Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 1, 2019, 219–232
DOI: 10.12775/TMNA.2019.038

INFINITELY MANY SOLUTIONS FOR A CLASS OF CRITICAL CHOQUARD EQUATION WITH ZERO MASS

Fashun Gao — Minbo Yang
Carlos Alberto Santos — Jiazheng Zhou

Abstract. In this paper we investigate the following nonlinear Choquard equation

$$-\Delta u = \bigg(\int_{\mathbb{R}^N} \frac{G(y,u)}{|x-y|^{\mu}}\,dy\bigg) g(x,u) \quad \text{in } \mathbb{R}^N,$$

where $0 < \mu < N$, $N \geq 3$, $g(x,u)$ is of critical growth in the sense of the Hardy–Littlewood–Sobolev inequality and $G(x,u) = \int_0^u g(x,s)\,ds$. By applying minimax procedure and perturbation technique, we obtain the existence of infinitely many solutions.

1. Introduction and main results

The aim of the present paper is to consider the following nonlinear critical Choquard equation with a subcritical nonlocal term

$$\begin{cases} -\Delta u = \bigg(\displaystyle\int_{\mathbb{R}^N} \frac{\delta |u(y)|^{2_\mu^*} + \lambda K(y)|u(y)|^p}{|x-y|^{\mu}}\,dy\bigg) \\ \qquad\qquad\qquad \bigg(\delta |u|^{2_\mu^*-2}u + \dfrac{p}{2_\mu^*}\lambda K(x)|u|^{p-2}u\bigg) \quad \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35J20, 35J60, 35A15.
Key words and phrases. Critical Choquard equation; Hardy–Littlewood–Sobolev inequality; infinitely many solutions.
Minbo Yang is the corresponding author who was partially supported by NSFC (11571317).

where $N \geq 3$, $0 < \mu < N$, $\max\{(2N-\mu)/2N, (\mu-4)/(N-2)\} < p < 1$, δ, λ are two positive parameters and $2_\mu^* = (2N-\mu)/(N-2)$ is the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. Concerning the function $K(x)$, we assume $0 \leq K(x) \in L^{p'}(\mathbb{R}^N)$, where $p' = 2^*/(2_\mu^* - p)$, $2^* = 2N/(N-2)$ is the critical exponent for the embedding $H^1(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$.

The nonlinear Choquard equation

$$-\Delta u + V(x)u = \big(|x|^{-\mu} * |u|^q\big)|u|^{q-2}u \quad \text{in } \mathbb{R}^N \tag{1.2}$$

arises in various domains of mathematical physics such as in the description of the quantum theory of a polaron at rest by S. Pekar in 1954 [19] and in the modeling of an electron trapped in its own hole in 1976 in the work of P. Choquard as a certain approximation to Hartree–Fock theory of one-component plasma [11], etc. The equation (1.2) is also known as the Schrödinger–Newton equation [20].

Lieb [11] proved the existence and uniqueness, up to translations, of the ground state for (1.2) with $\mu = 1$, $q = 2$ and V is a positive constant and Lions [13] showed the existence of a sequence of radially symmetric solutions via variational methods. Recently, a great deal of mathematical efforts have been devoted to the study of existence, multiplicity and properties of the solutions of the nonlinear Choquard equation (1.2). In [6], [15], [16], the authors showed the regularity, positivity and radial symmetry of the ground states and derived decay property at infinity as well. We also refer the readers to [1], [2], [5], [18] and [22] for the existence and concentration behavior of the semiclassical solutions for the singularly perturbed Choquard equation.

It is necessary to recall the well known Hardy-Littlewood-Sobolev inequality (see for instance [12]).

Proposition 1.1 (Hardy–Littlewood–Sobolev inequality). *Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}\,dx\,dy \leq C(t, N, \mu, r)|f|_t|h|_r, \tag{1.3}$$

where $|\cdot|_q$ stands for the $L^q(\mathbb{R}^N)$-norm for $q \in [1, \infty]$. If $t = r = 2N/(2N-\mu)$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\mu/2}\,\frac{\Gamma(N/2-\mu/2)}{\Gamma(N-\mu/2)}\left\{\frac{\Gamma(N/2)}{\Gamma(N)}\right\}^{-1+\mu/N}.$$

It is a consequence of Hardy-Littlewood-Sobolev inequality that the integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}\,dx\,dy, \quad \text{for } u \in H^1(\mathbb{R}^N),$$

is well defined if

$$\frac{2N-\mu}{N} \leq q \leq \frac{2N-\mu}{N-2}.$$

Then number $(2N-\mu)/N$ is called the lower critical exponent and $2^*_\mu = (2N-\mu)/(N-2)$ – the upper critical exponent. In [17], Moroz and Van Schaftingen considered the nonlinear Choquard equation (1.2) in $\mathbb{R}^N$ with lower critical exponent and obtained existence and nonexistence results if the potential $1-V$ does not decay to zero at infinity faster than the inverse square of $|x|$.

