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ON EXACT MULTIPLICITY FOR A SECOND ORDER EQUATION WITH RADIATION BOUNDARY CONDITIONS

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Abstract. A second order ordinary differential equation with a superlinear term $g(x,u)$ under radiation boundary conditions is studied. Using a shooting argument, all the results obtained in the previous work [2] for a Painlevé II equation are extended. It is proved that the uniqueness or multiplicity of solutions depend on the interaction between the mapping $\frac{\partial g}{\partial u}(\,\cdot\,,0)$ and the first eigenvalue of the associated linear operator. Furthermore, two open problems posed in [2] regarding, on the one hand, the existence of sign-changing solutions and, on the other hand, exact multiplicity are solved.

1. Introduction

In [2], the following problem arising on a two-ion electro-diffusion model (see [3], [5]) was studied:

$$u''(x) = Ku(x)^3 + L(x)u(x) + A \tag{1.1}$$

with

$$u'(0) = a_0u(0), \qquad u'(1) = a_1u(1). \tag{1.2}$$

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Here, K and A some given positive constants and $L(x) := a_0^2 + (a_1^2 - a_0^2)x$. Unlike the standard Robin condition, both coefficients a_0 and a_1 in the radiation boundary condition (1.2) are assumed to be positive.

It was proven that the problem has a negative solution; moreover, if $a_1 \le a_0$ then there are no other solutions. When $a_1 > a_0$, the solution is still unique for $A \gg 0$ but, if A is sufficiently small, then the problem has at least three solutions. Numerical evidence in [2] suggests that the number of solutions cannot be arbitrarily large and it was proven that, indeed, there exists exactly one negative solution, at most two positive solutions and that the set of solutions is bounded. It was conjectured that the maximum number of solutions is 3 (typically, one of them negative and the other two positive) but, however, none of the results in [2] prevents against the existence of many sign-changing solutions.

In this work, we study a generalization of the previous problem, namely the equation

$$u''(x) = g(x, u(x)) + p(x) \tag{1.3}$$

where $p \in C([0,1])$ and $g\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, of class C^1 with respect to u and superlinear, that is:

$$\lim_{|u| \to +\infty} \frac{g(x,u)}{u} = +\infty \tag{1.4}$$

uniformly in $x \in [0,1]$. Without loss of generality, we shall assume that $g(x,0) = 0$ for all $x \in [0,1]$. As before, we look for those solutions satisfying the radiation boundary condition (1.2) with $a_0, a_1 > 0$. In the spirit of [2], we shall assume that

$$g \text{ is strictly increasing in } u, \tag{1.5}$$

$$p(x) > 0 \quad \text{for all } x. \tag{1.6}$$

For general multiplicity results avoiding conditions (1.5) and (1.6) see [1]. In the present setting, we shall demonstrate that all the results in [2] can be retrieved in a simple manner; furthermore, we shall give an answer to two questions that were left open. Specifically, it shall be seen that the set of solutions is bounded and contains always a negative solution, which tends uniformly to $-\infty$ as $p \to +\infty$ uniformly. Moreover, we shall extend the uniqueness statement in [2] by imposing the condition that $-(\partial g/\partial u)(x,u)$ is smaller than the first eigenvalue λ_1 of the associated linear operator for all u. Under a weaker condition, it shall be proved that uniqueness holds also if p is large. As a complement of the uniqueness results, we shall also prove that if $-(\partial g/\partial u)(x,0)$ lies above λ_1 then the problem has at least three solutions, provided that $\|p\|_\infty$ is small. Furthermore, under an extra condition, which is fulfilled in (1.1), the multiplicity result is sharp. This extends the corresponding result for the particular problem (1.1) and gives an answer to a question, sustained by numerical evidence but not proven in [2]:

Theorem 1.1. *Assume that* (1.4)–(1.6) *hold. Then* (1.2)–(1.3) *has a negative solution. Moreover, if*

$$\frac{\partial g}{\partial u}(\,\cdot\,,0) \lneq -\lambda_1 \tag{1.7}$$

then there exists a constant $p_1 > 0$ *such that problem* (1.2)–(1.3) *has at least three solutions, one of them negative, one of them positive and another one sign-changing, when* $\|p\|_\infty < p_1$. *If furthermore*

$$\frac{\partial g}{\partial u}(x,u) > \frac{g(x,u)}{u}$$

for all $u \neq 0$ *and all* x, *then* (1.2)–(1.3) *has no other solutions, provided that* p_1 *is small enough.*

It follows that, under the previous assumptions, the number of solutions varies from 3 to 1 as $\|p\|_\infty$ gets large. Similarly, for each fixed p, if we take a_1 as a parameter then uniqueness or multiplicity of solutions vary according to its different values. In general terms, multiplicity arises when a_1 is sufficiently large and should not be expected if a_1 is small. More precisely:

Theorem 1.2. *Assume that* (1.4)–(1.6) *hold. Then there exist constants* $a^* > a_* > 0$ *such that*:

(a) *If* $a_1 > a^*$ *then problem* (1.2)–(1.3) *has at least three solutions, one of them negative and another one sign-changing.*
(b) *If* $0 < a_1 < a_*$ *then problem* (1.2)– (1.3) *has a unique* (*negative*) *solution.*

The paper is organized as follows. Section 2 is devoted to present several general aspects of the problem and state uniqueness and related results. In Section 3, we define a shooting-type operator that will be used to derive the proofs of Theorems 1.1 and 1.2. Some open questions are briefly exposed in a last section.

