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BLOW-UP SOLUTIONS FOR A p-LAPLACIAN ELLIPTIC EQUATION OF LOGISTIC TYPE WITH SINGULAR NONLINEARITY

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Abstract. In this paper, we deal with existence, uniqueness and exact rate of boundary behavior of blow-up solutions is for a class of logistic type quasilinear problems in a smooth bounded domain involving the p-Laplacian operator, where the nonlinearity can have a singular behavior. In the proof of the existence of solution, we have used the sub and super solution method in conjunction with variational techniques and comparison principles. Related to the rate on boundary and uniqueness, we combine comparison principle with our result of existence of solution.

1. Introduction

In this paper, we consider existence, uniqueness and exact rate of boundary behavior of blow-up (large or explosive) solutions for the following class of quasilinear problem of logistic type

$$(\mathrm{P})_\lambda \qquad \begin{cases} -\Delta_p u = \lambda a(x)g(u) - b(x)f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real parameter, Δ_p stands for the p-Laplacian operator given by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < +\infty$, $a, b\colon \Omega \to \mathbb{R}$ are appropriate functions with $b \gneqq 0$, a can change the signal and $f\colon [0,+\infty) \to [0,+\infty)$, $g\colon (0,+\infty) \to [0,+\infty)$ are continuous functions satisfying some technical conditions, which will be stated later on. We point out that the case $\lambda = 0$ is well known and our principal interest lies in the case when g is singular at 0, i.e. $g(s) \to +\infty$ as $s \to 0^+$.

We say that a function $u \in C^{1,\nu}_{\mathrm{loc}}(\Omega)$, for some $\nu \in (0,1)$, is a solution of problem $(\mathrm{P})_\lambda$, if

$$u(x) \to +\infty \quad \text{as } d(x) := \operatorname{dist}(x, \partial\Omega) \to 0,$$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \int_\Omega [\lambda a(x) g(u) - b(x) f(u)]\varphi, \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

where $d(x)$ stands for the distance of a point $x \in \Omega$ to $\partial\Omega$.

We have no intention to be too exhaustive in doing an overview of the context of our work, but since this class of problems seems to be very wide we present a number of works that motivated this paper even knowing that there are many important papers out of our list. We begin with the work of Delgado, López-Gómez and Suárez [2] from 2002 that showed existence of blow-up solution for the problem

$$\begin{cases} -\Delta u = \lambda u^{1/m} - b(x) u^{p/m} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}, 1 < m < p$ and $b(x) \geq 0$. Motivated by that paper, in 2004, the same authors studied in [3] the following problem

$$\begin{cases} -\Delta u = a(x) u^q - b(x) f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $0 < q < 1$, $a \in L^\infty(\Omega)$, $0 < b \in C^\mu(\overline{\Omega})$ for some $\mu \in (0,1)$ and f satisfies some technical conditions, such as, f is an increasing continuous and verifies the Keller–Osserman condition (with $p = 2$ see [11] and [17]), that is

$$\text{(KO)} \qquad \int_1^\infty F(t)^{-1/p}\,dt < \infty \quad \text{where } F(t) = \int_0^t f(\tau)\,d\tau.$$

In 2006, the same class of problem was considered by Du [5], with $q = 1$ and $f(u) = u^p$. In 2009, Feng in [7] showed that the problem

$$\begin{cases} -\Delta u = \lambda g(u) - b(x) f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

admits a unique solution for $\lambda \in \mathbb{R}$, $0 < b \in C^{\mu}(\overline{\Omega})$ for some $\mu \in (0,1)$ and f, g being increasing continuous functions satisfying additional conditions.

As an exception to the previous works, in 2010, Wei in [20] studied the problem $(\mathrm{P})_\lambda$ with negative exponents, more precisely, the singular problem

$$\begin{cases} -\Delta u = a(x)u^{-m} - b(x)u^{p} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $p > 1, m > 0$ and $a, b \in C^{\mu}(\overline{\Omega})$ for some $\mu \in (0,1)$ with b being a positive function.

Related to quasilinear problems, in 2012, Wei and Wang [21] worked with the following quasilinear boundary problem

$$\begin{cases} -\Delta_p u = a(x)u^{m} - b(x)u^{q} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $0 < m < p-1 < q$, $a \in L^{\infty}(\Omega)$ and b being a non-negative function.

One year later, in [1], Chen and Wang improved the results found in [21], because they showed that the problem

$$\begin{cases} -\Delta_p u = a(x)g(u) - b(x)f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

has a solution, by supposing that $a \in L^{\infty}(\Omega)$, $b \in C^{\mu}(\overline{\Omega})$ for some $0 < \mu < 1$, $b(x) \geq 0$, g is a nondecreasing and nonnegative continuous function with $g(0) = 0$ and $f \in C^1$ is an increasing function with $f(0) = 0$ and $f(s) > 0$ for $s > 0$. Moreover, $f(s)$ grows more slowly than s^q with $q > p-1$ and $g(s)$ does not grow faster than s^{p-1} at infinity.

Concerning the boundary behavior, in 2006, Ouyang and Xie [18] established a blow-up rate of the large positive solutions of the problem

$$\begin{cases} -\Delta u = \lambda u - b(\|x - x_0\|)u^{q} & \text{in } B, \\ u = +\infty & \text{on } \partial B, \end{cases}$$

where $B = B_R(x_0)$ stands for the ball centered at $x_0 \in \mathbb{R}^N$ with radius R, $b\colon [0,R] \to (0,\infty)$ is a continuous function, $q > 1$ and $\lambda \in \mathbb{R}$. Under additional conditions on b, they obtained a rate of boundary behavior accurate of the unique solution for the above problem.

In 2009, Feng [7] obtained the exact asymptotic behavior and uniqueness of solution for the problem

$$\begin{cases} -\Delta u = \lambda g(u) - b(x)f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda \in \mathbb{R}$, $0 \le b \in C^\mu(\overline{\Omega})$ for some $\mu \in (0,1)$, $b = 0$ on $\partial\Omega$ and there exists an increasing positive function $h \in C^1(0,\delta_0)$ for some $\delta_0 > 0$ verifying

$$\lim_{d(x)\to 0^+} \frac{b(x)}{h^2(d(x))} = c_0 > 0, \qquad \lim_{r\to 0^+} \frac{1}{h(r)}\bigg(\int_0^r h(s)\,ds\bigg) = 0$$

and

$$\lim_{r\to 0^+} \bigg[\frac{1}{h(r)}\bigg(\int_0^r h(s)\,ds\bigg)\bigg]' = l_1 > 0.$$

Related to f and g, it was assumed that $0 \le f, g \in C^1([0,+\infty))$, $f(0) = 0$, $f' \ge 0$, $f'(0) = 0$, $f(s)/s$, $s > 0$ increasing, f is RV_q with $q > 1$; g increasing with $\lim_{s\to 0^+} g'(s) > 0$, $g(s)/s$, $s > 0$ in non-increasing and g belongs to RV_q with $0 < q < 1$. In that paper, an arbitrary function $h\colon [s_o,\infty) \to (0,\infty)$, for some $s_0 > 0$, belongs to class RV_q, for some $q \in \mathbb{R}$, if

$$\lim_{s\to\infty} h(ts)/h(s) = t^q \quad \text{for all } t > 0.$$

Still in 2009, Melián [8] established an exact boundary behavior and uniqueness for the problem

$$\begin{cases} \Delta_p u = b(x)u^q & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $q > p-1 > 0$ with b satisfying

$$\lim_{x\to x_0} d(x)^{\gamma(x)} b(x) = Q(x_0) \quad \text{for } x_0 \in \partial\Omega,$$

for some $\gamma \in C^\mu(\overline{\Omega})$ with $0 < \mu < 1$, $\gamma(x) < 0$ and $Q(x) > 0$ for all $x \in \partial\Omega$.

In 2012, Li, Pang and Wang [13] also showed the boundary behavior and uniqueness for the problem

$$\begin{cases} -\Delta_p u = a(x)u^m - b(x)f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $0 < m < p-1$, $0 \le a \in L^\infty(\Omega)$, $f \in C^1([0,\infty)) \cap RV_q$, for some $q > p-1$, with $f(0) = 0$ and $f(s) > 0$ for $s > 0$, $f(s)/s^{p-1}$, $s > 0$ increasing, $b \in C^\mu(\overline{\Omega})$ for some $0 < \mu < 1$ with $b \ge 0$, $b(x) \not\equiv 0$ in Ω, $\overline{\Omega}_0 = \{x \in \overline{\Omega} \mid b(x) = 0\} \subset \Omega$ is a non-empty and connected set with C^2-boundary and some additional conditions on b.

In the same year, Chen and Wang [1] proved the boundary behavior and uniqueness of solution for the problem

$$\begin{cases} -\Delta_p u = a(x)g(u) - b(x)f(u) & \text{in}\Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$, $p > 1$, $0 \leq a \in L^\infty(\Omega)$, $g \in C([0,\infty)) \cap RV_q$, for some $q \leq p-1$ with g being nondecreasing, $g(s)/s^{p-1}$, $s > 0$ nonincreasing function satisfying $g(0) = 0$, f is such that $f(0) = 0$, $f(s) > 0$ for $s > 0$ and $f(s)/s^{p-1}$, $s > 0$ increasing, $0 \leq b \in C^\mu(\overline{\Omega})$ for some $0 < \mu < 1$ with $b \not\equiv 0$ in Ω and more hypotheses on b, f and g.

Still in 2012, Xie and Zhao [19] established the uniqueness and the blow-up rate of the large positive solution of the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} - b(\|x - x_0\|)f(u) & \text{in } B, \\ u = +\infty & \text{on } \partial B, \end{cases}$$

where $N \geq 2$, $2 \leq p < \infty$, $\lambda > 0$ is a parameter and the weight function $b\colon [0,R] \to (0,\infty)$ is a continuous function satisfying additional assumptions. Moreover, f is a locally Lipschitz continuous function with $f(s)/s^{p-1}$ increasing for $s \in (0,+\infty)$ and $f(s) \sim s^q$ for large $s > 0$ with $q > p-1$.

Motivated principally by the above papers and their results, we will study existence and uniqueness of blow-up solution and the exact boundary behavior rate. To do that, we denote by

$$a_0 := \operatorname*{essinf}_{\Omega} a, \qquad b_0 := \operatorname*{essinf}_{\Omega} b$$

and assume that f satisfies (KO) and the conditions:

(f_0) $0 < \liminf_{s\to+\infty} \dfrac{\inf\{f(t)/t^{p-1},\, t \geq s\}}{f(s)/s^{p-1}} \leq \infty$,

(f_1) (i) $\lim_{s\to 0^+} \dfrac{f(s)}{s^{p-1}} < \infty$ if $a_0 \geq 0$,

(ii) $\lim_{s\to+\infty} \dfrac{f(s)}{s^{p-1}} < \dfrac{1}{||b||_\infty}$ if $a_0 < 0$.

Remark 1.1. It follows from (KO) that

(f_2) $\lim_{s\to+\infty} \dfrac{f(s)}{s^{p-1}} = +\infty$

holds.

