Topological Methods in Nonlinear Analysis
Volume 54, No. 2A, 2019, 427–444
DOI: 10.12775/TMNA.2019.036



# TWO HOMOCLINIC ORBITS FOR SOME SECOND-ORDER HAMILTONIAN SYSTEMS

Patricio Cerda — Luiz F.O. Faria
Eduard Toon — Pedro Ubilla

Abstract. This paper is concerned with the existence of homoclinic orbits for a class of second order Hamiltonian systems considering a non-periodic potential and a weaker Ambrosetti–Rabinowitz condition. By considering an auxiliary problem, we show the existence of two different approximative sequences of periodic solutions, the first one of mountain pass type and the second one of local minima. We obtain two different homoclinic orbits by passing to the limit in such sequences. As a relevant application, we obtain another homoclinic solution for the Hamiltonian system studied in [5].

## 1. Introduction

The complex dynamical behavior of Hamiltonian systems has attracted mathematicians and physicists ever since Newton wrote the differential equations describing planetary motions and derived Kepler's ellipses as solutions.

It is well known that the existence of homoclinic solutions for Hamiltonian systems and their importance in the study of the behavior of dynamical systems

2010 *Mathematics Subject Classification.* 35J20, 35J70, 35J60.

*Key words and phrases.* Indefinite problem; concave-convex; variational methods.

P. Cerda was partially supported by Programa de Inserción de Capital Humano Avanzado en la Academia, CONICYT grant 79140015.

L.F.O. Faria was partially supported by CAPES/Brazil (Proc. 6129/2015-03) and by FAPEMIG CEX APQ 02374/17.

P. Ubilla was partially supported by FONDECYT grant 1120524 and partially supported by FAPEMIG CEX BPV 00853/15.

have been recognized by Poincaré [7]. In addition, homoclinic solutions may give the horseshoe chaos (see, for instance, [9] and the references therein).

A lot of attention has been devoted in the past twenty five years to finding the existence and multiplicity of homoclinic solutions of Hamiltonian systems. We would like to cite [3], [5], [6], [8], [1], [10] and the references therein.

In 1990, Rabinowitz [8] obtained the existence of one homoclinic orbit of the nonautonomous Hamiltonian system

$$\ddot{u} + H_u(t,u) = 0\,,$$

where the potential $H$ is given by

$$H(t,u) = -\frac{1}{2}(L(t)u,u) + M(t,u),$$

where $L$ is a continuous $T$-periodic matrix valued function such that $L(t)$ is positive definite and symmetric for all $t \in [0,T]$, and $M$ satisfies:

($\widetilde{\mathrm{H}}$) $M_u(t,u) = o(|u|)$, as $|u| \to 0$ uniformly with respect to $t$,

(AR) there is a constant $\mu > 2$ such that, for every $t \in \mathbb{R}$ and $u \in \mathbb{R}^n \setminus \{0\}$,

$$0 < \mu M(t,u) \leq \langle u, M_u(t,u)\rangle.$$

Rabinowitz obtained the existence of one homoclinic orbit, where the main key is to construct a sequence of periodic auxiliary systems to approximate the Hamiltonian system, by applying the Mountain Pass Theorem to obtain periodic solutions. Therefore, the homoclinic solution is obtained as the limit of those periodic solutions.

In 2005, Izydorek and Janczewska [5] considered the system studied by Rabinowitz in [8], perturbing it with a bounded time dependent force $f(t)$, i.e.

$$\ddot{u} + H_u(t,u) = f(t)$$

where $f\colon \mathbb{R} \to \mathbb{R}^n$ is a continuous bounded function such that its norm in $L^2$ space is small enough (can be considered $f \equiv 0$). Using the same ideas of [8], with the additional complication that $f$ is not periodic, the authors prove the existence of a homoclinic orbit for the perturbed problem.

Our study is motivated in part by the work of Izydorek and Janczewska [5]. First we observed that by using their hypotheses, we were able to obtain a second homoclinic orbit by minimization techniques. This motivated us to determine the class of non-periodic potentials for which it is possible to obtain two nontrivial homoclinic orbits. More precisely, in this paper we concentrate on the existence of two nontrivial homoclinic orbits for a class of second order systems of the form:

$$\text{(P)} \qquad \ddot{u} + V_u(t,u) = 0,$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, and the function $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies:

($H_1$) $V(t,u) = -K(t,u) + W(t,u)$ where $K, W\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ are $C^1$-maps, $W(t,0) = 0$ and $K(t,0) = 0$ for all $t \in \mathbb{R}$.

($H_2$) There exists a constant $b_1 > 0$ such that for all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$,

$$b_1|u|^2 \le K(t,u) \quad \text{and} \quad |K_u(t,u)| \le 2b_1|u|.$$

($H_3$) For all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$, $K(t,u) \le \langle u, K_u(t,u)\rangle$.

($H_4$) Given $M > 0$, there is a $C_M > 0$ such that $|W_u(t,u)| \le C_M$ for all $t \in \mathbb{R}$ and for all $|u| \le M$.

($H_5$) There exist constants $0 \le q < 1$, $1 < r < 2/(1-q)$, $u_1, \tau_1, c_0 > 0$, a continuous function $b \in L^r(\mathbb{R})$, bounded and not identically null, and a non-decreasing continuous function $F\colon \mathbb{R} \to [0,+\infty)$, $F(|u|) = o(|u|)$, as $|u| \to 0$, such that

$$|W_u(t,u)| \le F(|u|) + \lambda b(t)|u|^q, \quad \text{for all } |u| \le u_1 \text{ and for all } t \in \mathbb{R},$$

and $\lambda c_0|u|^{q+1} \le W(t,u)$, for all $|u| \le u_1$ and for all $|t| < \tau_1$, where $\lambda > 0$ is a real parameter.

$(\mathrm{AR})_l$ There exist constants $\mu > 2$, $u_0 > 0$ and $\tau_0 > 0$, such that $W(t,u) > 0$ for all $|t| \le \tau_0$ and for all $|u| \ge u_0$; and

$$\langle W_u(t,u), u\rangle - \mu W(t,u) \ge 0, \quad \text{for all } |u| \ge u_0 \text{ and for all } t \in \mathbb{R},$$

and there exists a function $g \in L^1(\mathbb{R})$, such that

$$\langle W_u(t,u), u\rangle - \mu W(t,u) \ge g(t), \quad \text{for all } |u| < u_0 \text{ and for all } t \in \mathbb{R}.$$

The contribution of this paper is to obtain two homoclinic orbits for a more general class of nonlinearities considering a weaker Ambrosetti-Rabinowitz condition $(\mathrm{AR})_l$.

Remark 1.1. In our equation we can consider a function

$$W(t,u) = a(t)G(u) + F(t,u)$$

where $G$ satisfies (AR) and the bounded non-null function $a \ge 0$ verifying $|\{t \in \mathbb{R};\ a(t) = 0\}| > 0$. Notice that the functions $a$ and $F$ might not to be periodic. We can also consider

$$W(t,u) = M(t,u) + F(t,u),$$

where $M$ as in [5] and

$$F(t,u) = |u(t)|^q\langle f(t), u(t)\rangle,$$

where $f \in L^r(\mathbb{R},\mathbb{R}^n)$ is bounded and not identically null, whose the particular case ($q = 0$) was studied in [5]. In such a case, as observed by Izydorek and Janczewska, $u(t) = 0$ is a solution of (P) only if $f(t) = 0$. For more details and more examples, see Section 6.