2. Uniqueness and related results

This section is devoted to introduce general results concerning problem (1.2)–(1.3) that shall be used in the proofs of the main results. In the first place, we observe that solutions are uniformly bounded:

Theorem 2.1. *Assume that* (1.4) *holds. Then there exists a constant* C *such that every solution* u *of* (1.2)–(1.3) *satisfies* $\|u\|_{C^2} \leq C$.

Proof. Let u be a solution. Multiply the equation by u and integrate to obtain

$$a_1 u(1)^2 - a_0 u(0)^2 = \int_0^1 \left[u'(x)^2 + g(x,u(x))u(x) + p(x)u(x)\right] dx.$$

Setting $\varphi(t) := [(a_1 - a_0)t + a_0]u(t)^2$, it is seen that

$$a_1 u(1)^2 - a_0 u(0)^2 = \int_0^1 \varphi'(x)\,dx \le \frac{C}{\varepsilon}\|u\|_{L^2}^2 + \varepsilon\|u'\|_{L^2}^2$$

for arbitrary $\varepsilon > 0$ and C depending on ε, a_1 and a_0. Choose for example $\varepsilon = 1/2$ and set $M > 2C + 1/2$, then by superlinearity there exists a constant γ (depending only on M and $\|p\|_{L^2}$) such that

$$\frac{1}{2}\|u'\|_{L^2}^2 + 2C\|u\|_{L^2}^2 \ge \|u'\|_{L^2}^2 + M\|u\|_{L^2}^2 - \gamma.$$

This implies $\|u\|_\infty \le \|u\|_{H^1} \le \sqrt{2\gamma}$ and the proof follows using (1.3). □

Next, we may state an uniqueness result in terms of the first eigenvalue λ_1 of the (self-adjoint) linear operator $-u''$ under the boundary conditions (1.2). To this end, let us simply recall that, by the standard Sturm–Liouville theory, λ_1 can be computed as the minimum of $-\int_0^1 u''u\,dx$ over all the smooth functions satisfying (1.2) such that $\|u\|_{L^2} = 1$.

Theorem 2.2. *Assume there exists an interval $I \subset \mathbb{R}$ such that, for all $u \in I$,*

$$\frac{\partial g}{\partial u}(x,u) \ge -\lambda_1 \quad \text{for all } x \in [0,1] \tag{2.1}$$

and the inequality is strict for some x independent of u. Then (1.2)–(1.3) *has at most one solution u such that $u(x) \in I$ for all x.*

Proof. Let $u_1, u_2\colon [0,1] \to I$ be solutions of (1.2)–(1.3) and define $w := u_1 - u_2$, then w satisfies the boundary condition and

$$w''(x) = g(x,u_1(x)) - g(x,u_2(x)) = \frac{\partial g}{\partial u}(x,\xi(x))w(x)$$

for some $\xi(x)$ between $u_1(x)$ and $u_2(x)$. Fix an open interval $J \ne \emptyset$ such that

$$\frac{\partial g}{\partial u}(x,\xi(x)) > -\lambda_1 \quad \text{for } x \in J$$

and suppose $w \not\equiv 0$ in J, then

$$0 = \int_0^1 \left(w''w - \frac{\partial g}{\partial u}(x,\xi(x))w^2\right)dx < \int_0^1 (w''w + \lambda_1 w^2)\,dx \le 0,$$

because λ_1 is the first eigenvalue. This contradiction proves that $w \equiv 0$ over J and consequently $w = 0$. □

Remark 2.3. As shown in [1], λ_1 is a strictly decreasing continuous function of a_1 and, moreover, $\lambda_1 \ge 0$ if and only if $a_1 \le a_0/(a_0+1)$. In particular, when (1.5) holds, the latter inequality is a sufficient condition for uniqueness. However, as proven in [2], the (sharp) condition for uniqueness in the particular case (1.1) is weaker, namely: $a_1 \le a_0$. This is due to fact that, in this specific case, it is verified that $\lambda_1 \ge -a_1^2$ and hence

$$-\lambda_1 \le a_1^2 \le L(x) + 3Ku^2 = \frac{\partial g}{\partial u}(x,u).$$

The next result shows that the failure of (2.1) does not necessarily imply multiplicity: this fact was already observed in [2] where, as mentioned, it was proven the solution of (1.1)–(1.2) is unique also when A is large. The latter property can be easily deduced from the general case with $p \equiv A$ as a consequence of the next two results. The first of them establishes that, for p large, solutions are negative; the second one proves that, under suitable assumptions, there cannot be two solutions with the same sign.

