)\delta > s \ge y_k
\end{aligned}$$

and hence $\Sigma(\widetilde{x}) \in Y_k^+$ and finally $\Sigma(F_k^+) \subset Y_k^+$. By analogy we can prove that $\Sigma(F_k^-) \subset Y_k^-$. Then $\Sigma^{-1}(Y_k)$ is a partition between the k-th faces F_k^- and F_k^+ of the "cube" $\widetilde{K}$.

Since $\bigcap_{k=1}^{n} Y_k = \big\{(y_k)_{k=1}^n\big\} \not\subset K+Z = \Sigma(\widetilde{K})$, the intersection $\bigcap_{k=1}^{n} \Sigma^{-1}(Y_k)$ is empty, which contradicts Proposition 2.1. □

Now we continue the proof of Theorem 2.2. By Claim 2.3, $[-s,s]^n \subset K+Z$. Taking into account that the set Z is finite, we conclude that the set $K = K_1+\ldots+K_n$ has non-empty interior in $\mathbb{R}^n$. Moreover, $[-s,s]^n \subset K+Z$ implies $(2s)^n = \lambda([-s,s]^n) \le \lambda(K+Z) \le \lambda(K)\cdot|Z| = \lambda(K)\cdot(2l+1)^n$ and hence $[2s/(2l+1)]^n \le \lambda(K)$. Then $\lambda(K) \ge 1$ by the choice of the numbers l and s. □

Remark 2.4. By Theorem 2.2 with $n=2$ we obtain that if K is a continuum in the plane which does not lie on a line, then the set $K-K$ contains an open set. This is the exactly [11, Proposition 1.3] proved by Kallman and Simmons to get a continuity criterion for automorphisms of the field of complex numbers.

The following corollary of Theorem 2.2 implies the first part of Theorem 1.3.

Corollary 2.5. *For a continuum $K \subset \mathbb{R}^n$ and its n-fold sum K^{+n} the following conditions are equivalent*:

(a) *K^{+n} has non-empty interior in $\mathbb{R}^n$;*
(b) *K^{+n} has positive Lebesgue measure in $\mathbb{R}^n$;*
(c) *K is not flat in $\mathbb{R}^n$;*
(d) *for any non-zero continuous linear functional $f\colon \mathbb{R}^n \to \mathbb{R}$ the image $f(K)$ has non-empty interior in $\mathbb{R}$;*
(e) *K belongs to the family $\mathcal{A}(\mathbb{R}^n)$;*
(f) *K belongs to the family $\mathcal{C}(\mathbb{R}^n)$.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. To prove that (c) $\Rightarrow$ (a), assume that K is not flat in $\mathbb{R}^n$. After a suitable shift, we can assume that K contains the origin of the vector space $\mathbb{R}^n$.

Let $E \subset K$ be a maximal linearly independent subset of K. Assuming that $|E| < n$ we would conclude that K is contained in the linear hull of the set E and hence is flat. So, $|E| = n$ and we can write E as $E = \{e_1, \dots, e_n\} \subset K$. Applying Theorem 2.2 to the continua $K_i := K$, $1 \le i \le n$, we conclude that the sum $K^{+n} = K_1 + \dots + K_n$ has non-empty interior in $\mathbb{R}^n$.

The equivalence (c) $\Leftrightarrow$ (d) is proved in Lemma 2.6 below, the implication (a) $\Rightarrow$ (e) follows from Theorem 1.1 and (e) $\Rightarrow$ (f) is trivial. To finish the proof, it suffices to show that (f) $\Rightarrow$ (c), which is equivalent to $\neg$(c) $\Rightarrow$ $\neg$(f). So, assume that the continuum K is flat in $\mathbb{R}^n$. Then $K \subset x + L$ for some $x \in \mathbb{R}^n$ and some linear subspace $L \subset \mathbb{R}^n$ of dimension $\dim(L) = n - 1$. Let $\mathbb{R}^n/L$ be the quotient space and $q\colon \mathbb{R}^n \to \mathbb{R}^n/L$ be the quotient linear operator.

Since the quotient space $\mathbb{R}^n/L$ is topologically isomorphic to $\mathbb{R}$, it admits a discontinuous additive function $a\colon \mathbb{R}^n/L \to \mathbb{R}$. Then $f = a \circ q\colon \mathbb{R}^n \to \mathbb{R}$ is a discontinuous additive function such that $f(K) \subset f(x + L) = \{f(x)\}$, which means that $K \notin \mathcal{C}(\mathbb{R}^n)$. □

Lemma 2.6. *A connected subset K of a locally convex topological vector space X is not flat if and only if for any non-zero linear continuous functional $f\colon X \to \mathbb{R}$ the image $f(K)$ has non-empty interior in X.*

Proof. If A is flat, then the affine hull of A in X is nowhere dense and hence $A \subset a + L$ for some $a \in A$ and some nowhere dense closed linear subspace of X. Using the Hahn–Banach Theorem, choose a non-zero linear continuous functional $f\colon X \to \mathbb{R}$ such that $L \subset f^{-1}(0)$. Then the image $f(A) \subset f(L + a) = \{f(a)\}$ is a singleton (which has empty interior in the real line).