Associated with $g\colon (0,+\infty) \to (0,+\infty)$, we assume that

(g_0) (i) $0 \leq \lim_{s\to 0^+} \dfrac{g(s)}{s^{p-1}} \leq \infty$,

(ii) $0 \leq \lim_{s\to+\infty} \dfrac{g(s)}{s^{p-1}} < +\infty$.

Our first result is the following.

Theorem 1.2. *Assume* $a, b \in L^\infty(\Omega)$ *with* $b_0 > 0$. *If* f *satisfies* (KO), (f_0), (f_1) *and* (g_0), *then there exist* $\lambda_* \in (0, +\infty]$ *and a real number* $\sigma_o > 0$ *such that the problem* $(\mathrm{P})_\lambda$ *has a solution* $u = u_\lambda \geq \sigma_o$ *for each* $0 < \lambda < \lambda_*$ *given. Moreover,* $\lambda_* = +\infty$, *if* $a_0 \geq 0$.

Related to condition (f_0), it is important to observe that:

- if $f(s)/s^{p-1}$, $s \geq s_0$ is nondecreasing, for some $s_o > 0$, then the limit at (f_0) is equal to 1,
- if

$$f(t) = \begin{cases} \sigma(t)t^{p-1} & \text{for } 0 < t < 1, \\ t^{p-1}e^{-t} & \text{for } t \geq 1, \end{cases}$$

  where $\sigma \geq 0$ is a continuous function satisfying $\sigma(1) = e^{-1}$ and $\lim_{t\to 0} \sigma(t) = 0$, then the limit at (f_0) is null and f does not satisfy (KO). This example shows the necessity of hypothesis (f_0),

To state our second result, we need of more specifically assumptions on f and g, namely:

$(\mathrm{f}_1)'$ $0 < \lim_{t\to+\infty} \dfrac{f(t)}{t^q} = f_\infty < +\infty$ for some $q > p - 1$,

$(\mathrm{g}_0)'$ $0 \leq \lim_{t\to+\infty} \dfrac{g(t)}{t^m} = g_\infty < +\infty$ for some $m \leq p - 1$,

and concerning to the potentials a and b, we will suppose that they are continuous and satisfy

(a) there exist $Q \in C(\overline{\Omega})$ and a $\gamma \in C^\mu(\overline{\Omega})$, for some $0 < \mu < 1$, such that

$$\lim_{x\to x_0} d(x)^{\gamma(x)} b(x) = Q(x_0), \quad \text{for each } x_0 \in \partial\Omega$$

with $Q(x) > 0$, $x \in \partial\Omega$ and $\gamma(x) \leq 0$ for all $x \in U_\delta$, for some $\delta > 0$,

(b) there exists $R \in C(\overline{\Omega})$ with $R(x) \geq 0$ on U_δ, for some $\delta > 0$, such that

$$\lim_{x\to x_0} d(x)^{\eta(x)} a(x) = R(x_0) \quad \text{for each } x_0 \in \partial\Omega,$$

where $\eta(x) = (p - 1 - m)(p - \gamma(x))/(q - p + 1) + p$ for $x \in \overline{\Omega}$ and $U_\delta := \{x \in \Omega \mid d(x) < \delta\}$.

Related to above notations, we have the following result.

Theorem 1.3. *Assume* $(\mathrm{f}_1)'$ *and* $(\mathrm{g}_0)'$. *Suppose that* $a, b \in L^\infty_{\mathrm{loc}}(\Omega)$ *with* $a \geq 0$ *almost everywhere in* U_δ, (a) *and* (b) *hold true for some* $\delta > 0$. *If* $u \in C^1(\Omega)$ *is a positive solution of* $(\mathrm{P})_\lambda$, *then*

$$\lim_{x\to x_0} d(x)^{\alpha(x)} u(x) = A(x_0) \quad \textit{for each } x_0 \in \partial\Omega,$$

where $\alpha(x) = (p - \gamma(x))/(q - p + 1)$, $x \in \overline{\Omega}$ *and* $A(x_0)$ *is the unique positive real number satisfying*

$$f_\infty Q(x_0)A^{q-m}(x_0) - (p-1)\alpha(x_0)^{p-1}(1 + \alpha(x_0))A^{p-m-1}(x_0) - \lambda g_\infty R(x_0) = 0.$$

Moreover, if $a_0, b_0 \geq 0$*,* $f(t)/t^{p-1}$ *is nondecreasing and* $g(t)/t^{p-1}$ *is nonincreasing for* $t \in (0, +\infty)$*, then the problem* $(\mathrm{P})_\lambda$ *has at most one solution.*

Related to assumption $(g_0)'$, we would like to detach that if g is $(p-1)$-sublinear at infinity, that is, if $g_\infty = 0$ with $m = p - 1$, then the behavior of the solution is unaffected by g.

To highlight our last result, we state the corollary below, whose boundary behavior's proof follows from Theorem 1.3 by taking $R(x) = \gamma(x) = 0$ for $x \in \overline{\Omega}$ at hypotheses (a) and (b).

COROLLARY 1.4. *Assume* $a \in L^\infty(\Omega)$ *and* $b \in C(\overline{\Omega})$ *with* $a_0 \geq 0$ *and* $b_0 > 0$*. If* $-\infty < m \leq p - 1 < q$*, then the quasilinear problem*

$$\begin{cases} -\Delta_p u = \lambda a(x) u^m - b(x) u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u = u_\lambda \in C^1(\Omega)$ *for each* $\lambda > 0$ *satisfying*

$$\lim_{x \to x_0} d(x)^{p/(q-p+1)} u(x) = b(x_0)^{-1/(q-p+1)} \left(\frac{(p-1)(q+1)p^{p-1}}{(q-p+1)^p} \right)^{-1/(q-p+1)},$$

for each $x_0 \in \partial\Omega$*.*

We would like point out that the main contributions of our results for this class of problem are the following:

About Theorem 1.2. Our result extends the principal result found in [20] to the context of the p-Laplacian operator. Here, we do not use the same approach explored [20], for example the degree theory, because in our context, it is not clear that some estimates used in [20] also hold for p-Laplacian operator. In our approach, one of the delicate points is to obtain a sub solution for a problem with boundary datum finite. Another point is to control by above a sequence of solutions of some problems with boundary datum finite. Even in the context of p-Laplacian operator, our result improves and complements the previous results principally because it does not requires that the term g be non-decreasing and any kind of monotonicity under f. In fact, we just assume local behaviors at zero and infinity of terms f and g permitting even singularity of g at zero.

About Theorem 1.3. This result improves some previous results for the context of singular logistic equation by assuming less hypotheses under the terms. In particular, it extends the principal result in [8]. This generalization is not straightforward. One point of much sensitive is the absence of comparison principle appropriate for this class of problems. Another one is related to structure of our logistic equation that does not permit us to use Poincaré–Bendixon's Theorem, as made in [8], to prove that the problem (4.1) has unique and explicit solution. This fact is essential in the proof of Theorem 2.2.

We now briefly outline the organization of the contents of this paper. In Section 2, by using a sub- and super-solution method in conjunction with variational method, we prove the existence of solution for two auxiliary blow-up problems. Section 3 is devoted to prove the existence of blow-up solution for $(\mathrm{P})_\lambda$, while in Sections 4 and 5, we study the rate boundary of the solutions.

2. Auxiliary problem

In this section, we are interested in the existence of solution for the following quasilinear problem

$$(\mathrm{P})_L \qquad \begin{cases} -\Delta_p u = \lambda a(x) g(u) - b(x) f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = L & \text{on } \partial\Omega, \end{cases}$$

where $L > 1$ is an appropriate real number. Associated with above problem, we have the following result.

Proposition 2.1. *Assume $a, b \in L^\infty(\Omega)$ with $b_0 > 0$. If* (f_1), (f_2) *and* (g_0) *hold, then there exist $\lambda_* \in (0, +\infty]$ and $\sigma_{\mathrm{o}} > 0$, which does not depend on $L > 0$, such that* $(\mathrm{P})_L$ *has a solution $u = u_{\lambda,L} \in C^1(\overline{\Omega})$ for each $0 < \lambda < \lambda_*$ and $L > L_0$ given, for some $L_0 > 0$, satisfying $\sigma_{\mathrm{o}} \le u(x) \le L$ for all $x \in \overline{\Omega}$. Moreover, $u_{L_1} \le u_{L_2}$ if $L_0 < L_1 \le L_2$ and $\lambda_* = +\infty$ if $a_0 \ge 0$.*

The proof of this proposition is based on the next two lemmas.

Lemma 2.2. *Assume $a, b \in L^\infty(\Omega)$ with $b_0 > 0$. If* (f_1) *and* (g_0) *hold, then there exist $\lambda_* \in (0, +\infty]$, $L_0 > 0$ and a $\sigma_{\mathrm{o}} > 0$, which does not depend on $L > 0$, such that* $(\mathrm{P})_L$ *has a sub solution $\underline{u} = \underline{u}_{\lambda,L} \in C^1(\overline{\Omega})$ for each $L > L_0$ and $\lambda \in (0, \lambda_*)$ given. Moreover, $\sigma_{\mathrm{o}} \le \underline{u}(x) \le L$ for all $x \in \overline{\Omega}$, and $\lambda_* = +\infty$ if $a_0 \ge 0$.*

Proof. In the sequel, we will divide our proof into two cases.

Case 1. $a_0 \ge 0$. From continuity of f and (f_1), the function

$$\widetilde{f}(s) = s^{p-1} \sup\left\{\frac{f(t)}{t^{p-1}},\ t \le s\right\} + s^p \quad \text{for } s \in (0, +\infty),$$

is continuous and verifies

(i) $\dfrac{\widetilde{f}(s)}{s^{p-1}}$, $s > 0$, is increasing,

(ii) $\widetilde{f}(s) \ge f(s)$, $s > 0$,

(iii) $\displaystyle\lim_{s\to 0^+}\frac{\widetilde{f}(s)}{s^{p-1}} < \infty$,

(iv) $\displaystyle\lim_{s\to +\infty}\frac{\widetilde{f}(s)}{s^{p-1}} = +\infty$.

Now, by using [4, Theorem 1.1], we know that the problem

$$\begin{cases} -\Delta_p v = -\|b\|_\infty \widetilde{f}(v) & \text{in } \Omega,\\ 0 < v \le 1 & \text{in } \Omega,\\ v = 1 & \text{on} \partial\Omega, \end{cases}$$

has a solution $\underline{u} \in C^1(\overline{\Omega})$. Since (iii) above holds, we are able to apply the strong maximum principle of Vazquez to conclude that $\underline{u}$ is positive in $\overline{\Omega}$. Besides this, $\underline{u} \le 1$ in Ω follows from the standard comparison principle. So, $\underline{u}$ satisfies

$$\begin{cases} -\Delta_p u \le \lambda a(x) g(u) - b(x) f(u) & \text{in } \Omega,\\ u \ge \gamma_1 > 0 & \text{in } \overline{\Omega},\\ u \le L & \text{on } \partial\Omega, \end{cases}$$

for all $L \ge 1$ and $\lambda > 0$ given (that is, $\lambda_* = \infty$), where $\gamma_1 = \min_{\overline{\Omega}} \underline{u} > 0$.

