Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 2A, 2019, 409–426
DOI: 10.12775/TMNA.2019.029

COERCIVE FUNCTIONALS AND THEIR RELATIONSHIP TO MULTIPLICITY OF SOLUTION TO NONLOCAL BOUNDARY VALUE PROBLEMS

Christopher S. Goodrich

Abstract. We consider perturbed Hammerstein integral equations of the form

$$y(t) = \gamma_1(t)H_1(\varphi_1(y)) + \gamma_2(t)H_2(\varphi_2(y)) + \lambda \int_0^1 G(t,s)f(s,y(s))\,ds$$

in the case where H_1 and H_2 are continuous functions, which can be either linear or nonlinear subject to some restrictions, and φ_1, φ_2 are linear functionals. We demonstrate that by using a specially constructed order cone one can equip φ_1 and φ_2 with coercivity conditions that are useful in proving existence of multiple positive solutions. In addition, we demonstrate that the methodology can be superior to competing methodologies. We provide an application to the modeling of the deflection of an elastic beam.

1. Introduction

In this paper we consider the perturbed Hammerstein integral equation

$$y(t) = \gamma_1(t)H_1(\varphi_1(y)) + \gamma_2(t)H_2(\varphi_2(y)) + \lambda \int_0^1 G(t,s)f(s,y(s))\,ds, \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary: 26A42, 45G10, 45M20; Secondary: 34B10, 34B18, 47B40, 47H14, 47H30.

Key words and phrases. Hammerstein integral equation; coercivity; positive solution; multiplicity of solution; beam deflection.

where $\varphi_i \colon \mathscr{C}\big([0,1]\big) \to \mathbb{R}$ are linear functionals, which are able to be represented in the form

$$\varphi_i(y) := \int_{[0,1]} y(t)\, d\alpha_i(t),$$

and $\gamma_i \colon [0,1] \to [0,+\infty)$, $H_i \colon \mathbb{R} \to [0,+\infty)$ are continuous functions. We have recently considered such problems [11]–[14] and have demonstrated in a few different contexts (e.g. the positone problem, the semipositone problem, and the case of a vanishing or sign-changing Green's function) that existence results in some situations can be improved if we look for solutions of (1.1) from the cone

$$\mathcal{K} := \big\{y \in \mathscr{C}([0,1]) : y \geq 0,\ \varphi_1(y) \geq C_0\|y\|,\ \varphi_2(y) \geq D_0\|y\|\big\},$$

where C_0 and D_0 are the constants defined by

$$C_0 := \inf_{s\in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t) \quad \text{and} \quad D_0 := \inf_{s\in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t)$$

with $\mathscr{G}(s) := \sup_{t\in[0,1]} |G(t,s)|$ and $S_0 \subseteq [0,1]$ a set of full measure to be detailed in Section 2. In other words, the cone $\mathcal{K}$ restricts our search for solutions of (1.1) to those maps that cause each of the functionals φ_1 and φ_2 to be coercive with coercivity constants C_0 and D_0. We have previously utilized a cone such as $\mathcal{K}$ when studying existence of positive solution to perturbed Hammerstein equations (see, for example, [13]–[15]), though not in any investigations of multiplicity of solution as we investigate here.

Particularly, when used in conjunction with the open set

$$\widehat{V}_{\rho,i} := \{y \in \mathcal{K} : \varphi_i(y) < \rho\}$$

this methodology can be more effective than existing methodologies in the literature; note that the above set is (relatively) open in the cone $\mathcal{K}$ – see Lemma 2.2. It is worth noting at this juncture that while problems such as (1.1) can be studied as a problem in pure mathematics, such nonlocal problems do arise naturally in various applications such as problems in beam deflection, chemical reactor theory, and thermostatics – see, respectively, Infante and Pietramala [22], Infante, Pietramala, and Tenuta [27], and Cabada, Infante, and Tojo [3]. In fact, we provide an example of such an application, namely a problem in modeling beam deflection, in Example 3.3 later in this paper. More specifically, we consider the differential equation

$$y^{(4)}(t) = f(t, y(t)), \quad t \in (0,1)$$

subject to the nonlocal boundary conditions

$$y(0) = H_1\big(\varphi_1(y)\big), \qquad y'(0) = 0, \qquad y''(1) = 0, \qquad y'''(1) = -H_2(\varphi_2(y)).$$

The physical meaning of this problem is described in more detail in Example 3.3. Succinctly, however, it describes the deflection of an elastic beam with controllers which affect the deflection at the left end ($t = 0$) and the shearing force at the right end ($t = 1$). This type of problem was also studied in the already mentioned paper of Infante and Pietramala [22].

The contribution of this note is twofold. First of all, we demonstrate how our methodology can generate quite easily conditions under which (1.1) will possess multiple positive solutions. Our results are stated more precisely in Section 2, but, loosely, one interpretation of one of the results is that if there are two positive numbers $\rho_1 \neq \rho_2$ such that, for $i = 1, 2$,

$$\frac{H(\rho_i)}{\rho_i} > \frac{1}{\varphi_1(\gamma_1)}$$

and, in addition, if $0 < \lambda < \lambda_0$ for some computable $\lambda_0 > 0$, then problem (1.1) has at least two positive solutions. In particular, by using $\widehat{V}_{\rho,i}$ and $\mathcal{K}$ in tandem, we can move the usual *interval*-type conditions on f (e.g. [1], [18], [20], [25], [30], [46], [47]) or *asymptotic-type* conditions (e.g. [8], [10], [46], [47]) to *pointwise*-type conditions on H. Note that by an "interval-type condition" we mean, for example, a condition of the form

$$\max_{z \in [a,b]} H(z) \leq C < +\infty,$$

whereas by an "asymptotic-type condition" we mean, for example, a condition of the form

$$\lim_{z \to +\infty} \frac{H(z)}{z} = +\infty.$$

Second of all, we also demonstrate that this methodology may be more effective in the special case of nonlocal elements associated to sign-changing Stieltjes measures. For example, in three very interesting and recent papers Cabada, Infante, and Tojo [3], [4] and Cianciaruso, Infante and Pietramala [6] utilize a new methodology for treating problems like (1.1). Among other facets of their theory, is the requirement that the linear functionals, such as φ_1, satisfy a sort-of upper boundedness condition of the form

$$H(\varphi_1(y)) \leq \alpha^\rho(y), \quad \text{for all } y \in \partial\Omega_\rho := \big\{y \in \mathscr{C}\big([0,1]\big) : \|y\| = \rho\big\},$$

where α^ρ is a linear functional and $\rho > 0$ is some fixed number – a similar lower boundedness condition is also required; it should be remarked that their methodology is somewhat more general than the above suggests, but it accurately captures the spirit of the condition they impose. However, the function α^ρ must be able to be written in the form (see, for example, [6, Lemma 2.3])

$$\int_0^1 y(t)\, dA^\rho(t),$$

where dA^ρ is a *positive* Stieltjes measure. This can create some issues if the function φ_1 above is associated to a *sign-changing* Stieltjes measure. We explicitly demonstrate this in Remark 3.2, where we argue that our methodology may be more naturally suited to these types of problems.

