Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 1, 2019, 153–175
DOI: 10.12775/TMNA.2019.032

SOLUTIONS FOR QUASILINEAR ELLIPTIC SYSTEMS WITH VANISHING POTENTIALS

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Abstract. In this paper, we study the following strongly coupled quasilinear elliptic system:

$$\begin{cases} -\Delta_p u + \lambda a(x)|u|^{p-2}u = \dfrac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x \in \mathbb{R}^N, \\ -\Delta_p v + \lambda b(x)|v|^{p-2}v = \dfrac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \mathbb{R}^N, \\ u, v \in D^{1,p}(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, $\lambda > 0$ is a parameter, $p < \alpha + \beta < p^* := Np/(N-p)$. Under some suitable conditions which are given in section 1, we use variational methods to obtain both the existence and multiplicity of solutions for the system on an appropriated space when the parameter λ is sufficiently large. Moreover, we study the asymptotic behavior of these solutions when $\lambda \to \infty$.

2010 *Mathematics Subject Classification.* Primary: 35H30, 35J20; Secondary: 35J45.
Key words and phrases. Positive solutions; potential well; quasilinear elliptic systems.
This work was partially supported by NSFC (No. 11301204, No. 11371159), self-determined research funds of CCNU from colleges' basic research and operation of MOE(CCNU16A05011).

1. Introduction and main results

In this paper, we study the existence, multiplicity and asymptotic behavior of solutions for the strongly coupled quasilinear elliptic system

$$
\begin{cases} -\Delta_p u + \lambda a(x)|u|^{p-2}u = \dfrac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x \in \mathbb{R}^N, \\ -\Delta_p v + \lambda b(x)|v|^{p-2}v = \dfrac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \mathbb{R}^N, \\ u, v \in D^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}
$$

where $N \geq 3, \lambda > 0$ is a parameter, $p < \alpha + \beta < p^* := pN/(N-p)$. The assumptions we imposed on $a(x)$ and $b(x)$ are as follows:

- (H_1) $a, b \in C^0(\mathbb{R}^N, [0,\infty)), \Omega_a := \operatorname{int} a^{-1}(0)$ and $\Omega_b := \operatorname{int} b^{-1}(0)$ have smooth boundaries, $\overline{\Omega}_a := a^{-1}(0), \overline{\Omega}_b := b^{-1}(0)$ and $\overline{\Omega}_a \cap \overline{\Omega}_b$ is a nonempty set;
- (H_2) there exists $M_0 > 0$ such that the set $F := \{x \in \mathbb{R}^N : a(x)b(x) \leq M_0\}$ has finite Lebesgue measure.

Since we do not assume any positive lower bounds for the potentials a and b, we can not expect to find solutions for (1.1) in the Sobolev space $W^{1,p}(\mathbb{R}^N)$. However, the strong coupling of the system and the assumption (H_2) suggest that we can use variational methods to investigate (1.1) by considering the corresponding functional defined in a proper product space. Noting that the sets Ω_a and Ω_b may be unbounded, $\Omega_a \cap \Omega_b$ is a nonempty set is very crucial for our results.

As we will see later, that the main results in this paper show that the quasilinear elliptic system

$$
\begin{cases} -\Delta_p u = \dfrac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega_a, \\ -\Delta_p v = \dfrac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega_b, \\ u \in W_0^{1,p}(\Omega_a), \quad v \in W_0^{1,p}(\Omega_b), \end{cases} \tag{1.2}
$$

may be seen as a limit problem for (1.1) when $\lambda \to \infty$ goes to infinity. We would like to emphasize that although Ω_a and Ω_b may be distinct open sets, (1.3) is variational. Moreover, (H_2) implies that Ω_a and Ω_b have finite Lebesgue measure. Therefore, we have the Sobolev compact imbedding

$$
W^{1,p}(\Omega_a) \times W^{1,p}(\Omega_b) \hookrightarrow L^{r_1}(\Omega_a) \times L^{r_2}(\Omega_b), \quad p-1 \leq r_1, r_2 < p^*.
$$

We say that the following system

$$
\begin{cases} -\Delta u = F_u(x,u,v), & x \in \Omega, \\ -\Delta v = F_v(x,u,v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, is a gradient system if $F\colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 class. The theory of gradient systems is sort of similar to that of scalar equations

$$-\Delta u = f(x,u) \quad \text{in } \Omega.$$

The system (1.3) is variational and its solutions correspond to the critical points of the following energy functional

$$\Phi(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega F(x,u,v), \tag{1.4}$$

for all $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$.

In [2], Alves, de Morais Filho and Souto studied the existence and non-existence of solutions for

$$\begin{cases} -\Delta u = au + bv + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, & x \in \Omega, \\ -\Delta v = bu + cv + \dfrac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

depending on the parameters $a,b,c \in \mathbb{R}$, $\alpha,\beta > 1$. They proved that when $\alpha+\beta = 2^*$, (1.5) had nontrivial solution. Moreover they proved

$$S_{\alpha,\beta}(\Omega) = \left(\left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\alpha}{\beta}\right)^{-\alpha/(\alpha+\beta)}\right)S,$$

where

$$S_{\alpha,\beta}(\Omega) = \inf_{u,v\in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx}{\left(\displaystyle\int_\Omega |u|^\alpha |v|^\beta \, dx\right)^{2/(\alpha+\beta)}},$$

and

$$S = \inf_{u\in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx}{\left(\displaystyle\int_\Omega |u|^{2^*} \, dx\right)^{2/2^*}}.$$

This result combined effectively $-\Delta u = u^{2^*-1}$ for $x \in \mathbb{R}^N$ with

$$\begin{cases} -\Delta u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}\, u|v|^\beta, & x \in \mathbb{R}^N, \\ -\Delta v = \dfrac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}\, v, & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $\alpha+\beta = 2^*$. Guo and Liu in [21] proved the uniqueness of positive solutions for (1.6).

In [22], Han studied the existence of solutions for the system

$$
\begin{cases} -\Delta u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + \lambda u, & x \in \Omega, \\ -\Delta v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \mu v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.7}
$$

on a non-contractible domain. He pointed out that when λ, μ were sufficiently small, (1.7) had at least one solution.

Let be a bounded domain in R^N with $N \geq 3$ satisfying

(i) $B_{1/\rho}(0) \setminus \overline{B}_\rho(0) \subset \Omega$,
(ii) $B_\rho(0) \not\subset \overline{\Omega}$,

and ρ is sufficiently small. In [24], He and Yang investigated the existence of positive solutions for the following system of elliptic equations

$$
\begin{cases} -\Delta u = \dfrac{p}{p+q}|u|^{p-2}u|v|^{q}, & x \in \Omega, \\ -\Delta v = \dfrac{q}{p+q}|u|^{p}|v|^{q-2}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.8}
$$

as well as

$$
\begin{cases} -\Delta u = \dfrac{p}{p+q}|u|^{p-2}u|v|^{q} + \varepsilon f(x), & x \in \Omega, \\ -\Delta v = \dfrac{q}{p+q}|u|^{p}|v|^{q-2}v + \varepsilon g(x), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.9}
$$

where $p > 1, q > 1$ satisfying $p+q = 2^*$, 2^* denotes the critical Sobolev exponent. $f, g \in C^1(\Omega)$, $f \not\equiv 0$, $g \not\equiv 0$. In [23], Han proved that when $\varepsilon > 0$ was small enough, (1.9) had two solutions.

When $p = 2$ in (1.1), we observe that there exists an extensive bibliography in the study of elliptic systems on bounded domains (see [12], [13], [15], [16], [25], [27], [31] and references therein). In the case of gradient systems in the whole $\mathbb{R}^N$, in [11] Costa proved the existence of a nonzero solution for

$$
\begin{cases} -\Delta u + a(x)u = F_u(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta v + b(x)v = F_v(x,u,v), & x \in \mathbb{R}^N, \end{cases}
$$

under the coercivity of the potentials $a(x)$ and $b(x)$, and a nonquadratic condition on the nonlinearity. A related result for noncoercive potentials is proved in [17] (see also [29] for the superlinear case). We should also mention [4], [28] where some existence results of positive solutions for weakly coupled system are established. We would like to emphasize that, instead of the aforementioned works, the coupling in our system (1.1) when $p = 2$ allows us to consider potentials which are not bounded from below by positive constants. We may have one of the potentials going to zero as $|x| \to \infty$ provided the other one goes to infinity at an appropriated rate.

The theory of gradient systems has also been considered in the framework p-Lapacians

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad p > 1.$$

For quasilinear elliptic systems, L. Baccardo and D.G. de Figueiredo in [9] studied the following system

$$\begin{cases} -\Delta_p u = F_u(x,u,v), & x \in \Omega, \\ -\Delta_q v = F_v(x,u,v), & x \in \Omega, \end{cases}$$

where p and q are real numbers larger than 1, Ω is some bounded domain in $\mathbb{R}^N$, u and v are real-valued functions defined in Ω and belonging to appropriate spaces of functions and F (sometimes referred as a potential) is a real-valued differentiable function with domain $\Omega \times \mathbb{R} \times \mathbb{R}$. They obtained nontrivial solutions in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ under the coercivity of F and some other technical conditions.