To the best of our knowledge, the hypotheses considered in the present paper are not found in previous literature. In this sense, we understand that this paper contributes to the study of this class of problems both in the results and the hypothesis. The lack of periodicity on the potential $V$ resulted on technical issues that we were able to solve.

Here and subsequently, we consider $\langle \cdot\,, \cdot \rangle$ and $|\cdot|$ the standard inner product and norm on $\mathbb{R}^n$, respectively. By a nontrivial homoclinic solution (to 0) of (P), we consider a orbit which connects the same equilibria 0. More precisely, we mean $u\colon \mathbb{R} \to \mathbb{R}^n$ satisfying:

(i) $u \in C^2(\mathbb{R}^n, \mathbb{R})$;
(ii) $u \not\equiv 0$, which solves (P);
(iii) $u(t) \to 0$, $\dot{u}(t) \to 0$, as $t \to \pm\infty$.

We state that our main result concerning (P) is the following:

Theorem 1.2. *Assuming conditions* $(\mathrm{H}_1)$–$(\mathrm{H}_4)$ *and* $(\mathrm{AR})_l$, *there exists* $\Lambda > 0$ *such that if* $(\mathrm{H}_5)$ *holds for all* $\lambda \in (0, \Lambda)$, *then problem* (P) *possesses at least two nontrivial homoclinic orbits.*

Remark 1.3. Notice that one can consider $(\widetilde{\mathrm{AR}})_l$ instead of hypothesis $(\mathrm{AR})_l$ considered above, where $(\widetilde{\mathrm{AR}})_l$ is given by: there exist constants $\mu > 2$, $u_0 > 0$ and $a, b \in \mathbb{R}$, such that $W(t, u) > 0$ for all $t \in [a, b]$ and for all $|u| \geq u_0$; and

$$\langle W_u(t, u), u\rangle - \mu W(t, u) \geq 0, \quad \text{for all } |u| \geq u_0 \text{ and for all } t \in \mathbb{R},$$

and there exists a function $g \in L^1(\mathbb{R})$, such that

$$\langle W_u(t, u), u\rangle - \mu W(t, u) \geq g(t), \quad \text{for all } |u| < u_0 \text{ and for all } t \in \mathbb{R}.$$

In the proof of Theorem 1.2, the first homoclinic orbit is obtained as a limit of a certain sequence of functions and using the Mountain Pass Theorem. It is important to note that, under these hypotheses, $u = (0, \ldots, 0) \in \mathbb{R}^n$ can be a trivial solution of the system $(P)$, which means that we can not directly use calculations of [5] to obtain the first homoclinic orbits. The second homoclinic orbits is obtained as a limit of a certain sequence of functions and by minimization methods into a small ball.

Remark 1.4. According to Remark 1.1, Theorem 1.2 establishes another homoclinic orbit for the problem treated in [5].

The paper is organized as follows. In Section 2, we gather the most relevant notations and known results we will use. In Section 3, we prove the existence of $2kT$-periodic solutions of an auxiliary system by using the Mountain Pass Theorem. In Section 4, we obtain the existence of $2kT$-periodic solutions of an auxiliary system through local minimization. In Section 5, we prove the

Theorem 1.2. These are the main results of this work. Finally, in Section 6, we show some applications of our results.

## 2. Preliminary results

For each $k \in \mathbb{N}$, let $E_k := W^{1,2}_{2kT}(\mathbb{R},\mathbb{R}^n)$ the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the product and norm are given, respectively, by

$$\langle u,v\rangle_{E_k} := \int_{-kT}^{kT} [\langle \dot u(t),\dot v(t)\rangle + \langle u(t),v(t)\rangle]\,dt, \qquad \|u\|_{E_k} := \langle u,u\rangle^{1/2}_{E_k}.$$

Remark 2.1. Notice that by $(\mathrm{H}_1)$ and $(\mathrm{H}_2)$, and using the Mean Value Theorem, we have

$$b_1|u|^2 \le K(t,u) \le 2b_1|u|^2 \quad \text{for all } (t,u)\in\mathbb{R}\times\mathbb{R}^n.$$

By Remark 2.1, there exist $\overline{b}_1$, $\overline{b}_2$ such that

$$\overline{b}_1\|u\|^2_{E_k} \le \int_{-kT}^{kT} \big[|\dot u(t)|^2 + 2K(t,u(t))\big]\,dt \le \overline{b}_2\|u\|^2_{E_k}. \tag{2.1}$$

Remark 2.2. By hypothesis $(\mathrm{H}_2)$, we have $\langle K_u(t,u),u\rangle \le 2K(t,u)$.

Let $L^\infty_{2kT}(\mathbb{R},\mathbb{R}^n)$ denote the space of $2kT$-periodic essentially bounded (measurable) functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_{L^\infty_{2kT}} := \operatorname{ess\,sup}\{|u(t)| : t\in[-kT,kT]\}.$$

We set $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R})}$, where

$$\|u\|_{L^\infty(\mathbb{R})} := \operatorname{ess\,sup}\{|u(t)| : t\in\mathbb{R}\}.$$

Let $L^\theta_{2kT}(\mathbb{R},\mathbb{R}^n)$, $\theta>1$, denote the Lebesgue space of $2kT$-periodic (measurable) functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_{L^\theta_{2kT}} := \bigg(\int_{kT}^{kT} |u(t)|^\theta\,dt\bigg)^{1/\theta}.$$

Remark 2.3. If $u\colon\mathbb{R}\to\mathbb{R}^n$, then $u\in E_k$ if and only if $u$ is an absolutely continuous function, $u(-kT)=u(kT)$ and $\dot u\in L^2([-kT,kT],\mathbb{R}^n)$.

Remark 2.4. Notice that $\phi(\eta)=W(t,\eta u)$ is a continuous function on the closed interval $[0,1]$, and differentiable on the open interval $(0,1)$, then by mean value theorem there exists $\xi\in(0,1)$ such that $W(t,u)=\langle W_u(t,\xi u),u\rangle$, and so by hypotheses $(\mathrm{H}_1)$ and $(\mathrm{H}_5)$, we have that:

$$|W(t,u)| \le G(|u|) + \lambda b(t)|u|^{q+1}, \quad \text{if } |u|\le u_1 \text{ and for al } t\in\mathbb{R}, \tag{2.2}$$

where $G(|u|)=F(|u|)|u|$.

The following result is a direct consequence of the estimations made by Rabinowitz in [8].

Proposition 2.5. *There is a positive constant $C$ (independent of $k \in \mathbb{N}$) such that for each $k \in \mathbb{N}$ and $u \in E_k$ the following inequality holds*:

$$\|u\|_{L^\infty_{2kT}} \leq C\|u\|_{E_k}. \tag{2.3}$$

The following lemma will be essential to our purposes. For a proof of the result, see [5, Fact 2.8].

