Topological Methods in Nonlinear Analysis
Volume 54, No. 2B, 2019, 887–906
DOI: 10.12775/TMNA.2019.031

EXISTENCE, LOCALIZATION AND STABILITY OF LIMIT-PERIODIC SOLUTIONS TO DIFFERENTIAL EQUATIONS INVOLVING CUBIC NONLINEARITIES

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Dedicated to the memory of Professor Ioan I. Vrabie

Abstract. We will prove, besides other things like localization and (in)stability, that the differential equations $x' + x^3 - \lambda x = \varepsilon r(t)$, $\lambda > 0$, and $x'' + x^3 - x = \varepsilon r(t)$, where $r\colon \mathbb{R} \to \mathbb{R}$ are uniformly limit-periodic functions, possess for sufficiently small values of $\varepsilon > 0$ uniformly limit-periodic solutions, provided r in the first-order equation is strictly positive. As far as we know, these are the first nontrivial effective criteria, obtained for limit-periodic solutions of nonlinear differential equations, in the lack of global lipschitzianity restrictions. A simple illustrative example is also indicated for difference equations.

1. Introduction

As our title indicates, the main aim of the present paper is to study limit-periodic solutions of the first-order and the second-order differential equations involving cubic nonlinearities and limit-periodic forcing terms. The investigation of limit-periodic nonlinear oscillations is a delicate problem, especially because the space of limit-periodic functions endowed with the sup-norm is complete, but not linear. That is why the related results are, unlike those for periodic

2010 *Mathematics Subject Classification.* 34C15, 34C27, 34K14.
Key words and phrases. Limit-periodic solutions; differential equations; cubic nonlinearity; existence of solutions; localization; (in)stability; essentiality.
The first author was supported by the grant IGA_PrF_2018_024 "Mathematical Models" of the Internal Grant Agency of Palacký University in Olomouc.

and almost-periodic oscillations, very rare (see e.g. [2], [6], [5], [19]). Moreover, many well known obstructions even for linear almost-periodic oscillations cannot be also avoided here (see e.g. [9], [16]). On the other hand, the class of limit-periodic functions is an important subclass of almost-periodic functions which generalizes in a natural way (like quasi-periodic functions) periodic functions. As pointed out in [17, p. 113], Harald Bohr was the first who paid an attention to it in [8].

For the first-order scalar differential equations of the form $x' = f(x)+p(t)$, an interesting generic result was obtained in [2], provided $f \in \mathrm{C}^2(\mathbb{R},\mathbb{R})$ is such that the set $\{x \in \mathbb{R} \mid f''(x) = 0\}$ is totally disconnected and p is limit-periodic. If the nonlinearities in given systems under consideration are globally lipschitzean with sufficiently small Lipschitz constants, then the Banach contraction principle can be easily applied to limit-periodic systems (see e.g. [6]). Otherwise, the situation becomes subtle, as documented by the theorem of Seifert [19, Theorem 2] which requires, besides other things, a finiteness of the number of entirely bounded solutions (cf. Proposition 2.7 below). This serious obstruction can be omitted for difference systems (see [5]). We also generalized in [5], under a slight growth restriction imposed on nonlinearities, Seifert's result by means of the Stepanov norms. Nevertheless, there are so far no nontrivial illustrative examples of the applications of the Seifert's theorem to our disposal. In fact, his unique illustrative example in [19] "only" says that the Duffing-type equation $x'' + \varepsilon x - x^3 = r(t)$, forced by a limit-periodic function r admits, for every sufficiently small $\varepsilon > 0$, an almost-periodic but not necessarily a limit-periodic solution. As far as we know, that is all what was done concerning nonlinear limit-periodic oscillations.

The same conclusion about almost-periodic solutions can be directly deduced, as particular cases of the results for the Duffing-type almost-periodic equations $x'' - x + x^3 = \varepsilon r(t)$, $\varepsilon > 0$ sufficiently small, and $x'' - x - x^3 = r(t)$ in [22] and [7], respectively, including the uniqueness result in [7]. Let us note that the existence of infinitely many harmonic as well as subharmonic (periodic) solutions was proved, in answering the Littlewood's problem, to the periodic equation $x'' + \varepsilon x + x^3 = r(t)$, $\varepsilon > 0$, (see [10, Chapter 10], and the references therein).

Despite the numerical simulations (see e.g. [13, pp. 501–503]) for the 2π-periodic equation $x'' - x + x^3 = 0.2\cos t$, resp. the equivalent planar system $x' = y$, $y' = x - x^3 + 0.2\cos t$, whose all orbits are bounded, we are able to extend in this paper the theorem of Zeng [22] in the sense that the limit-periodic equation $x'' - x + x^3 = \varepsilon r(t)$, $\varepsilon > 0$ sufficiently small, possesses a (uniformly) limit-periodic solution. Let us note that the presence of a transversal homoclinic point in the vicinity of the origin at the 2π-periodic system indicates a rather complicated behaviour near the homoclinic loops of the unforced equation $x'' - x + x^3 = 0$. In particular, the cross section of the orbit starting at $(0,0)$ does not appear to lie on a closed invariant curve which signalizes that the orbit might not be periodic or quasi-periodic.

The analogous extension can be done, via Seifert's approach (i.e. by means of Proposition 2.7 below) to the equation $x'' - x - x^3 = r(t)$, considered by Berger and Chen [7]. In view of the uniqueness theorem about at most one entirely bounded solution in [1, Theorem 2.4], which applies for $x'' - x^3 = r(t)$ ($\varepsilon = 0$), it is a question whether or not the limit-periodic equation $x'' + \varepsilon x - x^3 = r(t)$ considered by Seifert in [19] admits, for a sufficiently small $\varepsilon > 0$, a limit-periodic solution, too. However, because of infinitely many periodic solutions of the periodic equation $x'' + \varepsilon x + x^3 = r(t)$, $\varepsilon > 0$, one might not expect the applicability of the same approach to this Duffing-type equation.

The equation $x'' - x + x^3 = \varepsilon r(t)$, $\varepsilon > 0$, with a negative linear stiffness treated by Zeng [22] and ourselves, describes (according to [12, Chapter 2.2]) the dynamics of a buckled beam or plate, when one mode vibration is considered. In particular, it provides the simplest possible model for the forced vibration of a cantilever beam in the nonuniform field of two permanent magnets.

Furthermore, since the forced mathematical (undamped) pendulum equation $x'' + c\sin x = r(t)$, $c \neq 0$, is a paradigmatic object in the theory of nonlinear oscillations and classical mechanics, and since $\sin x$ was approximated just by the two first terms in the series $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots$ by Duffing [11], there is no doubt about the importance of the exploration of any of the Duffing-type equations of the form $x'' + ax + bx^3 = r(t)$, $b \neq 0$, including the limit-periodic case of r.

For the first-order differential equations, the situation seems to be partly more promissible, because we have to our disposal the exact multicity results about entirely bounded solutions due to Tineo [20], [21], required by Proposition 2.7 below. Moreover, the generic result of Alonso, Obaya and Ortega [2] encouraged us to consider again the equations with the "Duffing-type" cubic nonlinearities $f(x) = ax + bx^3$, $b \neq 0$, because the set $\{x \in \mathbb{R} \mid f''(x) = 6bx = 0\} = \{0\}$ is trivially totally disconnected.