In order to study the critical nonlocal equation with upper critical exponent 2^*_μ, we will use $S_{H,L}$ to denote the best constant defined by

$$S_{H,L} := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\displaystyle\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy\bigg)^{1/2^*_\mu}} \tag{1.4}$$

and note that $S_{H,L}$ is achieved if and only if

$$u(x) = C\bigg(\frac{b}{b^2+|x-a|^2}\bigg)^{(N-2)/2},$$

where $C>0$ is a fixed constant, $a\in\mathbb{R}^N$ and $b\in(0,\infty)$ are parameters. More,

$$S_{H,L} = \frac{S}{C(N,\mu)^{(N-2)/(2N-\mu)}},$$

where S is the best Sobolev constant and $C(N,\mu)$ is given in Proposition 1.1. See [8]. In [8], [9] the authors considered the Brézis–Nirenberg type problem

$$-\Delta u = \bigg(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u|^{2^*_\mu-2}u + \lambda u \quad \text{in } \Omega \tag{1.5}$$

and established the existence, multiplicity and nonexistence of solutions for the nonlinear Choquard equation in bounded domain by perturbation method. In [3], the authors studied the semiclassical limit problem for the singularly perturbed Choquard equation in $\mathbb{R}^3$ and characterized the concentration behavior by variational methods. Gao and Yang in [10] investigated the existence result for the strongly indefinite Choquard equation with upper critical exponent in the whole space. In the present paper we are interested in the existence of infinitely many solutions.

The main result reads as

THEOREM 1.2. *Assume* $\max\{(2N-\mu)/2N, (\mu-4)/(N-2)\} < p < 1$ *and the Lebesgue measure of* $\{x\in\mathbb{R}^N/K(x)>0\}$ *is positive. Then*:

(a) *for each* $\delta>0$*, there exists* λ_* *such that for any* $\lambda\in(0,\lambda_*)$*, problem* (1.1) *has a sequence of solutions* $\{u_m\}$ *with* $J_K(u_m)<0$ *and* $J_K(u_m)\to 0$ *as* $m\to\infty$*,*

(b) *for each* $\lambda>0$*, there exists* δ_* *such that for any* $\delta\in(0,\delta_*)$*, problem* (1.1) *has a sequence of solutions* $\{u_m\}$ *with* $J_K(u_m)<0$ *and* $J_K(u_m)\to 0$ *as* $m\to\infty$*.*

To apply variational methods, we introduce the energy functional associated to equation (1.1)

$$J_K(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx$$
$$-\frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\delta|u(x)|^{2_\mu^*}+\lambda K(x)|u(x)|^p)(\delta|u(y)|^{2_\mu^*}+\lambda K(y)|u(y)|^p)}{|x-y|^\mu}\,dx\,dy$$

and note that it is well defined on $D^{1,2}(\mathbb{R}^N)$ and belongs to $\mathcal{C}^1$ due to the assumption on $K(x)$ $(0 \le K(x) \in L^{p'}(\mathbb{R}^N))$ and to the Hardy–Littlewood–Sobolev inequality. Thus the weak solutions of (1.1) are precisely the critical points of the action functional J_K on $D^{1,2}(\mathbb{R}^N)$.

The paper is organized as follows. In Section 2, we introduce a concentration-compactness principle for nonlocal type problem and prove the (PS) condition. In Section 3, we prove the existence of infinitely many solutions for (1.1).

2. Variational setting

Throughout this paper we write $|\cdot|_q$ for the $L^q(\mathbb{R}^N)$-norm, $q \in [1,\infty]$, $0 < \mu < N$ and $N \ge 3$. Different positive constants are denoted by $C, C_1, C_2, \ldots$. Let

$$\|u\| := \bigg(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\bigg)^{1/2}$$

be the standard norm on $D^{1,2}(\mathbb{R}^N)$ and denote by

$$\|u\|_{NL} := \bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\bigg)^{1/(2\cdot 2_\mu^*)}.$$

The following splitting lemma was proved in Lemma 2.2 of [8].

Lemma 2.1. *Let $N \ge 3$ and $0 < \mu < N$. If $\{u_n\}$ is a bounded sequence in $L^{2N/(N-2)}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then*

$$\|u_n\|_{NL}^{2\cdot 2_\mu^*} - \|u_n - u\|_{NL}^{2\cdot 2_\mu^*} \to \|u\|_{NL}^{2\cdot 2_\mu^*} \quad \text{as } n \to \infty.$$

Since the lack of compactness also occurs when one considers the critical Choquard equation in unbounded domain, it is quite natural to apply the second concentration-compactness principle involving convolution type nonlinearities to overcome the difficulties. A version of the second concentration-compactness principle for nonlocal convolution case was proved in [7].

Lemma 2.2 (see [7]). *Let $\{u_n\}$ be a bounded sequence in $D^{1,2}(\mathbb{R}^N)$ converging weakly and almost everywhere to some u_0, $|\nabla u_n|^2 \rightharpoonup \omega$, $|u_n|^{2^*} \rightharpoonup \zeta$ weakly in the sense of measures, where ω and ζ are bounded non-negative measures on $\mathbb{R}^N$. Assume that*

$$\bigg(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,dy\bigg)|u_n(x)|^{2_\mu^*} \rightharpoonup \nu$$

weakly in the sense of measure, where ν is a bounded positive measure on $\mathbb{R}^N$ and define

$$\omega_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|\geq R} |\nabla u_n|^2\,dx,$$
$$\zeta_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|\geq R} |u_n|^{2^*}\,dx,$$
$$\nu_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|\geq R} \bigg(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_n(x)|^{2^*_\mu}\,dx.$$