Lemma 2.6. *Let $u\colon \mathbb{R} \to \mathbb{R}^n$ be a continuous mapping such that $\dot{u} \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$. For every $t \in \mathbb{R}$, the following inequality holds*

$$|u(t)| \leq \sqrt{2}\bigg(\int_{t-1/2}^{t+1/2} \big(|u(s)|^2 + |\dot{u}(s)|^2\big)\, ds\bigg)^{1/2}.$$

Now, consider the following sequence of Hamiltonian systems:

$$\ddot{u} + V_u^k(t, u) = 0, \tag{$P_k$}$$

where, for each $k \in \mathbb{N}$, $V_u^k\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $V^k\colon \mathbb{R}^{n+1} \to \mathbb{R}$ are, respectively, $2kT$-periodic extensions of the restriction of $V_u(t, u)$ and $V(t, u)$ to the interval $[-kT, kT)$ in the variable $t$.

Definition 2.7. By a non-trivial periodic solution of ($\mathrm{P}_k$), we consider $u_k\colon \mathbb{R} \to \mathbb{R}^n$ satisfying:

(a) $u_k \in C^1\big([-kT, kT], \mathbb{R}^n\big) \cap C^2\big((-kT, kT), \mathbb{R}^n\big)$;
(b) $\ddot{u} + V_u^k(t, u) = 0$, for all $t \in (-kT, kT)$;
(c) $u_k(-kT) - u_k(kT) = \dot{u}_k(-kT) - \dot{u}_k(kT) = 0$.

Let $I_{k,\lambda}\colon E_k \to \mathbb{R}$ be the associated functional defined by

$$I_{k,\lambda}(u) = \int_{-kT}^{kT} \bigg[\frac{1}{2}|\dot{u}(t)|^2 - V^k(t, u(t))\bigg]\, dt. \tag{2.4}$$

Then, $I_{k,\lambda} \in C^1(E_k, \mathbb{R})$ and it is easy to verify that

$$I'_{k,\lambda}(u)v = \int_{-kT}^{kT} \big[\langle \dot{u}(t), \dot{v}(t)\rangle - \langle V_u^k(t, u(t)), v(t)\rangle\big]\, dt. \tag{2.5}$$

The following lemma characterizes the functions that belong to the space $E_k$ using its Fourier series (see [4]).

Lemma 2.8. *If $u \in L^2_{2kT}(\mathbb{R}, \mathbb{R}^n)$ and*

$$u(t) = c_0 + \sum_{n=1}^{\infty} \bigg[c_n \cos\bigg(\frac{\pi n t}{kT}\bigg) + b_n \sin\bigg(\frac{\pi n t}{kT}\bigg)\bigg]$$

*is its associated Fourier series, then $u \in E_k$ if and only if*

$$\sum_{n=1}^{\infty} n^2\big(|c_n|^2 + |b_n|^2\big) < \infty.$$

The following result will be essential to our purposes. It says that if a function $u$ belongs to the space $E_k$ is a critical point of the functional (2.4), then $u$ is a solution of $(\mathrm{P}_k)$.

LEMMA 2.9. *If $u \in E_k$ is such that $I'_{k,\lambda}(u) = 0$, then $u$ is a solution of* $(\mathrm{P}_k)$.

PROOF. We assume $u \in E_k$ is a critical point of $I_{k,\lambda}$, then by (2.5) we have

$$\int_{-kT}^{kT} \dot{u}\dot{h}\,dt - \int_{-kT}^{kT} V_u(t,u(t))h(t)\,dt = 0, \quad \text{for all } h \in E_k. \tag{2.6}$$

Let us suppose that the Fourier series of $u$ is given by

$$u(t) = c_0 + \sum_{n=1}^{\infty} \left[ c_n \cos\left(\frac{\pi n t}{kT}\right) + b_n \sin\left(\frac{\pi n t}{kT}\right)\right],$$

and that the series of $V_u(t,u(t))$ is given by

$$V_u(t,u(t)) = \widetilde{c}_0 + \sum_{n=1}^{\infty} \left[ \widetilde{c}_n \cos\left(\frac{\pi n t}{kT}\right) + \widetilde{b}_n \sin\left(\frac{\pi n t}{kT}\right)\right].$$

Applying (2.6) with $h$ equal to the basic functions $\cos(\pi n t/kT)$, $\sin(\pi n t/kT)$ and 1, we obtain

$$\widetilde{b}_n = \left(\frac{\pi n}{kT}\right)^2 b_n, \quad \widetilde{c}_n = \left(\frac{\pi n}{kT}\right)^2 c_n \quad \text{and} \quad \widetilde{c}_0 = 0.$$

By Lemma 2.8, we have $\dot{u} \in E_k$ and, hence, $u \in C^1\big([-kT,kT],\mathbb{R}^n\big)$ and $\ddot{u} \in L^2_{2kT}(\mathbb{R},\mathbb{R}^n)$. Then, integrating by parts in (2.6) we obtain

$$\int_{-kT}^{kT} (\ddot{u} + V_u(t,u))h(t)\,dt = 0, \quad \text{for all } h \in E_k,$$

from where we conclude

$$\ddot{u} + V_u(t,u) = 0 \quad \text{for all } t \in (-kT,kT).$$

Notice that the second term in the left side is continuous so that

$$u \in C^2((-kT,kT),\mathbb{R}^n). \qquad \square$$

The next lemma is concerned with an important property of the function $W(t,u)$. We borrow some ideas from [5], [8], however, we consider hypothesis $(\mathrm{AR})_l$ instead of the classical (AR).

LEMMA 2.10. *Suppose that $(\mathrm{AR})_l$ is satisfied. Then, there exists $a_1 > 0$ such that for all $|t| < \tau_0$ and for all $|u| \ge u_0$*

$$W(t,u) \ge a_1 |u|^{\mu}. \tag{2.7}$$

PROOF. By using $(\mathrm{AR})_l$, notice that for every $|u| > u_0$ and $|t| < \tau_0$ the function $\psi\colon (0,|u|/u_0] \to \mathbb{R}$, given by $\psi(\tau) = W(t,\tau^{-1}u)\tau^{\mu}$, is non-increasing. Therefore, we can chose $a_1 > 0$ verifying (2.7) by the continuity of $W$. $\square$

The following lemma provides the energy of a solution obtained by the Mountain Pass Theorem (see Section 3).

Lemma 2.11. *Assume* (H$_1$), (H$_2$), (H$_5$) *and let* $\gamma \in (0,1)$. *There exists* $\Lambda > 0$, *such that for each* $\lambda \in (0,\Lambda)$, *one can find explicitly* $\alpha_\lambda > 0$ *such that*

$$I_{k,\lambda}(u) > \alpha_\lambda,$$

*for all* $u \in \chi(\lambda^\gamma) = \{u \in E_k : \|u\|_{E_k} = \lambda^\gamma\}$.