When $p = 2$, for the scalar case, in [6]–[8] it is considered the potential $c_\lambda(x) = \lambda c(x) + 1$ with c being such that the set $\{x \in \mathbb{R}^N : c(x) \le M_0\}$ has finite Lebesgue measure, for some $M_0 > 0$. In [8], Bartsch and Wang considered the Lusternik–Schnirelmann category of some set related with the limit problem.

Recently in [19], Furtado, Silva and Xavier studied the existence and multiplicity of solutions for the system when the parameter λ is sufficiently large,

$$\begin{cases} -\Delta u + \lambda a(x) u = \dfrac{p}{p+q}|u|^{p-2}u|v|^q, & x \in \mathbb{R}^N, \\ -\Delta v + \lambda b(x) v = \dfrac{q}{p+q}|u|^p|v|^{q-2}v, & x \in \mathbb{R}^N, \\ u, v \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.10}$$

where $N \ge 3, \lambda > 0$ is a parameter, $2 < p + q < 2^* := 2N/(N-2)$. $a(x)$ and $b(x)$ satisfy (H_1) and (H_2). They also studied the asymptotic behavior of these solutions when $\lambda \to \infty$. In this paper, we are mainly motivated by [19]. We want to extend the results of (1.10) to (1.1).

In order to state our main results later, we introduce the spaces:

$$X_a := \left\{ u \in D^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)|u|^p \, dx < \infty \right\}$$

and

$$X_b := \left\{ u \in D^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} b(x)|u|^p \, dx < \infty \right\}.$$

For any given $\lambda > 0$, we consider the Banach space $X := X_a \times X_b$ endowed with the norm

$$\|(u,v)\|_\lambda^p := \int_{\mathbb{R}^N} \big(|\nabla u|^p + |\nabla v|^p + \lambda a(x)|u|^p + \lambda b(x)|v|^p\big)\, dx.$$

Observe that $\|\cdot\|_0$ is the usual norm of the space $D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$.

The corresponding energy functional $I_\lambda \colon X \to \mathbb{R}$ for (1.1) is given by

$$I_\lambda(u,v) := \frac{1}{p}\, \|(u,v)\|_\lambda^p - \frac{1}{\alpha+\beta} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\, dx, \quad (u,v) \in X.$$

By (H_1) and (H_2), the functional I_λ is well defined and of class C^1.

Our main results are as follows:

Theorem 1.1. *Let* (H_1)–(H_2) *hold. Then there exists $\Lambda > 0$ such that, for all $\lambda \geq \Lambda$, the system* (1.1) *possesses a positive ground state solution z_λ. Moreover, if $(\lambda_n) \subset \mathbb{R}$ is such that $\lambda_n \to \infty$ and (z_{λ_n}) is a sequence of positive ground state solutions of* (1.1) *with $\lambda = \lambda_n$, then (z_{λ_n}) converges in $D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$ along a subsequence to a positive ground state solution of* (1.3).

A solution $z = (u,v)$ of (1.1) is called a ground state solution if it a solution with the least energy of the functional I_λ. Applying the symmetry of our problem, we obtain multiple solutions for large values of λ.

Theorem 1.2. *Let* (H_1)–(H_2) *hold. Then, for any given $m \in \mathbb{N}$, there exists $\Lambda_m > 0$ such that, for each $\lambda \geq \Lambda_m$, the system* (1.1) *possesses at least m pairs of nonzero solutions.*

Furthermore, we obtain the following concentration result.

Theorem 1.3. *Let $(\lambda_n) \subset \mathbb{R}$ be such that $\lambda_n \to \infty$ and (z_{λ_n}) be a sequence of solutions of* (1.1) *with $\lambda = \lambda_n$ such that $\liminf_{n\to\infty} I_{\lambda_n}(z_{\lambda_n}) < \infty$. Then (z_{λ_n}) converges in $D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$ along a subsequence to a solution of* (1.3).

The results presented in this article are motivated by that obtained in [19]. Theorems 1.1 to 1.3 extend the results in [19]. To the best knowledge of us, the results we obtain are new. However, in order to obtain our results, we have to overcome some difficulties. Firstly, since p-Laplacian is quasilinear, we have to use some different techniques to prove that the sequence of solutions of (1.1) converges in $D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$ along a subsequence to a solution of (1.3). Lemma 3.1 is very crucial in the whole proof. Secondly, preliminaries in Section 2 are very technical which are more complicated than [19]. We apply the symmetry of the nonlinearity to obtain the existence of multiple solutions as in [19]. We would like to point out that the coupling in our systems (1.3) allows us to consider potentials which are not bounded from below by positive constants. We may have one of the potentials going to zero as $|x| \to \infty$ provided the other one goes to infinity at an appropriated rate.

Before ending this section, we give some notations. B_R denotes the open ball in $\mathbb{R}^N$ of radius R and center at the origin. For any given set K, we set $K^C := \mathbb{R}^N \setminus K$ and we use $|K|$ for the Lebesgue of K whenever this set is measurable. $C_0^\infty(K)$ denotes the set of all functions $u\colon K \to \mathbb{R}$ of class C^∞ with compact support contained in the open set $K \subset \mathbb{R}^N$. If $u \in L^s(K)$, $s \geq 1$, we set $u_+ := \max\{u,0\}, u_- := \max\{-u,0\}$ and write $\|u\|_{L^s(K)}$ for L^s-norm of u. We write $\int_K u$ instead of $\int_K u\,dx$. We also omit the set K whenever $K = \mathbb{R}^N$. Finally, we use the symbols $c_i (i \in \mathbb{N}), C$ and $\widetilde{C}$ to represent positive constants. $u_n \to u$ in X denotes that u_n converges strongly to u in X and $u_n \rightharpoonup u$ in X denotes that u_n converges weakly to u in X.

The paper is organized as follows. In Section 2 we give some preliminary results which will be useful in our paper. We also study the behavior of the Palais–Smale sequences when λ goes to infinity. In Section 3 we prove Theorem 1.1. We give the proofs of Theorem 1.2 and 1.3 in Section 4.

2. Some preliminaries

In this section we give some preliminaries for the proof of Theorem 1.1.

Lemma 2.1. *For any given measurable set $K \subset \mathbb{R}^N$ there exists a constant $C > 0$ such that*

$$\int_K |u|^\alpha |v|^\beta \leq C\|(u,v)\|_0^{\alpha+\beta-p+p^*t/r} \bigg(\int_K |uv|^{p/2} \bigg)^\gamma, \quad \textit{for all } (u,v) \in X,$$

where $r = p^*/(p^* - (\alpha+\beta) + p) > 1$ and $t \in (0,1)$ satisfies $r = p^*t/p + (1-t)$ and $\gamma = (1-t)/r$.

Proof. Since $r = p^*/(p^* - (\alpha+\beta)+p)$, we have

$$\frac{\alpha - p/2}{p^*} + \frac{\beta - p/2}{p^*} + \frac{1}{r} = 1.$$

By Hölder inequality and the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, we have

$$\begin{aligned}
\text{(2.1)} \quad \int_K |u|^\alpha |v|^\beta &= \int_K |u|^{\alpha-p/2} |v|^{\beta-p/2} |uv|^{p/2} \\
&\leq \bigg(\int_K |u|^{p^*} \bigg)^{(\alpha-p/2)/p^*} \bigg(\int_K |v|^{p^*} \bigg)^{(\beta-p/2)/p^*} \bigg(\int_K |uv|^{rp/2} \bigg)^{1/r} \\
&\leq C_1 \|(u,v)\|_0^{\alpha+\beta-p} \bigg(\int_K |uv|^{rp/2} \bigg)^{1/r}.
\end{aligned}$$

Noting that $1 < r < p^*/p$, there exists $t \in (0,1)$ such that

$$r = \frac{p^*}{p} t + (1-t).$$

By Hölder inequality and the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ again, we have

$$
\begin{aligned}
\int_K |uv|^{rp/2} &= \int_K |uv|^{p^* t/2} |uv|^{(1-t)p/2} \\
&\le \bigg(\int_K |uv|^{p^*/2}\bigg)^t \bigg(\int_K |uv|^{p/2}\bigg)^{1-t} \\
&\le \bigg(\int_K \frac{|u|^{p^*} + |v|^{p^*}}{2}\bigg)^t \bigg(\int_K |uv|^{p/2}\bigg)^{1-t} \\
&\le C_2 \|(u,v)\|_0^{p^* t} \bigg(\int_K |uv|^{p/2}\bigg)^{1-t}.
\end{aligned} \tag{2.2}
$$