Then, there exists a countable sequence of points $\{z_i\}_{i\in I} \subset \mathbb{R}^N$ and families of positive numbers $\{\nu_i : i \in I\}$, $\{\zeta_i : i \in I\}$ and $\{\omega_i : i \in I\}$ such that

$$\nu = \bigg(\int_{\mathbb{R}^N} \frac{|u_0(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\bigg)|u_0(x)|^{2^*_\mu} + \sum_{i\in I} \nu_i\delta_{z_i}, \qquad \sum_{i\in I} \nu_i^{1/2^*_\mu} < \infty, \tag{2.1}$$

$$\omega \geq |\nabla u_0|^2 + \sum_{i\in I} \omega_i\delta_{z_i}, \tag{2.2}$$

$$\zeta \geq |u_0|^{2^*} + \sum_{i\in I} \zeta_i\delta_{z_i}, \tag{2.3}$$

and

$$S_{H,L}\nu_i^{1/2^*_\mu} \leq \omega_i, \qquad \nu_i^{N/(2N-\mu)} \leq C(N,\mu)^{N/(2N-\mu)}\zeta_i, \tag{2.4}$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$. Furthermore, we have

$$\overline{\lim}_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}}{|x-y|^\mu}\,dy\,dx = \nu_\infty + \int_{\mathbb{R}^N} d\nu \tag{2.5}$$

and

$$\begin{gathered} C(N,\mu)^{-2N/(2N-\mu)}\nu_\infty^{2N/(2N-\mu)} \leq \zeta_\infty\bigg(\int_{\mathbb{R}^N} d\zeta + \zeta_\infty\bigg), \\ S_{H,L}^2\nu_\infty^{2/2^*_\mu} \leq \omega_\infty\bigg(\int_{\mathbb{R}^N} d\omega + \omega_\infty\bigg). \end{gathered} \tag{2.6}$$

Moreover, if $u_0 = 0$ and $\int_{\mathbb{R}^N} d\omega = S_{H,L}(\int_{\mathbb{R}^N} d\nu)^{1/2^*_\mu}$, then ν is concentrated at a single point.

Let us show the following lemma.

LEMMA 2.3. *Suppose that* $\max\{(2N-\mu)/2N, (\mu-4)/(N-2)\} < p < 1$, *then*:

- (a) *for each fixed $\delta > 0$ and $c < 0$, there exists $\overline{\lambda} > 0$ such that for any* $(\mathrm{PS})_c$-*sequence $\{u_n\}$ contains a convergent subsequence for each $\lambda \in (0,\overline{\lambda})$ given,*
- (b) *for each fixed $\lambda > 0$ and $c < 0$, there exists $\overline{\delta} > 0$ such that any* $(\mathrm{PS})_c$-*sequence $\{u_n\}$ has a convergent subsequence for each $\delta \in (0,\overline{\delta})$ given.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_c$-sequence, i.e. $J_K(u_n) \to c$ and

$$\sup\big\{|\langle J'_K(u_n), \varphi\rangle| : \varphi \in E, \|\varphi\| = 1\big\} \to 0$$

as $n \to +\infty$. Then there exists a $C_1 > 0$ such that

$$|J_K(u_n)| \leq C_1 \quad \text{and} \quad |\langle J'_K(u_n), u_n/\|u_n\|\rangle| \leq C_1$$

for all n large. Since $(\mu-4)/(N-2) < p < 1$, we have $p+2^*_\mu > 2$. So, we obtain

$$\begin{aligned}
C_1(1+\|u_n\|) &\geq J_K(u_n) - \frac{1}{p+2^*_\mu}\langle J'_K(u_n), u_n\rangle \\
&= \bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)\|u_n\|^2 \\
&\quad + \bigg(\frac{1}{p+2^*_\mu} - \frac{1}{2\cdot 2^*_\mu}\bigg)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\delta|u_n(x)|^{2^*_\mu}\delta|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\
&\quad - \frac{1}{2\cdot 2^*_\mu}\bigg(1 - \frac{2p}{p+2^*_\mu}\bigg)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\lambda K(x)|u_n(x)|^p\lambda K(y)|u_n(y)|^p}{|x-y|^\mu}\,dx\,dy \\
&\geq \bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)\|u_n\|^2 - \frac{1}{2\cdot 2^*_\mu}\bigg(1 - \frac{2p}{p+2^*_\mu}\bigg)\lambda^2|K|^2_{p'}|u_n|^{2p}_{2^*} \\
&\geq \bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)\|u_n\|^2 - \frac{1}{2\cdot 2^*_\mu}\bigg(1 - \frac{2p}{p+2^*_\mu}\bigg)\lambda^2|K|^2_{p'}C_2\|u_n\|^{2p}
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$.

Since $D^{1,2}(\mathbb{R}^N)$ is reflexive, up to a subsequence, we may assume that there exists $u \in D^{1,2}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$, $u_n \to u$ almost everywhere in $\mathbb{R}^N$, $|\nabla u_n|^2$ converges weakly to some nonnegative measure ω, $(\int_{\mathbb{R}^N}|u_n(y)|^{2^*_\mu}/|x-y|^\mu\,dy)|u_n(x)|^{2^*_\mu}$ converges weakly to some nonnegative measure ν.

Let z_i be a singular point of measure ω and ν. By taking a function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N,[0,1])$ such that $\phi(x) = 1$ in $B_{z_i}(\varepsilon)$, $\phi(x) = 0$ in $\mathbb{R}^N \setminus B_{z_i}(2\varepsilon)$, $|\nabla\phi| \leq C/\varepsilon$ in $\mathbb{R}^N$, we infer that $\{\phi u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Evidently, $\langle J'_K(u_n), \phi u_n\rangle \to 0$, i.e.

$$\begin{aligned}
0 \leftarrow \langle J'_K(u_n), u_n\phi\rangle &= \int_{\mathbb{R}^N}|\nabla u_n|^2\phi dx + \int_{\mathbb{R}^N}u_n\nabla u_n\nabla\phi\,dx \\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^\mu}\big(\delta|u_n(x)|^{2^*_\mu} + \lambda K(x)|u_n(x)|^p\big) \\
&\quad\quad \times \bigg(\delta|u_n(y)|^{2^*_\mu}\phi(y) + \frac{p}{2^*_\mu}\lambda K(y)|u_n(y)|^p\phi(y)\bigg)\,dx\,dy.
\end{aligned}$$

as $n \to \infty$.