We conclude this section by mentioning some of the relevant literature. In addition to the papers already mentioned, there exists a substantial number of papers on nonlocal boundary value problems, whether with linear nonlocal or nonlinear nonlocal boundary conditions. Two papers of historical interest in the area and which are worth reading are those by Picone [38] and Whyburn [44]. A general and useful theory for nonlocal BVPs with linear nonlocal BCs was developed by Webb and Infante [41], [43], further refined by Infante and co-authors in many other subsequent papers such [42]. Somewhat earlier Karakostas and Tsamatos [30], [31] studied nonlocal BVPs as related to certain Fredholm integral equations, and, then, somewhat contemporaneously Yang [48], [49] also studied fairly general nonlocal BVPs. A number of related papers, both complementing these mentioned papers and extending them in various directions, have appeared, such as ones by Cianciaruso and Pietramala [7], Goodrich [9], Graef and Webb [17], Infante and Pietramala [21], [23], [24], [26], Infante, Minhós, and Pietramala [19], Infante, Pietramala, and Tenuta [27], Jankowski [28], Karakostas [29], and Webb [40]. Since the study of such BVPs is closely connected to the study of perturbed Hammerstein integral equations, one may consult papers by Lan [32], Lan and Lin [33], Liu and Wu [36], Xu and Yang [45], and Yang [50] for various applications of Hammerstein integral equations. In this paper, by contrast, we continue to demonstrate that in contrast to the existing methodologies in the literature, our approach of utilizing a nonstandard cone together with the $\widehat{V}$-type set can be advantageous in certain problems.

2. Preliminaries and main results

We begin this section by mentioning the basic hypotheses that we impose on the constituent parts of problem (1.1).

(H1) For each $i = 1, 2$ the functional φ_i has the form

$$\varphi_i(y) := \int_{[0,1]} y(t)\, d\alpha_i(t),$$

where $\alpha_i\colon [0,1] \to \mathbb{R}$ satisfies $\alpha_i \in \mathrm{BV}([0,1])$. In addition, let the constants $C_1, D_1 > 0$ be selected so that, for each $y \in \mathscr{C}([0,1])$,

$$|\varphi_1(y)| \le C_1\|y\| \quad \text{and} \quad |\varphi_2(y)| \le D_1\|y\|.$$

(H2) The kernel $G\colon [0,1] \times [0,1] \to [0,+\infty)$ appearing in (1.1) satisfies each of the following conditions:

(1) $G \in L^1([0,1] \times [0,1])$.

(2) For almost every $s \in [0,1]$ it follows that
$$\lim_{t \to \tau} |G(t,s) - G(\tau,s)| = 0 \quad \text{for each } \tau \in [0,1].$$

(3) It holds that $\mathscr{G}(s) := \sup_{t \in [0,1]} |G(t,s)| < +\infty$ for each $s \in [0,1]$, with the map $s \mapsto \mathscr{G}(s)$ not identically zero.

(H3) There exists a set S_0 of full measure (i.e. $|S_0| = 1$) such that the numbers C_0 and D_0, which are defined by
$$C_0 := \inf_{s \in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t)$$
and
$$D_0 := \inf_{s \in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t),$$
satisfy $C_0, D_0 \in (0, +\infty)$.

(H4) The functions $\gamma_1, \gamma_2 \colon [0,1] \to [0,+\infty)$, $H_1, H_2 \colon \mathbb{R} \to [0,+\infty)$, $f \colon [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous and, in addition, for each $i = 1,2$ it holds both that
$$\varphi_1(\gamma_i) \geq C_0 \|\gamma_i\| \quad \text{and} \quad \varphi_2(\gamma_i) \geq D_0 \|\gamma_i\|.$$

As mentioned in Section 1 we will subsequently use the cone $\mathcal{K} \subseteq \mathscr{C}\big([0,1]\big)$ defined by
$$\begin{aligned}\mathcal{K} := \bigg\{ y \in \mathscr{C}([0,1]) : y \geq 0,\ \varphi_1(y) &\geq \bigg(\inf_{s \in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t) \bigg) \|y\|, \\ \varphi_2(y) &\geq \bigg(\inf_{s \in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t) \bigg) \|y\| \bigg\}.\end{aligned}$$

Remark 2.1. We note that the cone $\mathcal{K}$ is inspired by related cones studied by Graef, Kong and Wang [16], Webb [40], and Ma and Zhong [37].

For convenience we include a statement of the properties of the set $\widehat{V}_{\rho,i}$ described in Section 1 – see [11] for additional details.

Lemma 2.2. *For $\rho > 0$ define the set $\widehat{V}_{\rho,i}$ by $\widehat{V}_{\rho,i} := \{y \in \mathcal{K} \colon \varphi_i(y) < \rho\}$. Then each of the following is true.*

(a) *If $y \in \partial\widehat{V}_{\rho,i}$, then $\varphi_i(y) = \rho$, where by $\partial\widehat{V}_{\rho,i}$ we mean the boundary relative to the cone $\mathcal{K}$.*

(b) *If $\rho_1 < \rho_2$, then $\widehat{V}_{\rho_1,i} \subset \widehat{V}_{\rho_2,i}$.*

(c) *For each $\rho > 0$ the set $\widehat{V}_{\rho,i}$ is (relatively) open in $\mathcal{K}$.*

Remark 2.3. We note that sets similar to the open set $\widehat{V}_{\rho,i}$, namely open sets involving a bound on an appropriate functional used in concert with topological fixed point methods and Hammerstein equations, can be found as far back as the well-known paper of Leggett and Williams [34].

Notation 2.4. For the functions f and H identified in condition (H3) above, we utilize the following notation.