Combining (2.1) and (2.2), we can complete the proof of the lemma. □

Lemma 2.2. *There exists a constant $\widetilde{C} > 0$ such that*

$$\int |u|^\alpha |v|^\beta \le \widetilde{C} \|(u,v)\|_1^{\alpha+\beta} \quad \textit{for all } (u,v) \in X.$$

Proof. It follows from Lemma 2.1 that

$$\int |u|^\alpha |v|^\beta \le C \|(u,v)\|_0^{\alpha+\beta-p+p^* t/r} \bigg(\int |uv|^{p/2}\bigg)^\gamma. \tag{2.3}$$

We recall that the set F given in (H_2) has finite measure and $a(x)b(x) > M_0$ in F^C. By Hölder's inequality, we have

$$
\begin{aligned}
\int |uv|^{p/2} &= \int_F |uv|^{p/2} + \int_{F^C} |uv|^{p/2} \\
&\le \bigg(\int_F |u|^{p^*}\bigg)^{p/2p^*} \bigg(\int_F |v|^{p^*}\bigg)^{p/2p^*} |F|^{1-p/p^*} \\
&\quad + \frac{1}{\sqrt{M_0}} \int_{F^C} \sqrt{a(x)} |u|^{p/2} \sqrt{b(x)} |v|^{p/2} \\
&\le C_3 \|(u,v)\|_0^p + \frac{1}{\sqrt{M_0}} \bigg(\int_{F^C} a(x)|u|^p\bigg)^{1/2} \bigg(\int_{F^C} b(x)|v|^p\bigg)^{1/2} \\
&\le C_4 \|(u,v)\|_1^p.
\end{aligned} \tag{2.4}
$$

By (2.3) and (2.4), we have

$$
\begin{aligned}
\int |u|^\alpha |v|^\beta &\le C \|(u,v)\|_0^{\alpha+\beta-p+p^* t/r} \big(C_4 \|(u,v)\|_1^p\big)^{(1-t)/r} \\
&\le C \|(u,v)\|_1^{\alpha+\beta-p+p(p^* t/p+(1-t))/r} = C \|(u,v)\|_1^{\alpha+\beta},
\end{aligned}
$$

where we have used that $r = p^* t/p + (1-t)$. □

Since we are interested in positive solutions of (1.1), we will work with a functional slightly different from that defined in the introduction. Precisely, we consider $\Phi_\lambda \colon X \to \mathbb{R}$ defined by

$$\Phi_\lambda(u,v) := \frac{1}{p} \|(u,v)\|_\lambda^p - \frac{1}{\alpha+\beta} \int (u_+)^\alpha (v_+)^\beta, \quad (u,v) \in X.$$

It follows from Lemma 2.2 that Φ_λ is well defined. Further, applying Lemma 2.2 and (H_2), we can verify that $\Phi_\lambda \in C^1(X,\mathbb{R})$ for any $\lambda > 0$.

Let E be a Banach space and $I \in C^1(E,\mathbb{R})$. First we recall that $(z_n) \subset E$ is a Palais–Smale sequence at level c ($(\mathrm{PS})_c$ sequence for short) if $I(z_n) \to c$ and $I'(z_n) \to 0$. I satisfies $(\mathrm{PS})_c$ if any $(\mathrm{PS})_c$ sequence possesses a convergent subsequence.

LEMMA 2.3. *Let $\lambda \geq 1$ and $(z_n) \subset X$ be a $(PS)_c$ sequence for Φ_λ.*

(a) *(z_n) is bounded in X;*

(b) $\displaystyle\lim_{n\to\infty} \|z_n\|_\lambda^p = \lim_{n\to\infty} \int (u_n)_+^\alpha (v_n)_+^\beta = c\bigg(\frac{1}{p} - \frac{1}{\alpha+\beta}\bigg)^{-1}$;

(c) *if $c \neq 0$, then $c \geq \gamma_0 > 0$ for some γ_0 independent of λ.*

PROOF. Since $(z_n) \subset X$ is a $(\mathrm{PS})_c$ sequence for Φ_λ, we have

$$\bigg(\frac{1}{p} - \frac{1}{\alpha+\beta}\bigg)\|z_n\|_\lambda^p = \Phi_\lambda(z_n) - \frac{1}{\alpha+\beta}\,\Phi'_\lambda(z_n)z_n = c + o(1)\|z_n\|_\lambda, \tag{2.5}$$

as $n \to \infty$ and hence (a) holds.

Meanwhile, as $n \to \infty$, we get

$$\begin{aligned}\bigg(\frac{1}{p} - \frac{1}{\alpha+\beta}\bigg)\|z_n\|_\lambda^p &= \Phi_\lambda(z_n) - \frac{1}{\alpha+\beta}\,\Phi'_\lambda(z_n)z_n = c + o(1)\|z_n\|_\lambda \\ &= \Phi_\lambda(z_n) - \frac{1}{p}\,\Phi'_\lambda(z_n)z_n = \bigg(\frac{1}{p} - \frac{1}{\alpha+\beta}\bigg)\int (u_n)_+^\alpha (v_n)_+^\beta,\end{aligned}$$

which implies that (b) holds.

By Lemma 2.2, for any $\lambda \geq 1$ we have

$$\Phi'_\lambda(z)z = \|z\|_\lambda^p - \int (u_+)^\alpha (v_+)^\beta \geq \|z\|_\lambda^p - \widetilde{C}\|z\|_\lambda^{\alpha+\beta} \geq \frac{1}{p}\|z\|_\lambda^p,$$

where $\|z\|_\lambda \leq \big((p-1)/\big(p\widetilde{C}\big)\big)^{1/(\alpha+\beta-p)} := \sqrt[p]{\delta}$.

Suppose that $c < \delta(1/p - 1/(\alpha+\beta))$. By (b), there exists $n_0 \in \mathbb{N}$ such that $\|z_n\|_\lambda < \sqrt[p]{\delta}$ for any $n \geq n_0$. Therefore,

$$\frac{1}{p}\|z_n\|_\lambda^p \leq \Phi'_\lambda(z_n)z_n \leq o(1)\|z_n\|_\lambda, \quad \text{as }, n \to \infty$$

and we infer that $z_n \to 0$ in X. Hence $\Phi_\lambda(z_n) \to 0 = c$ it follows that (c) holds for $\gamma_0 =: \delta(1/p - 1/(\alpha+\beta))$. □

LEMMA 2.4. *Given $\varepsilon > 0$ and $C_0 > 0$, there exist $\Lambda_\varepsilon = \Lambda(\varepsilon, C_0) > 0$ and $R_\varepsilon = R(\varepsilon, C_0)$ such that if $((u_n, v_n)) \subset X$ is a $(\mathrm{PS})_c$-sequence for Φ_λ with $c \leq C_0$ and $\lambda \geq \Lambda_\varepsilon$, then*

$$\limsup_{n\to\infty} \int_{B_{R_\varepsilon}^C} (u_n)_+^\alpha (v_n)_+^\beta \leq \varepsilon.$$

PROOF. Observing that $\|\cdot\|_0 \leq \|\cdot\|_\lambda$, by Lemma 2.1 and Lemma 2.3 (a) we have

$$\int_{B_R^C} (u_n)_+^\alpha (v_n)_+^\beta \leq C\bigg(\int_{B_R^C} |u_n v_n|^{p/2}\bigg)^\gamma, \quad \text{for any } R > 0. \tag{2.6}$$

Then by Yong and Hölder's inequalities, the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ and Lemma 2.3 (a), we get

$$\begin{aligned} \int_{B_R^C\cap F} |u_n v_n|^{p/2} &\leq \frac{1}{2}\int_{B_R^C\cap F} (|u_n|^p + |v_n|^p) \\ &\leq \frac{1}{2}|B_R^C \cap F|^{p/N}(\|u_n\|_{L^{p^*}}^p + \|v_n\|_{L^{p^*}}^p) \leq C|B_R^C \cap F|^{p/N}. \end{aligned} \tag{2.7}$$

On the other hand, since $((u_n, v_n))$ is bounded and $a(x)b(x) > M_0$ in $B_R^C \cap F^C$, we have

$$\begin{aligned} \int_{B_R^C\cap F^C} |u_n v_n|^{p/2} &\leq \frac{1}{\lambda\sqrt{M_0}}\int_{B_R^C\cap K^C} \sqrt{\lambda a(x)}|u_n|^{p/2}\sqrt{\lambda b(x)}|v_n|^{p/2} \\ &\leq \frac{1}{2\lambda M_0}\int_{B_R^C\cap K^C} (\lambda a(x)|u_n|^p + \lambda b(x)|v_n|^p) \leq \frac{C}{\lambda}. \end{aligned} \tag{2.8}$$

It follows from (2.6)–(2.8) that

$$\int_{B_R^C} |u_n v_n|^{p/2} \leq C\bigg(C\big|B_R^C \cap F\big|^{p/N} + \frac{C}{\lambda}\bigg)^\gamma. \tag{2.9}$$

Since F has finite Lebesgue measure, we have that $\big|B_R^C \cap F\big| \to 0$ as $R \to \infty$. Hence for R and λ sufficiently large, the right-hand of (2.8) is small. □

LEMMA 2.5. *There exist $\delta, \rho > 0$ and $z_0 \in X$, all of them independent of λ such that*

(a) $\Phi_\lambda(z) \geq \delta$ *for* $\|z\|_\lambda = \rho$.
(b) $\Phi_\lambda(z_0) \leq \Phi_\lambda(0) = 0$ *and* $\|z_0\| > \rho$.