Applying Hölder's inequality, we have, as $\varepsilon \to 0^+$,

$$\begin{aligned}
0 &\le \limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} u_n \nabla u_n \nabla\phi \, dx \right| \\
&\le \limsup_{n\to\infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^N} |u_n|^2 |\nabla \phi|^2 \, dx \right)^{1/2} \\
&\le C \biggl(\int_{B_{z_i}(2\varepsilon)} |u|^{2N/(N-2)} \, dx \biggr)^{(N-2)/2N} \biggl(\int_{B_{z_i}(2\varepsilon)} |\nabla\phi|^N \, dx \biggr)^{1/N} \\
&\le C \biggl(\int_{B_{z_i}(2\varepsilon)} |u|^{2N/(N-2)} \, dx \biggr)^{(N-2)/2N} \quad \to 0,
\end{aligned}$$

where we used the fact that $|\nabla\phi| \le C/\varepsilon$ and the sequence u_n is bounded. It follows from the Hölder inequality and the Sobolev inequality that

$$\begin{aligned}
\int_{B_{z_i}(2\varepsilon)} |u_n|^\sigma \, dx &\le |B_{z_i}(2\varepsilon)|^{1-\sigma/2^*} \biggl(\int_{B_{z_i}(2\varepsilon)} |u_n|^{2^*} \, dx \biggr)^{\sigma/2^*} \\
&\le C |B_{z_i}(2\varepsilon)|^{1-\sigma/2^*} \biggl(\int_{B_{z_i}(2\varepsilon)} |\nabla u_n|^2 \, dx \biggr)^{\sigma/2} = o(1)
\end{aligned}$$

as $\varepsilon \to 0^+$ holds for all $\sigma \in [0, 2^*)$. Hence, by the Hardy–Littlewood–Sobolev's inequality, there holds

$$\begin{aligned}
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} K(y) |u_n(y)|^p \phi(y)}{|x-y|^\mu} \, dx \, dy = o(1), \\
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|u_n(x)|^p K(y) |u_n(y)|^p \phi(y)}{|x-y|^\mu} \, dx \, dy = o(1), \\
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|u_n(x)|^p |u_n(y)|^{2^*_\mu} \phi(y)}{|x-y|^\mu} \, dx \, dy = o(1),
\end{aligned}$$

as $\varepsilon \to 0^+$. Therefore,

$$J'_K(u_n), u_n\phi\rangle = \int_{\mathbb{R}^N} |\nabla u_n|^2 \phi \, dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta |u_n(x)|^{2^*_\mu} \delta |u_n(y)|^{2^*_\mu} \phi(y)}{|x-y|^\mu} \, dx \, dy + o(1),$$

that is

$$\int_{\mathbb{R}^N} \phi \, d\omega - \delta^2 \int_{\mathbb{R}^N} \phi \, d\nu + o(1) = 0$$

as $n \to \infty$. Let $\varepsilon \to 0$, we obtain $\omega_i - \delta^2 \nu_i = 0$. Hence, it follows from (2.4) in Lemma 2.2 that either

(i) $\nu_i = 0$ or
(ii) $\nu_i \ge (\delta^{-2} S_{H,L})^{(2N-\mu)/(N-\mu+2)}$.

To examine a possible concentration of the sequence $\{u_n\}$ at infinity, we define $\phi_R \in \mathcal{C}_0^\infty(\mathbb{R}^N, [0,1])$ such that

$$\phi_R = 0 \quad \text{on } |x| < R \quad \text{and} \quad \phi_R = 1 \quad \text{on } |x| > R+1.$$

Since $\{u_n\phi_R\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$, we have $\langle J_K'(u_n), u_n\phi_R\rangle \to 0$, i.e.

$$\begin{aligned}0 \leftarrow \langle J_K'(u_n), u_n\phi_R\rangle &= \int_{\mathbb{R}^N} |\nabla u_n|^2 \phi_R\,dx + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \phi_R\,dx \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^\mu}\big(\delta|u_n(x)|^{2^*_\mu} + \lambda K(x)|u_n(x)|^p\big) \\ &\qquad \times \bigg(\delta|u_n(y)|^{2^*_\mu}\phi_R(y) + \frac{p}{2^*_\mu}\lambda K(y)|u_n(y)|^p\phi_R(y)\bigg)\,dx\,dy.\end{aligned}$$

Arguing as above, we have $\omega_\infty - \delta^2\nu_\infty = 0$. Again, by (2.6) in Lemma 2.2, we know

$$S_{H,L}^2 \nu_\infty^{2/2^*_\mu} \le \omega_\infty\bigg(\int_{\mathbb{R}^N} d\omega + \omega_\infty\bigg),$$

i.e.

$$\omega_\infty \ge \frac{1}{2}\bigg(\bigg(\bigg(\int_{\mathbb{R}^N} d\omega\bigg)^2 + 4S_{H,L}^2\nu_\infty^{2/2^*_\mu}\bigg)^{1/2} - \int_{\mathbb{R}^N} d\omega\bigg)$$

and so, we get either

(iii) $\nu_\infty = 0$; or

(iv) $\delta^4\nu_\infty + \delta^2 \displaystyle\int_{\mathbb{R}^N} d\omega \ge S_{H,L}^2 \nu_\infty^{(\mu-4)/(2N-\mu)}$.