In this way, we are able to extend the theorem of Haraux [14] (see Lemma 3.2 below) about the almost-periodic solutions of the equation $x' + x^3 - \lambda x = r(t)$, $\lambda > 0$, in the sense that, besides other things, this equation possesses, under the additional hypothesis $\inf_{t\in\mathbb{R}} r(t) > 0$ for a limit-periodic forcing r, a limit-periodic solution, provided the amplitude of r is sufficiently small. Because of the substitution $t := -\tau$, the same is also true for $x' - x^3 + \lambda x = -r(t)$, $\lambda > 0$.

The easiest situation apparently appears for the limit-periodic difference equations and systems, because as already pointed out the space of limit-periodic sequences endowed with the sup-norm is this time Banach, and subsequently the finiteness of the set of entirely bounded solutions is no longer required. In order to demonstrate its comparison with continuous analogies, we give in Concluding remarks an illustrative example to the equation $x_{t+1} - x_t^3 - \lambda x_t = r_t$, $\lambda > 1$.

Appart from the existence results, the localization and (in)stability analysis is supplied in all our main statements. A certain sort of structural stability called essentiality, which is important for the numerical stability, is investigated also by our technique (see Proposition 2.15 below) for the second-order differential equation and the first-order difference equation.

Let us finally note that the mentioned existence of a transversal homoclinic point of the equivalent system to $x'' - x + x^3 = 0.2\cos t$, for which our Theorem 4.5 below trivially applies, means the presence of deterministic chaos. For more details, see e.g. [12], [18].

2. Preliminaries and auxiliary results

We will recall the notion of a limit-periodic function and its basic properties. Hence, let $\mathrm{C}(\mathbb{R},\mathbb{R}^n)$ denote the set of continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$; $\mathrm{BC}(\mathbb{R},\mathbb{R}^n)$ be the set of continuous functions which are bounded in the sup-norm $\|\cdot\|_\infty\colon f \to \sup_{t\in\mathbb{R}} |f(t)|$; $\mathrm{C}_T(\mathbb{R},\mathbb{R}^n)$ be the set of continuous T-periodic ($T > 0$) functions. Then

$$\mathrm{Per}(\mathbb{R},\mathbb{R}^n) := \bigcup_{T>0} \mathrm{C}_T(\mathbb{R},\mathbb{R}^n)$$

denotes the subset of continuous (bounded) functions.

Definition 2.1. A function $f \in \mathrm{C}(\mathbb{R},\mathbb{R}^n)$ is said to be (*uniformly* or *Bohr*) *almost-periodic* (briefly: a.p.) if, for every $\varepsilon > 0$, there corresponds a relatively dense set $\{\tau\}_\varepsilon$ of ε-almost-periods τ of f (i.e. if there exists a number $l > 0$ such that every interval $[a, a+l]$, $a \in \mathbb{R}$, contains at least one point in $\{\tau\}_\varepsilon$) such that

$$\|f(\,\cdot\, + \tau) - f(\,\cdot\,)\|_\infty \le \varepsilon, \quad \text{for all } \tau \in \{\tau\}_\varepsilon.$$

Let us denote the Banach space of almost-periodic functions $f\colon \mathbb{R} \to \mathbb{R}^n$, endowed with the sup-norm $\|\cdot\|_\infty$, by $\mathrm{AP}(\mathbb{R},\mathbb{R}^n)$.

Definition 2.2. A (bounded) continuous function $f \in \mathrm{C}(\mathbb{R},\mathbb{R}^n)$ is said to be (*uniformly*) *limit-periodic* (l.p.) if there exists a sequence $\{f_k\}$ of continuous periodic functions $f_k \in \mathrm{Per}(\mathbb{R},\mathbb{R}^n)$, $k \in \mathbb{N}$, converging uniformly to f.

Let us denote the set of limit-periodic functions $f\colon \mathbb{R} \to \mathbb{R}^n$ by $\mathrm{LPer}(\mathbb{R},\mathbb{R}^n)$.

Definition 2.3. A continuous function $f \in \mathrm{C}(\mathbb{R},\mathbb{R}^n)$ is said to be *semi-periodic* (s.p.) if

$$\forall \varepsilon > 0 \quad \exists T > 0 \quad \forall j \in \mathbb{Z} \quad \forall t \in \mathbb{R} \quad |f(t+jT) - f(t)| \le \varepsilon.$$

Such a number T will be called an *ε-semi-period* of f.

Let us denote the set of semi-periodic functions $f\colon \mathbb{R} \to \mathbb{R}^n$ by $\mathcal{S}(\mathbb{R},\mathbb{R}^n)$.

It can be easily seen from the definition that every continuous periodic function is semi-periodic. Moreover, if f is semi-periodic, then it is uniformly (Bohr) almost-periodic (i.e. $f \in \mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$) and, in particular, it is bounded. Thus, we can rewrite Definition 2.3 as follows.

DEFINITION 2.4. A (bounded) continuous function $f \in \mathrm{C}(\mathbb{R}, \mathbb{R}^n)$ is said to be *semi-periodic* (s.p.) if

$$\forall \varepsilon > 0 \quad \exists T > 0 \quad \forall j \in \mathbb{Z} \quad \|f(\,\cdot\, + jT) - f(\,\cdot\,)\|_\infty \leq \varepsilon.$$

Definitions 2.2, 2.3 and 2.4 are equivalent (see e.g. [6], [17, p. 115]), i.e.

$$\mathcal{S}(\mathbb{R}, \mathbb{R}^n) = \mathrm{LPer}(\mathbb{R}, \mathbb{R}^n).$$

Moreover, we have

$$\mathrm{Per}(\mathbb{R}, \mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}, \mathbb{R}^n) = \mathrm{LPer}(\mathbb{R}, \mathbb{R}^n) \subset \mathrm{AP}(\mathbb{R}, \mathbb{R}^n) \subset \mathrm{BC}(\mathbb{R}, \mathbb{R}^n).$$

From this, we can also consider $\mathrm{LPer}(\mathbb{R}, \mathbb{R}^n)$ as a metric space with the metric

$$\mathrm{d}(f, g) := \sup_{t \in \mathbb{R}} |f(t) - g(t)| = \|f - g\|_\infty.$$

On the other hand, unfortunately, $\mathrm{LPer}(\mathbb{R}, \mathbb{R}^n)$ is not a linear space, but a complete metric space (see e.g. [6]) which brings some obstructions. Nevertheless, since $\mathrm{LPer}(\mathbb{R}, \mathbb{R}^n)$ is the closure of $\mathrm{Per}(\mathbb{R}, \mathbb{R}^n)$ in the sup-norm, we can define a limit-periodic function as the uniform limit of a uniformly convergent sequence of continuous periodic functions.

REMARK 2.5. We have proved in [6] that limit-periodic functions which are at the same time quasi-periodic are in fact periodic and that "pure" limit-periodic functions can be therefore characterized as those which are limit-periodic, but not periodic.

On this basis, we can already consider the differential system

$$x' = f(x) + p(t), \tag{2.1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function, i.e. $f \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, and $p\colon \mathbb{R} \to \mathbb{R}^n$ is a uniformly limit-periodic function, i.e. $p \in \mathrm{LPer}(\mathbb{R}, \mathbb{R}^n)$, jointly with the associated systems

$$x' = f(x) + p_N(t), \tag{2.2}$$

where the sequence $\{p_N\}$, $N \in \mathbb{N}$, of T_N-periodic functions $p_N \in \mathrm{C}_{T_N}(\mathbb{R}, \mathbb{R}^n)$ converges, according to Definition 2.2, uniformly to p.