Proof. Let $\zeta > 0$, by (H$_5$) there exists $\lambda > 0$ sufficiently small, such that if $u \in \chi(\lambda^\gamma)$ then $F(|u|) < \zeta|u|$. By (H$_1$), (H$_2$) and (H$_5$), we have

$$\begin{aligned}
I_{k,\lambda}(u) &\geq \frac{\overline{b}_1}{2}\|u\|_{E_k}^2 - \int_{-kT}^{kT} \big|W^k(t,u)\big|\,dt \\
&\geq \frac{\overline{b}_1}{2}\|u\|_{E_k}^2 - \int_{-kT}^{kT} G(|u(t)|)\,dt - \lambda\int_{-kT}^{kT} b(t)|u(t)|^{q+1}\,dt \\
&\geq \bigg(\frac{\overline{b}_1}{2} - \zeta\bigg)\|u\|_{E_k}^2 - \lambda c_1\|u\|_{E_k}^{q+1},
\end{aligned}$$

where $c_1 > 0$. Since $\|u\|_{E_k} = \lambda^\gamma$,

$$I_{k,\lambda}(u) \geq \bigg(\frac{\overline{b}_1}{2} - \zeta\bigg)\lambda^{2\gamma} - c_1\lambda^{\gamma(q+1)+1}.$$

Notice that $\gamma \in (0,1)$ implies $2\gamma < \gamma(q+1)+1$. Therefore, if $\zeta > 0$ such that $\overline{b}_1/2 - \zeta > 0$, we have

$$\bigg(\frac{\overline{b}_1}{2} - \zeta\bigg)\lambda^{2\gamma} - c_1\lambda^{\gamma(q+1)+1} := \alpha_\lambda > 0,$$

for $\lambda \in (0,\Lambda)$, where $\Lambda$ is small enough. Hence, we obtain

$$I_{k,\lambda}(u) \geq \alpha_\lambda, \quad \text{for all } \|u\|_{E_k} = \lambda^\gamma$$

and the result is proved. □

## 3. Mountain Pass solution

In this section, we are concerned with obtaining $2kT$-periodic solutions of the system (P$_k$), for each $k \in \mathbb{N}$. More precisely, we have the following theorem.

Theorem 3.1. *Assume that conditions* (H$_1$)–(H$_3$) *and* (AR)$_l$ *are satisfied. For every* $k \in \mathbb{N}$, *there exists* $\Lambda > 0$ *such that for each* $\lambda \in (0,\Lambda)$ *system* (P$_k$) *possesses a non-trivial* $2kT$*-periodic solution, denoted by* $u_{1,k}$, *such that* $I_{k,\lambda}(u_{1,k}) > 0$.

We obtain a critical point of $I_{k,\lambda}$ by the use of a standard version of the Mountain Pass Theorem (see [2]). In addition, we notice that such result provides a minimax characterization for the critical value. For this reason, we state this theorem.

Theorem 3.2 (see [2]). *Let $X$ be a real Banach space and $I\colon X \to \mathbb{R}$ be a $C^1$-smooth functional. If $I$ satisfies the following conditions*:

(a) $I(0) = 0$,

(b) *every sequence $\{u_j\}_{j\in\mathbb{N}}$ in $X$ such that $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and $I'(u_j) \to 0$ in $X^*$, as $j \to +\infty$, contains a convergent subsequence* (*the Palais–Smale condition*),

(c) *there exists constants $\delta,\ \alpha > 0$ such that $I|_{\partial B_\delta(0)} \geq \alpha$*,

(d) *there exists $e \in X \setminus \overline{B}_\delta(0)$ such that $I(e) \leq 0$*,

*where $B_\delta(0)$ is an open ball in $X$ of radius $\delta$ centered at* 0, *then $I$ possesses a critical value $c \geq \alpha$ given by*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

*where $\Gamma = \{g \in C([0,1],X) : g(0) = 0,\ g(1) = e\}$.*

Proof of Theorem 3.1. From now on, we assume $0 < \Lambda < 1$. Clearly $I_{k,\lambda}(0) = 0$. We show that $I_{k,\lambda}$ satisfies the Palais–Smale condition. Suppose that $\{u_j\}_{j\in\mathbb{N}}$ in $E_k$ is a sequence such that $\{I_{k,\lambda}(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'_{k,\lambda}(u_j) \to 0$ as $j \to +\infty$. Then, there exists a constant $C_k > 0$ such that, for every $j \in \mathbb{N}$,

$$|I_{k,\lambda}(u_j)| \leq C_k, \quad \|I'_{k,\lambda}(u_j)\|_{E_k^*} \leq C_k. \tag{3.1}$$

We first prove that $\{u_j\}_{j\in\mathbb{N}}$ is bounded. Notice that

$$\begin{aligned} I_{k,\lambda}(u_j) - \frac{1}{\mu} I'_{k,\lambda}(u_j)u_j &\geq \bar{b}_1\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|^2_{E_k} \\ &\quad + \frac{1}{\mu}\int_{-kT}^{kT} \left[\langle W_u^k(t,u_j(t)), u_j(t)\rangle - \mu W^k(t,u_j(t))\right] dt \end{aligned}$$

By $(\mathrm{AR})_l$ we have

$$\int_{-kT}^{kT} \left[\langle W_u^k(t,u_j(t)), u_j(t)\rangle - \mu W^k(t,u_j(t))\right] dt \geq A,$$

where $A$ is a constant that does not depend on $j$. Thus

$$b_1\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|^2_{E_k} \leq C_k\|u_j\|_{E_k} + D_k, \tag{3.2}$$

where $D_k$ is a constant. Since $\mu > 2$, there exists a constant $\widehat{C}_k > 0$ that does not depend on $\lambda$, such that

$$\|u_j\|_{E_k} < \widehat{C}_k, \tag{3.3}$$

which shows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $E_k$. Going if necessary to a subsequence, we can assume that there exists $u \in E_k$ such that $u_j \rightharpoonup u$ as $j \to +\infty$ in $E_k$, which implies $u_j \to u$ uniformly on $[-kT, kT]$.

On the other hand, note that

$$|I'_{k,\lambda}(u_j)(u_j - u)| \leq \varepsilon_j \|u_j - u\|_{E_k}$$

and $\varepsilon_j \to 0$ when $j \to +\infty$. By the continuity of $V_u$, it is easy to verify that

$$\int_{-kT}^{kT} \langle V_u^k(t, u_j(t)), u_j(t) - u(t)\rangle\, dt \to 0.$$

Therefore,

$$\eta_j = \int_{-kT}^{kT} \langle \dot{u}_j(t), \dot{u}_j(t) - \dot{u}(t)\rangle\, dt \to 0.$$

Thus,

$$\langle u_j, u_j - u\rangle_{E_k} = \eta_j + \int_{-kT}^{kT} \langle u_j(t), u_j(t) - u(t)\rangle \to 0.$$

Consequently, $\|u_j\|_{E_k} \to \|u\|_{E_k}$, and then, $u_j \to u$ in $E_k$.

Notice that the condition (c) of Theorem 3.2 follows from Lemma 2.11, where $\delta = \lambda^\gamma$ is obtained in such lemma.

It remains to show tha,t for every $k \in \mathbb{N}$, there exists $w_k \in E_k$ such that $\|w_k\|_{E_k} > \delta$ and $I_{k,\lambda}(w_k) \leq 0$. Let $w_1 \in E_1$ such that $|w_1(t)| \geq u_0$ for all $|t| < \tau_0/2$ and $w_1(t) = 0$ for all $\tau_0 < |t| < T$.