THEOREM 2.4. *Let* (1.4) *hold. Then there exists* p_0 *such that, if* $p(x) \geq p_0$ *for all* $x \in [0,1]$*, then all the solutions of* (1.2)–(1.3) *are negative.*

PROOF. Due to the superlinearity of g, for each $M \geq 0$ we may define the quantity

$$N_M := \inf_{x\in[0,1],u\geq 0} \{g(x,u) - Mu\} > -\infty.$$

Then

$$g(x,u) \geq Mu + N_M \tag{2.2}$$

for all $u \geq 0$. Let $M > 0$ to be determined, fix $p_0 > -N_M$ and let u be a solution of (1.2)–(1.3) such that $u(x) \geq 0$ for some $x \in [0,1]$. In view of (2.2), the inequality $u''(x) \geq g(x,u(x)) + p_0$ implies that

$$u''(x) > Mu(x) \tag{2.3}$$

whenever $u(x) \geq 0$. We deduce that, if $x_0 \in [0,1]$ is such that $u(x_0)$ and $u'(x_0)$ are nonnegative, then $u(x)$ and $u'(x)$ are strictly positive for $x > x_0$. Multiply (2.3) by u' and integrate to obtain, for $x > x_0$:

$$u'(x)^2 > u'(x_0)^2 + M(u(x)^2 - u(x_0)^2). \tag{2.4}$$

If $u(0) > 0$, then $u'(0) > 0$ and

$$u(1)^2 - u(0)^2 = \int_0^1 2u(x)u'(x)\,dx > 2a_0 u(0)^2.$$

Thus,

$$u(1)^2 - u(0)^2 > \frac{2a_0}{1+2a_0}\,u(1)^2 \tag{2.5}$$

and fixing $M = a_1^2(1+2a_0)/(2a_0)$ we obtain, from (2.4) and (2.5):

$$a_1^2 u(1)^2 > M\frac{2a_0}{1+2a_0}\,u(1)^2 = a_1^2 u(1)^2.$$

This contradiction proves that there are no positive solutions when $p_0 > -N_M$.

On the other hand, if $u(0) \leq 0$ then u vanishes at a (unique) value x_0, with $u'(x_0) \geq 0$. Fix $M = a_1^2$, then (2.4) yields

$$a_1^2 u(1)^2 = u'(1)^2 > u'(x_0)^2 + a_1^2 u(1)^2 \geq a_1^2 u(1)^2,$$

a contradiction. □

Theorem 2.5. *Assume there exists an interval $I \subset \mathbb{R}_{\neq 0}$ such that*

$$\frac{\partial g}{\partial u}(x,u) > \frac{g(x,u)+p(x)}{u} \tag{2.6}$$

holds for all $x \in [0,1]$ and $u \in I$. Then there exists at most one solution u of (1.2)–(1.3) such that $u(x) \in I$ for all x.

Proof. Let $u_1, u_2\colon [0,1] \to I$ be two different solutions, then $u_1(0) \neq u_2(0)$. Suppose for example that $u_1 < u_2 < 0$ over $[0,x_0)$, then

$$u_1''(x) = \frac{g(x,u_1(x))+p(x)}{u_1(x)}\, u_1(x) < \frac{g(x,u_2(x))+p(x)}{u_2(x)}\, u_1(x)$$

and hence

$$u_1''(x)u_2(x) > u_1(x)u_2''(x) \quad \text{for } x < x_0.$$

We conclude that

$$u_1'(x_0)u_2(x_0) > u_1(x_0)u_2'(x_0), \tag{2.7}$$

and a contradiction yields if $x_0 = 1$. Thus, we may assume that u_1 and u_2 meet for the first time at $x_0 < 1$, then $u_1(x_0) = u_2(x_0)$ and $u_1'(x_0) \geq u_2'(x_0)$. Again, this contradicts (2.7). The proof is analogous if we assume $u_2 > u_1 > 0$ over some interval $[0,x_0)$, obtaining in this case

$$u_1''(x)u_2(x) < u_1(x)u_2''(x) \quad \text{for } x < x_0. \qquad \square$$

Remark 2.6. Condition (2.6) implies that the function $(g(x,u)+p(x))/u$ increases in u when $I \subset \mathbb{R}_+$ and decreases when $I \subset \mathbb{R}_-$. Moreover, if $0 \in \partial I$ then $sp \leq 0$, where s denotes the sign of the elements of I. In particular, if the condition holds for all $u \neq 0$, then $p = 0$. This case is well known in the literature (see e.g. [4]) and implies that if $u_0 \neq 0$ is a critical point of the associated functional $\mathcal{J}$, then u_0 is transversal to the Nehari manifold, which was introduced after the pioneering work [6], namely:

$$\mathcal{N} := \big\{u \in H^1(0,1) \setminus \{0\} : D\mathcal{J}(u)(u) = 0\big\}.$$

Indeed, setting $\mathcal{I}(u) := D\mathcal{J}(u)(u)$ it is readily seen that $T_{u_0}\mathcal{N} = \ker(D\mathcal{I}(u_0))$ and $D\mathcal{I}(u_0)(u_0) > 0$. For the particular case of problem (1.1), condition (2.6) simply reads $A/u^3 < 2K$, so the previous result applies with $I = (-\infty, 0)$ and $I = \big(\sqrt[3]{A/(2K)}, +\infty\big)$.