Case 2. $a_0 < 0$. By the continuity of g and (g_0), the function

$$\widehat{g}(s) = s^{p-1} \sup\left\{\frac{g(t)}{t^{p-1}},\ t \ge s\right\} + 1, \quad s > 0,$$

is continuous and verifies

(i) $\dfrac{\widehat{g}(s)}{s^{p-1}}$, $s > 0$, is decreasing,

(ii) $\widehat{g}(s) > g(s)$, $s > 0$,

(iii) $\displaystyle\lim_{s\to 0^+}\frac{\widehat{g}(s)}{s^{p-1}} = \infty$,

(iv) $\displaystyle\lim_{s\to +\infty}\frac{\widehat{g}(s)}{s^{p-1}} < \infty$.

Next, we denote by $w \in C^{1,\mu}(\overline{\Omega})$ the unique positive solution of the problem

$$(\mathrm{P}_3) \qquad \begin{cases} \Delta_p u = u^{p-1} & \text{in } \Omega,\\ 0 < u \le 1 & \text{in } \Omega,\\ u = 1 & \text{on } \partial\Omega. \end{cases}$$

The existence of the above function can be found in [4].

Defining $w_0 = \min_{\overline{\Omega}} w > 0$ and

$$\varphi(M) = \frac{(Mw_0)^{p-1}}{\widehat{g}(Mw_0)}\left[1 - \|b\|_\infty \frac{\widetilde{f}(M)}{M^{p-1}}\right] \quad \text{for } M > 0,$$

it follows from the properties of $\widetilde{f}$ that

$$\lim_{M\to\infty}\varphi(M) = -\infty, \quad \lim_{M\to 0}\varphi(M) \geq 0 \quad \text{and} \quad \varphi(\widetilde{M}) > 0, \quad \text{for some } \widetilde{M} > 0,$$

where the last one is a consequence of (f_1) (ii). Thereby, there is $M_0 > 0$ such that $\varphi(M_0) = \sup\{\varphi(M) \mid M > 0\}$. In the sequel, we denote by λ_* the real number given by

$$\lambda_* := -\frac{\sup\{\varphi(M) \mid M > 0\}}{a_0} = -\frac{\varphi(M_0)}{a_0} > 0.$$

Thus, for $\lambda \in (0, \lambda_*)$,

$$1 - \|b\|_\infty \frac{\widetilde{f}(M_0)}{M_0^{p-1}} \geq (-\lambda a_0)\frac{\widehat{g}(M_0 w_0)}{(M_0 w_0)^{p-1}}.$$

Now, once $\widetilde{f}(s)/s^{p-1}$ is increasing and $\widehat{g}(s)/s^{p-1}$ is decreasing in the interval $(0, +\infty)$, we obtain

$$1 - \|b\|_\infty \frac{\widetilde{f}(M_0 w(x))}{(M_0 w(x))^{p-1}} \geq (-\lambda a_0)\frac{\widehat{g}(M_0 w(x))}{(M_0 w(x))^{p-1}}, \quad \text{for all } x \in \Omega.$$

Taking $\underline{u} = M_0 w \geq M_0 w_0 := \gamma_2 > 0$, the last inequality gives

$$\begin{cases} -\Delta_p(\underline{u}) \leq \lambda a_0 \widehat{g}(\underline{u}) - \|b\|_\infty \widetilde{f}(\underline{u}) \leq \lambda a(x) g(\underline{u}) - b(x) f(\underline{u}) & \text{in } \Omega, \\ \underline{u} \geq \gamma_2 & \text{in } \Omega, \\ \underline{u} < L & \text{on } \partial\Omega, \end{cases}$$

for all $L > M_0$. Hence, choosing $\sigma_{\mathrm{o}} = \min\{\gamma_1, \gamma_2\} > 0$ and $L_0 = \max\{1, M_0\}$, we get the desired result. □

For the super solution, our result is the following lemmaa.

Lemma 2.3. *Assume $a, b \in L^\infty(\Omega)$ with $b_0 > 0$. If (f_1), (f_2) and (g_0) hold, then $\overline{u}(x) := L$ in $\overline{\Omega}$ is a super solution of Problem $(\mathrm{P})_L$ satisfying $\underline{u} \leq \overline{u}$ for each $L > L_0$ given, where $L_0 \geq 1$ was given in Lemma 2.2.*

Proof. Let $\lambda \in (0, \lambda_*)$, where $\lambda_* > 0$ was given in Lemma 2.1. By (f_2) and (g_0) (ii), we can choose $0 < c_3 < c_4$ and $t_\infty > 1$ positive constants verifying

$$f(t) \geq c_4 t^{p-1} \quad \text{and} \quad g(t) \leq c_3 t^{p-1} \quad \text{for all } t \in (t_\infty, +\infty)$$

and $\lambda c_3 \|a\|_\infty - b_0 c_4 < 0$. Defining $\overline{u} = L$, with $L \geq \max\{L_0, t_\infty\}$, we derive

$$-\Delta_p \overline{u} = 0 > (\lambda \|a\|_\infty c_3 - b_0 c_4) L^{p-1} \geq \lambda \|a\|_\infty g(\overline{u}) - b_0 f(\overline{u}) \quad \text{in } \Omega.$$

Consequently, $\overline{u} \in C^1(\overline{\Omega})$ and it satisfies

$$\begin{cases} -\Delta_p \overline{u} \geq \lambda a(x) g(\overline{u}) - b(x) f(\overline{u}) & \text{in } \Omega, \\ \overline{u} \geq \underline{u} & \text{in } \Omega, \\ \overline{u} \geq L & \text{on } \partial\Omega, \end{cases}$$

for all $L \geq \max\{L_0, t_\infty\}$. □

Proof of Proposition 2.1 (a sketch). We just outline some lines of the proof, because it follows by applying well known arguments. First, as a consequence of Lemmas 2.1 and 2.2, we have that the functions $\underline{v} = \underline{u} - L \leq 0$ and $\overline{v} = 0$ are sub and super solutions of the problem

$$(\mathrm{P}_1) \qquad \begin{cases} -\Delta_p v = \lambda a(x) g(v+L) - b(x) f(v+L) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively, for each $L \geq L_0$ and $0 < \lambda < \lambda_*$ given.

Hereafter, we will consider the function $h \colon \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$h(x,t) = \begin{cases} \lambda a(x) g(\underline{v}+L) - b(x) f(\underline{v}+L) & \text{if } t \leq \underline{v}(x), \\ \lambda a(x) g(t+L) - b(x) f(t+L) & \text{if } \underline{v}(x) \leq t \leq 0, \\ \lambda a(x) g(\overline{v}+L) - b(x) f(\overline{v}+L) & \text{if } t \geq 0, \end{cases}$$

and the problem

$$(\mathrm{P}_2) \qquad \begin{cases} -\Delta_p v = h(x,v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Our goal is proving that problem (P_2) has a $W_0^{1,p}(\Omega)$-solution v_0 satisfying

$$(2.1) \qquad \underline{v}(x) \leq v_0(x) \leq 0 \quad \text{a.e. in } \Omega,$$

because it is enough to conclude that v_0 in a $W_0^{1,p}(\Omega)$-solution for (P_1). To do this, we define the energy functional associated with the above problem by

$$I(v) = \frac{1}{p} \int_\Omega |\nabla v|^p - \int_\Omega H(x,v), \quad v \in W_0^{1,p}(\Omega),$$

where $H(x,t) = \int_0^t h(x,\tau)\,d\tau$, and note that it is standard to show that I belongs to $C^1(W_0^{1,p}(\Omega),\mathbb{R})$, I is weak s.c.i and bounded from below in $W_0^{1,p}(\Omega)$. Then, there is $v_0 \in W_0^{1,p}(\Omega)$ such that $I(v_0) = \min\{I(v) \mid v \in W_0^{1,p}(\Omega)\}$, that is, $I'(v_0) = 0$. So, v_0 is a weak solution of (P_2) and by elliptic regularity theory $v_0 \in C^1(\overline{\Omega})$. Moreover, the monotonicity of the $-\Delta_p$ yields (2.1) occurs. Therefore, the function $u = u_{\lambda,L} = v_0 + L$ is a solution of $(\mathrm{P})_L$ with

$$0 < \sigma_o \leq \underline{u}(x) \leq u(x) \leq \overline{u}(x) = L \quad \text{a.e. in } \Omega.$$

Now, if $L_0 < L_1 \leq L_2$, we can repeat the above arguments by using $\underline{u} = u_{L_1}$ and $\overline{u} = L_2$ as sub and super solution of problem $(\mathrm{P})_L$, respectively, where u_{L_1} is the solution of the problem $(\mathrm{P})_{L_1}$. So, we complete the proof. □

3. Proof of the Theorem 1.2

In this section, we will finish the proof Theorem 1.2. To do that, we will need of two auxiliary results. The first result is due to Matero [15].

Lemma 3.1. *Assume that Ω is a smooth bounded domain in $\mathbb{R}^N$ and a function $h\colon (0,\infty)\to(0,\infty)$ is continuous, increasing and satisfies* (KO). *Then, the quasilinear problem*

$$\begin{cases} \Delta_p u = h(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

admits a positive solution $u \in C^1(\Omega)$.

The second one is a comparison principle appropriate as well for blow-up solutions and includes singular nonlinearities and unbounded potentials α and β, which is essential in our approach and its proof follows by exploiting arguments as those found in [14]. The following lemma complements some results in [13] and [6] by considering a more general hypothesis under h, and complements the comparision principle of [16] because we do not request $u_2 \in L^\infty(\Omega)$.

Lemma 3.2 (Comparison Principle). *Suppose that Ω is a bounded domain in $\mathbb{R}^N$ and that $\alpha,\beta\colon \Omega\to[0,\infty)$ are nonnegative continuous functions. Let $u_1,u_2\in C^1(\Omega)$ be positive functions verifying*

$$\begin{cases} -\Delta_p u_1 \geq \alpha(x)h(u_1) - \beta(x)k(u_1) & \text{in } \Omega, \\ -\Delta_p u_2 \leq \alpha(x)h(u_2) - \beta(x)k(u_2) & \text{in } \Omega, \\ -\infty \leq \limsup\limits_{d(x)\to 0}(u_2 - u_1) \leq 0, \end{cases}$$

in the sense of distributions, where $h,k\colon[0,\infty)\to[0,\infty)$ are continuous functions satisfying $h(t),k(t)>0$ for $t>0$. If

$$\int_{0<u_2(x)\leq 1} h(u_2)u_2\,dx < \infty \quad \text{or} \quad 0 < \liminf_{d(x)\to 0} u_1(x) \leq \infty,$$

and either

(a) *$h(s)/s^{p-1}$ is decreasing, $k(s)/s^{p-1}$ is non-decreasing and $\alpha\in L^\infty(\Omega)$ with $\alpha\not\equiv 0$, or*
(b) *$h(s)/s^{p-1}$ is non-increasing and $k(s)/s^{p-1}$ is increasing and $\beta\in L^\infty(\Omega)$ with $\beta\not\equiv 0$*

holds true, for all $s \in \Big(\inf\limits_{\Omega}\{u_1,u_2\}, \sup\limits_{\Omega}\{u_1,u_2\}\Big)$, then $u_1\geq u_2$ in Ω.