We have that, for every $\zeta > 1$, the following inequality holds by $(\mathrm{H}_2)$ and (2.7)

$$\begin{aligned} I_{1,\lambda}(\zeta w_1) &= \frac{1}{2}\int_{-T}^{T} \big[\zeta^2|\dot{w}_1(t)|^2 - 2V^1(t, \zeta w_1(t))\big]\, dt \\ &\leq \frac{\zeta^2\overline{b}_2}{2}\|w_1\|_{E_1}^2 - \zeta^\mu \overline{a}_1 \tau_0 u_0^\mu + B. \end{aligned} \tag{3.4}$$

Since $2 < \mu$, let $\zeta_1 > 1$, such that $I_{1,\lambda}(\zeta_1 w_1) < 0$. Take $\delta = \lambda^\gamma$ and $\tilde{\zeta} = \max\{\zeta_1, \zeta_2\}$, where $\zeta_2 \in \mathbb{R}\setminus\{0\}$ is such that $\|\zeta_2 w_1\|_{E_1} > \delta$. Define

$$\widetilde{w}_k(t) = \begin{cases} \widetilde{\zeta} w_1(t) & \text{if } |t| \leq T, \\ 0 & \text{if } T < |t| \leq kT, \end{cases} \tag{3.5}$$

for $k > 0$. Then $\widetilde{w}_k \in E_k$, $\|\widetilde{w}_k\|_{E_k} = \big\|\widetilde{\zeta} w_1\big\|_{E_1} > \delta$ and $I_{k,\lambda}(\widetilde{w}_k) = I_{1,\lambda}(\widetilde{\zeta} w_1) < 0$. Hence, by Theorem 3.2, $I_{k,\lambda}$ possesses a critical value $c_k \geq \alpha$ given by

$$c_k = \inf_{g\in\Gamma_k} \max_{s\in[0,1]} I_{k,\lambda}(g(s)), \tag{3.6}$$

where $\Gamma_k = \{g \in C([0,1], E_k) : g(0) = 0,\ g(1) = w_k\}$. Therefore, for every $k \in \mathbb{N}$, there is $u_{1,k} \in E_k$ such that

$$I_{k,\lambda}(u_{1,k}) = c_k, \quad {I'}_{k,\lambda}(u_{1,k}) = 0.$$

The function $u_{1,k}$ is the desired classical $2kT$-periodic solution of $(\mathrm{P}_k)$. Since $c_k > 0$, $u_{1,k}$ is a non-trivial solution. $\square$

## 4. Minimization solution

In this section, we are concerned with obtaining, for each $k \in \mathbb{N}$, a second $2kT$-periodic solutions $u_{2,k}$ of system $(\mathrm{P}_k)$ through local minimization.

Theorem 4.1. *Assume that hypothesis* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$, *and* $(\mathrm{H}_5)$ *holds, then for every* $k \in \mathbb{N}$ *the system* $(\mathrm{P}_k)$ *possesses a nontrivial* $2kT$*-periodic solution, denoted by* $u_{2,k}$, *such that* $I_{k,\lambda}(u_{2,k}) < 0$.

Proof. Firstly, we will show that there exists $\omega_k \in E_k$ such that

$$I_{k,\lambda}(\xi\omega_k) \le -\eta < 0, \quad \text{for all } \xi \in (0,\xi_0). \tag{4.1}$$

Let $w_1 \in E_1$ not identically null, such that $|w_1(t)| \le u_1$ for all $|t| \le \tau_1$ and $w_1(t) = 0$ for all $\tau_1 < |t| \le T$. Since $k \in \mathbb{N}$, define

$$w_k(t) = \begin{cases} w_1(t) & \text{if } |t| \le T, \\ 0 & \text{if } T < |t| \le kT. \end{cases}$$

Let $\xi_1 \in (0,1)$ be such that $\xi_1 \|w_1\|_{E_1} < \lambda^\gamma$. By (2.1) and $(\mathrm{H}_5)$, for all $\xi \in (0,\xi_1)$ we have

$$I_{k,\lambda}(\xi w_k(t)) \le \xi^2 c \|w_1\|_{E_1}^2 - \lambda \xi^{q+1} c_0 \int_{-\tau_1}^{\tau_1} |w_1(t)|^{q+1}\, dt. \tag{4.2}$$

Since $q+1 < 2$, there exists $\xi_2 > 0$ small enough such that

$$-\eta := \xi^2 c \|w_1\|_{E_1}^2 - \lambda \xi^{q+1} c_0 \int_{-\tau_1}^{\tau_1} |w_1(t)|^{q+1}\, dt < 0.$$

Thus, taking $\xi_0 = \min\{\xi_1,\xi_2\}$, (4.1) follows.

Now, consider the closed ball $B^*_{\lambda^\gamma} = \{u \in E_k : \|u\|_{E_k} \le \lambda^\gamma\}$. By Lemma 2.11, $I_{k,\lambda}(u) > \alpha_\lambda$ for all $u \in \{u \in E_k : \|u\|_{E_k} = \lambda^\gamma\}$. It follows from (4.1) that the minimum of the (weakly lower semicontinuous) functional $I_{k,\lambda}$ on $B^*_{\lambda^\gamma}$ is achieved in the corresponding open ball and, thus, yields a non-trivial solution $u_{2,k}$ of $(\mathrm{P}_k)$, with

$$I_{k,\lambda}(u_{2,k}) \le -\eta < 0 \quad \text{and} \quad \|u_{2,k}\| < \lambda^\gamma. \tag{4.3}$$

□

## 5. Proof of main results

In this section, we prove Theorem 1.2. Before proceeding to the proof, we need a technical lemma.

Lemma 5.1. *Let* $u_k$ *be a solution of Problem* $(\mathrm{P}_k)$ *for each* $k \in \mathbb{N}$. *If* $\|u_k\|_{L^\infty_{2kT}} \le M_1$, *where* $M_1$ *is a constant independent of* $k$, *and assume* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$ *and* $(\mathrm{H}_4)$ *holds, then there exists* $u_0 \in C^1(\mathbb{R},\mathbb{R}^n)$ *and a subsequence* $\{u_{j,j}\}_j$ *of* $\{u_k\}_k$ *converging to* $u_0$ *in* $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$. *Moreover, if* $\|u_k\|_{E_k} \le M_1$, *where* $M_1$ *is a constant independent of* $k$, *then* $u_0$ *is a solution of* $(\mathrm{P})$ *and*

$$u_0(t) \to 0 \quad \text{and} \quad \dot u_0(t) \to 0 \quad \text{as } t \to \pm\infty.$$

Proof. Let $j \in \mathbb{N}$ then there exist $k_j \in \mathbb{N}$ such that $[-j,j] \subset (-kT,kT)$ for all $k \geq k_j$. For each $k \geq k_j$, since $u_k$ is a solution of ($\mathrm{P}_k$), we have that $u_k \in C^2([-j,j],\mathbb{R}^n)$ and

$$\ddot{u}_k + V_u(t,u_k) = 0, \quad t \in [-j,j]. \tag{5.1}$$

Then, by ($\mathrm{H}_1$), ($\mathrm{H}_2$) and ($\mathrm{H}_4$), we have

$$|\ddot{u}_k(t)| \leq |V_u(t,u_k)| \leq M_3,$$

and it follows that

$$\|\ddot{u}_k\|_{L^\infty_{2j}} \leq M_3 \quad \text{for all } k \geq k_j, \tag{5.2}$$

where $M_3$ is a constant independent of $j$.