For each $N \in \mathbb{N}$, we assume that:

(H1) system (2.2) has a T_N-periodic solution φ_N such that $\|\varphi_N\|_\infty \leq D$, where $D > 0$ is a common constant, for all $N \in \mathbb{N}$, (H2) if $A_N(t)$ is the Jacobian matrix of f at $\varphi_N(t)$, then there exist a nonsingular matrix solution U_N of $y' = A_N(t)y$, and constants $K_N > 0$, $\alpha_N > 0$ such that, for every $N \in \mathbb{N}$,

$$
\begin{aligned}
|U_N(t) I_1 U_N^{-1}(s)|_{\mathcal{M}_{n_1}} &\leq K_N e^{-\alpha_N(t-s)}, \quad \text{for } t \geq s,\\
|U_N(t) I_2 U_N^{-1}(s)|_{\mathcal{M}_{n_2}} &\leq K_N e^{\alpha_N(t-s)}, \quad \text{for } t \leq s,
\end{aligned}
\tag{2.3}
$$

where $I_1 = \mathrm{diag}(E_1, 0)$, $I_2 = \mathrm{diag}(0, E_2)$, E_1 and E_2 are the $n_1 \times n_1$ and $n_2 \times n_2$ unit matrices, respectively, and $n_1 + n_2 = n$. For $n_1 = 0$ or $n_2 = 0$, the corresponding inequality in (2.3) can be omitted.

Substituting $x := y + \varphi_N(t)$ into (2.1), we obtain

$$y' = A_N(t)y + g(t, y, N) + h(t, N),$$

where $h(t, N) := p(t) - p_N(t)$ and $g(t, y, N) := f(y + \varphi_N(t)) - f(\varphi_N(t)) - A_N(t)y$. Observe that h is almost-periodic, but not necessarily limit-periodic, and g is T_N-periodic. Moreover, for $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|g(t, y, N)| \leq \varepsilon |y|, \tag{2.4}$$

and

$$|g(t, y, N) - g(t, z, N)| \leq \varepsilon |y - z|, \tag{2.5}$$

for $|y| \leq \delta(\varepsilon)$, $|z| \leq \delta(\varepsilon)$, $N \in \mathbb{N}$, $t \in \mathbb{R}$.

Remark 2.6. Let us note that condition (2.3) is called the *exponential dichotomy* for $y' = A_N(t)y$, and that A_N need not be periodic in general. For more details about exponential dichotomies, see e.g. [18].

The following theorem, which we state here in the form of a proposition, is due to Seifert (see [19, Theorem 2]).

Proposition 2.7. *Consider* (2.1), *where*

$$f \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n) \quad \textit{and} \quad p \in \mathrm{LPer}(\mathbb{R}, \mathbb{R}^n).$$

Let conditions (H1) *and* (H2) *hold for* (2.2), *for each fixed* $N \in \mathbb{N}$, *and suppose that, for all* N *sufficiently large,*

$$4K_N \|p_n - p\|_\infty \leq \alpha_n \widetilde{\delta}_N, \tag{2.6}$$

where $\widetilde{\delta}_N \to 0$ *as* $N \to \infty$, *and* $\widetilde{\delta}_N \leq \delta(\alpha_n / 4K_N)$, δ *being the function involved in* (2.4) *and* (2.5). *Then if there exists a* $D_1 > D$ *such that* (2.1) *has at most a finite number of solutions* φ *satisfying* $\|\varphi\|_\infty \leq D_1$, *then* (2.1) *possesses a uniformly limit-periodic solution.*

The following corollary allows us to avoid condition (2.6), provided the exponential dichotomy holds with common constants $K > 0$, $\alpha > 0$.

COROLLARY 2.8. *Consider* (2.1), *where*

$$f \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n) \quad \textit{and} \quad p \in \mathrm{LPer}(\mathbb{R}, \mathbb{R}^n).$$

Let the conditions (H1) *and* (H2) *hold for* (2.2), *for each fixed* $N \in \mathbb{N}$. *Suppose that, for all* N *sufficiently large, the exponential dichotomy* (2.3) *holds with common constants* $K > 0$ *and* $\alpha > 0$, *i.e. for sufficiently large values of* N *we can put* $K_N = K$ *and* $\alpha_N = \alpha$. *If system* (2.1) *has at most a finite number of entirely bounded solutions* φ *such that* $\|\varphi\|_\infty \le D_1$, $D_1 > D$, *then* (2.1) *possesses a uniformly limit-periodic solution.*

Now, let us briefly turn to limit-periodic and semi-periodic sequences.

DEFINITION 2.9. A sequence $\underline{x} := \{x_t\} \in (\mathbb{R}^n)^{\mathbb{Z}}$ is called (*uniformly*) *limit-periodic* (l.p.) if there exists a sequence of periodic sequences $\underline{x}^k := \{x_t^k\}$, $k \in \mathbb{N}$, such that $\lim\limits_{k\to\infty} x_t^k = x_t$, uniformly with respect to $t \in \mathbb{Z}$.

DEFINITION 2.10. A sequence $\underline{x} := \{x_t\} \in (\mathbb{R}^n)^{\mathbb{Z}}$ is called *semi-periodic* (s.p.) if

$$\forall\, \varepsilon > 0 \quad \exists\, T \in \mathbb{N} \quad \forall\, m \in \mathbb{Z} \quad \forall\, k \in \mathbb{Z} \quad |x_{k+mT} - x_k| \le \varepsilon.$$

The following proposition (see [6, Proposition 1]) relates the link between semi-periodic sequences and functions. Given a sequence $\underline{x} := \{x_t\}$, $t \in \mathbb{Z}$, consider the function $f_{\underline{x}} \colon \mathbb{R} \to \mathbb{R}^n$ such that its restriction to $\mathbb{Z}$ is $\underline{x}$ and which is linear on each interval $[k, k+1]$, $k \in \mathbb{Z}$, namely

$$f_{\underline{x}}(u) := \{u\}x_{t+1} + (1 - \{u\})x_t, \quad \text{for all } t \in \mathbb{Z},$$

where $\{u\}$ is the fractional part of u, i.e. $\{u\} \in [0, 1)$ and $u - \{u\} \in \mathbb{N} \cup \{0\}$.

PROPOSITION 2.11. *For* $\underline{x} \in (\mathbb{R}^n)^{\mathbb{Z}}$, *all the following statements are equivalent*:

(a) $f_{\underline{x}}$ *is semi-periodic with a semi-period in* $\mathbb{N}$,
(b) *there exists a semi-periodic function with a semi-period in* $\mathbb{N}$ *whose restriction to* $\mathbb{Z}$ *is* $\underline{x}$,
(c) $\underline{x}$ *is semi-periodic.*

REMARK 2.12. Quite analogously to the continuous case, one can prove the equivalence of Definitions 2.9 and 2.10 (cf. [6]). Observe that Definition 2.10 easily implies Definition 2.9, by Proposition 2.11.