(a) For each $[a,b] \subseteq [0,1]$ and $[c,d] \subseteq [0,+\infty)$ we put

$$f^M_{[a,b]\times[c,d]} := \max_{(t,y)\in[a,b]\times[c,d]} f(t,y).$$

(b) For given $i = 1,2$ and for each $[a,b] \subseteq [0,+\infty)$ we put

$$H^{M,i}_{[a,b]} := \max_{z\in[a,b]} H_i(z).$$

Now consider the operator $T\colon \mathscr{C}([0,1]) \to \mathscr{C}\big([0,1]\big)$ defined by

$$(Ty)(t) := \gamma_1(t)H_1(\varphi_1(y)) + \gamma_2(t)H_2(\varphi_2(y)) + \lambda \int_0^1 G(t,s)f(s,y(s))\,ds.$$

A solution of the operator equation $(Ty)(t) = y(t)$, for $t \in [0,1]$, is then a solution of the integral equation (1.1). With this in mind we state the main result of this section. We note that the proof methodology for the existence results is via fixed point index theory – see Zeidler [52] for more details.

Theorem 2.5. *Assume that conditions* (H1)–(H4) *hold. Assume that* $\|\gamma_1\|$, $\|\gamma_2\| \neq 0$*. If there exist numbers* $\rho_1, \rho_2, \rho_3 > 0$*, where either*

(a) $\rho_1 > \rho_2 > \rho_3$ *or*

(b) $\rho_3 > \rho_2 > \rho_1$

such that

(i) $\dfrac{H_1(\rho_1)}{\rho_1} > \dfrac{1}{\varphi_1(\gamma_1)}$;

(ii)

$$H_1(\rho_2)\varphi_1(\gamma_1) + \big(H^{M,2}_{[D_0\rho_2/C_1, D_1\rho_2/C_0]}\big)\varphi_1(\gamma_2) + \lambda\big(f^M_{[0,1]\times[0,\rho_2/C_0]}\big)\int_0^1\int_0^1 G(t,s)\,d\alpha_1(t)\,ds < \rho_2;$$

(iii) $\dfrac{H_1(\rho_3)}{\rho_3} > \dfrac{1}{\varphi_1(\gamma_1)}$

then problem (1.1) *has at least two positive solutions.*

Proof. It is standard to show that T is a completely continuous operator, and so, we omit this part of the proof. Moreover, it has been shown in other papers that $T(\mathcal{K}) \subseteq \mathcal{K}$, and so, we also omit that demonstration – see, for example, [12, Lemma 2.12], [13, Lemma 2.3], [14, Theorem 3.1], and [15, Lemma 2.5] for materially identical arguments of this type. Without loss of generality, throughout the proof we assume that $\rho_3 > \rho_2 > \rho_1$ – the case in which $\rho_1 > \rho_2 > \rho_3$ is treated essentially in an identical manner.

We demonstrate first that for each $y \in \partial\widehat{V}_{\rho_1,1}$ and for each $\mu \geq 0$ it holds that $y \neq Ty + \mu e$, where $e(t) := \gamma_1(t)$. So, for contradiction, suppose not. Then, there exists $y \in \partial\widehat{V}_{\rho_1,1}$ and a number $\mu \geq 0$ such that $y = Ty + \mu e$. Applying φ_1 to both sides of the operator equation implies that

$$\rho_1 > \varphi_1(\gamma_1)H_1(\rho_1) > \rho_1, \tag{2.1}$$

which is a contradiction; note that in (2.1) we use that (for each $i = 1, 2$)

$$\int_0^1 G(t,s)\, d\alpha_i(t) > 0,$$

for almost every $s \in [0,1]$. Inequality (2.1) implies that

$$i_{\mathcal{K}}\big(T, \widehat{V}_{\rho_1,1}\big) = 0. \tag{2.2}$$

In a completely similar manner, we obtain that

$$i_{\mathcal{K}}\big(T, \widehat{V}_{\rho_3,1}\big) = 0. \tag{2.3}$$

On the other hand, we show that for each $y \in \partial\widehat{V}_{\rho_2,1}$ it follows that $\mu y \neq Ty$ for each $\mu \geq 1$. Suppose not. Then, for some $y \in \partial\widehat{V}_{\rho_2,1}$ and some number $\mu \geq 1$, we have that $\mu y = Ty$, and so, applying φ_1 to both sides of the operator equation we obtain

$$\begin{aligned}\rho_2 < \varphi_1(\gamma_1)H_1(\rho_2) &+ \varphi_1(\gamma_2)H_2\big(\varphi_2(y)\big) \\ &+ \lambda \int_0^1 \int_0^1 G(t,s) f\big(s, y(s)\big)\, d\alpha_1(t)\, ds.\end{aligned} \tag{2.4}$$

Note that, since $\varphi_1(y) = \rho_2$, it follows that $c\rho_2/C_1 \leq \|y\| \leq \rho_2/C_0$. Consequently, since $D_0\|y\| \leq \varphi_2(y) \leq D_1\|y\|$, it follows that

$$\frac{D_0\rho_2}{C_1} \leq \varphi_2(y) \leq \frac{D_1\rho_2}{C_0}.$$

Therefore, we estimate both that

$$0 \leq f\big(s, y(s)\big) \leq f^M_{[0,1]\times[0,\rho_2/C_0]} \tag{2.5}$$

and that

$$0 \leq H_2\big(\varphi_2(y)\big) \leq H^{M,2}_{[D_0\rho_2/C_1, D_1\rho_2/C_0]}. \tag{2.6}$$

Then, putting (2.5)–(2.6) into inequality (2.4), we arrive at the inequality

$$\begin{aligned}\rho_2 &< \varphi_1(\gamma_1)H_1(\rho_2) + \varphi_1(\gamma_2)H_2(\varphi_2(y)) + \lambda \int_0^1 \int_0^1 G(t,s) f(s,y(s))\, d\alpha_1(t)\, ds \\ &\leq \varphi_1(\gamma_1)H_1(\rho_2) + \big(H^{M,2}_{[D_0\rho_2/C_1, D_1\rho_2/C_0]}\big)\varphi_1(\gamma_2) \\ &\quad + \lambda\big(f^M_{[0,1]\times[0,\rho_2/C_0]}\big) \int_0^1 \int_0^1 G(t,s)\, d\alpha_1(t)\, ds < \rho_2,\end{aligned}$$

which is a contradiction. Consequently, we conclude that

$$i_{\mathcal{K}}\big(T, \widehat{V}_{\rho_2,1}\big) = 1. \tag{2.7}$$

Finally, putting (2.2), (2.3), and (2.7) together and using the fact that $\rho_3 > \rho_2 > \rho_1 > 0$, we conclude that

$$i_{\mathcal{K}}\big(T, \widehat{V}_{\rho_2,1} \setminus \widehat{V}_{\rho_1,1}\big) = 1 \tag{2.8}$$

and that

$$i_{\mathcal{K}}\big(T, \widehat{V}_{\rho_3,1} \setminus \widehat{V}_{\rho_2,1}\big) = -1. \tag{2.9}$$