The next theorem generalizes another result from [2], concerning the behaviour of the solutions as p increases. We know that all solutions are negative if $p \geq p_0 \gg 0$ and it is readily verified (e.g. by the method of upper and lower solutions) that a solution always exists; however, if the assumptions of Theorem 2.2 or Theorem 2.5 are not satisfied, then there might be more than one negative solution. As we shall see, all possible solutions tend uniformly to $-\infty$

as p tends uniformly to $+\infty$. In order to emphasize the dependence on p, any solution shall be denoted u_p, despite the fact that it might not be unique.

THEOREM 2.7. *Assume that* (1.4) *holds and let* u_p *be a solution of* (1.2)–(1.3). *Then* $u_p \to -\infty$ *uniformly when* $p \to +\infty$ *uniformly.*

PROOF. Let $p \geq p_0$ for some large constant p_0. From Theorem 2.4, we may assume $u_p < 0$. Fix x_p such that $\max_{x \in [0,1]} u_p(x) = u_p(x_p)$, then $x_p < 1$. Suppose $u_p(x_p) > -M$ and fix p_0 large enough, such that,

$$g(x,u) + p_0 > Ma_0, \quad \text{for all } u \geq -(1+a_0)M. \tag{2.8}$$

It follows that $x_p = 0$. Consider the maximum value $\delta \leq 1$ such that $u_p''(x) \geq 0$ for all $x \in [0,\delta]$, then $u_p'(x) \geq u_p'(0) = a_0 u_p(0) > -Ma_0$, for $x \leq \delta$. Hence, $u_p(\delta) > u_p(0) - \delta M a_0 \geq -M(1+a_0)$ and by (2.8) we conclude that $u_p''(\delta) > 0$. Thus, $\delta = 1$ and, in particular, $u_p(x) > -M(1+a_0)$. Using (2.8) again, it follows that $u_p''(x) > Ma_0$ for all x. Then $u_p'(1) > u_p'(0) + Ma_0 > 0$, a contradiction. □

Combining the previous result with Theorems 2.2 and 2.5 we deduce that, in fact, the solution is typically unique when p is large. Indeed, due to superlinearity we observe that, on the one hand, $\frac{\partial g}{\partial u}(x,u)$ cannot remain bounded from above as $u \to -\infty$ and, on the other hand, the function $g(x,u)/u$ cannot be increasing in u over any interval $(-\infty, C)$. In other words, it is reasonable to expect that either condition (2.1) holds or

$$\frac{\partial g}{\partial u}(x,u) - \frac{g(x,u)}{u} \geq \frac{k}{u}$$

when $u \ll 0$. Any of these conditions, which are fulfilled in the particular case (1.1), ensures the applicability of Theorems 2.2 or 2.5 when p is large. Thus, the following corollary is obtained:

COROLLARY 2.8. *Assume that* (1.4) *holds. Moreover, assume there exists* $C \leq 0$ *such that one of the following conditions holds*:

(a) *Condition* (2.1) *holds for all* $u \leq C$,

(b) $\displaystyle \sup_{x \in [0,1],\, u \leq C} u \frac{\partial g}{\partial u}(x,u) - g(x,u) < +\infty.$

Then there exists p_0 *such that problem* (1.2)–(1.3) *has a unique solution, which is negative, for all* $p \geq p_0$.

PROOF. From Theorem 2.7, there exists $\tilde{p}$ such that if u is a solution for $p \geq \widetilde{p}$ then $u(x) \leq C$ for all x. If the first condition holds, then the proof follows directly from Theorem 2.2. Otherwise, there exists a constant M such that

$$\frac{\partial g}{\partial u}(x,u) - \frac{g(x,u)}{u} > \frac{M}{u}$$

for all $u \leq C$ and, by Theorem 2.5, the result follows by taking p_0 as the maximum value between $\widetilde{p}$ and M. $\square$

3. A shooting operator for problem (1.2)–(1.3)

This section is devoted to proof Theorems 1.1 and 1.2 by means of a shooting-type operator. To this end, let us firstly state the following lemma, which ensures that, if (1.5) holds, then the graphs of two different solutions of (1.3) with initial condition $u'(0) = a_0 u(0)$ do not intersect. More generally, observe that such solutions might blow up before the value $t = 1$; thus, the following lemma is established for an arbitrary value $b \leq 1$:

Lemma 3.1. *Let u_1 and u_2 be solutions of* (1.3) *defined over an interval* $[0, b]$ *such that $u_1(0) > u_2(0)$ and $u_1'(0) > u_2'(0)$ and assume that* (1.5) *holds. Then $u_1 > u_2$ and $u_1' > u_2'$ on* $[0, b]$.