PROOF. It follows from Lemma 2.2 that

$$\Phi_\lambda(z) = \frac{1}{p}\|z\|_\lambda^p - \frac{1}{\alpha+\beta}\int (u_+)^\alpha (v_+)^\beta \geq \frac{1}{p}\|z\|_\lambda^p - \frac{\widetilde{C}}{\alpha+\beta}\|z\|_\lambda^{\alpha+\beta} \geq \frac{1}{2p}\,\rho^p,$$

whenever $\|z\|_\lambda = \rho := \big((\alpha+\beta)/\big(2p\widetilde{C}\big)\big)^{1/(\alpha+\beta-p)}$.

However, if $\varphi \in C_0^\infty(\Omega_a \cap \Omega_b) \setminus \{0\}$ we have $a(x)\varphi \equiv b(x)\varphi \equiv 0$ on $\mathbb{R}^N$. Therefore,

$$\lim_{t\to\infty} \Phi_\lambda(t(\varphi,\varphi)) = \lim_{t\to\infty}\bigg(\frac{2t^p}{p}\int |\nabla\varphi|^p - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int (\varphi_+)^{\alpha+\beta}\bigg) = -\infty$$

uniformly on λ. It is sufficient to set $z_0 := t_0(\varphi,\varphi)$ with $t_0 > 0$ sufficiently large. □

REMARK 2.6. Let z_0 be given by Lemma 2.5. For each $\lambda > 0$, we may define the mountain pass level of Φ_λ as

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)),$$

where $\Gamma := \big\{\gamma \in C([0,1],X),\ \gamma(0) = 0,\ \gamma(1) = z_0\big\}$. For future reference we observe that

$$0 < \delta \le c_\lambda \le \xi_0 := \max_{t\in[0,1]} \Phi_\lambda(tz_0). \tag{2.10}$$

3. Least energy solutions

In this section, we mainly want to prove Theorem 1.1.

LEMMA 3.1. *If $\{z_n\} = \{(u_n, v_n)\}$ is a* (PS)$_c$ *sequence of I_λ, then $\{\nabla u_n\}$ and $\{\nabla v_n\}$ has subsequences which converge to $\{\nabla u\}$ and $\{\nabla v\}$ respectively almost everywhere for some $(u,v) \in X$ in $\mathbb{R}^N$.*

PROOF. Assume that $\{z_n\} = \{(u_n, v_n)\} \subset X$ is a $(PS)_c$ sequence of I_λ. Then

$$I_\lambda(z_n) \to c \quad \text{and} \quad \langle I'_\lambda(z_n), z_n\rangle \to 0. \tag{3.1}$$

Since $\{z_n\}$ is bounded in X, there exists a $z = (u,v) \in X$ such that

$$(u_n, v_n) \rightharpoonup (u,v) \quad \text{in } X, \tag{3.2}$$

$$(u_n, v_n) \to (u,v) \quad \text{a.e. in } \mathbb{R}^N, \tag{3.3}$$

$$(u_n, v_n) \to (u,v) \quad \text{strong in } L^r_{\mathrm{loc}}(\mathbb{R}^N) \times L^r_{\mathrm{loc}}(\mathbb{R}^N),\ r \in [p,p^*). \tag{3.4}$$

For each $R > 0$, fix η be a C^∞ function satisfying $\eta \equiv 1$ in $B_R(0)$ and $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_{2R}(0)$. By (3.1) and (3.2), we get $\langle I'_\lambda(z_n) - I'_\lambda(z), \eta(z_n - z)\rangle \to 0$, i.e.

$$\begin{aligned}
o(1) ={}& \langle I'_\lambda(z_n) - I'_\lambda(z), \eta(z_n - z)\rangle \\
={}& \bigg\{\int |\nabla u_n|^{p-2}\nabla u_n(\nabla\eta(u_n - u) + \eta\nabla(u_n - u)) \\
&+ \int |\nabla v_n|^{p-2}\nabla v_n(\nabla\eta(v_n - v) + \eta\nabla(v_n - v)) \\
&+ \int \big(\lambda a(x)|u_n|^{p-2}u_n\eta(u_n - u) + \lambda b(x)|v_n|^{p-2}v_n\eta(v_n - v)\big) \\
&- \frac{\alpha}{\alpha+\beta}\int |u_n|^{\alpha-2}u_n\eta(u_n - u)|v_n|^\beta \\
&- \frac{\beta}{\alpha+\beta}\int |u_n|^\alpha|v_n|^{\beta-2}v_n\eta(v_n - v)\bigg\} \\
&- \bigg\{\int |\nabla u|^{p-2}\nabla u(\nabla\eta(u_n - u) + \eta\nabla(u_n - u)) \\
&+ \int |\nabla v|^{p-2}\nabla v(\nabla\eta(v_n - v) + \eta\nabla(v_n - v))
\end{aligned} \tag{3.5}$$

$$+\int \big(\lambda a(x)|u|^{p-2}u\eta(u_n-u)+\lambda b(x)|v|^{p-2}v\eta(v_n-v)\big)$$
$$-\frac{\alpha}{\alpha+\beta}\int |u|^{\alpha-2}u\eta(u_n-u)|v|^{\beta}$$
$$-\frac{\beta}{\alpha+\beta}\int |u|^{\alpha}|v|^{\beta-2}v\eta(v_n-v)\bigg\}.$$

By (3.3), the boundedness of $\{z_n\}$ and Hölder's inequality, we can prove

$$\lim_{n\to\infty}\int a(x)\eta(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u) = \lim_{n\to\infty}\int b(x)\eta(|v_n|^{p-2}v_n-|v|^{p-2}v)(v_n-v)=0, \tag{3.6}$$

$$\lim_{n\to\infty}\int |\nabla u_n|^{p-2}\nabla u_n\nabla\eta(u_n-u)=\lim_{n\to\infty}\int |\nabla v_n|^{p-2}\nabla v_n\nabla\eta(v_n-v)=0 \tag{3.7}$$

and

$$\lim_{n\to\infty}\int |\nabla u|^{p-2}\nabla u\nabla\eta(u_n-u)=\lim_{n\to\infty}\int |\nabla v|^{p-2}\nabla v\nabla\eta(v_n-v)=0. \tag{3.8}$$

By (3.2), we have

$$\lim_{n\to\infty}\int \eta|\nabla u|^{p-2}\nabla u\nabla(u_n-u)=\lim_{n\to\infty}\int \eta|\nabla v|^{p-2}\nabla v\nabla(v_n-v)=0. \tag{3.9}$$

By (3.3), (3.4) and Lebesgue Theorem, we have

$$\lim_{n\to\infty}\int \eta|u_n|^{\alpha-2}u_n(u_n-u)|v_n|^{\beta}=\lim_{n\to\infty}\int \eta|u_n|^{\alpha}|v_n|^{\beta-2}v_n(v_n-v)=0 \tag{3.10}$$

and

$$\lim_{n\to\infty}\int \eta|u|^{\alpha-2}u(u_n-u)|v|^{\beta}=\lim_{n\to\infty}\int \eta|u|^{\alpha}|v|^{\beta-2}v(v_n-v)=0. \tag{3.11}$$

Hence, from (3.5) to (3.10), we get

$$\lim_{n\to\infty}\int \eta\big[|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n-\nabla u)+|\nabla v_n|^{p-2}\nabla v_n(\nabla v_n-\nabla v)\big]=0. \tag{3.12}$$

It follows from (3.9) and (3.12) that

$$\lim_{n\to\infty}\int \eta\big[\big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\big)(\nabla u_n-\nabla u) + \big(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v\big)(\nabla v_n-\nabla v)\big]=0.$$

By Lemma 2.2 in [1], we have

$$\langle |x|^{p-2}x-|y|^{p-2}y,x-y\rangle \ge |x-y|^p,\quad \text{for } p\ge 2,\ x,y\in\mathbb{R}^N.$$

Hence, we have

$$\lim_{n\to\infty}\int \eta\big(|\nabla u_n-\nabla u|^p+|\nabla v_n-\nabla v|^p\big)=0.$$