Since $\widehat{V}_{\rho_3,1} \supset \widehat{V}_{\rho_2,1} \supset \widehat{V}_{\rho_1,1}$, it follows from (2.8)–(2.9) that there exist maps

$$y_1 \in \widehat{V}_{\rho_2,1} \setminus \widehat{V}_{\rho_1,1} \quad \text{and} \quad y_2 \in \widehat{V}_{\rho_3,1} \setminus \widehat{V}_{\rho_2,1}$$

such that $Ty_i = y_i$ for each $i = 1, 2$. Since, in addition, for each $i = 1, 2$ we have that $\|y_i\| > 0$ and

$$\big(\widehat{V}_{\rho_2,1} \setminus \widehat{V}_{\rho_1,1}\big) \cap \big(\widehat{V}_{\rho_3,1} \setminus \widehat{V}_{\rho_2,1}\big) = \varnothing,$$

it follows that y_1 and y_2 are nontrivial, distinct positive solutions to equation (1.1). $\square$

REMARK 2.6. Notice that condition (ii) of Theorem 2.5 can be recast as requiring that the parameter λ satisfy $\lambda \in \big(0, \lambda_0\big)$, where

$$\begin{aligned} \lambda_0 := \Big[\rho_2 - \big(H_1(\rho_2)\varphi_1(\gamma_1) + \big(H^{M,2}_{[D_0\rho_2/C_1, D_1\rho_2/C_0]}\big)\varphi_1(\gamma_2)\big)\Big] \\ \times \left(f^M_{[0,1]\times[0,\rho_2/C_0]} \int_0^1 \int_0^1 G(t,s)\, d\alpha_1(t)\, ds\right)^{-1}. \end{aligned} \tag{2.10}$$

In other words, problem (1.1) has at least one positive solution provided that conditions (i) and (iii) are satisfied, and, in addition, the number λ is sufficiently small according to inequality (2.10) above.

Numerous slight variations of Theorem 2.5 may be provided. We state a representative example.

COROLLARY 2.7. *Assume that conditions* (H1)–(H4) *hold, that* $H_2(z) \equiv 0$, *and that* $\|\gamma_1\| \neq 0$. *If there exist numbers* ρ_1, ρ_2, $\rho_3 > 0$, *where either*

(a) $\rho_1 > \rho_2 > \rho_3$ *or*
(b) $\rho_3 > \rho_2 > \rho_1$;

such that

(i) $\dfrac{H_1(\rho_1)}{\rho_1} > \dfrac{1}{\varphi_1(\gamma_1)}$;

(ii) $H_1(\rho_2)\varphi_1(\gamma_1) + \lambda\big(f^M_{[0,1]\times[0,\rho_2/C_0]}\big) \displaystyle\int_0^1 \int_0^1 G(t,s)\, d\alpha_1(t)\, ds < \rho_2$; *and*

(iii) $\dfrac{H_1(\rho_3)}{\rho_3} > \dfrac{1}{\varphi_1(\gamma_1)}$

then problem (1.1) *has at least two positive solutions.*

Finally, we conclude this section with a remark.

REMARK 2.8. It is easy to see that the technique utilized in the proof of Theorem 2.5 can be replicated to derive theorems guaranteeing existence of n distinct positive solutions for problem (1.1) for any integer $n \geq 3$. However, we omit the statement of such results here.

3. Analysis and application of the methodology

In this section we begin by providing an example to illustrate the possible advantage of utilizing coercive linear functionals. The example is a modification of an example given in [13], and it also serves to illustrate the applicability of the theorems we have given in Section 2 of this paper.

EXAMPLE 3.1. Consider the functional

$$\varphi_1(y) := y\left(\frac{1}{40}\right).$$

If we set $H_1(z) := 121\big(z^3 + \sqrt{z}\big)/195$, then

$$H_1(\varphi_1(y)) = \frac{121}{195}\left(\left(y\left(\frac{1}{40}\right)\right)^3 + \sqrt{y\left(\frac{1}{40}\right)}\right).$$

Then we consider the boundary value problem

$$\begin{aligned} &-y'' = f\big(t, y(t)\big), \quad t \in (0,1) \\ &y(0) = \frac{121}{195}\left(\left(y\left(\frac{1}{40}\right)\right)^3 + \sqrt{y\left(\frac{1}{40}\right)}\right) \\ &y(1) = 0. \end{aligned} \tag{3.1}$$

In this case we select $\gamma_1(t) := 1-t$, so that $\varphi_1(\gamma_1) = 39/40$, so $1/\varphi_1(\gamma_1) = 40/39$. In addition, we note that the coercivity constant C_0 is here computed to be $C_0 = 1/40$ since

$$\inf_{s\in(0,1)} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\,d\alpha(t) = \inf_{s\in(0,1)} \begin{cases} \dfrac{1}{40s}, & 0 < \dfrac{1}{40} < s < 1, \\ \dfrac{39}{40(1-s)}, & 0 < s < \dfrac{1}{40} < 1, \end{cases} = \frac{1}{40},$$

where G is the Green's function defined by

$$G(t,s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{3.2}$$

Furthermore, observe that

$$\varphi_1(\gamma_1) = \frac{39}{40} \geq \frac{1}{40}\|\gamma_1\|.$$

Consider Corollary 2.7. Notice that if we put $\rho_1 := 10^{-4}$, say, then

$$\frac{H_1(\rho_1)}{\rho_1} = \frac{121}{195}\left(\rho_1^3 + \frac{1}{\sqrt{\rho_1}}\right) = \frac{121}{195}(10^{-12} + 100) > \frac{40}{39},$$

and so, condition (i) of the theorem is satisfied. Similarly, if we select $\rho_3 := 10$, then

$$\frac{H_1(\rho_3)}{\rho_3} = \frac{121}{195}\left(\rho_3^3 + \frac{1}{\sqrt{\rho_3}}\right) = \frac{121}{195}\left(1000 + \frac{1}{\sqrt{10}}\right) > \frac{40}{39},$$

and so, condition (iii) of the theorem is satisfied. We conclude that if there exists a number $\rho_2 \in (10^{-4}, 10)$ such that

$$\frac{121}{200}\big(\rho_2^3 + \sqrt{\rho_2}\big) + \lambda\big(f^M_{[0,1]\times[0,40\rho_2]}\big)\underbrace{\int_0^1\int_0^1 G(t,s)\,d\alpha(t)\,ds}_{=39/3200} < \rho_2, \tag{3.3}$$

then problem (3.1) will have at least two positive solutions. Regarding the choice of the constant ρ_2 note that inequality (3.3) is not so restrictive since condition (i) of Corollary 2.7 is, in fact, satisfied for all ρ_1 sufficiently small and, likewise, condition (iii) of the corollary is satisfied for all ρ_3 sufficiently large. Consequently, we have sufficient latitude in how we can choose the number ρ_2.