Now we claim both (ii) and (iv) can not occur, if δ, λ are chosen properly. In fact, from the weak lower semicontinuity of J_K, we obtain

$$\begin{aligned}0 > c &= \lim_{n\to\infty}\bigg[J_K(u_n) - \frac{1}{p+2^*_\mu}\langle J_K'(u_n), u_n\rangle\bigg] \\ &= \lim_{n\to\infty}\bigg[\bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)\|u_n\|^2 \\ &\quad + \bigg(\frac{1}{p+2^*_\mu} - \frac{1}{2\cdot 2^*_\mu}\bigg)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\delta|u_n(x)|^{2^*_\mu}\delta|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\ &\quad - \frac{1}{2\cdot 2^*_\mu}\bigg(1 - \frac{2p}{p+2^*_\mu}\bigg)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\lambda K(x)|u_n(x)|^p\lambda K(y)|u_n(y)|^p}{|x-y|^\mu}\,dx\,dy\bigg] \\ &\ge \lim_{n\to\infty}\bigg[\bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)\|u_n\|^2 - \frac{1}{2\cdot 2^*_\mu}\bigg(1 - \frac{2p}{p+2^*_\mu}\bigg)\lambda^2|K|_{p'}^2|u_n|_{2^*}^{2p}\bigg] \\ &\ge \bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)\|u\|^2 - \frac{1}{2\cdot 2^*_\mu}\bigg(1 - \frac{2p}{p+2^*_\mu}\bigg)\lambda^2|K|_{p'}^2|u|_{2^*}^{2p} \\ &\ge \bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)S|u|_{2^*}^2 - \frac{1}{2\cdot 2^*_\mu}\bigg(1 - \frac{2p}{p+2^*_\mu}\bigg)\lambda^2|K|_{p'}^2|u|_{2^*}^{2p},\end{aligned}$$

which implies $|u|_{2^*} \le C\lambda^{1/(1-p)}$. Hence

$$\begin{aligned}0 > c &= \lim_{n\to\infty}\bigg[J_K(u_n) - \frac{1}{p+2^*_\mu}\langle J_K'(u_n), u_n\rangle\bigg] \\ &\ge \lim_{n\to\infty}\bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)\int_{\mathbb{R}^N} |\nabla u_n|^2\phi_R\,dx - \frac{1}{2\cdot 2^*_\mu}\bigg(1 - \frac{2p}{p+2^*_\mu}\bigg)\lambda^2|K|_{p'}^2|u|_{2^*}^{2p}\end{aligned}$$

$$\geq \bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)\omega_\infty - C\lambda^{2p/(1-p)}$$
$$\geq \bigg(\frac{1}{2} - \frac{1}{p+2^*_\mu}\bigg)\frac{1}{2}\bigg(\bigg(\bigg(\int_{\mathbb{R}^N} d\omega\bigg)^2 + 4S^2_{H,L}\nu_\infty^{2/2^*_\mu}\bigg)^{1/2} - \int_{\mathbb{R}^N} d\omega\bigg) - C\lambda^{2p/(1-p)}.$$

Combining this with (iv), for a given $\lambda > 0$ given, we obtain $\overline{\delta} > 0$ such that for every $0 < \delta < \overline{\delta}$ the last term on the right-hand side above is greater than 0, which is a contradiction. Similarly, for each δ fixed, there exists a $\overline{\lambda}$ so small that for every $0 < \lambda < \overline{\lambda}$ the last term on the right-hand side above is greater than 0 as well, thus we have $\nu_\infty = 0$. Arguing with a similar process, we can prove $\nu_i = 0,\ i \in I$.

Now, applying Lemma 2.2 again, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy$$

and we can deduce from Lemma 2.1 that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(u_n-u)(x)|^{2^*_\mu}|(u_n-u)(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy = 0. \tag{2.7}$$

Now we are ready to prove that $u_n \to u$ strongly in $D^{1,2}(\mathbb{R}^N)$. Notice that

$$\begin{aligned}
\langle J'_K(u_n) &- J'_K(u), u_n - u\rangle \\
&= \int_{\mathbb{R}^N}|\nabla(u_n-u)|^2\,dx \\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\delta^2|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu-2}u_n(y)(u_n-u)(y)}{|x-y|^\mu}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\delta|u_n(x)|^{2^*_\mu}(p/2^*_\mu)\lambda K(y)|u_n(y)|^{p-2}u_n(y)(u_n-u)(y)}{|x-y|^\mu}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\lambda K(x)|u_n(x)|^p\delta|u_n(y)|^{2^*_\mu-2}u_n(y)(u_n-u)(y)}{|x-y|^\mu}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\lambda^2 K(x)|u_n(x)|^p(p/2^*_\mu)K(y)|u_n(y)|^{p-2}u_n(y)(u_n-u)(y)}{|x-y|^\mu}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\delta^2|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu-2}u(y)(u_n-u)(y)}{|x-y|^\mu}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\delta|u(x)|^{2^*_\mu}(p/2^*_\mu)\lambda K(y)|u(y)|^{p-2}u(y)(u_n-u)(y)}{|x-y|^\mu}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\lambda K(x)|u(x)|^p\delta|u(y)|^{2^*_\mu-2}u(y)(u_n-u)(y)}{|x-y|^\mu}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\lambda^2 K(x)|u(x)|^p(p/2^*_\mu)K(y)|u(y)|^{p-2}u(y)(u_n-u)(y)}{|x-y|^\mu}\,dx\,dy,
\end{aligned}$$