Proof. Set $u(x) = u_1(x) - u_2(x)$, then

$$u''(x) = \theta(x)u(x) \quad \text{on } [0, b), \text{ where } \theta(x) := \frac{\partial g}{\partial u}(x, \xi(x)) > 0.$$

Thus, the result follows since $u(0), u'(0) > 0$. $\square$

Next, we define our shooting operator as follows. For each fixed $\lambda \in \mathbb{R}$, let u_λ be the unique solution of problem

$$\begin{cases} u''(x) = g(x, u(x)) + p(x), \\ u(0) = \lambda, \quad u'(0) = a_0\lambda \end{cases} \tag{3.1}$$

and define the function $T\colon \mathcal{D} \to \mathbb{R}$, by

$$T(\lambda) = \frac{u_\lambda'(1)}{u_\lambda(1)},$$

where $\mathcal{D} \subset \mathbb{R}$ is the set of values of λ such that the corresponding solution u_λ of (3.1) is defined on $[0, 1]$, with $u_\lambda(1) \neq 0$. Thus, solutions of(1.2)–(1.3) that do not vanish on $x = 1$ can be characterized as functions u_λ, where $\lambda \in \mathcal{D}$ is such that $T(\lambda) = a_1$. By continuity arguments, it is easy to verify that, for each $s \in \mathbb{R}$, there exists λ such that $u_\lambda(1) = s$. By Lemma 3.1, this value of λ is unique; in particular, there exists a unique λ_0 such that $u_{\lambda_0}(1) = 0$. Thus, we conclude that

$$\mathcal{D} = (\lambda_*, \lambda_0) \cup (\lambda_0, \lambda^*) \quad \text{for some } \lambda_* \geq -\infty \text{ and } \lambda^* \leq +\infty.$$

From (1.6), it follows that $\lambda_0 < 0$ and, furthermore: if $\lambda > 0$ then u_λ is positive and if $\lambda_0 \leq \lambda \leq 0$ then u_λ vanishes exactly once in $[0, 1]$. In particular, $u_{\lambda_0} < 0$ in $[0, 1)$ and, since $u_{\lambda_0}''(x) > 0$ when x is close to 1, we conclude that $u_{\lambda_0}'(1) > 0$. Hence,

$$\lim_{\lambda \to \lambda_0^-} T(\lambda) = -\infty, \qquad \lim_{\lambda \to \lambda_0^+} T(\lambda) = +\infty.$$

We claim that also

$$\lim_{\lambda\to(\lambda^*)^-} T(\lambda) = +\infty, \qquad \lim_{\lambda\to(\lambda_*)^+} T(\lambda) = +\infty.$$

Indeed, observe firstly that, because solutions of (3.1) do not cross each other, $\lim_{\lambda\to(\lambda^*)^-} u_\lambda(1) = +\infty$. On the other hand, multiplying (1.3) by u' it is easy to see, given $M > 0$ that

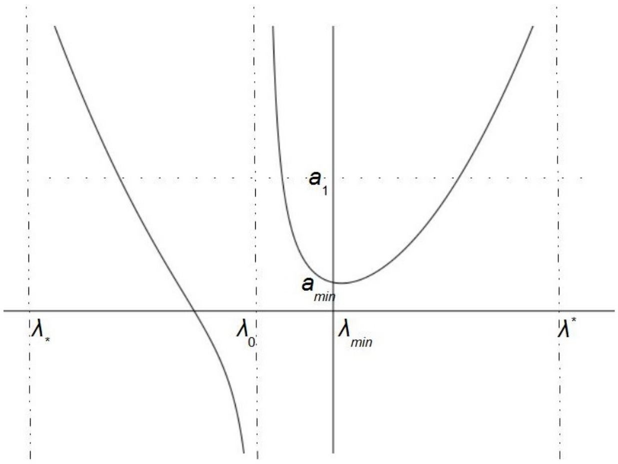
$$|u'_\lambda(1)| \geq \sqrt{M}\mathcal{O}(|u_\lambda(1)|)$$

for $|\lambda|$ sufficiently large. This implies that

$$|T(\lambda)| = \left|\frac{u'_\lambda(1)}{u_\lambda(1)}\right| > \sqrt{M}$$

and the claim follows.