Since $u_k, \dot{u}_k \in E_k$ are absolutely continuous functions for each $k \in \mathbb{N}$ (see Remark 2.3), by the Mean Value Theorem, for each $k \in \mathbb{N}$ and $t \in \mathbb{R}$, there exists $\tau_k \in [t-1,t]$ such that

$$\dot{u}_k(\tau_k) = \int_{t-1}^{t} \dot{u}_k(s)\,ds = u_k(t) - u_k(t-1).$$

Consequently,

$$|\dot{u}_k(t)| = \left| \int_{\tau_k}^{t} \ddot{u}_k(s)\,ds + \dot{u}_k(\tau_k) \right| \leq 2M_1 + M_3 =: M_2,$$

which means that for each $k \geq k_j$,

$$\|\dot{u}_k\|_{L^\infty_{2j}} \leq M_2. \tag{5.3}$$

One can easily see that $u_k$ and $\dot{u}_k$ verify the Lipschitz condition for each $k \geq k_j$, with constants independent of $j \in \mathbb{N}$. In other words,

$$|u_k(t) - u_k(t_0)| \leq M_2|t-t_0| \quad \text{and} \quad |\dot{u}_k(t) - \dot{u}_k(t_0)| \leq M_3|t-t_0| \quad \text{for all } k \geq k_j.$$

It follows from the Arzela–Ascoli Theorem that there exists a subsequence $\{u_{j,k}\}_k$ of $\{u_k\}_k$ converging to $u_{j,0}$ in $C^1([-j,j],\mathbb{R}^n)$.

By equation (5.1), it follows that $\ddot{u}_{j,k} \to w$ uniformly in $[-j,j]$ and, then,

$$w(t) + V_u(t,u_{j,0}) = 0, \quad t \in [-j,j].$$

Since the function $\ddot{u}_{j,k}$ is continuous on $[-j,j]$ for each $k \geq k_j$, from [5, Fact 2.7], it follows that $\ddot{u}_{j,k}$ is a derivative of $\dot{u}_{j,k}$ in $[-j,j]$. Since $\ddot{u}_{j,k} \to w$ and $\dot{u}_{j,k} \to \dot{u}_{j,0}$ uniformly, it follows that $\{u_{j,k}\}_k$ converge to $u_{j,0}$ in the topology $C^2([-j,j],\mathbb{R}^n)$. In particular, we get $u_{j,0} \in C^2([-j,j],\mathbb{R}^n)$.

By a diagonal argument, there exists a subsequence $\{u_{j,j}\}_j$ of $\{u_{j,k}\}_k$ converging to $u_0$ in $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$, where $u_0(t) = \lim_{j\to\infty} u_{j,0}(t)$. Moreover, note that we have actually proved that $\{u_k\}_{k\in\widetilde{\mathbb{N}}}$ converges to $u_0$ in the topology of $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$. In particular, we get $u_0 \in C^2(\mathbb{R},\mathbb{R}^n)$.

It remains to show that $u_0(t) \to 0$ and $\dot u_0(t) \to 0$ as $t \to \pm\infty$. Let us prove the first assertion. Notice that

$$\int_{-\infty}^{+\infty} \big(|\dot u_0(t)|^2 + |u_0(t)|^2\big)\,dt = \lim_{i\to+\infty} \int_{-iT}^{iT} \big(|\dot u_0(t)|^2 + |u_0(t)|^2\big)\,dt$$
$$= \lim_{i\to+\infty} \lim_{k\to+\infty} \int_{-iT}^{iT} \big(|\dot u_k(t)|^2 + |u_k(t)|^2\big)\,dt.$$

For each $i \in \mathbb{N}$, there exists $k_i \in \widetilde{\mathbb{N}}$, such that, for every $k \geq k_i$,

$$\int_{-iT}^{iT} \big(|\dot u_k(t)|^2 + |u_k(t)|^2\big)\,dt \leq \|u_k\|_{E_k}^2 \leq M_1^2.$$

Since the constant $M_1$ is independent of $k$, if we take $i \to +\infty$, it follows that

$$\int_{-\infty}^{+\infty} \big(|\dot u_0(t)|^2 + |u_0(t)|^2\big)\,dt \leq M_1^2,$$

and then

$$\int_{|t|>r} \big(|\dot u_0(t)|^2 + |u_0(t)|^2\big)\,dt \to 0,$$

as $r \to +\infty$. By Lemma 2.6, we know that

$$|u(t)| \leq \sqrt{2}\bigg(\int_{t-1/2}^{t+1/2} \big(|\dot u(t)|^2 + |u(t)|^2\big)\,dt\bigg)^{1/2} \tag{5.4}$$

and it follows that

$$u_0(t) \to 0 \quad \text{as } t \to \pm\infty. \tag{5.5}$$

Let us prove the second assertion, i.e. $\dot u_0(t) \to 0$ whenever $t \to \pm\infty$. By (5.4), we have

$$\begin{aligned} |\dot u_0(t)|^2 &\leq 2\bigg(\int_{t-1/2}^{t+1/2} (|\ddot u_0(t)|^2 + |\dot u_0(t)|^2)\,dt\bigg) \\ &\leq 2\bigg(\int_{t-1/2}^{t+1/2} \big(|\dot u_0(t)|^2 + |u_0(t)|^2\big)\,dt\bigg) + 2\int_{t-1/2}^{t+1/2} |\ddot u_0(t)|^2\,dt. \end{aligned} \tag{5.6}$$

Since $u_0$ is a solution of (P), it follows that

$$\int_{t-1/2}^{t+1/2} |\ddot u_0(t)|^2\,dt = \int_{t-1/2}^{t+1/2} V_u(t,u_0)^2\,dt.$$

By $(\mathrm{H}_1)$, $(\mathrm{H}_2)$ and (5.5), it follows that

$$\int_{t-1/2}^{t+1/2} |\ddot u_0(t)|^2\,dt \to 0 \quad \text{as } t \to \pm\infty.$$

By (5.6), we obtain $\dot u_0(t) \to 0$ as $t \to \pm\infty$, and we have the desired result. $\square$

Proof of Theorem 1.2. For each $k \in \mathbb{R}$, let $u_k := u_{1,k}$, where $u_{1,k}$ is the solution obtained in Theorem 3.1. For every $k \in \mathbb{N}$, let $g_k\colon [0,1] \to E_k$ be the curve given by $g_k(s) = s\widetilde{w}_k$ where $\widetilde{w}_k$ is determined by (3.5). Then $g_k \in \Gamma_k$ and $I_{k,\lambda}(g_k(s)) = I_{1,\lambda}(g_1(s))$ for all $k \in \mathbb{N}$ and $s \in [0,1]$. Therefore, by (3.6),

$$c_k \leq \max_{s\in[0,1]} I_{1,\lambda}(g_1(s)) \equiv M_0, \tag{5.7}$$

where $M_0$ is independent of $k \in \mathbb{N}$. Notice that

$$\begin{aligned}
M_0 &\geq I_{k,\lambda}(u_{1,k}) - \frac{1}{\mu} I'_{k,\lambda}(u_{1,k})u_{1,k} \\
&\geq \overline{b}_1\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_{1,k}\|^2_{E_k} \\
&\quad + \frac{1}{\mu}\int_{-kT}^{kT} \big[\langle W_u(t,u_{1,k}(t)), u_{1,k}(t)\rangle - \mu W(t,u_{1,k}(t))\big]\,dt \\
&\geq \overline{b}_1\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_{1,k}\|^2_{E_k} + A,
\end{aligned}$$

then there exists a constant $C_0 > 0$ that does not depend on $\lambda$ and $k$, such that

$$\|u_{1,k}\|_{E_k} \leq C_0. \tag{5.8}$$

By Lemma 5.1, there exists a subsequence of $\{u_k\}$ which we still will denote by $\{u_k\}$, and $u_1\colon \mathbb{R} \to \mathbb{R}^n$ such that $u_k \to u_1$ in $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$, where $u_1$ is a homoclinic solution of (P).