Therefore, we get

$$\lim_{n\to\infty}\int_{B_R(0)}\big(|\nabla u_n-\nabla u|^p+|\nabla v_n-\nabla v|^p\big)=0,$$

i.e.

$$\nabla u_n\to\nabla u\quad\text{in } L^p(B_R(0))\quad\text{and}\quad\nabla u_n\to\nabla u\quad\text{in } L^p(B_R(0)).$$

Hence up to subsequences, there exists a $(u,v)\in X$ such that

$$(\nabla u_n,\nabla v_n)\to(\nabla u,\nabla v)\quad\text{a.e. in }\mathbb{R}^N. \qquad\square$$

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\varepsilon>0$ to be chosen later, $C_{0:}=\xi_0$ given in (2.10) and consider $Lambda_\varepsilon, R_\varepsilon$ provided in Lemma 2.4. By Remark 2.6, for any fixed $\lambda\geq\Lambda_\varepsilon$, there exists a sequence $(z_n)\subset X$ such that

$$\Phi_\lambda(z_n)\to c_\lambda\geq\delta\quad\text{and}\quad\Phi'_\lambda(z_n)\to 0.$$

It follows from Lemma 2.3 (a) that (z_n) is bounded. Then, up to a subsequence, we have that $z_n\rightharpoonup z_\lambda:=(u_\lambda,v_\lambda)$ weakly in X.

We shall prove that $\Phi'_\lambda(z_\lambda)=0$. Let $\phi\in C_0^\infty(\mathbb{R}^N)$ and denote by K be the support of ϕ. By the compact embedding $D^{1,p}(\mathbb{R}^N)\hookrightarrow L^{\alpha+\beta-1}_{\rm loc}(\mathbb{R}^N)$, we have

$$\begin{aligned}
&(u_n,v_n)\to(u_\lambda,v_\lambda) &&\text{in } L^{\alpha+\beta-1}(K)\times L^{\alpha+\beta-1}(K),\\
&(u_n,v_n)\to(u_\lambda,v_\lambda) &&\text{a.e. in } K,\\
&|u_n|,|v_n|\leq h_K(x)\in L^{\alpha+\beta-1}(K) &&\text{a.e. in } K.
\end{aligned}\tag{3.13}$$

Therefore, almost everywhere in K,

$$(u_n)_+^{\alpha-1}(v_n)_+^{\beta-1}|\phi|\leq|u_n|^{\alpha-1}|v_n|^{\beta}|\phi|\leq h_K^{\alpha+\beta-1}|\phi|\in L^1(K).\tag{3.14}$$

By (3.14) and the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int(u_n)_+^{\alpha-1}(v_n)_+^{\beta}\phi=\int(u_\lambda)_+^{\alpha-1}(v_\lambda)_+^{\beta}\phi,\quad\text{for all }\phi\in C_0^\infty(\mathbb{R}^N).\tag{3.15}$$

Similarly, we get

$$\lim_{n\to\infty}\int(u_n)_+^{\alpha}(v_n)_+^{\beta-1}\psi=\int(u_\lambda)_+^{\alpha}(v_\lambda)_+^{\beta-1}\psi,\quad\text{for all }\psi\in C_0^\infty(\mathbb{R}^N).\tag{3.16}$$

On one hand, as (u_n,v_n) is bounded in $L^p_{\rm loc}(\mathbb{R}^N)\times L^p_{\rm loc}(\mathbb{R}^N)$, it follows from a result due to Brezis and Lieb (see [10]),

$$\lambda\int a(x)|u_n|^{p-2}u_n\varphi\to\lambda\int a(x)|u_\lambda|^{p-2}u_\lambda\varphi,\quad\text{for }\varphi\in C_0^\infty(\mathbb{R}^N),\tag{3.17}$$

$$\lambda\int b(x)|v_n|^{p-2}v_n\psi\to\lambda\int b(x)|v_\lambda|^{p-2}v_\lambda\psi,\quad\text{for }\psi\in C_0^\infty(\mathbb{R}^N).\tag{3.18}$$

Similar to the proof of Lemma 3.1, we can prove

$$\nabla u_n\to\nabla u_\lambda\quad\text{in } L^p(\operatorname{supp}\varphi)\quad\text{and}\quad\nabla v_n\to\nabla v_\lambda\quad\text{in } L^p(\operatorname{supp}\psi).$$

Therefore, we have

$$\int |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi \to \int |\nabla u_\lambda|^{p-2}\nabla u_\lambda \nabla\varphi \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^N), \tag{3.19}$$

$$\int |\nabla v_n|^{p-2}\nabla v_n \nabla\psi \to \int |\nabla v_\lambda|^{p-2}\nabla v_\lambda \nabla\psi \quad \text{for } \psi \in C_0^\infty(\mathbb{R}^N). \tag{3.20}$$

As a result, for each $(\varphi,\psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$, there holds

$$0 = \lim_{n\to\infty} I'_\lambda(z_n)(\varphi,\psi) = I'_\lambda(z_\lambda)(\varphi,\psi).$$

Therefore z_λ is a critical point of Φ_λ.

Suppose that $z_\lambda \equiv 0$. Since $u_n, v_n \to 0$ in $L^p(B_{R_\varepsilon})$, we may use Lemma 2.1 the boundedness of z_n in X and Young's inequality, to get

$$\int_{B_{R_\varepsilon}} (u_n)_+^\alpha (v_n)_+^\beta \le C\bigg(\int_{B_{R_\varepsilon}} (|u_n||v_n|)^{\frac{p}{2}}\bigg)^\gamma \tag{3.21}$$

$$\le C\bigg(\int_{B_{R_\varepsilon}} |u_n|^p + |v_n|^p\bigg)^\gamma \to 0, \tag{3.22}$$

as $n \to \infty$. It follows from Lemma 2.3 (b) and Lemma 2.4 that, for $\lambda \ge \Lambda_\varepsilon$,

$$\begin{aligned} c_\lambda\bigg(\frac{1}{p} - \frac{1}{\alpha+\beta}\bigg)^{-1} &= \lim_{n\to\infty} \int (u_n)_+^\alpha (v_n)_+^\beta \\ &= \lim_{n\to\infty} \bigg(\int_{B_{R_\varepsilon}} (u_n)_+^\alpha (v_n)_+^\beta + \int_{B_{R_\varepsilon}^C} (u_n)_+^\alpha (v_n)_+^\beta\bigg) \le \varepsilon. \end{aligned}$$

If we choose $\varepsilon > 0$ sufficiently small, then we conclude that $c_\lambda = 0$, contradicting $c_\lambda > 0$. This shows that $z_\lambda \not\equiv 0$.

By Fatou's Lemma, we get

$$\begin{aligned} c_\lambda &= \lim_{n\to\infty}\bigg(I_\lambda(z_n) - \frac{1}{p}I'_\lambda(z_n)z_n\bigg) = \lim_{n\to\infty}\bigg(\frac{1}{p} - \frac{1}{\alpha+\beta}\bigg)\int (u_n)_+^\alpha (v_n)_+^\beta \\ &\ge \bigg(\frac{1}{p} - \frac{1}{\alpha+\beta}\bigg)\int (u_\lambda)^\alpha (v_\lambda)^\beta = I_\lambda(z_\lambda) \ge c_\lambda, \end{aligned}$$

which implies that $I_\lambda(z_\lambda) = c_\lambda$. Hence, z_λ is a ground state solution.

Since $I'_\lambda(z_\lambda)((u_\lambda)_-, (v_\lambda)_-) = \|((u_\lambda)_-, (v_\lambda)_-)\|_\lambda^p = 0$, we have that $u_\lambda, v_\lambda \ge 0$ in $\mathbb{R}^N$. Furthermore, by the Vasquez Maximum Principle (see [34]) for the p-Laplacian equation in each equation of (1.1) we conclude that $u_\lambda, v_\lambda > 0$ in $\mathbb{R}^N$. This proves the first part of Theorem 1.1.

Now we consider the concentration behavior of the solutions. Suppose that $(\lambda_n) \subset \mathbb{R}$ is such that $\lambda_n \to \infty$ and let $z_{\lambda_n} = (u_{\lambda_n}, v_{\lambda_n})$ be the associated solution of (1.1) with $\lambda = \lambda_n$ such that $I_{\lambda_n}(z_{\lambda_n}) = c_{\lambda_n}$. In what follows, we write only z_n, u_n and v_n to denote $z_{\lambda_n}, u_{\lambda_n}$ and v_{λ_n}, respectively.