It is well known that (unlike in the continuous case) the set of uniformly limit-periodic sequences $\underline{x} := \{x_t\} \in (\mathbb{R}^n)^{\mathbb{Z}}$, endowed with the sup-norm $\|\cdot\|_\infty$, i.e.

$$\|\underline{x}\|_\infty = \|\{x_t\}\|_\infty := \sup_{t\in\mathbb{Z}} |x_t|,$$

is a Banach space (see e.g. [6], and the references therein).

Remark 2.13. We have proved in [6] that (like in the continuous case) limit-periodic sequences which are at the same time quasi-periodic are in fact periodic and that "pure" limit-periodic sequences can be therefore characterized as those which are limit-periodic, but not periodic.

In Concluding remarks, we will indicate an easier application of a discrete analogy of Corollary 2.8 to a scalar limit-periodic difference equation.

It will be also convenient to recall the notion of essentiality introduced by Fort, Jr. (see e.g. [4], and the references therein).

Definition 2.14. Let $X = (X, \mathrm{d})$ be a metric space, $f\colon X \to X$ be a continuous mapping and x_0 be an isolated fixed point of f, i.e. $x_0 = f(x_0)$. We say that $x_0 \in X$ is an *essential fixed point* of f if, for every open ε-neighbourhood ($\varepsilon > 0$) $\mathcal{U}$ of x_0, there exists $\delta = \delta(\varepsilon) > 0$ such that any continuous map $g\colon X \to X$ which is δ-near to f, i.e. $\sup_{x\in X} \mathrm{d}(f(x), g(x)) < \delta$, has a fixed point in $\mathcal{U}$.

The following proposition was given in [4, Theorem 3].

Proposition 2.15. *Let X be a metric absolute retract, i.e. up to a homeomorphism and up to a retraction a convex set, and $f\colon X \to X$ be a compact* (*continuous*) *mapping. Assume that the set* $\mathrm{Fix}(f)$ *of fixed points of f satisfies* $\dim \mathrm{Fix}(f) = 0$, *where* $\dim$ *stands for the topological* (*coverging*) *dimension. Then f admits an essential fixed point.*

3. Limit-periodic solutions of the first-order equation

At first, we will show that the equation $x' + x^3 - \lambda x = r(t)$, $\lambda > 0$, admits, for a strictly positive limit-periodic forcing r such that $\|r\|_\infty$ is sufficiently small (namely $\|r\|_\infty < 2 \cdot 3^{-3/2} \cdot \lambda^{3/2}$), an unstable limit-periodic solution. Hence, consider the equation

$$x' + x^3 - \lambda x = r(t), \tag{3.1}$$

where $r\colon \mathbb{R} \to \mathbb{R}$ is a continuous bounded function.

Let us note that the related case of the equation $x' - x^3 + \lambda x = -r(t)$, $\lambda > 0$, will be also considered in Remark 3.9 below. The following Ambrosetti–Prodi type lemma is a particular case of [21, Theorem 5.1 and Corollary 1.1] due to Tineo.

Lemma 3.1. *Let $r\colon \mathbb{R} \to \mathbb{R}$ be a bounded continuous function such that*

$$\inf_{t\in\mathbb{R}} r(t) > 0. \tag{3.2}$$

Then there exists a critical value $\lambda_* \in \mathbb{R}$ such that equation (3.1) *has*

(a) *exactly three separate, entirely bounded solutions u_1, u_2, u_3 such that*

$$u_3(t) < u_2(t) < 0 < u_1(t), \quad t \in \mathbb{R}, \ \text{for } \lambda > \lambda_*, \tag{3.3}$$

(b) *exactly two separate, entirely bounded solutions* u_1, u_2 *such that*

$$u_2(t) < 0 < u_1(t), \quad t \in \mathbb{R}, \text{ for } \lambda = \lambda_*,$$

(c) *exactly one entirely bounded solution* u_1 *such that*

$$0 < u_1(t), \quad t \in \mathbb{R}, \text{ for } \lambda < \lambda_*.$$

In particular, equation (3.1) *has, for any* $\lambda \in \mathbb{R}$, *a finite number of entirely bounded solutions.*

For $\lambda > 0$, and a uniformly almost-periodic (a.p.) forcing r, the following result was obtained by Haraux in [14, Example 5.1].

LEMMA 3.2. *Let* $r\colon \mathbb{R} \to \mathbb{R}$ *be a uniformly almost-periodic function such that*

$$\|r\|_\infty < \frac{2}{3}\lambda\sqrt{\frac{\lambda}{3}}, \quad \lambda > 0. \tag{3.4}$$

Then equation (3.1) *has exactly three uniformly* (*Bohr*) *almost-periodic solutions* u_1, u_2, u_3. *These solutions satisfy the inequalities*

$$u_3(t) \le -\sqrt{\frac{\lambda}{3}} \le u_2(t) \le \sqrt{\frac{\lambda}{3}} \le u_1(t), \quad t \in \mathbb{R}. \tag{3.5}$$

Moreover, u_1 *and* u_3 *are* (*positively*) *stable, while* u_2 *is* (*positively*) *unstable. If* r *is, in particular,* T*-periodic* ($T > 0$), *then the solutions* u_1, u_2, u_3 *are also* T*-periodic.*

REMARK 3.3. In view of Lemma 3.1, the critical value λ_* must be, under (3.2) and (3.4), non-positive, i.e. $\lambda_* \le 0$.

Lemmas 3.1 and Lemma 3.2 can be easily matched as follows.

LEMMA 3.4. *Let* $r\colon \mathbb{R} \to \mathbb{R}$ *be a uniformly almost-periodic function, satisfying* (3.2) *and* (3.4). *Then equation* (3.1) *has exactly three separate, uniformly* (*Bohr*) *almost periodic solutions* u_1, u_2, u_3. *These solutions satisfy at the same time the inequalities* (3.3) *and* (3.5). *In particular,* u_2 *satisfies the inequalities*

$$-\sqrt{\frac{\lambda}{3}} \le u_2(t) < 0 < \sqrt{\frac{\lambda}{3}}, \quad t \in \mathbb{R}. \tag{3.6}$$

Moreover, u_1 *and* u_3 *are* (*positively*) *stable, while* u_2 *is* (*positively*) *unstable. If* r *is still* T*-periodic* ($T > 0$), *then the solutions* u_1, u_2, u_3 *are also* T*-periodic.*

Now, consider the linear homogenous equation

$$x' + [3q_T^2(t) - \lambda]x = 0, \quad \lambda > 0, \tag{3.7}$$

where $q_T := u_2$ is a T-periodic solution of (3.1), guaranteed by Lemma 3.4.

We will prove that the equation (3.7) possesses, under the above assumptions (3.4) and (3.6), an exponential dichotomy with suitable constants $K > 0$ and $\alpha > 0$.

Setting $a(t) := 3q_T^2(t) - \lambda$, we have $a(t) \le 0$, for $t \in [0,T]$, because $\|q_T\|_\infty \le \sqrt{\lambda/3}$, according to (3.6). If $a = 0$, on a nondegenerated subinterval $J \subset [0,T]$, then

$$r(t) := q_T^3(t) - \lambda q(t) = -\frac{2}{3}\,\lambda\sqrt{\frac{\lambda}{3}}$$

on J, which is impossible, because $\|r\|_\infty < (2/3)\,\lambda\sqrt{\lambda/3}$, according to (3.4). Thus, the mean value $\mathcal{M}\{a\}$ of a is negative, i.e. $\mathcal{M}\{a\} < 0$.