The previous considerations show the existence of $\lambda_{\min} \in (\lambda_0, \lambda^*)$ such that $T(\lambda_{\min}) \leq T(\lambda)$ for all $\lambda \in (\lambda_0, \lambda^*)$. The value $a_{\min} := T(\lambda_{\min}) > 0$ depends on p and, in this context, Theorem 2.4 simply states that if $p \geq p_0$ for some large enough constant p_0 then $a_{\min} > a_1$. Also, we easily deduce some of the conclusions of Theorems 1.1 and 1.2, as shown in the following figure.



By continuity, there exists $\lambda < \lambda_0$ such that $T(\lambda) = a_1$; the corresponding u_λ is a negative solution. Uniqueness of negative solutions does not follow directly from this setting, unless an extra assumption like (2.6) is assumed for $u < 0$ (see Proposition 3.5 below). However, recall (see Remark 2.3) that if $a_1 < a_0/(a_0+1)$, then $\lambda_1 > 0$ so (2.1) is satisfied; thus uniqueness holds if a_1 is small.

Proof of Theorem 1.2. From the previous considerations, the problem has a negative solution, which is unique if a_1 is sufficiently small. Moreover, the equation $T(\lambda) = a_1$ has, over the interval (λ_0, λ^*), no solutions when $a_1 < a_{\min}$

and at least two solutions when $a_1 > a_{\min}$. Finally, observe that, as the value of a_1 increases, at least one of these solutions is located in $(\lambda_0, 0)$.

REMARK 3.2. If $\lambda^* > 0$ or, equivalently, if u_0 is defined on $[0,1]$, then we deduce that the problem has also a positive solution when $a_1 \gg 0$.

Under appropriate conditions, a lower bound for $a_{\min}$ is easily obtained as follows:

PROPOSITION 3.3. *Assume that* (1.5) *and* (1.6) *hold. If there exists* $r \le a_0$ *such that* $g(x,u) + p(x) > r^2 u$ *for all* $u \ge 0$ *and all* x, *then* $a_{\min} > r$.

PROOF. Fix $\lambda \in (\lambda_0, \lambda^*)$ and let $v(x) := e^{rx}$. Define x_0 as the minimum value such that u_λ is positive after x_0 and observe that

$$v(x)u_\lambda''(x) > v(x)r^2u_\lambda(x) = v''(x)u_\lambda(x)$$

for $x > x_0$. Thus,

$$v(1)[u_\lambda'(1) - ru_\lambda(1)] > v(x_0)[u_\lambda'(x_0) - ru_\lambda(x_0)] \ge 0$$

and we conclude that $T(\lambda) = u_\lambda'(1)/u_\lambda(1) > r$. □

REMARK 3.4. For problem (1.1), the previous proposition implies that $a_{\min} > \min\{a_0, a_1\}$, which provides an alternative proof of the fact that the problem has no positive nor sign-changing solutions when $a_1 \le a_0$.

In order to complete the proof of Theorem 1.1, let us make a more careful description of the graph of T. With this aim, compute

$$T'(\lambda) = \frac{\partial}{\partial\lambda}\left(\frac{u_\lambda'(1)}{u_\lambda(1)}\right) = \frac{u_\lambda(1)\,\dfrac{\partial u_\lambda'}{\partial\lambda}(1) - u_\lambda'(1)\,\dfrac{\partial u_\lambda}{\partial\lambda}(1)}{u_\lambda(1)^2}$$

and set $w_\lambda := \partial u_\lambda/\partial\lambda$, then

$$T'(\lambda) = \frac{u_\lambda(1)w_\lambda'(1) - u_\lambda'(1)w_\lambda(1)}{u_\lambda(1)^2}.$$

Moreover, observe that w_λ solves the linear problem

$$\begin{cases} w_\lambda''(x) = \dfrac{\partial g}{\partial u}(x, u_\lambda(x))w_\lambda(x), \\ w_\lambda(0) = 1, \quad w_\lambda'(0) = a_0, \end{cases} \tag{3.2}$$

and hence

$$\begin{aligned} u_\lambda(1)w_\lambda'(1) - u_\lambda'(1)w_\lambda(1) &= \int_0^1 \big(u_\lambda(x)w_\lambda''(x) - u_\lambda''(x)w_\lambda(x)\big)dx \\ &= \int_0^1 \bigg(u_\lambda(x)\,\frac{\partial g}{\partial u}(x,u_\lambda(x)) - g(x,u_\lambda(x)) - p(x)\bigg)w_\lambda(x)\,dx. \end{aligned} \tag{3.3}$$

Taking into account that $w_\lambda(x) > 0$ for all x and that u_λ is negative for $\lambda < \lambda_0$ and positive for $\lambda > 0$, the following proposition is obtained:

PROPOSITION 3.5. *Assume that* (1.4)–(1.6) *hold. Then*:

(a) *T is strictly decreasing for $\lambda < \lambda_0$, provided that* (2.6) *holds for $u < 0$.*
(b) *T is strictly increasing for $\lambda > C$, provided that* (2.6) *holds for $u > C \geq 0$.*

REMARK 3.6. In particular, the previous proposition shows that, when (1.5) and (1.6) are assumed, the conclusions of Theorem 2.5 are retrieved in a simple manner.