On one hand, from the fact that $\{u_n\}$ is bounded in $L^{2^*}(\mathbb{R}^N)$ we have

$$|u_n|^{2^*_\mu} \rightharpoonup |u|^{2^*_\mu} \quad \text{in } L^{2N/(2N-\mu)}(\mathbb{R}^N)$$

as $n \to +\infty$. By the Hardy–Littlewood–Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{2N/(2N-\mu)}(\mathbb{R}^N)$ to $L^{2N/\mu}(\mathbb{R}^N)$, we know that

$$\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dy \rightharpoonup \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy \quad \text{in } L^{2N/\mu}(\mathbb{R}^N)$$

as $n \to +\infty$. Combining this with the fact that

$$|u_n|^{2^*_\mu-2}u_n \rightharpoonup |u|^{2^*_\mu-2}u \quad \text{in } L^{2N/(N-\mu+2)}(\mathbb{R}^N)$$

as $n \to +\infty$, we have

$$\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy |u_n(x)|^{2^*_\mu-2}u_n(x) \rightharpoonup \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy |u(x)|^{2^*_\mu-2}u(x)$$

in $L^{2N/(N+2)}(\mathbb{R}^N)$ as $n \to +\infty$. Since (2.7), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu-2}u_n(y)(u_n-u)(y)}{|x-y|^\mu}\, dx\, dy \to 0.$$

Similarly, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu}K(y)|u_n(y)|^{p-2}u_n(y)(u_n-u)(y)}{|x-y|^\mu}\, dx\, dy \to 0,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)|u_n(x)|^{p}|u_n(y)|^{2^*_\mu-2}u_n(y)(u_n-u)(y)}{|x-y|^\mu}\, dx\, dy \to 0,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)|u_n(x)|^{p}K(y)|u_n(y)|^{p-2}u_n(y)(u_n-u)(y)}{|x-y|^\mu}\, dx\, dy \to 0,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu-2}u(y)(u_n-u)(y)}{|x-y|^\mu}\, dx\, dy \to 0,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}K(y)|u(y)|^{p-2}u(y)(u_n-u)(y)}{|x-y|^\mu}\, dx\, dy \to 0,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)|u(x)|^{p}|u(y)|^{2^*_\mu-2}u(y)(u_n-u)(y)}{|x-y|^\mu}\, dx\, dy \to 0,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)|u(x)|^{p}K(y)|u(y)|^{p-2}u(y)(u_n-u)(y)}{|x-y|^\mu}\, dx\, dy \to 0.$$

Since $J'_K(u_n) \to 0$ and $u_n \rightharpoonup u$, we have $\langle J'_K(u_n) - J'_K(u), u_n - u\rangle \to 0$, as $n \to \infty$, that is

$$\int_{\mathbb{R}^N} |\nabla(u_n-u)|^2 dx \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

3. Infinitely many solutions for problem (1.1)

In this section we will use minimax procedure and perturbation technique to prove the existence of infinitely many solutions of problem (1.1). Let X be a Banach space and $\Sigma = \{A \subset X \setminus \{0\},\ A$ is closed in X and symmetric with respect to the orgin$\}$. For $A \in \Sigma$, we define the genus γ as

$$\gamma(A) := \inf\{m \in \mathbb{N} : \exists\, \varphi \in C(A, \mathbb{R}^m \setminus \{0\}), -\varphi(x) = \varphi(-x)\}.$$

If there is no mapping φ as above for any $m \in N$, then $\gamma(A) := +\infty$. For future use, we list some properties of the genus firstly.

PROPOSITION 3.1 (see [21]). *Let $A, B \in \Sigma$. Then*:

(a) *if there exists an odd map $f \in C(A, B)$, then $\gamma(A) \le \gamma(B)$,*
(b) *if $A \subset B$ then $\gamma(A) \le \gamma(B)$,*
(c) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$,
(d) *if $\mathcal{S}$ is a sphere centered at the origin in $\mathbb{R}^m$, then $\gamma(\mathcal{S}) = m$,*
(e) *if A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $N_\delta(A) \in \Sigma$ and $\gamma(A) = \gamma(N_\delta(A))$, where $N_\delta(A) = \{x \in X : \|x - A\| \le \delta\}$.*

The technique has been used in [4], [14]. Applying Sobolev inequality and the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned} J_K(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^\mu}\big(\delta|u(x)|^{2^*_\mu} + \lambda K(x)|u(x)|^p\big) \\ \times \big(\delta|u(y)|^{2^*_\mu} + \lambda K(y)|u(y)|^p\big)\,dx\,dy \\ \ge \frac{1}{2}\|u\|^2 - \delta^2 C_1\|u\|^{2\cdot 2^*_\mu} - \lambda^2 C_2\|u\|^{2p}. \end{aligned}$$

Given $\delta > 0$, set $t = \|u\|$. So,

$$J_K(u) \ge Q(t) := \frac{1}{2}t^2 - \delta^2 C_1 t^{2\cdot 2^*_\mu} - \lambda^2 C_2 t^{2p} = t^{2p}\bigg(\frac{1}{2}t^{2-2p} - \delta^2 C_1 t^{2\cdot 2^*_\mu - 2p} - \lambda^2 C_2\bigg)$$

and so there exists $\lambda_* < \overline{\lambda}$ so small that for every $\lambda \in (0, \lambda_*)$, there exit $0 < R_0 < R_1$ such that $Q(t) < 0$ for $0 < t < R_0$, $Q(t) > 0$ for $R_0 < t < R_1$, $Q(t) < 0$ for $t > R_1$. Clearly, $Q(R_0) = Q(R_1) = 0$.