Every solution x of (3.7) takes the form

$$x(t) = x(s)\exp\bigg(-\int_s^t a(u)\,du\bigg).$$

Let us put

$$F(t) := \int_0^t [\mathcal{M}\{a\} - a(u)]\,du.$$

Since $\mathcal{M}\{a\} - a$ is periodic with the mean value equal to zero, F must be bounded. Subsequently,

$$-\int_s^t a(u)\,du = F(t) - F(s) - \mathcal{M}\{a\}(t-s) \le 2\|F\|_\infty - \mathcal{M}\{a\}(t-s),$$

for $t < s$, and so

$$|x(t)| \le |x(s)|e^{2\|F\|_\infty}e^{-\mathcal{M}\{a\}(t-s)}, \quad \text{for } t < s.$$

In this way, the exponential dichotomy for (3.7) occurs with the constants $K = e^{2\|F\|_\infty}$, $\alpha = -\mathcal{M}\{a\} > 0$, and $I_2 = 1$. Therefore, we can formulate the following lemma.

Lemma 3.5. *Equation* (3.7), *where* $q_T\colon \mathbb{R} \to \mathbb{R}$ *is a continuous* T*-periodic* ($T > 0$) *function such that* (*cf.* (3.6))

$$-\sqrt{\frac{\lambda}{3}} \le q_T(t) < 0, \quad t \in [0,T],$$

exhibits an exponential dichotomy with the constants $K = e^{2\|F\|_\infty}$, $\alpha = -\mathcal{M}\{a\}$, *defined above.*

Since it is not evident whether the constants K, α, characterizing the exponential dichotomy (since a depends on T, so do $\|F\|_\infty$ and $\mathcal{M}\{a\}$), can be taken uniformly, for any q_T such that

$$\|q_T\|_\infty \le \sqrt{\frac{\lambda}{3}}, \quad T > 0,$$

we cannot apply Corollary 2.8. On the other hand, we can apply Proposition 2.7, for which we need, however, to make some further calculations.

Hence, let us consider the one-parameter family of equations

$$x' + x^3 - \lambda x = r_k(t), \quad k \in \mathbb{N},\ \lambda > 0, \tag{3.8}$$

where $r_k\colon \mathbb{R}\to\mathbb{R}$ are continuous T_k-periodic functions such that

$$\lim_{k\to\infty}\|r-r_k\|_\infty = 0, \tag{3.9}$$

where $r\colon \mathbb{R}\to\mathbb{R}$ is a limit-periodic function in (3.1), satisfying (3.2) and (3.4). It follows from the definition of a limit-periodic function r that if (3.2) and (3.4) hold, then there always exists a sequence $\{r_k\}$ of T_k-periodic functions, converging uniformly to r, i.e. (3.9), such that

$$\sup_{k\in\mathbb{N}}\inf_{t\in\mathbb{R}} r_k(t) > 0, \tag{3.10}$$

$$\sup_{k\in\mathbb{N}}\|r_k\|_\infty < \frac{2}{3}\,\lambda\sqrt{\frac{\lambda}{3}}, \qquad \text{i.e. } \sup_{k\in\mathbb{N}}\max_{t\in[0,T_k]}|r_k(t)| < \frac{2}{3}\,\lambda\sqrt{\frac{\lambda}{3}}. \tag{3.11}$$

Thus, according to Lemma 3.4, equation (3.8) admits, for every $k\in\mathbb{N}$, a T_k-periodic solution φ_k such that $\|\varphi_k\|_\infty \le \sqrt{\lambda/3}$.

Now, after the substitution $x := y+\varphi_k$ into (3.1), we receive the equation

$$y' = -3\varphi_k^2(t)y + \lambda y - g(t,y,k) + h(t,k), \tag{3.12}$$

where $g(t,y,k) := y^3 + 3y^2\varphi_k(t)$ and $h(t,k) := r(t) - r_k(t)$, and so

$$\begin{aligned}
|g(t,y,k)| &\le |y|(y^2 + 3|y|\,\|\varphi_k\|_\infty),\\
|g(t,y,k) - g(t,z,k)| &= |(y-z)(y^2+yz+z^2) + 3\varphi_k(t)(y+z)(y-z)|\\
&\le |y-z|(y^2+|yz|+z^2+3|\varphi_k(t)|\,|y+z|).
\end{aligned}$$

Subsequently,

$$\begin{aligned}
|g(t,y,k)| &\le |y|\delta\big(\delta+\sqrt{3\lambda}\big), && \text{for } \|\varphi_k\|_\infty \le \sqrt{\frac{\lambda}{3}} \text{ and } |y|\le\delta,\\
|g(t,z,k)| &\le |z|\delta\big(\delta+\sqrt{3\lambda}\big), && \text{for } \|\varphi_k\|_\infty \le \sqrt{\frac{\lambda}{3}} \text{ and } |z|\le\delta,\\
|g(t,y,k)-g(t,z,k)| &\le |y-z|\delta\big(3\delta+2\sqrt{3\lambda}\big), && \text{for } \|\varphi_k\|_\infty \le \sqrt{\frac{\lambda}{3}}\\
& && \text{and } |y|\le\delta,\ |z|\le\delta.
\end{aligned}$$

Setting, for $\varepsilon>0$, $\delta=\delta(\varepsilon)>0$ in order $\varepsilon = \delta\big(3\delta+2\sqrt{3\lambda}\big)$ to be satisfied, i.e.

$$\delta = \frac{\sqrt{3}}{3}\big(\sqrt{\lambda+\varepsilon}-\sqrt{\lambda}\big),$$

we can see that $\delta\to 0$, when $\varepsilon\to 0$. Moreover,

$$\|r-r_k\|_\infty \le \varepsilon\delta(\varepsilon) = \delta^2(\varepsilon)\big(3\delta(\varepsilon)+2\sqrt{3\lambda}\big) = \frac{\sqrt{3}}{3}\,\varepsilon\left(\sqrt{\lambda+\varepsilon}-\sqrt{\lambda}\right)$$

holds, for a given $\varepsilon>0$, whenever $k\ge k_\varepsilon$, when $k_\varepsilon\in\mathbb{N}$ is a sufficiently large integer. Summing up, we can formulate the following lemma.

Lemma 3.6. *For a given limit-periodic function* $r\colon \mathbb{R} \to \mathbb{R}$*, satisfying* (3.2) *and* (3.4)*, there exist a sequence* $\{r_k\}$ *of* T_k*-periodic continuous functions, satisfying* (3.9)–(3.11)*, and a sequence* $\{\varphi_k\}$ *of related* T_k*-periodic solutions of* (3.8)*, satisfying*

$$\max_{t\in[0,T_k]} |\varphi_k(t)| \leq \sqrt{\frac{\lambda}{3}},$$

for all $k \in \mathbb{N}$*. Furthermore, for a given* $\varepsilon > 0$*, the inequalities*

$$\big|y^3 + 3y^2\varphi_k(t)\big| \leq \frac{\varepsilon}{2}|y| \quad \textit{and} \quad \big|y^3 - z^2 + 3\varphi_k(t)\big(y^2 - z^2\big)\big| \leq \varepsilon|y - z|$$