In particular, note that condition (3.3) is the same as requiring that

$$\lambda < \frac{3200}{39}\big(f^M_{[0,1]\times[0,40\rho_2]}\big)^{-1}\left(\rho_2 - \frac{121}{200}\big(\rho_2^3 + \sqrt{\rho_2}\,\big)\right). \tag{3.4}$$

Evidently, for (3.4) to be sensible, we must require that

$$\rho_2 - \frac{121}{200}\big(\rho_2^3 + \sqrt{\rho_2}\big) > 0. \tag{3.5}$$

But inequality (3.5) is, to four decimal places of accuracy, satisfied for $0.5375 \lessapprox \rho_2 \lessapprox 0.6124$. Therefore, if we take, say,

$$\rho_1 = 10^{-4}, \qquad \rho_2 = 0.5756, \qquad \rho_3 = 10,$$

then provided that the inequality, where we have rounded

$$\frac{3200}{39}\left(\rho_2 - \frac{121}{200}\big(\rho_2^3 + \sqrt{\rho_2}\big)\right)$$

to five decimal places of accuracy,

$$0 < \lambda < 0.10011\big(f^M_{[0,1]\times[0,23.024]}\big)^{-1}$$

is satisfied, by Corollary 2.7 we deduce the existence of at least two positive solutions, y_1 and y_2, to problem (3.1) such that y_1 and y_2 satisfy the localization:

$$y_1 \in \widehat{V}_{0.5756} \setminus \widehat{V}_{0.0001}, \qquad y_2 \in \widehat{V}_{10} \setminus \widehat{V}_{0.5756}.$$

We note that the value of ρ_2 selected above has the property that it maximizes the upper bound for λ (at least if we use four decimal places of accuracy).

REMARK 3.2. Whereas Example 3.1 demonstrated the application of one of our results, we now wish to compare and contrast our methodology with that presented by Cianciaruso, Infante, and Pietramala [6]. To this end suppose that the boundary conditions in (3.1) were replaced by the boundary conditions

$$y(0) = \frac{197}{195}\, y\left(\frac{1}{40}\right) - y\left(\frac{1}{50}\right), \qquad y(1) = 0.$$

It can be shown that

$$\int_0^1 \int_0^1 G(t,s)\, d\alpha(t)\, ds = \frac{201}{80000} > 0,$$
$$C_0 = \frac{1}{200} > 0 \quad \text{and} \quad \varphi(\gamma) = \frac{1}{200} > 0,$$

where, recall, $\gamma(t) := 1-t$. It can then be shown that if we use the nonlinear function $H(z) = z^3 + \sqrt{z}$, then the conditions in, for example, Corollary 2.7 are satisfied for any $\rho_1, \rho_3 \in (0, +\infty)$ and, approximately, any $\rho_2 \in (0.000025, 14.1327)$. All in all, then, our results can be reasonably applied to this problem.

We now consider what happens were to use instead the methodology by Cianciaruso, Infante and Pietramala [6]. In this eventuality we might begin by writing

$$\psi(y) := \left(\frac{197}{195}\, y\left(\frac{1}{40}\right) - y\left(\frac{1}{50}\right)\right)^3 + \sqrt{\frac{197}{195}\, y\left(\frac{1}{40}\right) - y\left(\frac{1}{50}\right)},$$

where ψ is a nonlinear functional, and, among conditions, we would need to find a linear functional $\alpha^{\rho}(y)$ such that (see [6, Lemma 2.3])

$$\psi(y) \geq \alpha^{\rho}(y), \tag{3.6}$$

whenever $y \in \partial\Omega_{\rho} \cap \mathcal{K}' := \{y \in \mathscr{C}([0,1]) : \|y\| = \rho\} \cap \mathcal{K}'$, for some $\rho > 0$ and where

$$\mathcal{K}' := \Big\{ y \in \mathscr{C}\big([0,1]\big) : y \geq 0,\ \min_{t \in [a,b]} y(t) \geq \eta_0 \|y\| \Big\}.$$

The constant η_0 appearing in the definition of $\mathcal{K}'$ will generally depend on a and b, i.e. $\eta_0 := \eta_0(a,b)$. For example, for G defined by (3.2) it is known that $\eta_0 := \min\{a, 1-b\}$.

Note that, as described in [6, Lemma 2.3], the functional α^{ρ} must be associated to a *positive* Stieltjes measure. Consequently, the natural choice of

$$\alpha^{\rho}(y) := A\left(\frac{197}{195}\, y\left(\frac{1}{40}\right) - y\left(\frac{1}{50}\right)\right),$$

for some constant $A > 0$, is *not* admissible since this functional is associated to a signed Stieltjes measure. Were the above functional admissible, then verifying (3.6) would be straightforward since it would amount to determining whether there is $z > 0$ such that $z^3 + \sqrt{z} \geq Az$ for some $A \geq 0$. But this we cannot do on account of the restricted nature of α^{ρ}. Instead one must try less straightforward choices for α^{ρ}.