Proof. It follows from the hypotheses about u_i, that

$$-\int_\Omega [|\nabla u_2|^{p-2}\nabla u_2\nabla\varphi_2 - |\nabla u_1|^{p-2}\nabla u_1\nabla\varphi_1] \geq \int_\Omega \alpha(x)[h(u_1)\varphi_1 - h(u_2)\varphi_2] + \int_\Omega \beta(x)[k(u_2)\varphi_2 - k(u_1)\varphi_1], \tag{3.1}$$

for all $0 \leq \varphi_1, \varphi_2 \in C_0^\infty(\Omega)$. Now, by considering the set

$$\Omega_\varepsilon = \left\{x \in \Omega \;\middle|\; u_2(x) + \frac{\varepsilon}{2} > u_1(x) + \varepsilon\right\},$$

it follows from the hypothesis about the behaviors of u_i on the boundary of Ω, that $\overline{\Omega}_\varepsilon \subset \Omega$ for each $\varepsilon > 0$ given. Then, by density, we can consider the functions $v_1, v_2 \in W_0^{1,p}(\Omega)$ given by

$$v_1 = \frac{[(u_2+\varepsilon/2)^p - (u_1+\varepsilon)^p]^+}{(u_1+\varepsilon)^{p-1}} \quad\text{and}\quad v_2 = \frac{[(u_2+\varepsilon/2)^p - (u_1+\varepsilon)^p]^+}{(u_2+\varepsilon/2)^{p-1}},$$

with $\varepsilon > 0$, as test functions in (3.1). Since

$$\nabla v_1 = -\left[1 + (p-1)\left(\frac{u_2+\varepsilon/2}{u_1+\varepsilon}\right)^p\right]\nabla u_1 + p\left(\frac{u_2+\varepsilon/2}{u_1+\varepsilon}\right)^{p-1}\nabla u_2,$$

and

$$\nabla v_2 = \left[1 + (p-1)\left(\frac{u_1+\varepsilon}{u_2+\varepsilon/2}\right)^p\right]\nabla u_2 - p\left(\frac{u_1+\varepsilon}{u_2+\varepsilon/2}\right)^{p-1}\nabla u_1 \quad \text{in } \Omega_{\varepsilon,\delta},$$

we obtain that

$$\begin{aligned} I :&= |\nabla u_2|^{p-2}\nabla u_2\nabla v_2 - |\nabla u_1|^{p-2}\nabla u_1\nabla v_1 \\ &= \left\{\left[1 + (p-1)\left(\frac{u_1+\varepsilon}{u_2+\varepsilon/2}\right)^p\right]|\nabla u_2|^p \right. \\ &\quad \left. + \left[1 + (p-1)\left(\frac{u_2+\varepsilon/2}{u_1+\varepsilon}\right)^p\right]|\nabla u_1|^p\right\} \\ &\quad - p\left(\frac{u_1+\varepsilon}{u_2+\varepsilon/2}\right)^{p-1}|\nabla u_2|^{p-2}\nabla u_1\nabla u_2 \\ &\quad - p\left(\frac{u_2+\varepsilon/2}{u_1+\varepsilon}\right)^{p-1}|\nabla u_1|^{p-2}\nabla u_1\nabla u_2 \end{aligned} \tag{3.2}$$

in Ω_ε. Now, setting $w_1 = u_1 + \varepsilon$, $w_2 = u_2 + \varepsilon/2$, $V_1 := \nabla \ln(w_1) = (\nabla u_1)/w_1$ and $V_2 := \nabla \ln(w_2) = (\nabla u_2)/w_2$ in Ω_ε, it follows from (3.2), that

$$\begin{aligned} I &= \{w_2^p|V_2|^p + (p-1)w_1^p|V_2|^p + w_1^p|V_1|^p + (p-1)w_2^p|V_1|^p\} \\ &\quad - pw_1^p|V_2|^{p-2}V_1V_2 - pw_2^p|V_1|^{p-2}V_1V_2 \\ &= w_2^p(|V_2|^p - |V_1|^p - p|V_1|^{p-2}V_1(V_2 - V_1)) \\ &\quad + w_1^p(|V_1|^p - |V_2|^p - p|V_2|^{p-2}V_2(V_1 - V_2)). \end{aligned} \tag{3.3}$$

Then, from (3.3) and [14, Lemma 4.2],

$$\begin{aligned} I &\geq c(p)\frac{|V_2-V_1|^2}{(|V_2|+|V_1|)^{2-p}}\, w_2^p + c(p)\frac{|V_1-V_2|^2}{(|V_1|+|V_2|)^{2-p}}\, w_1^p \\ &= c(p)(w_1^p+w_2^p)\,\frac{|V_1-V_2|^2}{(|V_1|+|V_2|)^{2-p}}, \end{aligned}$$

if $1<p<2$ and

$$I \geq w_2^p\,\frac{|V_2-V_1|^p}{2^{p-1}-1} + w_1^p\,\frac{|V_1-V_2|^p}{2^{p-1}-1} = \frac{1}{2^{p-1}-1}\,(w_1^p+w_2^p)|V_1-V_2|^p,$$

if $p\geq 2$, where $c(p)$ is a real positive constant depending on just p, that is, by gathering these above informations, we obtain

$$I \geq C(p)(w_1^p+w_2^p)\frac{|V_1-V_2|^{p+(2-p)^+}}{(|V_1|+|V_2|)^{(2-p)^+}} \ \text{for all } p>1, \tag{3.4}$$

for some $C(p)$ positive.

Now, (3.4) combined with (3.1) gives

$$\begin{aligned} C(p)\int_{\Omega_\varepsilon}(w_1^p+w_2^p)&\frac{|V_1-V_2|^{p+(2-p)^+}}{(|V_1|+|V_2|)^{(2-p)^+}} \\ &+\int_{\Omega_\varepsilon}\alpha(x)\left[\frac{h(u_1)}{w_1^{p-1}}-\frac{h(u_2)}{w_2^{p-1}}\right](w_2^p-w_1^p) \\ &\qquad\leq \int_{\Omega_\varepsilon}\beta(x)\left[\frac{k(u_1)}{w_1^{p-1}}-\frac{k(u_2)}{w_2^{p-1}}\right](w_2^p-w_1^p)\leq 0, \end{aligned} \tag{3.5}$$

where we used the hypothesis under $k(s)$, to obtain

$$\frac{k(u_1)}{w_1^{p-1}}-\frac{k(u_2)}{w_2^{p-1}} \leq \frac{k(u_2)}{u_2^{p-1}}\left[\frac{u_1^{p-1}}{w_1^{p-1}}-\frac{u_2^{p-1}}{w_2^{p-1}}\right]\leq 0 \quad \text{in } \Omega_\varepsilon. \tag{3.6}$$

Now, we should consider two cases. If $\int_{u_2(x)\leq 1} h(u_2)u_2<\infty$, let us split Ω_ε in

$$D_1(\varepsilon)=\Omega_\varepsilon\cap\{u_2\leq 1\} \quad \text{and} \quad D_2(\varepsilon)=\Omega_\varepsilon\cap\{u_2>1\},$$

that is, $\Omega_\varepsilon=D_1(\varepsilon)\cup D_2(\varepsilon)$. First, note that

$$\alpha(x)\left[\frac{h(u_1)}{w_1^{p-1}}-\frac{h(u_2)}{w_2^{p-1}}\right](w_2^p-w_1^p)\geq -2p\alpha(x)h(u_2)u_2 \quad \text{in } D_1(\varepsilon). \tag{3.7}$$

About $D_2(\varepsilon)$, let us show that there exists a $K_1>0$, independent of $\varepsilon>0$, such that

$$\frac{h(u_1)}{w_1^{p-1}}-\frac{h(u_2)}{w_2^{p-1}}\geq -K_1 \quad \text{in } D_2(\varepsilon). \tag{3.8}$$

In fact, if the last inequality does not occur, there would be $\varepsilon_n\in(0,1]$ and $x_n\in\Omega_{\varepsilon_n}$ verifying

$$\frac{h(u_1(x_n))}{w_1^{p-1}(x_n)}-\frac{h(u_2(x_n))}{w_2^{p-1}(x_n)}\to-\infty, \quad \text{when } n\to\infty,$$

that is, we would have that the limit $h(u_2(x_n))/w_2^{p-1}(x_n) \to +\infty$, which leads to

$$\frac{h(u_2(x_n))}{u_2^{p-1}(x_n)} = \frac{h(u_2(x_n))}{w_2^{p-1}(x_n)} \frac{w_2^{p-1}(x_n)}{u_2^{p-1}(x_n)} \to +\infty,$$

implying that $u_2(x_n) \to 0$, but this is a contradiction by definition of $D_2(\varepsilon)$.

Next, we must prove that there is $M > 0$, which does not depend on $\varepsilon \in (0,1)$, such that

$$0 < w_2^p(x) - w_1^p(x) \le M \quad \text{for all } x \in \Omega_\varepsilon \text{ and for all } \varepsilon \in (0,1). \tag{3.9}$$

Indeed, arguing by contradiction, we assume that there are $\varepsilon_n \in (0,1]$ and $x_n \in \Omega_{\varepsilon_n}$, such that

$$M_n = (u_2(x_n) + \varepsilon_n/2)^p - (u_1(x_n) + \varepsilon_n)^p \to +\infty.$$

The above limit gives $u_2(x_n) \to +\infty$, and thus, $d(x_n) = d(x_n, \partial\Omega) \to 0$. Rewriting M_n as

$$M_n = \left(1 + \frac{\varepsilon_n}{2u_2(x_n)}\right)^p [u_2(x_n)^p - u_1(x_n)^p] + \left[\left(1 + \frac{\varepsilon_n}{2u_2(x_n)}\right)^p - \left(1 + \frac{\varepsilon_n}{u_1(x_n)}\right)^p\right] u_1(x_n)^p$$

and using the inequality $u_1(x_n) \le u_2(x_n)$ in Ω_{ε_n}, together with $\limsup_{x \to \partial\Omega} (u_2 - u_1) \le 0$, we are led to $\limsup_{n\to\infty} M_n \le 0$, which is a contradiction again.

By (3.8) and (3.9), have

$$\alpha(x)\left[\frac{h(u_1)}{w_1^{p-1}} - \frac{h(u_2)}{w_2^{p-1}}\right](w_2^p - w_1^p) \ge -\alpha(x) K_1 M \text{ in } D_2(\varepsilon). \tag{3.10}$$

Now, assume $0 < \liminf_{d(x)\to 0} u_1(x) \le \infty$. So, it follows from this assumption and definition of Ω_ε that there exists $K_2 > 0$, which does not depend on $\varepsilon > 0$, such that

$$\frac{h(u_1)}{w_1^{p-1}} - \frac{h(u_2)}{w_2^{p-1}} \ge -K_2 \quad \text{in } \Omega_0, \tag{3.11}$$

where $\Omega_0 = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$. So, it follows from (3.9) and (3.11), that

$$\alpha(x)\left[\frac{h(u_1)}{w_1^{p-1}} - \frac{h(u_2)}{w_2^{p-1}}\right]\left(w_2^p - w_1^p\right) \ge -\alpha(x) K_2 M \quad \text{in } \Omega_\varepsilon \tag{3.12}$$

holds true.