Now assuming that A is not flat, we shall prove that for any non-zero linear continuous functional $f\colon X \to \mathbb{R}$ the image $f(A)$ has non-empty interior in $\mathbb{R}$. Since A is not flat, its affine hull is dense in X and hence the affine hull of $f(A)$ is dense in $\mathbb{R}$. This implies that $f(A)$ contains two distinct points $a < b$. By the connectedness of $f(A)$ (which follows from the connectedness of A and the continuity of f), the set $f(A)$ contains the interval $[a,b]$ and hence has non-empty interior in $\mathbb{R}$. □

PROBLEM 2.7. Is there a compact subset $K \subset \mathbb{R}^2$ such that $K+K$ has empty interior in $\mathbb{R}^2$ but for any non-zero linear continuous functional $f\colon \mathbb{R}^2 \to \mathbb{R}$ the image $f(K)$ has non-empty interior in $\mathbb{R}$?

3. Collectively nowhere flat subsets in $\mathbb{R}^n$

Subsets $A_1, \dots, A_n$ of $\mathbb{R}^n$ will be called *collectively nowhere flat in* $\mathbb{R}^n$ if any non-empty relatively open subsets $U_1 \subset A_1, \dots, U_n \subset A_n$ contain points $a_1, b_1 \in U_1, \dots, a_n, b_n \in U_n$ such that the vectors $b_1 - a_1, \dots, b_n - a_n$ form a basis of the linear space $\mathbb{R}^n$. For example, for a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ with $n \geq 2$ the closed intervals $[0,1]\cdot e_1, \dots [0,1]\cdot e_n$ are collectively nowhere flat; yet each set $[0,1]\cdot e_i$ separately is flat.

It is easy to see that a subspace $A \subset \mathbb{R}^n$ is nowhere flat in $\mathbb{R}^n$ if and only if the sequence of n its copies $A_1 = A, \dots, A_n = A$ is collectively nowhere flat in $\mathbb{R}^n$.

THEOREM 3.1. *Let $K_1, \dots, K_n$ be collectively nowhere flat locally connected subspaces of $\mathbb{R}^n$. For every non-meager subsets $B_1, \dots, B_n$ in $K_1, \dots, K_n$ the algebraic sum $B_1 + \dots + B_n$ is non-meager in $\mathbb{R}^n$.*

PROOF. To derive a contradiction, assume that the sum $B_1 + \dots + B_n$ is meager and hence is contained in the countable union $\bigcup_{i\in\omega} F_i$ of closed nowhere dense subsets of $\mathbb{R}^n$. Consider the continuous map

$$\Sigma\colon (\mathbb{R}^n)^n \to \mathbb{R}^n, \qquad \Sigma\colon (x_k)_{k=1}^n \mapsto \sum_{k=1}^n x_k.$$

Taking into account that for every $i \leq n$ the subset B_i is not meager in K_i, we can apply a classical result of Banach [16, §10.V] and find a non-empty open set $W_i \subset K_i$ such that the intersection $W_i \cap B_i$ is a dense Baire subspace of W_i. Replacing W_i by a smaller open subset of W_i, we can assume that the set W_i is bounded in $\mathbb{R}^n$. Replacing B_i by $B_i \cap W_i$, we can assume that B_i is a dense Baire space in W_i.

By [12, 8.44], the product $\prod_{k=1}^n B_k$ of second countable Baire spaces B_k is Baire. Since $\prod_{k=1}^n B_k \subset \bigcup_{i\in\omega} \Sigma^{-1}(F_i)$, we can apply Baire Theorem and find $i \in \omega$ such that the set $\Sigma^{-1}(F_i) \cap \prod_{k=1}^n B_k$ has non-empty interior in $\prod_{k=1}^n B_k$. Then we can find non-empty open sets $V_1 \subset W_1, \dots, V_n \subset W_n$ such that

$$\prod_{k=1}^n (B_k \cap V_k) \subset \Sigma^{-1}(F_i).$$

Since the spaces $K_1, \dots, K_n$ are locally connected, we can additionally assume that each set V_k is connected and hence has compact connected closure $\overline{V}_k$ in $\mathbb{R}^n$. The set $\Sigma^{-1}(F_i)$ is closed and hence contains the closure $\prod_{k=1}^{n} \overline{V}_k$ of the set $\prod_{k=1}^{n} (V_k \cap B_k)$ in $(\mathbb{R}^n)^n$. Then the set

$$\overline{V}_1 + \dots + \overline{V}_n = \Sigma\bigg(\prod_{k=1}^{n} \overline{V}_k\bigg) \subset F_i$$

is nowhere dense in $\mathbb{R}^n$. On the other hand, taking into account that the sets $K_1, \dots, K_n$ are collectively nowhere flat in $\mathbb{R}^n$, in each set $\overline{V}_k$ we can choose two points a_k, b_k such that the vectors $b_1 - a_1, \dots, b_n - a_n$ form a basis of the vector space $\mathbb{R}^n$. Applying Theorem 2.2, we can conclude that the set $\overline{V}_1 + \dots + \overline{V}_n$ has non-empty interior and hence cannot be contained in the nowhere dense set F_i. This contradiction completes the proof. $\square$

The following corollary of Theorem 3.1 yields the second part of Theorem 1.3.