For example, we could choose $\alpha^{\rho}(y) := Ay(1/40)$, for some constant $A > 0$, as a natural alternative choice. Then we would have to show that

$$Ay\left(\frac{1}{40}\right) \leq \left(\frac{197}{195}\,y\left(\frac{1}{40}\right) - y\left(\frac{1}{50}\right)\right)^3 + \sqrt{\frac{197}{195}\,y\left(\frac{1}{40}\right) - y\left(\frac{1}{50}\right)} \tag{3.7}$$

for all $y \in \partial\Omega_{\rho} \cap \mathcal{K}'$. In fact, one can study inequality (3.7) numerically by, say, putting $\xi_1 := y(1/40)$ and $\xi_2 := y(1/50)$ and then studying the problem

$$A\xi_1 \leq \left(\frac{197}{195}\,\xi_1 - \xi_2\right)^3 + \sqrt{\frac{197}{195}\,\xi_1 - \xi_2}.$$

One then needs to show, in addition, that the relative values of ξ_1 and ξ_2 are such that y is an element of $\mathcal{K}'$. While, in principle, this can be accomplished, it is more technical and complicated. So, we see that when dealing with a sign-changing measure, the method utilized here works somewhat more naturally than that of [6]. Therefore, it may be simpler to check the three, straightforward numerical conditions of, say, Corollary 2.7 rather than embarking on a sequence of technical calculations requiring, in part, the numerical solver of a computer algebra system.

To conclude this paper we provide an example of the application of the results to a model problem in beam deflection in order to demonstrate that our methodology can be applied to a problem arising from modeling. This example will also illustrate how our theory can treat a Green's function different from the examples just given.

Before presenting the example let us note that some additional background on the modeling of elastic beams with nonlocal controllers can be found in the paper by Infante and Pietramala [22]. In particular, they consider the problem

$$u^{(4)}(t) = g(t)f\big(t, u(t)\big), \quad t \in (0,1)$$

subject to the nonlocal BCs

$$u(0) = u'(0) = u''(1) = 0 \quad \text{and} \quad u'''(1) + k_0 + B\big(\alpha[u]\big) = 0,$$

where k_0 is a constant, B is a continuous function, and α is a functional that represents a nonlocal element. The condition

$$u'''(1) + k_0 + B\big(\alpha[u]\big) = 0$$

describes some sort of controller that affects the shearing force at the right end of the beam. For example, if the condition reads $u'''(1) + B(u(\eta)) = 0$ for some $0 < \eta < 1$, then this would describe a sort of feedback mechanism in which the spring reacts to the displacement from equilibrium at the point η units along the length of the beam.

So that our example is related to [22] we will also study the situation of a controller at the right end ($t = 1$) of the beam such that the controller affects the shearing force. In addition, we will impose a controller at the left end ($t = 0$) that affects the displacement of the beam at this point.

Example 3.3. Let us consider the problem

$$y^{(4)}(t) = f\big(t, y(t)\big), \quad t \in (0,1) \tag{3.8}$$

subject to the boundary conditions

$$y(0) = y'(0) = 0 = y''(1) = y'''(1). \tag{3.9}$$

As mentioned in Cianciaruso, Infante, and Pietramala [6] this can model the stationary states of the deflection of an elastic beam – see also the exposition regarding this problem in [2], [5], [35], [39], [51]. In the case of boundary conditions (3.9) the beam would be assumed to be clamped at the left end (i.e. neither position deflection nor nonzero derivative) and with free deflection at the right end but subjected to a zero bending moment (namely, $y''(1) = 0$) and a zero shearing force (namely, $y'''(1) = 0$). It is known that the Green's function associated to the problem with the homogeneous boundary conditions (3.9) is

$$G(t,s) := \begin{cases} \dfrac{1}{6}\big(3t^2 s - t^3\big), & s \geq t, \\ \dfrac{1}{6}\big(3s^2 t - s^3\big), & s \leq t. \end{cases} \tag{3.10}$$

For this problem it can be shown (see, for example, Cianciaruso, Infante, and Pietramala [6] or Infante and Pietramala [22]) that for this Green's function the function $s \mapsto \mathscr{G}(s)$ is

$$\mathscr{G}(s) := \frac{1}{2}\, s^2 - \frac{1}{6}\, s^3. \tag{3.11}$$

To provide an example of how our theory can be applied to a problem similar to (3.8)–(3.9), let us consider problem (3.8) but now subject to the following nonlocal boundary conditions:

$$y(0) = H_1(\varphi_1(y)), \quad y'(0) = 0, \quad y''(1) = 0, \quad y'''(1) = -H_2(\varphi_2(y)). \tag{3.12}$$

The boundary conditions in (3.12) indicate that the deflection of the bar at the left end is affected by a controller, mathematically described as the nonlocal element $H_1\big(\varphi_1(y)\big)$, whereas there exists a different controller, mathematically described as the nonlocal element $H_2\big(\varphi_2(y)\big)$, affecting the shearing force at the right end of the beam. Here we will assume that

$$\varphi_1(y) := y\left(\frac{1}{40}\right), \qquad \varphi_2(y) := y\left(\frac{3}{4}\right).$$

Of course, more complicated nonlocal elements could also be utilized.

In order to see how our theory could apply to problem (3.8), (3.12) we first calculate the coercivity constants C_0 and D_0. Using (3.10)–(3.11) together with the definition of φ_1 we note that for $0 \le s \le 1/40$ we have that

$$\frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t) = \frac{\frac{1}{6}\left(\frac{3}{40} s^2 - s^3\right)}{\frac{1}{2} s^2 - \frac{1}{6} s^3},$$

whereas for $1/40 \le s \le 1$ we have that

$$\frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t) = \frac{\frac{1}{6}\left(\frac{3}{1600} s - \frac{1}{64000}\right)}{\frac{1}{2} s^2 - \frac{1}{6} s^3}.$$

Then we see by direct computation that

$$C_0 := \inf_{s\in(0,1]} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t) = \left[\frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t)\right]_{s=1} = \frac{119}{128000}.$$

In a similar way, we compute D_0. To this end we note that, if $0 \le s \le 3/4$, then

$$\frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t) = \frac{\frac{1}{6}\left(\frac{9}{4} s^2 - s^3\right)}{\frac{1}{2} s^2 - \frac{1}{6} s^3}$$

whereas, if $3/4 \le s \le 1$, then

$$\frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t) = \frac{\frac{1}{6}\left(\frac{27}{16} s - \frac{27}{64}\right)}{\frac{1}{2} s^2 - \frac{1}{6} s^3}.$$

Then direct computation again yields

$$D_0 := \inf_{s\in(0,1]} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t) = \frac{81}{128}.$$

Let us now consider the integral equation

$$y(t) = \gamma_1(t) H_1(\varphi_1(y)) + \gamma_2(t) H_2(\varphi_2(y)) + \int_0^1 G(t,s) f(s, y(s))\, ds, \tag{3.13}$$

where we define the functions γ_1 and γ_2 as follows:

$$\gamma_1(t) \equiv 1, \qquad \gamma_2(t) := \frac{1}{6}\big(3t^2 - t^3\big).$$

Then a solution of problem (3.15) is a solution of problem (3.8), (3.12).