hold, for $t \in [0, T_k]$*, whenever*

$$|y| \leq \frac{\sqrt{3}}{3}\big(\sqrt{\lambda+\varepsilon} - \sqrt{\lambda}\,\big), \quad |z| \leq \frac{\sqrt{3}}{3}\big(\sqrt{\lambda+\varepsilon} - \sqrt{\lambda}\,\big),$$

and a positive integer k_ε *exists such*

$$\|r - r_k\|_\infty \leq \frac{\sqrt{3}}{3}\,\varepsilon\big(\sqrt{\lambda+\varepsilon} - \sqrt{\lambda}\,\big) \tag{3.13}$$

is satisfied, for all $k \geq k_\varepsilon$*.*

Remark 3.7. As a consequence of Lemma 3.6, i.e. under (3.2) and (3.4), for $k \geq k_\varepsilon$, there exists a sequence $\{y_k\}$ of limit-periodic solutions to (3.12) such that

$$\|y_k\|_\infty \leq \delta(\varepsilon) = \frac{\sqrt{3}}{3}\big(\sqrt{\lambda+\varepsilon} - \sqrt{\lambda}\,\big),$$

and so $\lim_{k\to\infty} \|y_k\|_\infty = 0$, for $\varepsilon \to 0$ ($\Rightarrow \delta(\varepsilon) \to 0$). For more details, see the arguments in the proof of [19, Theorem 2]. Consequently, equation (3.1) has a uniformly limit-periodic solution

$$u = \lim_{k\to\infty} y_k + \lim_{k\to\infty} \varphi_k = \lim_{k\to\infty} \varphi_k$$

such that $\|u\|_\infty \leq \sqrt{\lambda/3}$.

We are ready to formulate the first main theorem.

Theorem 3.8. *Equation* (3.1)*, where* $r\colon \mathbb{R} \to \mathbb{R}$ *is a uniformly limit-periodic function such that* (3.2) *and* (3.4) *is satisfied, i.e.*

$$\inf_{t\in\mathbb{R}} r(t) > 0 \quad \textit{and} \quad \|r\|_\infty < \frac{2}{3}\,\lambda\sqrt{\frac{\lambda}{3}},$$

admits a uniformly almost-periodic solution u *such that* (3.6) *holds, i.e.*

$$-\sqrt{\frac{\lambda}{3}} \leq u(t) < 0 < \sqrt{\frac{\lambda}{3}}, \quad t \in \mathbb{R}.$$

This solution is positively unstable, but negatively asymptotically stable. Moreover, equation (3.1) *admits exactly two further entirely bounded solutions* u_-, u_+ *such that*

$$u_-(t) \le -\sqrt{\frac{\lambda}{3}}, \quad u_-(t) < u(t) \quad \text{and} \quad \sqrt{\frac{\lambda}{3}} \le u_+, \quad t \in \mathbb{R}.$$

These solutions are both uniformly almost-periodic and positively stable.

PROOF. Since we apply Proposition 2.7, let us check its assumptions. It follows from Definition 2.9 that, to a given r, a sequence $\{r_k\}$ of T_k-periodic functions r_k exists such that conditions (3.9)–(3.11) are satisfied. Then hypothesis (H1) is implied by Lemma 3.4, because T_k-periodic solutions u_k of (3.8) exist such that $\|u_k\|_\infty \le \sqrt{\lambda/3} := D$, for all related $k \in \mathbb{N}$. Hypothesis (H2) is ensured by Lemma 3.5. Condition (2.6), which takes here the form (3.13), is implied by Lemma 3.6. At most a finite number of entirely bounded solutions v of (3.1), satisfying $\|v\|_\infty \le D_1$, $D_1 > 0$, follows trivially from Lemma 3.1. Thus, in view of Remark 3.7, equation (3.1) possesses, according to Proposition 2.7, a uniformly limit-periodic solution u such that $\|u\|_\infty \le \sqrt{\lambda/3}$.

According to Lemma 3.4, the localization of u can be still improved (see (3.6)), and completed by the exact multiplicity result (see (3.3) and (3.5)). The (in)stability properties follow from Lemma 3.2 and the analysis performed before Lemma 3.5. □

REMARK 3.9. Because of the substitution $t := -\tau$ into (3.1), one can easily check that, under the assumptions of Theorem 3.8, the same conclusion, except (in)stability, is true for the equation $x' - x^3 + \lambda x = -r(t)$, $\lambda > 0$, because then the solution u becomes positively asymptotically stable and negatively unstable.

4. Limit-periodic solutions of the second-order equation

Now, we will show that the limit-periodic equation $x'' + x^3 - x = r(t)$ admits, for a sufficiently small value of $\|r\|_\infty$ (namely $\|r\|_\infty < 8/27$), an essential limit-periodic solution. Hence, consider the Duffing-type differential equation

$$x'' + x^3 - x = r(t), \tag{4.1}$$

where $r\colon \mathbb{R} \to \mathbb{R}$ is a continuous bounded function. Consider also the one-parameter family of linearized equations

$$x'' + \varphi^2(t)x - x = r(t), \tag{4.2}$$

obtained from (4.1) by the Schauder-like parametrization, for

$$\varphi \in \mathcal{B}_1 := \left\{ p_1 \in \mathrm{C}(\mathbb{R}, \mathbb{R}) \;\middle|\; \|p_1\|_\infty \le \frac{1}{3} \right\},$$

jointly with the equivalent linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-\varphi^2(t) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ r(t) \end{pmatrix}. \tag{4.3}$$

Finally, consider the homogeneous system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-C\varphi^2(t) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad C > 0, \tag{4.4}$$

which is, for $C = 1$, associated with (4.3).

The following lemma is a slight generalization of [22, Theorem 1], and its proof is quite analogous to the one, for $C = 1$, done by Zeng.

Lemma 4.1. *System* (4.4), *where* $9 > C > 0$, *exhibits, for any* $\varphi \in \mathcal{B}_1$, *an exponential dichotomy on* $\mathbb{R}$ *with common constants* K_C, α_C *such that* $2K_1/\alpha_1 \le 9/8$ $(C = 1)$.

The following lemma is a slight modification of [22, Theorem 2].

Lemma 4.2. *Equation* (4.1) *possesses an entirely bounded solution* u *such that* $\|(u,u')\|_\infty \le 1/3 - 9\Delta/8$, *when* $\|r\|_\infty \le 8/27 - \Delta$, *where* $(8/27 >)$ $\Delta > 0$ *is a small constant. This solution is unique in the ball* $\mathcal{B}_1$.