By (2.10), we have

$$\bigg(\frac{1}{p} - \frac{1}{\alpha+\beta}\bigg)\|z_n\|_{\lambda_n}^p = I_{\lambda_n}(z_n) = c_{\lambda_n} \le \xi_0. \tag{3.23}$$

Thus, up to a subsequence, we have that $z_n \rightharpoonup \overline{z} = (\overline{u}, \overline{v})$ weakly in $D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$ and $z_n(x) \to \overline{z}(x)$ almost everywhere in $\mathbb{R}^N$. Given $\varphi \in C_0^\infty(\Omega_a)$, recalling that $a \equiv 0$ in Ω_a and using $(\varphi, 0)$ as a test function, we get

$$\int |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi = \frac{\alpha}{\alpha+\beta} \int (u_n)_+^{\alpha-1} (v_n)_+^\beta \varphi. \tag{3.24}$$

Since φ has compact support, we may take the limit in (3.24) and argue as in the proof (3.15) to get

$$\int_{\Omega_a \cup \Omega_b} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi = \frac{\alpha}{\alpha+\beta} \int_{\Omega_a \cup \Omega_b} (\overline{u})_+^{\alpha-1} (\overline{v})_+^\beta \varphi, \quad \text{for all } \varphi \in C_0^\infty(\Omega_a). \tag{3.25}$$

Similarly, for all $\psi \in C_0^\infty(\Omega_b)$, we have

$$\int_{\Omega_a \cup \Omega_b} |\nabla \overline{v}|^{p-2} \nabla \overline{v} \nabla \psi = \frac{\beta}{\alpha+\beta} \int_{\Omega_a \cup \Omega_b} (\overline{u})_+^{\alpha} (\overline{v})_+^{\beta-1} \psi. \tag{3.26}$$

We claim that $\overline{u} \equiv 0$ in Ω_a^C. In order to see this, we take $j \in \mathbb{N}$, denote

$$C_j := \left\{ x \in B_j(0),\ a(x) > \frac{1}{j} \right\}$$

and, by (3.24),

$$0 \le \int_{C_j} |u_n|^p \le \frac{j}{\lambda_n} \int_{C_j} \lambda_n a(x) |u_n|^p \le \frac{j}{\lambda_n} \|z_n\|_{\lambda_n}^p \to 0,$$

as $n \to \infty$. Noting that C_j is bounded and $u_n \to \overline{u}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, we conclude that $\int_{C_j} |\overline{u}|^p \, dx = 0$ for all $j \in \mathbb{N}$. Thus $\overline{u} \equiv 0$ almost everywhere in $\Omega_a^C = \bigcup_{j=1}^{n} C_j$. Recalling that Ω_a has smooth boundary, we infer that $\overline{u} \in W_0^{1,p}(\Omega_a)$. Similarly, $\overline{v} \in W_0^{1,p}(\Omega_b)$. Thus $\overline{u}, \overline{v}$ is a solution of the limiting problem (1.3).

In order to check that $\overline{z} \ne 0$, we define

$$m := \inf_{z \in \mathcal{N}} J(z),$$

where $J \colon W_0^{1,p}(\Omega_a) \times W_0^{1,p}(\Omega_b) \to \mathbb{R}$ is given by

$$J(u,v) := \frac{1}{p} \int_{\Omega_a \cup \Omega_b} \big(|\nabla u|^p + |\nabla v|^p \big) - \frac{1}{\alpha+\beta} \int_{\Omega_a \cup \Omega_b} (u_+)^\alpha (v_+)^\beta$$

and $\mathcal{N}$ is the Nahari manifold of J, namely

$$\mathcal{N} := \big\{ (u,v) \in W_0^{1,p}(\Omega_a) \times W_0^{1,p}(\Omega_b) : (u,v) \ne (0,0), J'(u,v)(u,v) = 0 \big\}.$$

Since $W_0^{1,p}(\Omega_a) \times W_0^{1,p}(\Omega_b)$ can be viewed as a subspace of X, we have that $c_\lambda \le m$, for all λ. On the other hand,

$$m \ge c_{\lambda_n} = I_{\lambda_n}(z_n) - \frac{1}{p} I'_{\lambda_n}(z_n) z_n = \left(\frac{1}{p} - \frac{1}{\alpha+\beta} \right) \int (u_n)_+^\alpha (v_n)_+^\beta. \tag{3.27}$$

Taking $n \to \infty$, applying Fatou's lemma and $J'(\overline{u}, \overline{v}) = 0$ we obtain

$$\begin{aligned} m &\geq \lim_{n\to\infty} \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right) \int (u_n)_+^\alpha (v_n)_+^\beta \\ &\geq \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right) \int_{\Omega_a \cup \Omega_b} (u_+)^\alpha (v_+)^\beta = J(\overline{u}, \overline{v}) \geq m. \end{aligned} \tag{3.28}$$

Hence $J(\overline{u}, \overline{v}) = m$ and therefore $\overline{z} \neq 0$ is a ground state solution of (1.3). By (3.26) and (3.27) we obtain $\|(\overline{u}_-, \overline{v}_-)\|_0 = 0$. Thus, $\overline{u}, \overline{v} \geq 0$ and a Harnack-type inequality given by Serrin (Theorem 5 in [30]) together with (3.26) and (3.27) implies that $\overline{u} > 0$ in Ω_a and $\overline{v} > 0$ in Ω_b.

In order to complete the proof, by Lemma 3.1 and Brezis–Lieb lemma, the fact that z_n is a solution of (1.1) with $\lambda = \lambda_n$, (3.28) and $(\overline{u}, \overline{v}) \in \mathcal{N}$, we get

$$\begin{aligned} &\|z_n - \overline{z}\|_\lambda^p \\ &= \int \big(|\nabla(u_n - \overline{u})|^p + |\nabla(v_n - \overline{v})|^p + \lambda_n a(x)|u_n - \overline{u}|^p + \lambda_n b(x)|v_n - \overline{v}|^p\big) \\ &= \int \big(|\nabla u_n|^p + |\nabla v_n|^p + \lambda_n a(x)|u_n|^p + \lambda_n b(x)|v_n|^p\big) \\ &\quad - \int \big(|\nabla \overline{u}|^p + |\nabla \overline{v}|^p\big) + o(1) \\ &= \int (u_n)_+^\alpha (v_n)_+^\beta - \int \big(|\nabla \overline{u}|^p + |\nabla \overline{v}|^p\big) + o(1) \\ &= \int (\overline{u})_+^\alpha (\overline{v})_+^\beta - \int \big(|\nabla \overline{u}|^p + |\nabla \overline{v}|^p\big) + o(1) = o(1), \end{aligned}$$

as $n \to \infty$. Since $\|\cdot\|_0 \leq \|\cdot\|_{\lambda_n}$, it follows that $z_n \to \overline{z}$ in $D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$. This finishes the proof of Theorem 1.1. □

4. Multiplicity of bound state solutions

In this section, we mainly prove Theorems 1.2 and 1.3. Since we are not concerned the sign of solutions, we redefine the functional I_λ given by

$$I_\lambda(u, v) := \frac{1}{p}\|(u, v)\|_\lambda^p - \frac{1}{\alpha+\beta} \int |u|^\alpha |v|^\beta, \quad (u, v) \in X.$$

As in Section 2, the functional is of class C^1 and its critical points are the weak solutions of (1.1). For future reference, first we give the following inequalities:

$$\begin{aligned} &\int_{B_R^C} |u|^{\alpha-1}|v|^{\beta-1}|\varphi\psi| \\ &\qquad \leq C\|u\|_{L^{p^*}(B_R^C)}^{\alpha-1} \|v\|_{L^{p^*}(B_R^C)}^{\beta-1} \|(\varphi, \psi)\|_0^{2-p+p^* t/r} \bigg(\int_{B_R^C} |\varphi\psi|^{p/2}\bigg)^{(1-t)/r} \end{aligned} \tag{4.1}$$

and

$$\int_{B_R^C}|\varphi\psi|^{p/2}\leq C\|(\varphi,\psi)\|_0^p|B_R^C\cap F|^{p/N} + \frac{1}{\sqrt{M_0}}\bigg(\int_{B_R^C\cap F^C}a(x)|\varphi|^p\bigg)^{1/2}\bigg(\int_{B_R^C\cap F^C}b(x)|\psi|^p\bigg)^{1/2}, \tag{4.2}$$

for any $R>0$ and $(u,v),(\varphi,\psi)\in X$. Here $r>1, t\in(0,1)$ and $\gamma>0$ are given by Lemma 2.1. In fact, we have