Now note the following:

$$\begin{aligned}
\varphi_1(\gamma_1) &= \gamma_1\left(\frac{1}{40}\right) = 1 \geq \frac{119}{128000}\cdot 1 = C_0\|\gamma_1\|,\\
\varphi_1(\gamma_2) &= \gamma_2\left(\frac{1}{40}\right) = \frac{119}{384000} = \frac{119}{128000}\cdot\frac{1}{3} = C_0\|\gamma_2\|,\\
\varphi_2(\gamma_1) &= \gamma_1\left(\frac{3}{4}\right) = 1 \geq \frac{81}{128}\cdot 1 = D_0\|\gamma_1\|,\\
\varphi_2(\gamma_2) &= \gamma_2\left(\frac{3}{4}\right) = \frac{27}{128} = \frac{81}{128}\cdot\frac{1}{3} = D_0\|\gamma_2\|.
\end{aligned}$$

Therefore, condition (H3) is verified. Condition (H2) has already been verified by calculating the coercivity constants as above. Condition (H1) is obvious. Finally, condition (H4) is also easy to verify since we have already provided the formula for the map $s \mapsto \mathscr{G}(s)$ and parts (1)–(2) of (H2) are obvious. Therefore, conditions (H1)–(H4) are satisfied.

Finally, depending upon the choice of the functions H_1, H_2, and f we may then apply our previous existence theorems. For example, if, say,

$$H_1(z) := \frac{1}{20}\big(z^3 + \sqrt{z}\big), \qquad H_2(z) := \sqrt{z},$$

then we will be able to apply Theorem 2.5 and thus deduce that problem (3.8), (3.12) will have at least two positive solutions provided that there exist numbers, say, $\rho_1 > \rho_2 > \rho_3$ such that

$$\rho_1^2 + \frac{1}{\sqrt{\rho_1}} > 20 \tag{3.14}$$

and, recalling that $\lambda = 1$ here in light of the right-hand side of equation (3.8),

$$\begin{aligned}
\frac{1}{20}\big(\rho_2^3 + \sqrt{\rho_2}\big) + \frac{119}{384000}\sqrt{\frac{128000}{119}\rho_2}&\\
+ \big(f^M_{[0,1]\times[0,128000/119\rho_2]}\big)\int_0^1\int_0^1 G(t,s)\,d\alpha_1(t)\,ds &< \rho_2
\end{aligned} \tag{3.15}$$

and

$$\rho_3^2 + \frac{1}{\sqrt{\rho_3}} > 20. \tag{3.16}$$

Similar statements may be made for other choices of the functions H_1 and H_2. Now, conditions (3.14) and (3.16) are clearly able to be satisfied either for either $\rho_1 > 0$ or $\rho_3 > 0$ sufficiently small or, alternatively, for either $\rho_1 > 0$ or $\rho_3 > 0$ sufficiently large. Moreover, since

$$\frac{1}{20}\big(\rho_2^3 + \sqrt{\rho_2}\big) + \frac{119}{384000}\sqrt{\frac{128000}{119}\rho_2} < \rho_2$$

is true for (approximately) $0.00362 < \rho_2 < 4.40759$, it follows that inequality (3.15) is not vacuous. All in all, then, the existence theorems of Section 2 can be readily applied to problem (3.8), (3.12).

Acknowledgments. I would like to thank the five anonymous referees for their constructive comments, which led to an improved presentation of the results.

REFERENCES

[1] D.R. ANDERSON, *Existence of three solutions for a first-order problem with nonlinear nonlocal boundary conditions*, J. Math. Anal. Appl. **408** (2013), 318–323.

[2] D.R. ANDERSON AND J. HOFFACKER, *Existence of solutions for a cantilever beam problem*, J. Math. Anal. Appl. **323** (2006), 958–973.

[3] A. CABADA, G. INFANTE AND F.A.F. TOJO, *Nonzero solutions of perturbed Hammerstein integral equations with deviated arguments and applications*, Topol. Methods Nonlinear Anal. **47** (2016), 265–287.

[4] A. CABADA, G. INFANTE AND F.A.F. TOJO, *Nonlinear perturbed integral equations related to nonlocal boundary value problems*, Fixed Point Theory **19** (2018), 65–92.

[5] A. CABADA AND S. TERSIAN, *Multiplicity of solutions to a two-point boundary value problem for a fourth-order equation*, Appl. Math. Comput. **219** (2013), 5261–5267.

[6] F. CIANCIARUSO, G. INFANTE AND P. PIETRAMALA, *Solutions of perturbed Hammerstein integral equations with applications*, Nonlinear Anal. Real World Appl. **33** (2017), 317–347.

[7] F. CIANCIARUSO AND P. PIETRAMALA, *Multiple positive solutions of a (p_1, p_2)-Laplacian system with nonlinear BCs*, Bound. Value Probl. **163** (2015), 18 pp.

[8] D.R. DUNNINGER AND H. WANG, *Multiplicity of positive solutions for a nonlinear differential equation with nonlinear boundary conditions*, Ann. Polon. Math. **69** (1998), 155–165.

[9] C.S. GOODRICH, *Positive solutions to boundary value problems with nonlinear boundary conditions*, Nonlinear Anal. **75** (2012), 417–432.

[10] C.S. GOODRICH, *On nonlinear boundary conditions involving decomposable linear functionals*, Proc. Edinb. Math. Soc. (2) **58** (2015), 421–439.

[11] C.S. GOODRICH, *Coercive nonlocal elements in fractional differential equations*, Positivity **21** (2017), 377–394.

[12] C.S. GOODRICH, *A new coercivity condition applied to semipositone integral equations with nonpositive, unbounded nonlinearities and applications to nonlocal BVPs*, J. Fixed Point Theory Appl. **19** (2017), 1905–1938.

[13] C.S. GOODRICH, *The effect of a nonstandard cone on existence theorem applicability in nonlocal boundary value problems*, J. Fixed Point Theory Appl. **19** (2017), 2629–2646.

[14] C.S. GOODRICH, *New Harnack inequalities and existence theorems for radially symmetric solutions of elliptic PDEs with sign changing or vanishing Green's function*, J. Differential Equations **264** (2018), 236–262.

[15] C.S. GOODRICH, *Radially symmetric solutions of elliptic PDEs with uniformly negative weight*, Ann. Mat. Pura Appl. (4) **197** (2018), 1585–1611.