Proof (sketch). The (unique) entirely bounded solution u of (4.1) can be equivalently considered as the first component of the (unique) fixed point of the solution operator $T\colon \mathcal{B} \to \mathcal{B}$, associated with (4.3), where

$$\mathcal{B} := \left\{ (p_1, p_2) \in \mathrm{C}(\mathbb{R}, \mathbb{R}^2) \;\middle|\; \|p_1\|_\infty \le \frac{1}{3},\ \|p_2\|_\infty \le \frac{2K_1}{\alpha_1}\|r\|_\infty \right\},$$

and $T\colon \psi \to T\psi$, where

$$T\psi := \int_{-\infty}^{t} X_\psi(t) P_\psi X_\psi^{-1}(s) \begin{pmatrix} 0 \\ r(s) \end{pmatrix} ds - \int_t^\infty X_\psi(t)(I - P_\psi) X_\psi^{-1}(s) \begin{pmatrix} 0 \\ r(s) \end{pmatrix} ds,$$

X_ψ is the fundamental matrix to (4.3), I is the unit matrix and P_ψ $(= P_\psi^2)$ is the projection. In view of the exponential dichotomy for (4.4), where $C = 1$, this solution operator prescribes, for any $\psi \in \mathcal{B}$, a unique bounded solution (x, y) of (4.3). Furthermore, T is compact in the Fréchet topology and such that

$$T(\mathcal{B}) \subset \mathcal{B}_\Delta := \left\{ p \in \mathrm{C}(\mathbb{R}, \mathbb{R}^2) \;\middle|\; \|p\|_\infty \le \frac{1}{3} - \frac{9}{8}\Delta \right\} \ (\subset \mathcal{B}).$$

Therefore, according to the Schauder–Tikhonov fixed point theorem (see e.g. [3, Theorem I.1.38]), it has a fixed point $\bar{p} = (u, u')$ in $\mathcal{B}_\Delta$, i.e. $\|(u,u')\|_\infty \le 1/3 - 9\Delta/8$, as claimed.

Its uniqueness on $\mathcal{B}$ can be proved by a contradiction, when assuming that we have two different fixed points, say (u_1, u_1') and (u_2, u_2'), where

$$u_i(t) = \int_{-\infty}^{\infty} G(t,s)\big[r(s) - u_i^3(s)\big]\, ds, \quad i = 1, 2, \tag{4.5}$$

G denotes the Green function associated to the homogenous equation $x'' - x = 0$ such that (see e.g. [3, p. 553])

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} |G(t,s)|\, ds \le 2.$$

Thus,

$$\begin{aligned}\|u_1 - u_2\|_\infty &\le \|u_1^3 - u_2^3\|_\infty \int_{-\infty}^{\infty} \big|G(t,s)\big|\, ds \\ &\le 2\|u_1 - u_2\|_\infty \big(\|u_1^2\|_\infty + \|u_1 u_2\|_\infty + \|u_2^2\|_\infty\big) \le \frac{2}{3}\|u_1 - u_2\|_\infty,\end{aligned}$$

which is a contradiction. □

The existence part of Lemma 4.2 can be easily reformulated for a T-periodic forcing r as follows.

LEMMA 4.3. *Equation* (4.1)*, where r is a T-periodic* ($T > 0$) *continuous function satisfying $\|r\|_\infty \le 8/27 - \Delta$,* ($8/27 >$) $\Delta > 0$*, possesses a T-periodic solution u such that*

$$\max_{t\in[0,T]} \big|(u(t), u'(t))\big| \le \frac{1}{3} - \frac{9}{8}\,\Delta.$$

PROOF (sketch). We can proceed quite analogously as in the proof of Lemma 4.2 but, instead of the Schauder–Tikhonov theorem, it is enough to apply the Schauder fixed point theorem. For more details, see e.g. [3, Chapter III.5]. □

REMARK 4.4. As already pointed out, it follows from the definition of a limit-periodic function, say r, that if $\|r\|_\infty < R$, then there exists a sequence $\{r_k\}$ of T_k-periodic functions, converging uniformly to r, such that

$$\sup_{k\in\mathbb{N}} \|r_k\|_\infty < R, \quad \text{i.e.} \ \sup_{k\in\mathbb{N}} \max_{t\in[0,T_k]} |r_k(t)| < R.$$

We are ready to formulate the second main theorem.

THEOREM 4.5. *Equation* (4.1)*, where $r\colon \mathbb{R} \to \mathbb{R}$ is a uniformly limit-periodic function such that $\|r\|_\infty < 8/27$, admits a uniformly limit-periodic solution u, satisfying $\|u\|_\infty < 1/3$, which is positively as well as negatively unstable. This solution is unique on the ball*

$$\mathcal{B}_1 := \left\{ p_1 \in \mathrm{C}(\mathbb{R},\mathbb{R}) \;\middle|\; \|p_1\|_\infty \le \frac{1}{3} \right\},$$

and subsequently it is essential with respect to the integral operator $T|_{\mathcal{B}_1}\colon \mathcal{B}_1 \to \mathcal{B}_1$, where

$$T\varphi := \int_{-\infty}^{\infty} G(t,s)\big[r(s) - \varphi^3(s)\big]\, ds$$

was defined in (4.5).

Proof. Since we apply Corollary 2.8, let us check its assumptions. It follows from Definition 2.9 that, to a given r, a sufficiently small constant $(8/27 >)\ \Delta > 0$ and a sequence $\{r_k\}$ of T_k-periodic functions r_k exist such that $\|r_k\|_\infty \leq 8/27-\Delta$ holds, for all related $k \in \mathbb{N}$ (see Remark 4.4). Hypothesis (H1) is, therefore, implied by Lemma 4.3, where $D := 1/3 - 9\Delta/8$. Hypothesis (H2) is ensured by Lemma 4.1. At most a finite number of entirely bounded solutions v of (4.1), satisfying $\|(v,v')\|_\infty \leq D_1 := 1/3 > D := 1/3 - 9\Delta/8$, follows from Lemma 4.2. Thus, equation (4.1) possesses, according to Corollary 2.8, a uniformly limit-periodic solution u such that $\|u\|_\infty < 1/3$. Since the exponential dichotomy (2.3) holds for (4.4) with $n_1 = n_2 = 1$ $(n = n_1 + n_2 = 2)$, this solution is positively as well as negatively unstable.

Its uniqueness on the ball $\mathcal{B}_1$ is ensured by Lemma 4.2, and its essentiality subsequently by Proposition 2.15. □

Remark 4.6. It can be checked, when following the arguments of the proof of Theorem 4.5, that the analogous conclusion also holds for the equation $x'' - x^3 - x = r(t)$, which justifies the main result in [7] in the particular case of a limit-periodic forcing r.

5. Concluding remarks (discrete case)

As already pointed out, the situation is significantly easier in the discrete case of difference systems, because the assumption about a finite number of entirely bounded solutions of given systems is no longer required. For the proof of the following discrete analogy of Corollary 2.8, see [6, Corollary 3.3].

Hence, consider the difference system

$$x_{t+1} = f(x_t) + p_t, \tag{5.1}$$

where $f \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\{p_t\} \in (\mathbb{R}^n)^{\mathbb{Z}}$ is a limit-periodic sequence, jointly with the associated systems

$$x_{t+1} = f(x_t) + p_t^N, \tag{5.2}$$

where the family $\{p_t^N\} \in (\mathbb{R}^n)^{\mathbb{Z}}$, $N \in \mathbb{N}$, of T_N-periodic $(T_N > 0)$ sequences converges, according to Definition 2.9, uniformly to $\{p_t\}$.

Proposition 5.1. *Assume still that*

(a) *for each fixed N, system (5.2) admits a T_N-periodic solution $\{x_t^N\}$,*
(b) $\sup_{N\in\mathbb{N}} \big\|\{x_t^N\}\big\|_\infty < \infty$,
(c) *if A_t^N is the Jacobian matrix of f at x_t^N, then there exists a non-singular solution of the homogeneous system $y_{t+1} = A_t^N y_t$ which satisfies the exponential dichotomy, for all sufficiently large values of N, with common constants K and α.*

Then system (5.1) *possesses a uniformly limit-periodic solution.*

REMARK 5.2. The exponential dichotomy for $y_{t+1} = A_t^N y_t$ can be defined quite analogously as in the continuous case (see e.g. [6], [15], [18]).