Now, let $\chi\colon \mathbb{R}^+ \to [0,1]$ be a nonincreasing $\mathcal{C}^\infty$ function such that $\chi(t) = 1$ if $t \le R_0$ and $\chi(t) = 0$ if $t \ge R_1$. Let define $\phi(u) = \chi(\|u\|)$ and consider the perturbation of $J_K(u)$ given by

$$\begin{aligned} J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2\cdot 2^*_\mu}\phi(u)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\delta^2|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\ - \frac{1}{2^*_\mu}\phi(u)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\delta|u(x)|^{2^*_\mu}\lambda K(y)|u(y)|^p}{|x-y|^\mu}\,dx\,dy \\ - \frac{1}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\lambda^2 K(x)|u(x)|^p K(y)|u(y)|^p}{|x-y|^\mu}\,dx\,dy. \end{aligned}$$

Similarly, for each $\lambda > 0$, we can find $\delta_* > 0$, R_0, R_1 as above for each $0 < \delta < \delta_*$, and define the perturbation of $J_K(u)$ as well.

From Lemma 2.3 and the discussion above, we have the following:

LEMMA 3.2. *If $J(u)$ is defined as above, then*:

(a) $J \in \mathcal{C}^1(D^{1,2}(\mathbb{R}^N), \mathbb{R})$, *$J$ is even and bounded from below.*

(b) *If $J(u) \le 0$, then we have either $\|u\| \le R_0$ or $\|u\| \ge R_1$. Furthermore, $J_K(u) = J(u)$ if $\|u\| \le R_0$.*

(c) *for any fixed δ there exists λ_* such that for any $\lambda \in (0, \lambda_*)$, J satisfies a* (PS) *condition at $c < 0$.*

(d) *for any fixed λ there exists δ_* such that for any $\delta \in (0, \delta_*)$, J satisfies a* (PS) *condition at $c < 0$.*

PROOF. It is easy to see (a) and (b). Conditiona (c) and (d) are consequences of (b) and Lemma 2.3. □

PROPOSITION 3.3. *Denote by $J^c = \{u \in D^{1,2}(\mathbb{R}^N)/J(u) \le c\}$. Then, for any $k \in \mathbb{N}$, there exists $\sigma(k) > 0$ such that $\gamma(J^{-\sigma(k)}) \ge k$.*

PROOF. Firstly, given $k \in N$, let E_k be a k-dimensional subspace of $D^{1,2}(\mathbb{R}^N)$. From the assumption on $K(x)$, there exist $d_k > 0$ such that

$$\inf_{u \in E_k,\, \|u\|=1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|u(x)|^p K(y)|u(y)|^p}{|x-y|^\mu}\, dx\, dy \ge d_k.$$

So, by using this information, the Hardy–Littlewood–Sobolev and Hölder's inequalities, we have

$$\begin{aligned} J(\rho u) = J_K(\rho u) &= \frac{\rho^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{\rho^{2\cdot 2^*_\mu}}{2 \cdot 2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^2 |u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu}\, dx\, dy \\ &\quad - \frac{\rho^{2^*_\mu + p}}{2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta |u(x)|^{2^*_\mu} \lambda K(y)|u(y)|^p}{|x-y|^\mu}\, dx\, dy \\ &\quad - \frac{\rho^{2p}}{2 \cdot 2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\lambda^2 K(x)|u(x)|^p K(y)|u(y)|^p}{|x-y|^\mu}\, dx\, dy \\ &\le \frac{\rho^2}{2} - \delta^2 \rho^{2\cdot 2^*_\mu} C_1 - \lambda^2 C_2 d_k \rho^{2p} \le -\sigma(k) < 0 \end{aligned}$$

for each $\rho < R_0$ small enough (see definition of $\phi(u)$) and $\|u\| = 1$. That is,

$$\{u \in E_k, \|u\| = \rho\} \subset \{u \in D^{1,2}(\mathbb{R}^N), J(u) \le -\sigma(k)\},$$

and consequently, we have $\gamma(J^{-\sigma(k)}) \ge k$. □

REMARK 3.4. By Lemma 3.2 and Proposition 3.3 we can define

$$\Gamma_k = \{A \in \Sigma : \gamma(A) \ge k\}$$

and let $c_k = \inf\limits_{A \in \Gamma_k} \sup\limits_{u \in A} J(u)$. It is obvious that $-\infty < c_k \le -\sigma(k)$, since $J^{-\sigma(k)} \in \Gamma_k$ and J is bounded from below.

Proposition 3.5. *Let δ, λ be as in* (c) *and* (d) *of Lemma* 3.2. *Then all c_m given by Remark* 3.4 *are critical values of J, and $c_m \to 0$.*

Proof. It is clear that $c_m \leq c_{m+1}, c_m < 0$, hence $c_m \to \overline{c} \leq 0$. Moreover, since all c_m are critical values of J, (refer to [21]), we claim $\overline{c} = 0$. If $\overline{c} < 0$, then $K_{\overline{c}} = \{J'(u) = 0, J(u) = \overline{c}\}$ is compact and $K_{\overline{c}} \in \Sigma$. Hence $\gamma(K_{\overline{c}}) = m_0 < \infty$, and there exists $\sigma > 0$, such that $\gamma(K_{\overline{c}}) = \gamma(N_\sigma(K_{\overline{c}})) = m_0$. By the deformation lemma, there exists $\varepsilon > 0$ such that $\overline{c} + \varepsilon < 0$ and an odd homeomorphism $\eta : D^{1,2}(\mathbb{R}^N) \to D^{1,2}(\mathbb{R}^N)$ such that $\eta(J^{\overline{c}+\varepsilon} \setminus N_\sigma(K_{\overline{c}})) \subset J^{\overline{c}-\varepsilon}$.