[16] J.R. GRAEF, L. KONG AND H. WANG, *A periodic boundary value problem with vanishing Green's function*, Appl. Math. Leet. **21** (2008), 176–180.

[17] J. GRAEF AND J.R.L. WEBB, *Third order boundary value problems with nonlocal boundary conditions*, Nonlinear Anal. **71** (2009), 1542–1551.

[18] G. Infante, *Nonlocal boundary value problems with two nonlinear boundary conditions*, Commun. Appl. Anal. **12** (2008), 279–288.

[19] G. Infante, F. Minhós and P. Pietramala, *Non-negative solutions of systems of ODEs with coupled boundary conditions*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012), 4952–4960.

[20] G. Infante and P. Pietramala, *Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations*, Nonlinear Anal. **71** (2009), 1301–1310.

[21] G. Infante and P. Pietramala, *Eigenvalues and non-negative solutions of a system with nonlocal BCs*, Nonlinear Stud. **16** (2009), 187–196.

[22] G. Infante and P. Pietramala, *A cantilever equation with nonlinear boundary conditions*, Electron. J. Qual. Theory Differ. Equ. 2009, Special Edition I, No. 15, 14 pp.

[23] G. Infante and P. Pietramala, *Perturbed Hammerstein integral inclusions with solutions that change sign*, Comment. Math. Univ. Carolin. **50** (2009), 591–605.

[24] G. Infante and P. Pietramala, *A third order boundary value problem subject to nonlinear boundary conditions*, Math. Bohem. **135** (2010), 113–121.

[25] G. Infante and P. Pietramala, *Multiple nonnegative solutions of systems with coupled nonlinear boundary conditions*, Math. Methods Appl. Sci. **37** (2014), 2080–2090.

[26] G. Infante and P. Pietramala, *Nonzero radial solutions for a class of elliptic systems with nonlocal BCs on annular domains*, NoDEA Nonlinear Differential Equations Appl. **22** (2015), 979–1003.

[27] G. Infante, P. Pietramala and M. Tenuta, *Existence and localization of positive solutions for a nonlocal BVP arising in chemical reactor theory*, Commun. Nonlinear Sci. Numer. Simul. **19** (2014), 2245–2251.

[28] T. Jankowski, *Positive solutions to fractional differential equations involving Stieltjes integral conditions*, Appl. Math. Comput. **241** (2014), 200–213.

[29] G.L. Karakostas, *Existence of solutions for an n-dimensional operator equation and applications to BVPs*, Electron. J. Differential Equations (2014), no. 71, 17 pp.

[30] G.L. Karakostas and P.Ch. Tsamatos, *Existence of multiple positive solutions for a nonlocal boundary value problem*, Topol. Methods Nonlinear Anal. **19** (2002), 109–121.

[31] G.L. Karakostas and P.Ch. Tsamatos, *Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems*, Electron. J. Differential Equations (2002), no. 30, 17 pp.

[32] K.Q. Lan, *Multiple positive solutions of semilinear differential equations with singularities*, J. Lond. Math. Soc. (2) **63** (2001), 690–704.

[33] K.Q. Lan and W. Lin, *Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations*, J. Lond. Math. Soc. (2) **83** (2011), 449–469.

[34] R.W. Leggett and L.R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. **28** (1979), 673–688.

[35] Y. Li, *Existence of positive solutions for the cantilever beam equations with fully nonlinear terms*, Nonlinear Anal. Real World Appl. **27** (2016), 221–237.

[36] X. Liu and J. Wu, *Positive solutions for a Hammerstein integral equation with a parameter*, Appl. Math. Lett. **22** (2009), 490–494.

[37] R. Ma and C. Zhong, *Existence of positive solutions for integral equations with vanishing kernels*, Commun. Appl. Anal. **15** (2011), 529–538.

[38] M. Picone, *Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10** (1908), 1–95.

[39] Y. SONG, *A nonlinear boundary value problem for fourth-order elastic beam equations*, Bound. Value Probl. **191** (2014), 11 pp.

[40] J.R.L. WEBB, *Boundary value problems with vanishing Green's function*, Commun. Appl. Anal. **13** (2009), 587–595.

[41] J.R.L. WEBB AND G. INFANTE, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. Lond. Math. Soc. (2) **74** (2006), 673–693.

[42] J.R.L. WEBB AND G. INFANTE, *Positive solutions of nonlocal boundary value problems involving integral conditions*, NoDEA Nonlinear Differential Equations Appl. **15** (2008), 45–67.

[43] J.R.L. WEBB AND G. INFANTE, *Nonlocal boundary value problems of arbitrary order*, J. Lond. Math. Soc. (2) **79** (2009), 238–258.

[44] W.M. WHYBURN, *Differential equations with general boundary conditions*, Bull. Amer. Math. Soc. **48** (1942), 692–704.

[45] J. XU AND Z. YANG, *Positive solutions for a system of nonlinear Hammerstein integral equations and applications*, J. Integral Equations Appl. **24** (2012), 131–147.

[46] Z. YANG, *Positive solutions to a system of second-order nonlocal boundary value problems*, Nonlinear Anal. **62** (2005), 1251–1265.

[47] Z. YANG, *Positive solutions of a second-order integral boundary value problem*, J. Math. Anal. Appl. **321** (2006), 751–765.

[48] Z. YANG, *Existence and nonexistence results for positive solutions of an integral boundary value problem*, Nonlinear Anal. **65** (2006), 1489–1511.

[49] Z. YANG, *Existence of nontrivial solutions for a nonlinear Sturm–Liouville problem with integral boundary conditions*, Nonlinear Anal. **68** (2008), 216–225.

[50] Z. YANG, *Positive solutions for a system of nonlinear Hammerstein integral equations and applications*, Appl. Math. Comput. **218** (2012), 11138–11150.

[51] Q. YAO, *Monotonically iterative method of nonlinear cantilever beam equations*, Appl. Math. Comput. **205** (2008), 432–437.

[52] E. ZEIDLER, *Nonlinear Functional Analysis and Its Applications*, I: *Fixed-Point Theorems*, Springer, New York, 1986.

Manuscript received November 23, 2017

accepted February 23, 2019

CHRISTOPHER S. GOODRICH
School of Mathematics and Statistics
UNSW Australia
Sydney, NSW 2052, AUSTRALIA

E-mail address: cgoodrich@creightonprep.org
c.goodrich@unsw.edu.au