To finish the proof, first assume that (a) holds true. It follows from (3.7) and either (3.10) or (3.12), that we are able to use the Fatou's Lemma at (3.5), to obtain

$$0 \le C(p) \int_{\Omega_0} \big[u_1^p + u_2^p\big] \frac{|\nabla \ln u_1 - \nabla \ln u_2|^{p+(2-p)^+}}{(|\nabla \ln u_1| + |\nabla \ln u_2|)^{(2-p)^+}} + \int_{\Omega_0} \alpha(x) \bigg[\frac{h(u_1)}{u_1^{p-1}} - \frac{h(u_2)}{u_2^{p-1}}\bigg](u_2^p - u_1^p) \le 0,$$

where we used (3.6) and the hypothesis $k(s)/s^{p-1}$ being non-decreasing in $(0,+\infty)$ to infer the last inequality. So,

$$\nabla \ln u_1 - \nabla \ln u_2 \equiv 0 \quad \text{and} \quad \alpha(x) \equiv 0 \quad \text{in } \Omega_0,$$

that is, $u_2 = cu_1$ in Ω_0 for some $c > 1$, because $u_2(x) > u_1(x)$ for $x \in \Omega$. Since $\alpha(x) \equiv 0$ in Ω_0 and $\alpha \neq 0$ in Ω, we obtain that $\Omega_0 \subsetneq \Omega$. Since $u_1 = cu_2$ and $u_1 = u_2$ on on $\partial\Omega_0$, we must have $c = 1$. This is a contradiction.

Finally, assuming (b) and applying the Fatou's Lemma in (3.5), we are led to inequality

$$0 \le C(p) \int_{\Omega_0} (u_1^p + u_2^p) \frac{|\nabla \ln u_1 - \nabla \ln u_2|^{p+(2-p)^+}}{(|\nabla \ln u_1| + |\nabla \ln u_2|)^{(2-p)^+}} + \int_{\Omega_0} \beta(x) \bigg[\frac{k(u_2)}{u_2^{p-1}} - \frac{k(u_1)}{u_1^{p-1}}\bigg](u_2^p - u_1^p) \le 0,$$

which permits us to apply the same arguments as done in (a) to finish the proof of Lemma 3.2. □

Proof of Theorem 1.2 (completed). First of all, we consider the following auxiliary blow-up problem

$$(\mathrm{P}_6) \qquad \begin{cases} -\Delta_p u = \lambda \|a\|_\infty \widehat{g}(u) - b_0 \widehat{f}(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\widehat{g}$ was fixed in the proof of Lemma 2.2 and $\widehat{f}$ is given by

$$\widehat{f}(s) = s^{p-1} \inf\bigg\{\frac{f(t)}{t^{p-1}},\ t \ge s\bigg\} \quad \text{for } s > 0.$$

Combining the continuity of f with (f_1), (g_0) and (KO), we derive that $\widehat{f}$ is continuous and satisfies:

(iv) $\dfrac{\widehat{f}(s)}{s^{p-1}}$, $s > 0$ is nondecreasing,

(v) $\widehat{f}(s) \le f(s)$, $s > 0$,

(vi) $\displaystyle\lim_{s \to 0^+} \frac{\widehat{f}(s)}{s^{p-1}} = 0$,

(vii) $\lim_{s\to+\infty} \dfrac{\widehat{f}(s)}{s^{p-1}} = \infty.$

Next, we fix the function $h\colon (0,+\infty) \to \mathbb{R}$ by

$$h(t) = b_0\widehat{f}(t) - \lambda\,\|a\|_\infty \widehat{g}(t) = t^{p-1}\left[b_0\,\frac{\widehat{f}(t)}{t^{p-1}} - \lambda\|a\|_\infty \frac{\widehat{g}(t)}{t^{p-1}}\right].$$

Using the properties on $\widehat{f}$ and $\widehat{g}$, we see that

$$h(t_0) < 0 \quad \text{for some } t_0 > 0, \qquad \lim_{t\to+\infty} h(t) = +\infty.$$

Moreover, h is increasing in $(t_1,+\infty)$, where $t_1 > 0$ is the unique number verifying $h(t_1) = 0$, or equivalently,

$$b_0\,\frac{\widehat{f}(t_1)}{{t_1}^{p-1}} = \lambda\|a\|_\infty \frac{\widehat{g}(t_1)}{{t_1}^{p-1}}.$$

Considering $\widetilde{h}(t) = h(t+t_1)$ for $t \in (0,+\infty)$, we have that $\widetilde{h}$ is a continuous, positive and increasing function verifying (KO). In fact, $\widetilde{h}$ satisfies the hypothesis (KO), because

$$\lim_{s\to+\infty} \frac{\widehat{f}(s+t_1)}{(s+t_1)^{p-1}} = +\infty \quad \text{and} \quad \lim_{s\to+\infty} \frac{\widehat{g}(s+t_1)}{(s+t_1)^{p-1}} = 0,$$

that is, there exists a $s_0 > 0$ such that

$$\frac{\widehat{g}(s+t_1)}{(s+t_1)^{p-1}} < \frac{b_0}{2\lambda\|a\|_\infty}\,\frac{\widehat{f}(s+t_1)}{(s+t_1)^{p-1}} \quad \text{for all } s > s_0.$$

Consequently,

$$\begin{aligned}
\widetilde{H}(t) &:= \int_0^t \widetilde{h}(s)\,ds = \int_0^{s_0} \widetilde{h}(s)ds + \int_{s_0}^t [b_0\widehat{f}(s+t_1) - \lambda\|a\|_\infty \widehat{g}(s+t_1)]\,ds \\
&> \int_{s_0}^t (s+t_1)^{p-1}\left[b_0\,\frac{\widehat{f}(s+t_1)}{(s+t_1)^{p-1}} - \lambda\|a\|_\infty \frac{\widehat{g}(s+t_1)}{(s+t_1)^{p-1}}\right] ds \\
&> \int_{s_0}^t (s+t_1)^{p-1}\,\frac{b_0}{2}\,\frac{\widehat{f}(s+t_1)}{(s+t_1)^{p-1}}\,ds = \frac{b_0}{2}\int_{s_0}^t \widehat{f}(s+t_1)\,ds,
\end{aligned}$$

for all $t > s_0$. Thus,

$$\begin{aligned}
\int_{s_0}^{+\infty} \widetilde{H}(t)^{-1/p}\,dt &< \left(\frac{2}{b_0}\right)^{1/p} \int_{s_0}^{+\infty} \left(\int_{s_0+t_1}^{t+t_1} \widehat{f}(\tau)d\tau\right)^{-1/p} dt \\
&< \left(\frac{2}{b_0}\right)^{1/p} \int_{s_0}^{+\infty} \left(\int_{s_0+t_1}^{t} \widehat{f}(\tau)\,d\tau\right)^{-1/p} dt < +\infty.
\end{aligned}$$

Here, we have used (f_0) and the fact that $\widehat{f}$ verifies (KO). Therefore, by Lemma 3.1, the blow-up problem

$$\begin{cases} \Delta_p u = \widetilde{h}(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases}$$

admits a solution $\xi \in C^1(\Omega)$. Thus, $w = \xi + t_1$ is a solution of (P_6).

In the sequel, we fix an unbounded sequence $(L_n) \subset (0, +\infty)$ satisfying $L_n < L_{n+1}$ for all $n \in \mathbb{N}$ with $L_1 = L_0 + 1$, where L_0 was given in Proposition 2.1. So, it follows from Proposition 2.1 that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^1(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta_p u_n = \lambda a(x) g(u_n) - b(x) f(u_n) & \text{in } \Omega, \\ u_n \geq u_{n-1} \geq \gamma_0 & \text{in } \Omega, \\ u_n = L_n & \text{on } \partial\Omega. \end{cases}$$

Consequently,

$$\begin{cases} -\Delta_p u_n \leq \lambda \|a\|_\infty \widehat{g}(u_n) - b_0 \widehat{f}(u_n) & \text{in } \Omega, \\ -\Delta_p w = \lambda \|a\|_\infty \widehat{g}(w) - b_0 \widehat{f}(w) & \text{in } \Omega, \\ \limsup\limits_{d(x,\partial\Omega)\to 0} (u_n - w) = -\infty. \end{cases}$$

Then, by Lemma 3.2, $\gamma_0 < u_1 \leq u_2 \leq \ldots \leq u_n \leq u_{n+1} \leq \cdots \leq w$.

Now, by using standard arguments, we are able to show that there exists a $u \in C^1(\Omega)$ such that $u_n \to u$ in $C^1_{\text{loc}}(\Omega)$ and u is a solution of $(P)_\lambda$. This completes the proof of Theorem 1.2. □

4. Proof of the Theorem 1.3

The proof of Theorem 1.3 is a consequence of the three technical lemmas below. The proof of the first and last ones were inpired in ideas found in [8] for a particular case of $(\mathrm{P})_\lambda$, more specifically, for $\lambda = 0$ and $f(u) = u^q$, $u \geq 0$ with $q > p - 1$.

The first of them establishes the behavior of the solution near of the boundary. More exactly, the results that we will use have the following statement:

Lemma 4.1. *Assume $a, b \in L^\infty_{\text{loc}}(\Omega)$ and that* (a), (b), $(\mathrm{f}_1)'$ *and* $(\mathrm{g}_0)'$ *hold. If $u \in C^1(\Omega)$ is a solution of* $(\mathrm{P})_\lambda$*, then there exist positive constants c_1, c_2, δ, such that*

$$c_1 d(x)^{-\alpha(x)} \leq u(x) \leq c_2 d(x)^{-\alpha(x)}, \quad x \in U_\delta,$$

where $\alpha(x) = (p - \gamma(x))/(q - p + 1)$ for all $x \in U_\delta$ and U_δ was defined in hypothesis (b).

The second one proves an exact rate boundary behavior for an one-dimensional problem.

Lemma 4.2. *Let $-\infty < m \leq p - 1 < q$, $\gamma \leq 0$, and $\eta = [(p - 1 - m)(p - \gamma)]/(q - p + 1) + p > p$ be a real numbers. If $Q, R > 0$ are real constants and $u \in C^1(0, +\infty)$ is a solution of problem*

$$\begin{cases} -(|u'|^{p-2}u')' = Rx^{-\eta}u^m - Qx^{-\gamma}u^q & \text{in } (0,\infty), \\ u > 0 & \text{in } (0,\infty),\ u(x) \overset{x\to 0}{\longrightarrow} \infty, \end{cases} \tag{4.1}$$

then $u(x) = Ax^{-\alpha}$ *for* $x > 0$*, where* $\alpha = (p-\gamma)/(q-p+1)$ *and* $A > 0$ *is the unique solution of*

$$QA^{q-m} - \alpha^{p-1}(1+\alpha)(p-1)A^{p-m-1} - R = 0. \tag{4.2}$$

Finally, the last lemma studies the behavior of the solution for a class of problem in the half space $D = \{x \in \mathbb{R}^N,\ x_1 > 0\}$.