Let us prove that $u_1 \not\equiv 0$. Firstly, we prove that there exists a constant $\sigma > 0$, independent of $k$, such that $\|u_k\|_{E_k} > \sigma$ for all $k \in \mathbb{N}$. Suppose that, on the contrary, there exists a subsequence of $\{u_k\}$ which we still will denote by $\{u_k\}$, such that $\|u_k\|_{E_k} \to 0$, as $k \to \infty$. From Lemma 2.11, we have $I_{k,\lambda}(u_k) > \alpha_\lambda > 0$ for all $k \in \mathbb{N}$. Thus, for $k$ large enough by Proposition 2.5, (2.2), $(\mathrm{H}_1)$, $(\mathrm{H}_2)$ and $(\mathrm{H}_5)$, we obtain

$$\begin{aligned}
0 < \alpha_\lambda &< I_{k,\lambda}(u_k) \\
&\leq \frac{1}{2}\,\overline{b}_2\|u_k\|^2_{E_k} + \int_{-kT}^{kT} F(|u_k(t)|)|u_k(t)|\,dt + \lambda\int_{-kT}^{kT} |b(t)||u_k(t)|^{q+1}\,dt \\
&\leq \frac{1}{2}\,\overline{b}_2\|u_k\|^2_{E_k} + \int_{-kT}^{kT} |u_k(t)|^2\,dt + \lambda\int_{-kT}^{kT} |b(t)||u_k(t)|^{q+1}\,dt \\
&\leq \left(\frac{1}{2}\,\overline{b}_2 + 1\right)\|u_k\|^2_{E_k} + \lambda\|u_k\|^{\varepsilon}_{L^\infty_{2kT}}\int_{-kT}^{kT} |b(t)||u_k(t)|^{q+1-\varepsilon}\,dt,
\end{aligned}$$

for $r > 0$ in hypothesis $(\mathrm{H}_5)$. By the Höder inequality, we get

$$0 < \alpha_\lambda < \left(\frac{1}{2}\,\overline{b}_2 + 1\right)\|u_k\|^2_{E_k} + \lambda\|u_k\|^{\varepsilon}_{L^\infty_{2kT}}\|b(t)\|_{L^{2/(1-q+\varepsilon)}_{2kT}}\|u_k(t)|\|^{q+1-\varepsilon}_{L^2_{2kT}}.$$

By Proposition 2.5 and taking $\varepsilon = q - 1 + 2/r$, we obtain

$$0 < \alpha_\lambda < \left(\frac{1}{2}\,\bar{b}_2 + 1\right)\|u_k\|_{E_k}^2 + \lambda\|b\|_{L^r(\mathbb{R})}\|u_k\|_{E_k}^{q+1}.$$

Thus, there exists $\sigma > 0$ such that $\|u_k\|_{E_k} > \sigma$ for all $k \in \mathbb{N}$.

Now, we will verify that there exists $c > 0$ such that $\|u_k\|_{L^\infty_{2kT}} > c$. Assume by contradiction that $\|u_k\|_{L^\infty_{2kT}} \to 0$ as $k \to \infty$. Since $I'_{k,\lambda}(u_k)u_k = 0$, (2.1) and (2.5), we have

$$\begin{aligned} (5.9)\qquad &\int_{-kT}^{kT} \big[\langle W_u(t,u_k(t)), u_k(t)\rangle\big]\,dt \\ &= \int_{-kT}^{kT} \big[\langle \dot{u}_k(t), \dot{u}_k(t)\rangle + \langle K_u(t,u_k(t)), u_k(t)\rangle\big]\,dt \geq \min\{1,b_1\}\|u_k\|_{E_k}^2. \end{aligned}$$

By hypothesis ($\mathrm{H}_5$), for any $\delta > 0$, there exist $k_0 > 0$ such that $F(|u_k|) \leq \delta|u_k|$ for all $k > k_0$. Then we have

$$\begin{aligned} (5.10)\qquad \int_{-kT}^{kT} \langle W_u(t,u_k(t)), u_k(t)\rangle\,dt &\leq \int_{-kT}^{kT} |W_u(t,u_k(t))||u_k(t)|\,dt \\ \leq \int_{-kT}^{kT} F(|u_k(t)|)|u_k(t)| &+ \lambda\|u_k\|_{L^\infty_{2kT}}^{\varepsilon}\int_{-kT}^{kT} b(t)|u_k(t)|^{q+1-\varepsilon}\,dt \\ &\leq \delta\|u_k\|_{E_k}^2\,dt + \lambda\|u_k\|_{L^\infty_{2kT}}^{\varepsilon}\|b\|_{L^r(\mathbb{R})}\|u_k\|_{E_k}^{q+1-\varepsilon}. \end{aligned}$$

By (5.9) and (5.10), we obtain

$$\delta\|u_k\|_{E_k}^2 + \lambda C_2\|u_k\|_{L^\infty_{2kT}}^{\varepsilon}\|u_k\|_{E_k}^{q+1-\varepsilon} \geq \min\{1,b_1\}\|u_k\|_{E_k}^2.$$

Then, we have

$$\lambda\|u_k\|_{L^\infty_{2kT}}^{\varepsilon} \geq \frac{\big(\min\{1,b_1\} - \delta\big)\sigma^{1-q+\varepsilon}}{C_2} > 0,$$

where we take $\delta > 0$, such that $\min\{1,b_1\} - \delta > 0$, and it is a contradiction. Thus, there exists $c > 0$ such that $\|u_k\|_{L^\infty_{2kT}} > c$ for all $k \in \mathbb{N}$ and $\lambda \in (0,\Lambda)$, where $c$ is independent of $k$. By the periodicity of $u_k$, its maximum is achieved in $[-T,T]$. This shows that $u_1 \not\equiv 0$.

Now, let $u_k := u_{2,k}$, where $u_{2,k}$ is the solution obtained in Theorem 4.1, for each $k \in \mathbb{R}$. By (4.3), we know that $\|u_k\|_{E_k} \leq \lambda^\gamma$. By Lemma 5.1, there exists a subsequence of $\{u_k\}$ which we still will denote by $\{u_k\}$, and $u_2\colon \mathbb{R} \to \mathbb{R}^n$ such that $u_k \to u_2$ in $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$, where $u_2$ is a homoclinic solution of (P).