We will finally give a simple illustrative example of the application of Proposition 5.1.

EXAMPLE 5.3. Consider the difference equation

$$x_{t+1} - x_t^3 - \lambda x_t = r_t, \quad \lambda > 1, \tag{5.3}$$

where $\underline{r} = \{r_t\} \colon \mathbb{Z} \to \mathbb{R}$ is a limit-periodic sequence, and its Schauder-like parametrization

$$x_{t+1} - \big(q_t^2 + \lambda\big)x_t = r_t, \quad \lambda > 1, \tag{5.4}$$

where $\underline{q} \in Q := \big\{\underline{p} \in \mathbb{R}^{\mathbb{Z}} \bigm| \|\underline{p}\|_\infty \leq D\big\}$, $D > 0$ is a suitable constant which will be specified below.

Consider still the homogeneous equation

$$x_{t+1} = (Cq_t^2 + \lambda)x_t, \quad C > 0,\ \lambda > 1, \tag{5.5}$$

associated for $C = 1$ with (5.4), where $\{r_t\} \equiv 0$. Since $Cq_t^2 + \lambda \geq \lambda > 1$, for all $t \in \mathbb{Z}$, we obtain that

$$|x_t| = |x_s| \prod_{j=s}^{t-1} (Cq_j^2 + \lambda) \geq \lambda^{t-s}|x_s|, \quad \text{for all } t \geq s. \tag{5.6}$$

Therefore, $|x_s| \leq \lambda^{-(t-s)}|x_t|$, for $s \leq t$. Hence, an exponential dichotomy holds for (5.5) on $\mathbb{Z}$ with the constants $K = K(\lambda) = 1$, $\alpha = \alpha(\lambda) = 1/\lambda$, which are independent of $C > 0$ (see e.g. [15], [18]), and subsequently (5.4) possesses a unique entirely bounded solution $\underline{u}$ such that

$$\|\underline{u}\|_\infty \leq K\,\frac{1+\alpha}{1-\alpha}\|\underline{r}\|_\infty. \tag{5.7}$$

This solution takes the form

$$u_t = \sum_{l \in \mathbb{Z}} G_q(t,l)\, r_{l-1},$$

where G_q denotes the Green function for (5.5), when $C = 1$. Moreover, $\underline{u}$ becomes T_k-periodic, $T_k \in \mathbb{N}$, $k \in \mathbb{N}$, whenever $\underline{r}$ is so.

Furthermore, the operator $\mathcal{T}_k \colon Q_k \to \mathbb{R}^{\mathbb{Z} \cap T_k}$, $k \in \mathbb{N}$, where

$$Q_k := \big\{\underline{p} \in \mathbb{R}^{\mathbb{Z}\cap T_k} \bigm| \|\underline{p}\|_{\mathbb{R}^{\mathbb{Z}\cap T_k}} \leq D\big\}, \qquad \mathcal{T}_k(q) := \sum_{l \in \mathbb{Z}\cap T_k} G_q(t,l) r_{l-1},$$

can be easily checked, by the standard arguments, to be compact and such that $\mathcal{T}_k(Q_k) \subset Q_k$, provided $\underline{r}$ satisfies (cf. (5.7))

$$\|\underline{r}\|_\infty = \|\underline{r}\|_{\mathbb{R}^{\mathbb{Z}\cap T_k}} < \frac{1-\alpha}{1+\alpha}\frac{D}{K} := R. \tag{5.8}$$

In other words, for a given $\underline{r}$ such that $\|\underline{r}\|_\infty < R$, D in the definitions of Q and Q_k can be always taken as $D = RK(1+\alpha)/(1-\alpha)$, in order (5.8) to be satisfied. Hence, applying the Brouwer fixed point theorem, $\mathcal{T}_k$ admits a fixed point $\{\varphi_k\} \in Q_k$, $k \in \mathbb{N}$, representing a T_k-periodic solution of (5.3), where $\underline{r}$ is T_k-periodic. Moreover,

$$\sup_{k\in\mathbb{N}} \|\{\varphi_k\}\|_\infty \leq D.$$

Now, taking $\underline{q} := \underline{\varphi}_k$, $k \in \mathbb{N}$, equation (5.5) with $C = 3$ was already shown to exhibit an exponential dichotomy with common constants $K = 1$, $\alpha = 1/\lambda$, for all $k \in \mathbb{N}$. Let us note that, by the definition of a limit-periodic sequence $\underline{r}$ satisfying $\|\underline{r}\|_\infty < R$, there exists a sequence $\{\underline{r}_k\}$ of T_k-periodic sequences $\underline{r}_k$, converging uniformly to $\underline{r}$, such that

$$\sup_{k\in\mathbb{N}} \|\underline{r}_k\|_\infty < R.$$

Since all the assumptions of Proposition 5.1 are in this way (i.e. for any $\lambda > 1$ and any $\underline{r}$) satisfied, (5.3) possesses accordingly a uniformly limit-periodic solution, say $\underline{z}$, such that

$$\|\underline{z}\|_\infty \leq RK\,\frac{1+\alpha}{1-\alpha} = R\,\frac{\lambda+1}{\lambda-1} = D.$$

Moreover, this solution is positively unstable and negatively asymptotically stable (see (5.6)).

Remark 5.4. Since

$$\begin{aligned}\|\underline{\varphi}_1 - \underline{\varphi}_2\|_\infty &\leq \|\underline{\varphi}_1^3 - \underline{\varphi}_2^3\|_\infty \sum_{l\in\mathbb{Z}} |G_0(t,l)| \\ &\leq \|\underline{\varphi}_1 - \underline{\varphi}_2\|_\infty\, 3D^2K\,\frac{1+\alpha}{1-\alpha} \leq \|\underline{\varphi}_1 - \underline{\varphi}_2\|_\infty\, 3R^2K^3\left(\frac{1+\alpha}{1-\alpha}\right)^3\end{aligned}$$

holds, for all $\underline{\varphi}_1, \underline{\varphi}_2 \in Q$, where G_0 is the Green function for the equation $x_{t+1} = \lambda x_t$, equation (5.3) has a unique limit-periodic solution on Q, provided

$$3R^2K^3\left(\frac{1+\alpha}{1-\alpha}\right)^3 < 1,$$

i.e. whenever

$$R < \sqrt{\frac{K^{-3}}{3}\left(\frac{1-\alpha}{1+\alpha}\right)^3}$$

holds. Thus, if

$$\|\underline{r}\|_\infty < \sqrt{\frac{K^{-3}}{3}\left(\frac{1-\alpha}{1+\alpha}\right)^3} = \sqrt{\frac{1}{3}\left(\frac{\lambda-1}{\lambda+1}\right)^3},$$

then equation (5.3) has a unique uniformly limit-periodic solution on Q. Consequently, this solution is, according to Proposition 2.15, essential on Q with respect to the operator $\mathcal{T}\varphi|_Q\colon Q \to Q$, where

$$\mathcal{T}\varphi := \sum_{l\in\mathbb{Z}} G_0(t,l)\big(r_{l-1} + \varphi_{l-1}^3\big)$$

and G_0 stands for the Green function of $x_{t+1} = \lambda x_t$, $\lambda > 1$.

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Manuscript received April 5, 2018
accepted October 6, 2018

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