Lemma 4.3. *Let* $-\infty < m \le p-1 < q$*,* $\gamma \le 0$*, and* $\eta > p$ *be a real numbers as above. If* $Q, R > 0$ *are real constants and* $u \in C^1(D)$ *is a solution of the problem*

$$\begin{cases} -\Delta_p u = Rx_1^{-\eta}u^m - Qx_1^{-\gamma}u^q & \text{in } D, \\ u > 0 & \text{in } D \\ u = +\infty & \text{on } \partial D, \end{cases} \tag{4.3}$$

then $u(x) = Ax_1^{-\alpha}$ *for* $x \in D$*, where* α *and* A *were obtained in Lemma* 4.2.

Proof of Theorem 1.3 (conclusion). Next, we will divide our proof in two parts. The first one is related to behavior of the solution near of the boundary, while the second one is associated with the uniqueness.

Part 1. Behavior near to boundary. Consider $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$ and $\nu(x_0) = -e_1$, where $\nu(x_0)$ stands for the exterior normal derivative at x_0 and e_1 is the first vector of canonical basis of $\mathbb{R}^N$. Take $x_n \subset \Omega$ such that $x_n \to x_0 = 0$ and denote by $\xi_n = x_n - t_ne_1$, where $t_n > 0$ is such that $\xi_n \in \partial\Omega$. Now, fixing $z_n = \xi_n - t_n\nu(\xi_n)$, we have that $d(z_n) = t_n$, where $d_n := d(z_n) = \inf\{|z_n - \xi| \mid \xi \in \partial\Omega\} = |z_n - \xi_n|$, for $n \in \mathbb{N}$.

Now, fixing $\alpha_n = \alpha(z_n)$ and

$$v_n(y) = d_n^{\alpha_n}u(\xi_n + d_ny), \quad y \in \Omega_n = \{y \in \mathbb{R}^N,\ \xi_n + d_ny \in U_\delta\},$$

where U_δ is a neighbourhood of $\partial\Omega$ given in Lemma 4.1, it follows that

$$|\nabla v_n(y)|^{p-2}\nabla v_n(y) = d_n^{(\alpha_n+1)(p-1)}|\nabla u(\xi_n + d_ny)|^{p-2}\nabla u(\xi_n + d_ny),$$

for $\in \Omega_n$. By change variable $z = \xi_n + d_ny$, we have that $y \in \Omega_n$ if and only if $z \in U_\delta$, and so,

$$\begin{aligned} &\int_{\Omega_n} |\nabla v_n(y)|^{p-2}\nabla v_n(y)\nabla\phi(y)\,dy \\ &\quad= d_n^{\alpha_n(p-1)+p-N}\int_{U_\delta} |\nabla u(z)|^{p-2}\nabla u(z)\nabla\phi\bigg(\frac{z-\xi_n}{d_n}\bigg)\,dz \\ &\quad= d_n^{\alpha_n(p-1)+p-N}\int_{U_\delta} [\lambda a(z)g(u(z)) - b(z)f(u(z))]\,\phi\bigg(\frac{z-\xi_n}{d_n}\bigg)\,dz \end{aligned}$$

$$= d_n^{-N} \int_{U_\delta} \big[\lambda d_n^{\eta(z_n)} a(\xi_n + d_n y) d_n^{m\alpha_n} g(u(\xi_n + d_n y)) - d_n^{\gamma(z_n)} b(\xi_n + d_n y) d_n^{q\alpha_n} f(u(\xi_n + d_n y))\big]\, \phi\bigg(\frac{z-\xi_n}{d_n}\bigg)\, dz,$$

that is

$$\begin{aligned} (4.4)\quad & \int_{\Omega_n} |\nabla v_n(y)|^{p-2}\nabla v_n(y)\nabla\phi(y)\, dy \\ &= d_n^{-N} \int_{U_\delta} \big[\lambda d_n^{\eta(z_n)-\eta(\xi_n+d_n y)} d_n^{\eta(\xi_n+d_n y)} \\ &\quad \cdot a(\xi_n + d_n y) d_n^{m\alpha_n} g(u(\xi_n + d_n y)) \\ &\quad - d_n^{\gamma(z_n)-\gamma(\xi_n+d_n y)} d_n^{\gamma(\xi_n+d_n y)} \\ &\quad \cdot b(\xi_n + d_n y) d_n^{q\alpha_n} f(u(\xi_n + d_n y))\big]\phi\bigg(\frac{z-\xi_n}{d_n}\bigg)\, dz \\ &= \int_{\Omega_n} \big[\lambda d_n^{\eta(z_n)-\eta(\xi_n+d_n y)} d_n^{\eta(\xi_n+d_n y)} a(\xi_n + d_n y) d_n^{m\alpha_n} g(u(\xi_n + d_n y)) \\ &\quad - d_n^{\gamma(z_n)-\gamma(\xi_n+d_n y)}\, d_n^{\gamma(\xi_n+d_n y)} \\ &\quad \cdot b(\xi_n + d_n y) d_n^{q\alpha_n} f(u(\xi_n + d_n y))]\phi(y)\, dy\big], \end{aligned}$$

for each $\phi \in C_0^\infty(\Omega_n)$. Since $\Omega_n \to D := \{y \in \mathbb{R}^N,\ y_1 > 0\}$ when $n \to +\infty$, it follows that there exists an $n_0 \in \mathbb{N}$ such that $K \subset\subset \Omega_n$ and $\xi_n + d_n y \in U_\delta$ for all $y \in K$ and $n > n_0$, for each compact set $K \subset\subset D$ given. Thus, from the regularity of distance function, see for instance [9, Lemma 14.16],

$$\begin{aligned} (4.5)\quad \frac{d(\xi_n + d_n y)}{d(z_n)} = \frac{d(\xi_n + d_n y) - d(\xi_n)}{d(z_n)} &= \frac{\langle \nabla d(\varsigma_n), d_n y\rangle}{d_n} \\ &\to \langle \nabla d(0), y\rangle = \langle e_1, y\rangle = y_1, \end{aligned}$$

uniformly in $y \in K$, for some ς_n between $\xi_n + d_n y$ and ξ_n.

Thereby, the hypothesis (b) combined with the above convergences gives

$$\begin{aligned} (4.6)\quad & d_n^{\eta(\xi_n+d_n y)} a(\xi_n + d_n y) \\ &= \bigg(\frac{d(z_n)}{d(\xi_n + d_n y)}\bigg)^{\eta(\xi_n+d_n y)} d(\xi_n + d_n y)^{\eta(\xi_n+d_n y)} a(\xi_n + d_n y) \to y_1^{-\eta(0)} R(0), \end{aligned}$$

for $y \in K$. With the same type of arguments, by combining (a) with the convergence at (4.5), we see that, for $y \in K$,

$$\begin{aligned} (4.7)\quad & d_n^{\gamma(\xi_n+d_n y)} b(\xi_n + d_n y) \\ &= \bigg(\frac{d(z_n)}{d(\xi_n + d_n y)}\bigg)^{\gamma(\xi_n+d_n y)} d(\xi_n + d_n y)^{\gamma(\xi_n+d_n y)} b(\xi_n + d_n y) \to y_1^{-\gamma(0)} Q(0). \end{aligned}$$

To complete our analysis of the convergence, we note that applying Lemma 4.1, we get

$$\begin{aligned}v_n(y) &\leq c_2 d_n^{\alpha_n} d(\xi_n + d_n y)^{-\alpha(\xi_n+d_n y)}\\ &= c_2\bigg(\frac{d_n}{d(\xi_n+d_n y)}\bigg)^{\alpha(\xi_n+d_n y)} d_n^{\alpha_n-\alpha(\xi_n+d_n y)},\end{aligned}$$

for $y \in K$, and

$$\begin{aligned}v_n(y) &\geq c_1 d_n^{\alpha_n} d(\xi_n + d_n y)^{-\alpha(\xi_n+d_n y)}\\ &= c_1\bigg(\frac{d_n}{d(\xi_n+d_n y)}\bigg)^{\alpha(\xi_n+d_n y)} d_n^{\alpha_n-\alpha(\xi_n+d_n y)},\end{aligned}$$

for $y \in K$. Furthermore, from (b), we have

$$\big|\ln d_n^{\alpha_n-\alpha(\xi_n+d_n y)}\big| = \big|(\alpha(z_n) - \alpha(\xi_n+d_n y))\ln d_n\big| \leq \widehat{c} d_n^{\mu}|\ln d_n| \to 0$$

uniformly in $y \in K$, for some $\widehat{c} > 0$, implying that

$$d_n^{\alpha_n-\alpha(\xi_n+d_n y)} \to 1 \quad \text{uniformly in } y \in K. \tag{4.8}$$

Gathering (4.5), (4.8), the regularity of the distance function with the fact that (v_n) is uniformly bounded on compacts set in D, we derive that there is a function v such that $v_n(y) \to v(y)$ and $c_1 y_1^{-\alpha_0} \leq v(y) \leq c_2 y_1^{-\alpha_0}$, for each $y \in D$. After that, by $(g_0)'$, we get

$$\begin{aligned}d_n^{m\alpha_n} g(u(\xi_n+d_n y)) &= d_n^{m\alpha_n} u^m(\xi_n+d_n y) u^{-m}(\xi_n+d_n y) g(u(\xi_n+d_n y))\\ &= v_n^m(y) u^{-m}(\xi_n+d_n y) g(u(\xi_n+d_n y)) \to g_\infty v(y)^m,\end{aligned} \tag{4.9}$$

for $y \in D$, and by $(\mathrm{f}_1)'$, for $y \in D$,

$$\begin{aligned}d_n^{q\alpha_n} f(u(\xi_n+d_n y)) &= d_n^{q\alpha_n} u^q(\xi_n+d_n y) u^{-q}(\xi_n+d_n y) f(u(\xi_n+d_n y))\\ &= v_n^q(y) u^{-q}(\xi_n+d_n y) f(u(\xi_n+d_n y)) \to f_\infty v(y)^q.\end{aligned} \tag{4.10}$$

Finally, as $\eta, \gamma \in C^\mu(\overline{\Omega})$ for some $0 < \mu < 1$, the same arguments used in the proof of (4.8) can be used to deduce that

$$d_n^{\eta(z_n)-\eta(\xi_n+d_n y)},\ d_n^{\gamma(z_n)-\gamma(\xi_n+d_n y)} \to 1 \quad \text{with } n \to +\infty, \text{ for each } y \in D.$$

Now, given $\phi \in C_0^\infty(D)$ and recalling that $\Omega_n \to D$, we have $\overline{\operatorname{supp}\phi} \subset \Omega_n$ for n large enough. Thereby, passing the limits in (4.4), and using (4.6), (4.7), (4.9) and (4.10), we conclude that $v_n \to v$ in $C^1_{\mathrm{loc}}(D)$ and v is a solution of the problem

$$\begin{cases}-\Delta_p u = \lambda g_\infty R(0) y_1^{-\eta(0)} u^m - f_\infty Q(0) y_1^{-\gamma(0)} u^q & \text{in } D,\\ c_1 y_1^{-\alpha(0)} \leq u \leq c_2 y_1^{-\alpha(0)} & \text{in } D,\\ u = +\infty & \text{on } \partial D.\end{cases}$$