$$\begin{aligned}
&\int_{B_R^C}|u|^{\alpha-1}|v|^{\beta-1}|\varphi\psi| \\
&\leq\bigg(\int_{B_R^C}|u|^{p^*}\bigg)^{(\alpha-1)/p^*}\bigg(\int_{B_R^C}|v|^{p^*}\bigg)^{(\beta-1)/p^*}\bigg(\int_{B_R^C}|\varphi\psi|^{\theta}\bigg)^{1/\theta} \\
&\leq\|u\|^{\alpha-1}_{L^{p^*}(B_R^C)}\|v\|^{\beta-1}_{L^{p^*}(B_R^C)}\bigg(\int_{B_R^C}|\varphi\psi|^{\theta_1}|\varphi\psi|^{\theta_2}\bigg)^{1/\theta} \\
&\leq\|u\|^{\alpha-1}_{L^{p^*}(B_R^C)}\|v\|^{\beta-1}_{L^{p^*}(B_R^C)}\bigg(\int_{B_R^C}|\varphi\psi|^{p^*/2}\bigg)^{(2-p)/p^*+t/r}\bigg(\int_{B_R^C}|\varphi\psi|^{p/2}\bigg)^{(1-t)/r} \\
&\leq\|u\|^{\alpha-1}_{L^{p^*}(B_R^C)}\|v\|^{\beta-1}_{L^{p^*}(B_R^C)}\bigg(\int_{B_R^C}|\varphi|^{p^*}\bigg)^{(2-p)/(2p^*)+t/(2r)} \\
&\quad\cdot\bigg(\int_{B_R^C}|\psi|^{p^*}\bigg)^{(2-p)/(2p^*)+t/(2r)}\bigg(\int_{B_R^C}|\varphi\psi|^{p/2}\bigg)^{(1-t)/r} \\
&\leq C\|u\|^{\alpha-1}_{L^{p^*}(B_R^C)}\|v\|^{\beta-1}_{L^{p^*}(B_R^C)}\|(\varphi,\psi)\|_0^{2-p+p^*t/r}\bigg(\int|\varphi\psi|^{p/2}\bigg)^{(1-t)/r},
\end{aligned}$$

where

$$\theta=\frac{p^*}{p^*-(\alpha+\beta-2)},\qquad \theta_1=\bigg(\frac{2-p}{2}+\frac{p^*t}{2r}\bigg)\theta,\qquad \theta_2=\frac{p(1-t)}{2r}\,\theta.$$

Meanwhile, we get

$$\begin{aligned}
\int_{B_R^C}|\varphi\psi|^{p/2}\leq&\bigg(\int_{B_R^C\cap F}|\varphi|^{p^*}\bigg)^{p/(2p^*)}\bigg(\int_{B_R^C\cap F}|\psi|^{p^*}\bigg)^{p/(2p^*)}\big|B_R^C\cap F\big|^{1-p/p^*} \\
&+\frac{1}{\sqrt{M}}\int_{B_R^C\cap F^C}\sqrt{a(x)}|\varphi|^{p/2}\sqrt{b(x)}|\psi|^{p/2} \\
\leq&\, C\|(\varphi,\psi)\|_0^p\big|B_R^C\cap F\big|^{p/N} \\
&+\frac{1}{\sqrt{M}}\bigg(\int_{B_R^C\cap F^C}a(x)|\varphi|^p\bigg)^{1/2}\bigg(\int_{B_R^C\cap F^C}b(x)|\psi|^p\bigg)^{1/2}.
\end{aligned}$$

In order to obtain multiple critical points for I_λ we shall use the following version of the Symmetric Mountain Pass Theorem [5] (see also [32, Theorem 2.1]).

Proposition 4.1. *Let E be a real Banach space and $W \subset E$ be a finite dimensional subspace. Suppose that $I \in C^1(E, \mathbb{R})$ is an even functional satisfying $I(0) = 0$ and*

(a) *there exists a constant $\rho > 0$ such that $I|_{\partial B_\rho(0)} \geq 0$;*
(b) *there exists $M > 0$ such that $\sup_{z \in W} I(z) < M$.*

If I satisfies $(\mathrm{PS})_c$ for any $0 < c < M$, then I possesses at least dimW pairs of nontrivial critical points.

Now we give a similar Brezis–Lieb type lemma (see [10]).

Lemma 4.2. *Let $((u_n, v_n)) \subset X$ be such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in X. Then*

$$\lim_{n\to\infty} \int \big(|u_n|^\alpha |v_n|^\beta - |u_n - u|^\alpha |v_n - v|^\beta\big) = \int |u|^\alpha |v|^\beta.$$

Proof. By (4.1) and (4.2), we can finish the proof by the same argument of Lemma 4.2 in [19]. Here we omit its proof.

Lemma 4.3. *Let $z_n = ((u_n, v_n)) \subset X$ be a $(\mathrm{PS})_c$ sequence for I_λ. Then, up to a subsequence, $z_n \rightharpoonup z := (u, v)$ weakly in X, where z is a critical point of I_λ. Furthermore, $\widetilde{z}_n := z_n - z$ is a $(\mathrm{PS})_{c'}$ sequence for I_λ, with $c' = c - I_\lambda(z)$.*

Proof. Since z_n is bounded in X, up to a subsequence, $z_n \rightharpoonup z := (u, v)$ weakly in X. Arguing as in the proof of Theorem 1.1 we can show that $I'_\lambda(z) = 0$. By Lemma 3.1, Brezis–Lieb Lemma and Lemma 4.2, we have

$$\begin{aligned}
&I_\lambda(z_n - z)\\
&\quad= \int \big(|\nabla(u_n - u)|^p + |\nabla(v_n - v)|^p + \lambda a(x)|u_n - u|^p + \lambda b(x)|v_n - v|^p\big)\\
&\qquad - \frac{1}{\alpha+\beta} \int |u_n - u|^\alpha |v_n - v|^\beta\\
&\quad= \int \big(|\nabla u_n|^p + |\nabla v_n|^p + \lambda a(x)|u_n|^p + \lambda b(x)|v_n|^p\big) - \frac{1}{\alpha+\beta} \int |u_n|^\alpha |v_n|^\beta\\
&\qquad - \int \big(|\nabla u|^p + |\nabla v|^p + \lambda a(x)|u|^p + \lambda b(x)|v|^p\big) + \frac{1}{\alpha+\beta} \int |u|^\alpha |v|^\beta + o(1)\\
&\quad= I_\lambda(z_n) - I_\lambda(z) + o(1) = c - I_\lambda(z) + o(1),
\end{aligned}$$

as $n \to \infty$.

It remains to show that $I'_\lambda(z_n - z) \to 0$. First, for any given $(\varphi, \psi) \in X$ such that $\|(\varphi, \psi)\|_\lambda \leq 1$, we have

$$\begin{aligned}
&I'_\lambda(z_n - z)(\varphi, \psi)\\
&\quad= I'_\lambda(z_n)(\varphi, \psi) - I'_\lambda(z)(\varphi, \psi) - \frac{\alpha}{\alpha+\beta} \int f_n \varphi - \frac{\beta}{\alpha+\beta}\, g_n \psi\\
&\qquad + \int \big(|\nabla(u_n - u)|^{p-2}\nabla(u_n - u) - |\nabla u_n|^{p-2}\nabla u_n + |\nabla u|^{p-2}\nabla u\big)\nabla\varphi
\end{aligned}$$

$$
\begin{aligned}
&+\int \big(|\nabla(v_n-v)|^{p-2}\nabla(v_n-v)-|\nabla v_n|^{p-2}\nabla v_n+|\nabla v|^{p-2}\nabla v\big)\nabla\psi\\
&+\lambda\int a(x)\big(|u_n-u|^{p-2}(u_n-u)-|u_n|^{p-2}u_n+|u|^{p-2}u\big)\varphi\\
&+\lambda\int b(x)\big(|v_n-v|^{p-2}(v_n-v)-|v_n|^{p-2}v_n+|v|^{p-2}v\big)\psi,
\end{aligned}
$$

where

$$
\begin{aligned}
f_n(x)&:=|u_n-u|^{\alpha-2}(u_n-u)|v_n-v|^{\beta}-|u_n|^{\alpha-2}u_n|v_n|^{\beta}+|u|^{\alpha-2}u|v|^{\beta},\\
g_n(x)&:=|u_n-u|^{\alpha}|v_n-v|^{\beta-2}(v_n-v)-|u_n|^{\alpha}|v_n|^{\beta-2}v_n+|u|^{\alpha}|v|^{\beta-2}v.
\end{aligned}
$$

By Lemma 3.1 and Lemma 3.2 in [1], we can check that

$$
\bigg(\int \big||\nabla(u_n-u)|^{p-2}\nabla(u_n-u)-|\nabla u_n|^{p-2}\nabla u_n+|\nabla u|^{p-2}\nabla u\big|^{p/(p-1)}\bigg)^{(p-1)/p} = o_n(1),
$$

$$
\bigg(\int \big||\nabla(v_n-v)|^{p-2}\nabla(v_n-v)-|\nabla v_n|^{p-2}\nabla v_n+|\nabla v|^{p-2}\nabla v\big|^{p/(p-1)}\bigg)^{(p-1)/p} = o_n(1),
$$

$$
\begin{aligned}
&\int a(x)\big||(u_n-u)|^{p-2}(u_n-u)-|u_n|^{p-2}u_n+|u|^{p-2}u\big|^{p/(p-1)}=o_n(1),\\
&\int b(x)\big||(v_n-v)|^{p-2}(v_n-v)-|v_n|^{p-2}v_n+|v|^{p-2}v\big|^{p/(p-1)}=o_n(1).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
I'_\lambda(z_n-z)(\varphi,\psi)=I'_\lambda(z_n)(\varphi,\psi)-I'_\lambda(z)(\varphi,\psi)&\\
-\frac{\alpha}{\alpha+\beta}\int f_n\varphi-\frac{\beta}{\alpha+\beta}\int g_n\psi+o_n(1).&
\end{aligned}
$$