Let us prove that $u_2 \not\equiv 0$. As above, suppose that there exists a subsequence of $\{u_k\}$ which we still will denote by $\{u_k\}$, such that $\|u_k\|_{E_k} \to 0$, as $k \to \infty$. By (4.3), there exists $\eta > 0$, independent of $k$, such that $I_{k,\lambda}(u_k) \leq -\eta < 0$. Therefore, by (2.1), we have

$$\frac{\bar{b}_1}{2}\|u_k\|_{E_k}^2 - \int_{-kT}^{kT} |W(t,u_k)|\,dt \leq -\eta.$$

Notice that, by $(\mathrm{H}_5)$,

$$-\int_{-kT}^{kT} |W(t,u_k)|\,dt \geq -\|u_k\|_{E_k}^2 - \lambda \|b\|_{L^r(\mathbb{R})} \|u_k\|_{E_k}^{q+1},$$

where $\widehat{c}_0$ is a positive constant. Hence,

$$\bigg(\frac{\overline{b}_1}{2} - 1\bigg)\|u_k\|_{E_k}^2 - \lambda \|b\|_{L^r(\mathbb{R})} \|u_k\|_{E_k}^{q+1} \leq -\eta.$$

Thus, we arrive at

$$\bigg(\frac{\overline{b}_1}{2} - 1\bigg)\|u_k\|_{E_k}^2 - \lambda \widetilde{c} \|u_k\|_{E_k}^{q+1} + \eta \leq 0,$$

where $\widetilde{c} > 0$ is a constant that do not depend on $k$. Using the same arguments as above, we conclude that $u_2 \not\equiv 0$.

By Lemma 5.1 we notice that $u_1(\pm\infty) = \dot{u}_1(\pm\infty) = 0$ and $u_2(\pm\infty) = \dot{u}_2(\pm\infty) = 0$, which is the desired result. $\square$

## 6. Applications and concluding remarks

In this section we provide examples of where we may apply our main result, which is Theorem 1.2.

Example 6.1. Suppose that $W(t,u) = \overline{W}(t,u) + \lambda\langle f(t), u\rangle$, where $|f| \in L^r(\mathbb{R})$ and $\overline{W}$ satisfies the classical (AR) condition, then $\overline{W}$ is superquadratic, i.e., there exists constants $a_1 > 0$ and $a_2 > 0$, such that

$$\overline{W}(t,u) \geq a_1 |u|^\mu \quad \text{if } |u| \geq 1, \tag{6.1}$$

$$\overline{W}(t,u) \leq a_2 |u|^\mu \quad \text{if } |u| \leq 1, \tag{6.2}$$

see [5, Fact 2.1]. Then, as a relevant application, we have that, assuming the hypotheses of the result proved in the paper of Izydorek and Janczewska, i.e. assuming the conditions $(\mathrm{H}_1)$–$(\mathrm{H}_4)$ of [5], and (AR) are satisfied, by using similar ideas to those in Theorem 1.2, is possible to prove that there exists $\Lambda > 0$ such that for each $\lambda \in (0, \Lambda)$, problem (P) possesses at least two nontrivial homoclinic orbits. In other words, we have established another homoclinic orbit for the problem studied in [5], which correspond to the case $q = 0$.

Example 6.2. Consider the hamiltonian system

$$\begin{gathered} \ddot{u} - K_u(t,u) + a(t,u)|u|^{p-1}u + \lambda b(t)|u|^{q-1}u = 0, \\ u(\pm\infty) = \dot{u}(\pm\infty) = 0, \end{gathered} \tag{6.3}$$

where $p > 1$, $a\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a $C^1$-map and bounded function. In addition, suppose that $A(t,s) \leq a(t,s)s$ for all $|s| > s_0$ where, $A(t,s) = \int_0^s a(t,\tau)\,d\tau$, the function $K$ verifies conditions $(\mathrm{H}_1)$–$(\mathrm{H}_3)$ and on the function $f$ is such that $|f| \in L^r(\mathbb{R})$. Then, since

$$W(t,u) = \frac{A(t,|u|^{p+1})}{p+1} + \lambda b(t)\,\frac{|u|^{q+1}}{q+1},$$

verifies the hypotheses $(\mathrm{H}_4)$, $(\mathrm{H}_5)$ and $(\mathrm{AR})_l$ with $\mu = p+1$, we may apply Theorem 1.2 to obtain two nontrivial homoclinic orbits. Note that if the function $a$ changes sign the classical Ambrosetti–Rabinowitz condition (AR) does not satisfy.

Example 6.3. Consider $W(t,u) = a(t)G(|u|) + \lambda b(t)|u|^{q+1}$, where $a$, $b$ are continuous functions verifying $a \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $b \in L^r(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, where $G \in C^1(0,+\infty)$ verifies:

$$G(s) = \begin{cases} s^{p+1} & \text{for } 0 \le s < 1/4, \\ \log(1+s) & \text{for } 1/2 < s < 3/4, \\ s^{p+1} & \text{for } 1 < s, \end{cases}$$

with $p > 1$. In addition, suppose that the function $a$ is nonnegative and there exist $\alpha, \beta > 0$, such that $b(t) > \delta_0 > 0$ in $(\alpha,\beta)$. One can easily verify that $W$ satisfies hypothesis $(\mathrm{H}_1)$–$(\mathrm{H}_5)$ and $(\mathrm{AR})_l$ but it does not verify the classical (AR).

**Acknowledgements.** Part of this article was done while the authors Luiz Fernando Faria and Pedro Ubilla visited the University of British Columbia and Universidade Federal de Campina Grande respectively, whose hospitality they gratefully acknowledges. Finally, the authors want to thank the anonymous referee whose report has greatly helped them to improve this paper.

## References

[1] A. Ambrosetti and V. Coti Zelati, *Multiple homoclinic orbits for a class of conservative systems*, Rend. Sem. Mat. Univ. Padova **89** (1993), 177–194.

[2] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[3] V. Coti Zelati and P.H. Rabinowitz, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potenials*, J. Amer. Math. Soc. **4** (1991), 693–727.

[4] P.L. Felmer, *Variational methods in Hamiltonian systems*, Dynamical Systems (Temuco, 1991/1992), Travaux en Cours, vol. 52, Hermann, Paris, 1996, 151–178.

[5] M. Izydorek and J. Janczewska, *Homoclinic solutions for a class of the second order Hamiltonian systems*, J. Differential Equations **219** (2005), 375–389.

[6] M. Izydorek and J. Janczewska, *Homoclinic solutions for nonautonomous second order Hamiltonian systems with a coercive potential*, J. Math. Anal. Appl. **335**, 1119–1127.

[7] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Gauthier–Villars, Pairs, 1897–1899.

[8] P.H. Rabinowitz, *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburgh Sect. A **114** (1990), 33–38.

[9] L.P. Shil'nikov, *Homoclinic trajectories*: *From Poincaré to the present*, Mathematical Events of the Twentieth Century, Springer, Berlin, 2006, 347–370.

[10] Ye, Yiwei and Tang, *Chun-Lei Multiple homoclinic solutions for second-order perturbed Hamiltonian systems*, Stud. Appl. Math. **132** (2014), no. 2, 112–137.

*Manuscript received April 19, 2018*

*accepted September 11, 2018*

Patricio Cerda and Pedro Ubilla
Departamento de Matematica y C. C.
Universidad de Santiago de Chile
Casilla 307, Correo 2
Santiago, CHILE

*E-mail address*: patricio.cerda@usach.cl, pedro.ubilla@usach.cl

Luiz F.O. Faria and Eduard Toon
Departamento de Matemática
Universidade Federal de Juiz de Fora
Juiz de Fora, CEP 36036-330
Minas Gerais, BRAZIL

*E-mail address*: luiz.faria@ufjf.edu.br, eduard.toon@ufjf.edu.br