Hence, fixing $Q = f_\infty Q(0)$, $R = \lambda g_\infty R(0)$ and $\alpha = \alpha(0)$, it follows from Lemma 4.3, that

$$v(y) = Ay_1^{-\alpha(0)}, \quad y \in D,$$

where $A = A(0) > 0$ is the unique solution of

$$f_\infty Q(0)A^{q-m} - (p-1)\alpha(0)^{p-1}(1+\alpha(0))A^{p-m-1} - \lambda g_\infty R(0) = 0.$$

Now, by taking $y = e_1$ and using the definition of v_n, we obtain that

$$\lim_{n\to+\infty} d_n^{\alpha_n} u(x_n) = A. \tag{4.11}$$

To complete our proof, let us note that

$$d_n^{-\alpha_n} d_n^{\alpha(x_n)},\ d_n^{-\alpha(x_n)} d(x_n)^{\alpha(x_n)} \to 1 \tag{4.12}$$

hold true by following the same arguments like those used to prove (4.8), because $\alpha \in C^\mu(\overline{\Omega})$ for some $0 < \mu < 1$, and $d(x_n)/d_n \to 1$, is true as well, by repeating the same ideas used to prove (4.5). Therefore, from (4.11) and (4.12)

$$\lim_{n\to+\infty} d(x_n)^{\alpha(x_n)} u(x_n) = \lim_{n\to+\infty} \big[d_n^{-\alpha_n} d_n^{\alpha(x_n)}\big]\big[d_n^{-\alpha(x_n)} d(x_n)^{\alpha(x_n)}\big]\big[d_n^{\alpha_n} u(x_n)\big] = A.$$

Part 2. Uniqueness. Let u, v be two solutions of (P_λ). By the above information,

$$\lim_{x\to x_0} \frac{u(x)}{v(x)} = 1 \quad \text{for each } x_0 \in \partial\Omega,$$

that is, by combining this limit with the compactness of $\partial\Omega$, there exists a $\delta > 0$ such that

$$(1-\varepsilon)v(x) < u(x) < (1+\varepsilon)v(x), \quad x \in U_\delta. \tag{4.13}$$

for each $\varepsilon > 0$ given.

Besides this, using that $f(t)/t^{p-1}$ is nondecreasing and $g(t)/t^{p-1}$ is nonincreasing in the interval $(0,+\infty)$, we deduce that $(1-2\varepsilon)v$ and $(1+2\varepsilon)v$ are sub and super solutions of the problem

$$\begin{cases} -\Delta_p w = \lambda a(x)g(w) - b(x)f(w) & \text{in } U^\delta, \\ w = u & \text{on } \partial U^\delta, \end{cases} \tag{4.14}$$

where $U^\delta := \{x \in \Omega,\ d(x) > \delta\}$. Since u is a solution of (4.14) as well, the $\limsup_{x\to x_0} [u-(1+2\varepsilon)v] = -\varepsilon v(x_0) < 0$, and the $\limsup_{x\to x_0} [(1-2\varepsilon)v-u] = -\varepsilon v(x_0) < 0$ for each $x_0 \in \partial U^\delta$, it follows from Lemma 3.2,

$$(1-2\varepsilon)v(x) \le u(x) \le (1+2\varepsilon)v(x), \quad x \in U^\delta.$$

Now, combining the last inequality with (4.13), we are led to

$$(1-2\varepsilon)v(x) \le u(x) \le (1+2\varepsilon)v(x), \quad x \in \Omega.$$

So, taking $\varepsilon \to 0$, we obtain $u = v$ in Ω. $\square$

5. Proofs of lemmas

Proof of Lemma 4.1. For each $x \in U_\delta$, where $\delta > 0$ is given by hypotheses (a) and (b), define the function

$$v(y) = v_x(y) = d(x)^{\alpha(x)} u(x + d(x)y), \quad y \in B_{1/2}(0).$$

As $u \in C^1(\Omega)$ is a solution of $(\mathrm{P})_\lambda$, the change of variable $z = x + d(x)y$ leads to

$$\begin{aligned}\int_{B_{1/2}(0)} & |\nabla v|^{p-2} \nabla v \nabla \varphi(y)\, dy \\ &= d(x)^{\alpha(x)(p-1)+p-N} \int_{B_{d(x)/2}(x)} |\nabla u(z)|^{p-2} \nabla u(z) \nabla \varphi\bigg(\frac{1}{d(x)}(z-x)\bigg)\, dz \\ &= d(x)^{\alpha(x)(p-1)+p-N} \int_{B_{d(x)/2}(x)} [\lambda a(z) g(u(z)) - b(z) f(u(z))] \varphi\bigg(\frac{1}{d(x)}(z-x)\bigg) dz,\end{aligned}$$

for each $\varphi \in C_0^\infty(B_{1/2}(0))$.

Now, gathering the compactness of $\partial\Omega$, $(\mathrm{f}_1)'$ and $(\mathrm{g}_0)'$, we derive that

$$\begin{aligned}(5.1)\quad \int_{B_{1/2}(0)} & |\nabla v|^{p-2} \nabla v \nabla \varphi(y)\, dy \\ &\leq d(x)^{-N} \int_{B_{d(x)/2}(x)} d(x)^{\alpha(x)(p-1)+p} \big[\lambda a(x + d(x)y) D_2 u^m(x + d(x)y) \\ &\quad - b(x + d(x)y) D_1' u^q(x + d(x)y)\big] \varphi\bigg(\frac{1}{d(x)}(z-x)\bigg)\, dz \\ &= d(x)^{-N} \int_{B_{d(x)/2}(x)} \big[\lambda D_2 a(x + d(x)y) d(x)^{\alpha(x)(p-1)+p} d(x)^{-m\alpha(x)} v^m(y) \\ &\quad - D_1' b(x + d(x)y) d(x)^{\alpha(x)(p-1)+p} d(x)^{-q\alpha(x)} v^q(y)\big] \varphi\bigg(\frac{1}{d(x)}(z-x)\bigg)\, dz \\ &= d(x)^{-N} \int_{B_{d(x)/2}(x)} \big[\lambda D_2 a(x + d(x)y) d(x)^{\eta(x)} v^m(y) \\ &\quad - D_1' b(x + d(x)y) d(x)^{\gamma(x)} v^q(y)\big] \varphi\bigg(\frac{1}{d(x)}(z-x)\bigg)\, dz,\end{aligned}$$

for all $\varphi \in C_0^\infty(B_{1/2}(0))$ with $\varphi \geq 0$ and for all $x \in U_\delta$, where $\delta > 0$ is such that

$$g(u(x)) \leq D_2 u(x)^m \quad \text{and} \quad f(u(x)) \geq D_1' u(x)^q \quad \text{for all } x \in U_\delta,$$

for some real constants $D_2, D_1' > 0$. Here, we have used that $u(x) \to \infty$ as $d(x) \to 0$. Moreover, the inequality

$$d(x)/2 \leq d(x + d(x)y) \leq 3d(x)/2, \quad \text{for all } x \in U_\delta$$

together with (a) and (b) gives

$$\begin{aligned} b(x + d(x)y) &\geq \widetilde{C} d(x + d(x)y)^{-\gamma(x+d(x)y)} \geq C d(x)^{-\gamma(x+d(x)y)}, \\ a(x + d(x)y) &\leq \widetilde{D} d(x + d(x)y)^{-\eta(x+d(x)y)} \leq D d(x)^{-\eta(x+d(x)y)},\end{aligned}$$

for $x \in U_\delta$, for suitable $\delta > 0$ and some positive constants $\widetilde{C}$, $\widetilde{D}$, C and D. Therefore,

$$\begin{aligned}\lambda D_2 a(x+d(x)y)d(x)^{\eta(x)}v^m(y) - D_1' b(x+d(x)y)d(x)^{\gamma(x)}v^q(y)\\ \le \lambda D_3 d(x)^{\eta(x)-\eta(x+d(x)y)}v^m(y) - D_1 d(x)^{\gamma(x)-\gamma(x+d(x)y)}v^q(y),\end{aligned}$$

for $x \in U_\delta$, $y \in B_{1/2}(0)$ and $D_1, D_3 > 0$.

Now, substituting this inequality in (5.1) and returning to the variable y in $B_{1/2}(0)$, we obtain

$$\begin{aligned}&\int_{B_{1/2}(0)} |\nabla v|^{p-2}\nabla v \nabla\varphi(y)\,dy\\ &\le \int_{B_{1/2}(0)} \big[\lambda D_3 d(x)^{\eta(x)-\eta(x+d(x)y)}v^m(y) - D_1 d(x)^{\gamma(x)-\gamma(x+d(x)y)}v^q(y)\big]\varphi(y)\,dy,\end{aligned}$$

that is,

$$\int_{B_{1/2}(0)} |\nabla v|^{p-2}\nabla v \nabla\varphi(y)\,dy \le \int_{B_{1/2}(0)} \bigg[\lambda\,\frac{3D_3}{2}\,v^m(y) - \frac{D_1}{2}\,v^q(y)\bigg]\varphi(y)\,dy,$$

for $x \in U_\delta$, for some suitable $\delta > 0$, because

$$d(x)^{\eta(x)-\eta(x+d(x)y)},\ d(x)^{\gamma(x)-\gamma(x+d(x)y)} \to 1 \quad \text{as } d(x) \to 0.$$

On the other hand, from Theorem 1.2, there exists $U \in C^1(B_{1/2}(0))$ satisfying

$$\begin{cases} -\Delta_p U = \lambda\,\dfrac{3D_3}{2}\,U^m - \dfrac{D_1}{2}\,U^q & \text{in } B_{1/2}(0),\\ U > 0 & \text{in} B_{1/2}(0),\\ U = +\infty & \text{on } \partial B_{1/2}(0).\end{cases}$$

Then, by Lemma 3.2, $v(y) \le U(y)$ in $B_{1/2}(0)$, that is

$$d(x)^{\alpha(x)}u(x+d(x)y) \le U(y) \quad \text{for all } y \in B_{1/2}(0) \text{ and } x \in U_\delta,$$

showing that

$$u(x) \le U(0)d(x)^{-\alpha(x)} \quad \text{for } x \in U_\delta. \tag{5.2}$$

Now, let us prove the other inequality. Denote by $\overline{x} \in \partial\Omega$ the point that carries out the distance of x on $\partial\Omega$, and fix $z_x = \overline{x} + d(x)\nu(\overline{x})$, where $\nu(\overline{x})$ is the exterior unity normal vector to the $\partial\Omega$ at $\overline{x}$. Since $\partial\Omega$ is smooth, we have that $z_x \in \Omega^c$ for $x \in U_{\delta/2}$ for some $\delta > 0$. This way, we can define

$$w(y) := d(x)^{\alpha(x)}u(z_x + d(x)y), \qquad y \in Q_x = \{y \in A \mid z_x + d(x)y \in U_\delta\},$$

where $A = \{y \in \mathbb{R}^N \mid 1 < |y| < 3\}$.