Since $I'_\lambda(z_n)\to 0$ and $I'_\lambda(z)=0$, it is sufficient to show that

$$
\lim_{n\to\infty}\sup_{\|\varphi\|_{X_a}\le 1}\int |f_n||\varphi| = 0 = \lim_{n\to\infty}\sup_{\|\psi\|_{X_b}\le 1}\int |g_n||\psi|, \tag{4.3}
$$

where we are denoting

$$
\|\varphi\|_{X_a}^p:=\int(|\nabla\varphi|^p+\lambda a(x)|\varphi|^p),\qquad \|\psi\|_{X_b}^p:=\int(|\nabla\psi|^p+\lambda b(x)|\psi|^p).
$$

Since we can prove that (4.3) is true by the same argument of (4.12) in Lemma 4.3 in [19], we omit the detailed proof. □

The following result is a local compactness property for the functional I_λ.

Lemma 4.4. *For any given C_0 there exists $\Lambda=\Lambda(\alpha,\beta,C_0)>0$ such that I_λ satisfies* (PS)$_c$ *for any $c\le C_0$ and $\lambda\ge\Lambda$.*

Proof. Let γ_0 be given by Lemma 2.3 (c) and fix $\varepsilon > 0$ such that

$$\varepsilon < \frac{\gamma_0}{p}\left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)^{-1}.$$

Fixing $C_0 > 0$, let Λ_ε and R_ε be given by Lemma 2.4. We will prove that the this lemma holds for $\Lambda := \Lambda_\varepsilon$. Let $(z_n) = ((u_n, v_n)) \subset X$ be a $(\mathrm{PS})_c$ sequence for I_λ with $c \leq C_0$ and $\lambda \geq \Lambda$. By Lemma 4.3, we may suppose that $(u_n, v_n) \rightharpoonup z := (u, v)$ weakly in X and $\widetilde{z}_n := (u_n - u, v_n - v)$ is a $(\mathrm{PS})_{c'}$ sequence for I_λ, with $c' = c - I_\lambda(z)$. We claim that $c' = 0$. If this is true, it follows from Lemma 2.3 (b) that

$$\lim_{n\to\infty} \|\widetilde{z}_n\|_\lambda^p = c'\left(\frac{1}{p} - \frac{2}{\alpha+\beta}\right)^{-1} = 0,$$

that is $z_n \to z$ in X.

Suppose, by contradiction, that $c' \neq 0$. Lemma 2.3 (c) implies that $c' \geq \gamma_0 > 0$. Since $\widetilde{u}_n, \widetilde{v}_n \to 0$ in $L^p(B_{R_\varepsilon})$, we may use Lemma 2.3 (b), Lemma 2.4, the same calculation of (3.21) and the choice of $\varepsilon > 0$ to get

$$\begin{aligned}\gamma_0\left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)^{-1} &\leq c'\left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)^{-1} = \lim_{n\to\infty}\int |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta \\ &\leq \lim_{n\to\infty}\left(\int_{B_{R_\varepsilon}} |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta + \int_{B^C_{R_\varepsilon}} |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta\right) \\ &\leq \varepsilon \leq \frac{\gamma_0}{p}\left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)^{-1},\end{aligned}$$

which contradicts $\gamma_0 > 0$. □

Now we are in ready to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Take a bounded open smooth set $\Omega \subset \Omega_a \cap \Omega_b$. Given $m \in \mathbb{N}$ we set $H := \operatorname{span}\{(\phi_1, \phi_1), \ldots, (\phi_m, \phi_m)\}$, where ϕ_i is an eigenfunction corresponding to the i-th eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ (see [27]). For each $i = 1, \ldots, m$, we have that

$$\lim_{t\to\infty} I_\lambda(t(\phi_i, \phi_i)) = \lim_{t\to\infty}\left(\frac{2t^p}{p}\int |\nabla\phi_i|^p - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int |\phi_i|^{\alpha+\beta}\right) = -\infty,$$

uniformly on λ. Since $\dim(H) < \infty$, we obtain $M_m > 0$ independent of $\lambda > 0$, such that

$$\sup_{z\in H} I_\lambda(z) < M_m.$$

Meanwhile, as in the proof of Lemma 2.5, we may obtain $\rho > 0$, independent of $\lambda > 0$, such that

$$I_\lambda(z) \geq 0 \quad \text{for any } \|z\|_\lambda = \rho.$$

By Lemma 4.4 there exists $\Lambda_m > 0$ such that I_λ satisfies $(\mathrm{PS})_c$ for any $c \le M_m$ and $\lambda \ge \Lambda_m$. Therefore, for any fixed $\lambda \ge \Lambda_m$ we may apply Theorem 1.1 to get m pairs of nontrivial solutions. □

PROOF OF THEOREM 1.3. Noting that

$$\Big(\frac{1}{p} - \frac{1}{\alpha+\beta}\Big)\|z_{\lambda_n}\|_{\lambda_n}^p = I_{\lambda_n}(z_{\lambda_n}) - \frac{1}{\alpha+\beta} I'_{\lambda_n}(z_{\lambda_n})z_{\lambda_n} = I_{\lambda_n}(z_{\lambda_n}),$$

since $\liminf\limits_{n\to\infty} I_\lambda(z_{\lambda_n}) < \infty$ we may assume, up to a subsequence, that (z_{λ_n}) is bounded. Thus, up to a subsequence, we have that

$$\begin{aligned} &z_{\lambda_n} \rightharpoonup \overline{z} := (\overline{u},\overline{v}) &&\text{weakly in } D^{1,p}(\mathbb{R}^N)\times D^{1,p}(\mathbb{R}^N),\\ &(u_n,v_n) \to (\overline{u},\overline{v}) &&\text{strongly in } L^\alpha_{\mathrm{loc}}(\mathbb{R}^N)\times L^\beta_{\mathrm{loc}}(\mathbb{R}^N),\\ &(u_n,v_n) \to (\overline{u},\overline{v}) &&\text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{4.4}$$

Given $\varepsilon > 0$ we can argue as in the proof of Lemma 2.4 to conclude that, for some $R > 0$ large, there holds

$$\limsup_{n\to\infty} \int_{(B_R(0))^C} |u_n|^\alpha |v_n|^\beta \le \varepsilon.$$

By taking R larger if necessary, we may suppose that

$$\int_{(B_R(0))^C} |u|^\alpha |v|^\beta \le \varepsilon.$$

Meanwhile, (4.4) and the Lebesgue Dominated Convergence Theorem imply that

$$\int_{(B_R(0))^C} |u_n|^\alpha |v_n|^\beta \to \int_{(B_R(0))^C} |u|^\alpha |v|^\beta$$

as $n \to \infty$. Noting that

$$\begin{aligned}\bigg|\int \big(|u_n|^\alpha|v_n|^\beta - |\overline{u}|^\alpha|\overline{v}|^\beta\big)\bigg| \le{}& \int_{(B_R(0))^C} |u_n|^\alpha|v_n|^\beta \\ &+ \int_{(B_R(0))^C} |\overline{u}|^\alpha|\overline{v}|^\beta + \bigg|\int_{B_R(0)} \big(|u_n|^\alpha|v_n|^\beta - |\overline{u}|^\alpha|\overline{v}|^\beta\big)\bigg|,\end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \bigg|\int \big(|u_n|^\alpha|v_n|^\beta - |\overline{u}|^\alpha|\overline{v}|^\beta\big)\bigg| \le 2\varepsilon$$

and therefore

$$\lim_{n\to\infty} \int |u_n|^\alpha|v_n|^\beta = \int |\overline{u}|^\alpha|\overline{v}|^\beta.$$

Hence, we can argue as in the end of the proof of Theorem 1.1 to conclude that $\|z_{\lambda_n} - \overline{z}\|_0 \le \|z_{\lambda_n} - \overline{z}\|_{\lambda_n} \to 0$ as $n \to \infty$. Therefore, $z_{\lambda_n} \to \overline{z}$ strongly in $\mathcal{D}^{1,p}(\mathbb{R}^N)\times\mathcal{D}^{1,p}(\mathbb{R}^N)$ and the theorem is proved. □

Acknowledgments. The author sincerely thanks C. Wang for helpful discussions and suggestions.

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Manuscript received April 8, 2018
accepted July 24, 2018

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