Since c_m is increasing and converges to $\overline{c}$, there exists $m \in N$ such that $c_m > \overline{c} - \varepsilon$ and $c_{m+m_0} \leq \overline{c}$, and there is $A \in \Gamma_{m+m_0}$ such that $\sup_{u \in A} J(u) < \overline{c} + \varepsilon$, i.e. $A \subset J^{\overline{c}+\varepsilon}$. Now it follows from the properties of the genus,

$$\gamma(\overline{A \setminus N_\sigma(K_{\overline{c}})}) \geq \gamma(A) - \gamma(N_\sigma(K_{\overline{c}})) \geq m,$$

hence $\gamma(\overline{A \setminus N_\sigma(K_{\overline{c}})}) \geq m$, and therefore $\eta(\overline{A \setminus N_\sigma(K_{\overline{c}})}) \in \Gamma_m$. Consequently

$$\sup_{u \in \eta(\overline{A \setminus N_\sigma(K_{\overline{c}})})} J(u) \geq c_m > \overline{c} - \varepsilon$$

but, by on the other hand, we have

$$\eta(A \setminus N_\sigma(K_{\overline{c}})) \subset \eta\big(J^{\overline{c}+\varepsilon} \setminus N_\sigma(K_{\overline{c}})\big) \subset J^{\overline{c}-\varepsilon},$$

which is a contradiction. $\square$

Proof of Theorem 1.2. Now, by Lemma 3.2, Remark 3.4 and Proposition 3.5, it is easy to prove Theorem 1.2. $\square$

References

[1] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z. **248** (2004), 423–443.

[2] C.O. Alves, D. Cassani, C. Tarsi and M. Yang, *Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in* $\mathbb{R}^2$, J. Differential Equations **261** (2014), 1933–1972.

[3] C.O. Alves, F. Gao, M. Squassina and M. Yang, *Singularly perturbed critical Choquard equations*, J. Differential Equations **263** (2017), 3943–3988.

[4] J. Bonder and J. Rossi, *A fourth order elliptic equation with nonlinear boundary conditions*, Nonlinear Anal. **49** (2002), 1037–1047.

[5] B. Buffoni, L. Jeanjean and C.A. Stuart, *Existence of a nontrivial solution to a strongly indefinite semilinear equation*, Proc. Amer. Math. Soc. **119** (1993), 179–186.

[6] S. Cingolani, M. Clapp and S. Secchi, *Multiple solutions to a magnetic nonlinear Choquard equation*, Z. Angew. Math. Phys. **63** (2012), 233–248.

[7] F. Gao, E. Silva, M. Yang and J. Zhou, *Existence of solutions for critical Choquard equations via the concentration compactness method*, arXiv:1712.08264

[8] F. Gao and M. Yang, *Brezis–Nirenberg type critical problem for nonlinear Choquard equation*, Sci China Math, DOI: 10.1007/s11425-016-9067-5.

[9] F. Gao and M. Yang, *On nonlocal Choquard equations with Hardy–Littlewood–Sobolev critical exponents*, J. Math. Anal. Appl. **448** (2017), 1006–1041.

[10] F. Gao and M. Yang, *A strongly indefinite Choquard equation with critical exponent due to Hardy–Littlewood–Sobolev inequality,* Commun. Contemp. Math. **20** (2018), 1750037, pp. 22.

[11] E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57** (1976/77), 93–105.

[12] E.H. Lieb and M. Loss, *Analysis*, *Gradute Studies in Mathematics*, Amer. Math. Soc., Providence, Rhode Island, 2001.

[13] P.L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** (1980), 1063–1072.

[14] S. Li and W. Zou, *Remarks on a class of elliptic problems with critical exponent*, Nonlinear Anal. **32** (1998), 769–774.

[15] L. Ma and L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal. **195** (2010), 455–467.

[16] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations*: *Existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), 153–184.

[17] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equation*: *Hardy–Littlewood–Sobolev critical exponent*, Commun. Contemp. Math. **17** (2015), 1550005, pp. 12.

[18] V. Moroz and J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. **367** (2015), 6557–6579.

[19] S. Pekar, *Untersuchungüber die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin, 1954.

[20] R. Penrose, *On gravity's role in quantum state reduction,* Gen. Relativ. Gravitat. **28** (1996), 581–600.

[21] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., vol. 65, AMS, RI, 1986.

[22] J.C. Wei and M. Winter, *Strongly interacting bumps for the Schrödinger–Newton equations*, J. Math. Phys. **50** (2009), 012905, pp. 22.

Manuscript received May 1, 2018

accepted June 16, 2018

Fashun Gao
Department of Mathematics and Physics
Henan University of Urban Construction
Pingdingshan, Henan, 467044, P.R. CHINA

E-mail address: fsgao@zjnu.edu.cn

Minbo Yang
Department of Mathematics
Zhejiang Normal University
Jinhua, Zhejiang, 321004, P.R. CHINA

E-mail address: mbyang@zjnu.edu.cn

Carlos Alberto Santos and Jiazheng Zhou
Universidade de Brasília
Departamento de Matemática
70910-900, Brasília DF, BRAZIL

E-mail address: csantos@unb.br, jiazzheng@gmail.com