Corollary 3.2. *Let K be a nowhere flat locally connected subset of $\mathbb{R}^n$ and A be a non-meager analytic subspace of K. Then the n-fold sum A^{+n} of A is a non-meager analytic subset of $\mathbb{R}^n$, the $2n$-fold sum A^{+2n} has non-empty interior and the set $(A - A)^{+n}$ is a neighbourhood of zero in $\mathbb{R}^n$. Moreover, the set A belongs to the family $\mathcal{A}(\mathbb{R}^n)$.*

Proof. Since the set A is non-meager in K, by [16, §10.V], there exists a non-empty open set $V \subset K$ such that $V \cap A$ is a dense Baire subspace of V. Since K is locally connected, we can assume that V is connected and so is its closure $\overline{V}$ in K. Replacing K by $\overline{V}$ and A by $A \cap V$, we can assume that A is a dense Baire subspace of K. By Theorem 3.1, the n-fold sum A^{+n} of A is a non-meager subset of $\mathbb{R}^n$. The subspace A^{+n} is analytic, being a continuous image of the analytic space A^n. Applying Pettis–Piccard Theorem [21, Corollary 5], [22] (also [13, Theorem 1]), we can conclude that the sum $A^{+n} + A^{+n} = A^{+2n}$ has non-empty interior and the difference $A^{+n} - A^{+n}$ is a neighbourhood of zero in $\mathbb{R}^n$. Since A^{+2n} has non-empty interior in $\mathbb{R}^n$, we can apply Theorem 1.1 and conclude that $A \in \mathcal{A}(\mathbb{R}^n)$. $\square$

Writing down the definition of the class $\mathcal{A}(\mathbb{R}^n)$ and applying Corollaries 2.5 and 3.2, we obtain the following characterization.

Corollary 3.3. *For a mid-convex function $f \colon D \to \mathbb{R}$ defined on a non-empty open convex set $D \subset \mathbb{R}^n$, the following conditions are equivalent*:

(a) *f is continuous*;
(b) *f is upper bounded on some non-flat continuum $K \subset D$*;

(c) *f is upper bounded on some non-meager analytic subspace of a nowhere flat locally connected subset of D.*

Now we present an example showing that the condition of (collective) nowhere flatness in Corollaries 3.2, 3.3 (and Theorem 3.1) is essential.

EXAMPLE 3.4. Let

$$C = \left\{ \sum_{n=1}^{\infty} \frac{2x_n}{3^n} : (x_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}} \right\}$$

be the standard Cantor set in the interval $[0,1]$. Let $f\colon C \to [0,1]$ be the continuous map assigning to each point $\sum\limits_{n=1}^{\infty} 2x_n/3^n$ of the Cantor set C the real number $\sum\limits_{n=1}^{\infty} x_n/2^n$. Let $\overline{f}\colon [0,1] \to [0,1]$ be the unique monotone function extending f. The function $\overline{f}$ is known as *Cantor ladders*. It is uniquely defined by the condition $\overline{f}^{-1}(y) = \mathrm{conv}(f^{-1}(y))$ for $y \in [0,1]$.

Let $\Gamma_f := \{(x, f(x)) : x \in C\}$ and $\Gamma_{\overline{f}} := \big\{(x, \overline{f}(x)) : x \in [0,1]\big\}$ be the graphs of the functions f and $\overline{f}$. The set Γ_f is nowhere flat and zero-dimensional, and the set $\Gamma_{\overline{f}}$ is connected but not nowhere flat in the plane $\mathbb{R} \times \mathbb{R}$.

It is easy to see that $A := \Gamma_{\overline{f}} \setminus \Gamma_f$ is an open dense subset of $\Gamma_{\overline{f}}$. By Corollary 2.5 (see also the proof of Theorem 9.5.2 in [14]), the sum $S := \Gamma_{\overline{f}} + \Gamma_{\overline{f}}$ has non-empty interior in the plane $\mathbb{R}^2$. On the other hand, the sum $A + A$ is meager in $\mathbb{R}^2$. Moreover, since $A + A$ is contained in the union of countably many parallel lines in $\mathbb{R} \times \mathbb{R}$, the $\mathbb{Q}$-linear hull of A has uncountable codimension in $\mathbb{R}^2$, which allows us to construct a discontinuous additive function $a\colon \mathbb{R}^2 \to \mathbb{R}$ such that $a(A) = \{0\}$, witnessing that $A \notin \mathcal{C}(\mathbb{R}^2)$.

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Taras Banakh
Ivan Franko University of Lviv, UKRAINE
and
Jan Kochanowski University in Kielce, POLAND
E-mail address: t.o.banakh@gmail.com

Eliza Jabłońska and Wojciech Jabłoński
Institute of Mathematics
Pedagogical University of Cracow
Podchorążych 2
30-084 Kraków, POLAND
E-mail address: eliza.jablonska@up.krakow.pl
wojciech.jablonski@up.krakow.pl