Assume firstly that $p = 0$. Although (1.6) obviously fails, the operator T is well defined, with $\lambda_0 = 0$. Moreover, using the L'Hôpital rule we deduce that

$$\lim_{\lambda\to 0} T(\lambda) = \lim_{\lambda\to 0} \frac{u'_\lambda(1)}{u_\lambda(1)} = \lim_{\lambda\to 0} \frac{w'_\lambda(1)}{w_\lambda(1)} = \frac{\Phi'(1)}{\Phi(1)},$$

where $\Phi := w_0$, that is, the unique solution of the linear initial value problem

$$\Phi''(x) = \frac{\partial g}{\partial u}(x,0)\Phi(x), \qquad \Phi'(0) = a_0\Phi(0) = a_0. \tag{3.4}$$

Thus, T can be extended continuously to a positive function defined over (λ_*, λ^*), which tends to $+\infty$ as $\lambda \to (\lambda_*)^+$ or $\lambda \to (\lambda^*)^-$. Furthermore, if (2.6) holds for $u \neq 0$ then it decreases strictly on $(\lambda_*, 0)$ and increases strictly on $(0, \lambda^*)$.

We are now in condition of completing the proof of Theorem 1.1. To this end, we shall need the following lemma:

LEMMA 3.7. *Assume that* (1.5) *and* (1.7) *holds. Then* $\Phi'(1) < a_1\Phi(1)$.

PROOF. Let φ_1 be the (unique) eigenfunction corresponding to λ_1 such that $\varphi_1(0) = 1$, then it is readily verified that $\varphi_1(x) > 0$ for all x. Moreover, it is seen from (3.4) that also $\Phi(x) > 0$ for all x. Then

$$\varphi_1(x)\Phi''(x) = \frac{\partial g}{\partial u}(x,0)\Phi(x)\varphi_1(x) \leq -\lambda_1\Phi(x)\varphi_1(x) = \Phi(x)\varphi_1''(x)$$

and the inequality is strict for some x. Integration yields

$$\varphi(1)\Phi'(1) < \Phi(1)\varphi'(1) = a_1\Phi(1)\varphi(1)$$

and the proof follows. □

PROOF OF THEOREM 1.1. In view of the previous lemma, the proof is an immediate corollary of the following proposition, slightly more general:

PROPOSITION 3.8. *Assume that* (1.4)–(1.6) *hold and that* $\Phi'(1) < a_1\Phi(1)$. *Then there exists a constant $p_1 > 0$ such that problem* (1.3)–(1.2) *has at least three solutions when $\|p\|_\infty < p_1$. Moreover, one of the solutions is negative, one of them positive and another one sign-changing. If furthermore* (2.6) *holds with $p = 0$ for all $u \neq 0$, then there exists p_1 such that the problem has exactly three solutions, provided that $\|p\|_\infty < p_1$. Moreover, exactly one of the solutions is negative and another one changes sign.*

Proof. From the previous considerations we know that, if p is small, then $\lambda^* > 0$ and $T(0) < a_1$; thus, the existence of at least three solutions follows. Clearly, one of the solutions is negative, another one is positive and another one changes sign.

From now on, assume that (2.6) with $p = 0$ holds for all $u \neq 0$. Consider, for arbitrary p, the mapping

$$R_p(\lambda) := u'_\lambda(1) - a_1 u_\lambda(1).$$

Let us firstly take $p = 0$. From the previous computations, we know that

$$\operatorname{sgn}(T'(\lambda)) = \operatorname{sgn}(u_\lambda) = \operatorname{sgn}(\lambda) \quad \text{for } \lambda \neq 0 \quad \text{and} \quad T(0) = \frac{\Phi'(1)}{\Phi(1)} < a_1,$$

whence R_0 has exactly three roots $\{0, \lambda_\pm\}$ with $\lambda_- < 0 < \lambda_+$. Moreover, write as before

$$T'(\lambda) = \frac{u_\lambda(1)w'_\lambda(1) - u'_\lambda(1)w_\lambda(1)}{u_\lambda(1)^2} = \frac{w_\lambda(1)}{u_\lambda(1)}\left(\frac{w'_\lambda(1)}{w_\lambda(1)} - T(\lambda)\right)$$

to deduce that

$$\frac{w'_{\lambda_\pm}(1)}{w_{\lambda_\pm}(1)} > T(\lambda_\pm) = a_1.$$

Next, observe that

$$R'_0(\lambda) = w'_\lambda(1) - a_1 w_\lambda(1) = w_\lambda(1)\left(\frac{w'_\lambda(1)}{w_\lambda(1)} - a_1\right),$$

so $R'_0(\lambda_\pm) > 0$. On the other hand, $R'_0(0) = \Phi'(1) - a_1\Phi(1) < 0$ and, by continuity, we conclude that if p is close to 0 then R_p has exactly three roots. Furthermore, $T(0)$ is close to $\Phi'(1)/\Phi(1) < a_1$, so the equation $T(\lambda) = a_1$ has at least one solution in $(\lambda_0, 0)$. Finally, observe that if p is small then u_0 is defined in $[0, 1]$; thus, $\lambda^* > 0$ and letting p be smaller if necessary we conclude that the equation $T(\lambda) = a_1$ has also a solution in $(0, \lambda^*)$. $\square$

Remark 3.9. In particular, all the assumptions of the previous proposition are fulfilled for problem (1.1) if (and only if) $a_1 > a_0$. Indeed, in this case it is readily seen that $\lambda_1 < -a_1^2$ and hence

$$\frac{\partial g}{\partial u}(x, 0) = L(x) \leq a_1^2 < -\lambda_1.$$

4. Open questions

(1) Numerical experiments for the particular case (1.1) suggest that $T'' > 0$ for $\lambda > \lambda_0$. If this is true, then an exact multiplicity result yields for arbitrary p, depending on whether $a_{\min}$ is smaller, equal or larger than a_1. It would be

interesting to investigate if this fact could be verified for the general case, under appropriate conditions, using the differential equation for

$$z_\lambda := \frac{\partial w_\lambda}{\partial \lambda} = \frac{\partial^2 u_\lambda}{\partial \lambda^2},$$

namely

$$\begin{aligned} z_\lambda''(x) &= \frac{\partial g}{\partial u}(x, u_\lambda(x)) z_\lambda(x) + \frac{\partial^2 g}{\partial u^2}(x, u_\lambda(x)) w_\lambda(x)^2, \\ z_\lambda(0) &= z_\lambda'(0) = 0. \end{aligned}$$

(2) Is it possible to obtain an exact multiplicity result also for a_1 large? Observe that, in such a case, the behaviour of T can be controlled near λ_0, but it is not easy to see what happens as λ gets closer to λ_* or λ^*. In more precise terms, we may set $\varepsilon := 1/a_1$ and

$$R_\varepsilon(\lambda) := \varepsilon u_\lambda'(1) - u_\lambda(1).$$

Then $R_0(\lambda) = -u_\lambda(1)$ decreases from $+\infty$ to $-\infty$ over (λ_*, λ^*). Furthermore, $R_0'(\lambda) = -w_\lambda(1) < 0$ for all λ; thus, if ε is small, then R_ε has, near λ_0, a unique root. However, for $\varepsilon \neq 0$ the graph of R_ε bends in such a way that it tends to $\pm\infty$ as λ gets closer to λ^* and λ_*, respectively. This ensures the existence of at least three solutions for ε small, although there might be more. Clearly, there exists λ_1 such that R_ε increases with ε for $\lambda > \lambda_1$ and decreases when $\lambda < \lambda_1$; moreover, if $K \subset (\lambda_*, \lambda^*)$ is a compact neighbourhood of λ_0, then R_ε vanishes exactly once in K when $\varepsilon = \varepsilon(K)$ is small. This is due to the fact that R_ε tends to R_0 over K for the C^1 norm. However, it is not clear which condition would be appropriate in order to prevent against a possible 'strange' behaviour of R_ε outside compact sets. For example, taking into account the superlinearity, we might impose the assumption that $(\partial g/\partial u)(x,u)$ tends uniformly to $+\infty$ as $|u| \to +\infty$. This would ensure that R_ε has positive derivative near the endpoints of its domain but, still, it might change sign many times.

(3) How does the graph of T vary with respect to p? Suppose for simplicity that p is constant and let $y_p := \partial u_\lambda/\partial p$. Then

$$y_p'' = \frac{\partial g}{\partial u}(x, u_\lambda(x)) y_p + 1, \qquad y_p(0) = y_p'(0) = 0$$

and the sign of $\partial T/\partial p$ coincides with the sign of the integral

$$\int_0^1 \left(u_\lambda(x) \frac{\partial g}{\partial u}(x, u_\lambda(x)) - g(x, u_\lambda(x)) - p \right) y_p(x) + u_\lambda(x)\, dx.$$

If (2.6) holds for $u < 0$, then $\partial T/\partial p < 0$ for $\lambda < \lambda_0$. In particular, the (unique) value $\lambda < \lambda_0$ for which $T(\lambda) = a_1$ moves to the left as p increases. This is consistent with the fact that the negative solution tends uniformly to $-\infty$ as $p \to +\infty$. It seems difficult to obtain similar conclusions for $\lambda \in (\lambda_0, 0)$ since u_λ

changes sign but, in general, if (2.6) is satisfied for $u > C \geq 0$, then $\partial T/\partial p > 0$ for $\lambda \geq C$. For example, this is the case in problem (1.1), with $C = \sqrt[3]{A/(2K)}$.

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