From the hypotheses (a) we can fix $\delta > 0$ small enough such that

$$b(z_x + d(x)y) \le C_1 d(z_x + d(x)y)^{-\gamma(z_x+d(x)y)}, \tag{5.3}$$

$$1/2 \le d(x)^{\eta(x)-\eta(z_x+d(x)y)}, \qquad d(x)^{\gamma(x)-\gamma(z_x+d(x)y)} \le 3/2, \tag{5.4}$$

for all $x \in U_{\delta/2}$ and some $C_1 > 0$. In the sequel, by using $(f_1)'$ and the fact that $u(x) \to \infty$ as $|x| \to \infty$, we can also fix $C_2 > 0$ verifying

$$f(u(x)) \leq C_2 u(x)^q \quad \text{for all } x \in U_{\delta/2}. \tag{5.5}$$

Thus, given $\varphi \in C_0^\infty(Q_x)$ with $\varphi \geq 0$, (5.3) together with (5.5) and the positivity of a on U_δ yield

$$\begin{aligned}
&\int_{Q_x} |\nabla w|^{p-2} \nabla w \nabla \varphi \, dy \\
&= d(x)^{(\alpha(x)+1)(p-1)} \int_{Q_x} |\nabla u(z_x + d(x)y)|^{p-2} \nabla u(z_x + d(x)y) \nabla \varphi \, dy \\
&\geq - \int_{Q_x} C_1 C_2 d(x)^{\gamma(x) - \gamma(z_x + d(x)y)} w^q(y) \varphi \, dy.
\end{aligned} \tag{5.6}$$

From (5.4) and (5.6),

$$\int_{Q_x} |\nabla w|^{p-2} \nabla w \nabla \varphi(y) \, dy \geq - \int_{Q_x} C_3 w^q(y) \varphi(y) \, dy \tag{5.7}$$

for $x \in U_{\delta/2}$ and some $C_3 > 0$.

On the other hand, set $\widehat{Z} \in C^1(1,3)$ denotes the positive solution of

$$\begin{cases} -(r^{N-1}|Z'|^{p-2}Z')' = -C_3 r^{N-1} Z^q & \text{in } (1,3), \\ Z > 0 & \text{in } (1,3), \\ Z(1) = K, \quad Z(3) = 0, \end{cases}$$

then $Z(y) = \widehat{Z}(|y|) \in C^1(A)$ is a radially-symmetric solution of the problem

$$\begin{cases} -\Delta_p Z = -C_3 Z^q & \text{in } A, \\ Z > 0 & \text{in } (1,3), \\ Z(1) = K, \quad Z(3) = 0. \end{cases} \tag{5.8}$$

Since $Q_x \subset A$, it follows that $Z(y) < w(y)$, $y \in \partial Q_x$. So, the inequality (5.7) combined with (5.8) and Lemma 3.2 give

$$d(x)^{\alpha(x)} u(z_x + d(x)y) = w(y) \geq Z(y) \quad \text{in } Q_x, \text{ for all } x \in U_{\delta/2},$$

that is, taking $y = -2\nu(\overline{x})$ and remembering that $x = z_x - 2d(x)\nu(\overline{x})$, we obtain

$$u(x) \geq Z(-2\nu(\overline{x})) d(x)^{-\alpha(x)} = \widehat{Z}(2) d(x)^{-\alpha(x)}, \quad x \in U_{\delta/2}. \tag{5.9}$$

Now, the lemma follows gathering (5.2) and (5.9) by considering the smallest $\delta > 0$ that we have considered in this proof. □

The proof of Lemma 4.2 is based upon ideas found in [10]. Here, we are able to prove that the solutions of the problem (4.1) are of the form $u(x) = Ax^{-\alpha}$, with A verifying (4.2), by using a result of [10] instead of the Poincaré–Bendixon's Theorem as used in [8].

Given positive numbers T_1, T_2 and h, we let $X := \{w \in C^1([T_1,T_2]) \mid w \geq h\}$ and the continuous function $H\colon [T_1,T_2] \to \mathbb{R}$ defined by

$$H(s) := s^{N-1}\big[\big|\big(w_2^{1/p}\big)'\big|^{p-2}\big(w_2^{1/p}\big)'w_2^{(1-p)/p} - \big|\big(w_1^{1/p}\big)'\big|^{p-2}\big(w_1^{1/p}\big)'w_1^{(1-p)/p}\big](w_1-w_2)(s)$$

for w_1, $w_2 \in X$ given. In [10], it was proved the following result

Lemma 5.1. *Assume that $w_1, w_2 \in X$, then*

$$H(U)-H(S) \leq \int_S^U \bigg[\frac{\big(r^{N-1}\big|(w_2^{1/p})'\big|^{p-2}\big(w_2^{1/p}\big)'\big)'}{w_2^{(p-1)/p}} - \frac{\big(r^{N-1}\big|\big(w_1^{1/p}\big)'\big|^{p-2}\big(w_1^{1/p}\big)'\big)'}{w_1^{(p-1)/p}}\bigg](w_1-w_2)\,dr$$

for all U, S such that $T_1 \leq S \leq U \leq T_2$ hold.

Proof of Lemma 4.2. It is easy to check that $u_0(x) := Ax^{-\alpha}$, $x>0$ is a solution of (4.1), where $A>0$ is the unique solution of (4.2). In the sequel, we will show that u_0 is a maximal solution for (4.1). To see why, our first step is showing that if $u \in C^1(0,\infty)$ is a solution of (4.1), then

$$u(x) \leq cx^{-\alpha}, \quad \text{for all } x>0, \tag{5.10}$$

for some positive constant c. Fixed $x>0$, define $v(y) = v_x(y) = x^\alpha u(x+xy)$ for $|y|<1/2$, and note that v satisfies

$$\begin{cases} -(|v'|^{p-2}v')' = R(1+y)^{-\eta}v^m - Q(1+y)^{-\gamma}v^q & \text{if } |y|<1/2,\\ v>0 & \text{if } |y|<1/2,\\ v(1/2) = x^\alpha u(3x/2) \text{ and } v(-1/2) = x^\alpha u(x/2). \end{cases} \tag{5.11}$$

On the other hand, it follows from Theorem 1.2 that there exists U in $C^1(-1/2,1/2)$ satisfying

$$\begin{cases} -(|U'|^{p-2}U')' = R(1+y)^{-\eta}U^m - Q(1+y)^{-\gamma}U^q, & \text{if } ||y|<1/2,\\ U>0 & \text{if } |y|<1/2,\\ U(1/2) = U(-1/2) = +\infty. \end{cases} \tag{5.12}$$

By combining (5.11) with (5.12) and Lemma 3.2, we deduce that $v(y) \leq U(y)$ for $|y|<1/2$, and a consequence of this, by taking $y=0$, we obtain that $u(x) \leq U(0)x^{-\alpha}$ for $x>0$ ($c = U(0)>0$), proving (5.10).

After the previous study, we are able to prove that

$$u(x) \leq u_0(x) \quad \text{for all } x>0. \tag{5.13}$$

To this end, we assume that there exists $\tau_0>0$ such that $u \leq \zeta_{\tau_0}$ does not hold in (τ_0,∞), where $\zeta_\tau(x) := u_0(x-\tau)$ for $x>\tau$ for each $\tau>0$ given.

Thereby, there exist $t_0 \in [\tau_0, \infty)$ and $s_0 \in [t_0, \infty]$ such that $u(t_0) = \zeta_{\tau_0}(t_0)$, $u(s_0) = \zeta_{\tau_0}(s_0)$, if $s_0 < \infty$ and $u(x) > \zeta_{\tau_0}(x)$ in (t_0, s_0).

A straightforward computation gives that ζ_{τ_0} satisfies

$$-\big(|\zeta'_{\tau_0}(x)|^{p-2}\zeta'_{\tau_0}(x)\big)' \geq Rx^{-\eta}\zeta^m_{\tau_0} - Qx^{-\gamma}\zeta^q_{\tau_0} \quad \text{in } (t_0, s_0). \tag{5.14}$$

Putting $N = 1$, $w_1^{1/p} = u$ and $w_2^{1/p} = \zeta$ into Lemma 5.1, (5.14) together with the fact that u is a solution of (4.1) yields

$$H(s_2) - H(s_1) \leq \int_{s_1}^{s_2} \bigg[\frac{(|\zeta'_{\tau_0}|^{p-2}\zeta'_{\tau_0})'}{\zeta^{p-1}_{\tau_0}} - \frac{(|u'|^{p-2}u')'}{u^{p-1}}\bigg](u^p - \zeta^p_{\tau_0})\,dx$$

$$\leq \int_{s_1}^{s_2} \bigg[\frac{Qx^{-\gamma}\zeta^q_{\tau_0} - Rx^{-\eta}\zeta^m_{\tau_0}}{\zeta^{p-1}_{\tau_0}} - \frac{Qx^{-\gamma}u^q - Rx^{-\eta}u^m}{u^{p-1}}\bigg](u^p - \zeta^p_{\tau_0})\,dx$$

$$= \int_{s_1}^{s_2} \big[Qx^{-\gamma}(\zeta^{q-p+1}_{\tau_0} - u^{q-p+1}) + Rx^{-\eta}(u^{m-p+1} - \zeta^{m-p+1}_{\tau_0})\big](u^p - \zeta^p_{\tau_0})\,dx < 0,$$

for all $t_0 \leq s_1 < s_2 < s_0$, where

$$H(x) = \big[|\zeta'_{\tau_0}|^{p-2}\zeta'_{\tau_0}\zeta^{(1-p)}_{\tau_0} - |u'|^{p-2}u'u^{(1-p)}\big](u^p(x) - \zeta^p_{\tau_0}(x)), \tag{5.15}$$

for $x \in (t_0, s_0)$. The above inequality implies that H is decreasing in (t_0, s_0). Thus, if $s_0 < +\infty$, then $H(t_0) = H(s_0) = 0$, that is impossible. If $s_0 = +\infty$, then $\lim_{x\to+\infty} H(x) = H_\infty \in [-\infty, 0)$, because $H(t_0) = 0$ and H is decreasing.

Moreover, by combining the definition of ζ_{τ_0} and (5.10), we obtain

$$\lim_{x\to+\infty} |\zeta'_{\tau_0}|^{p-2}\zeta'_{\tau_0}\zeta^{(1-p)}_{\tau_0}(x) = \lim_{x\to+\infty} (u^p - \zeta^p_{\tau_0})(x) = 0.$$

Then, by (5.15) and $H_\infty \in [-\infty, 0)$,

$$\lim_{x\to+\infty} |u'|^{p-2}u'u^{(1-p)}(x) = +\infty,$$

showing that $u' > 0$ for x large enough, which is impossible, because $u > 0$ in $[0, \infty)$ and $u(x) \xrightarrow{x\to\infty} 0$. Hence,

$$u(x) \leq \zeta_\tau(x) \quad \text{for all } x \in (\tau, +\infty), \text{ for all } \tau > 0,$$

implying that

$$u(x) \leq \lim_{\tau\to 0} \zeta_\tau(x) = u_0(x) \quad \text{for all } x \in (0, +\infty),$$

showing (5.13), and thus, u_0 is a maximal solution for (4.1).

To complete the proof of Lemma 4.2, we will show that u_0 is also a minimal solution for (4.1). In the sequel, we define $\xi_\varepsilon(x) = u_0(x + \varepsilon)$ in $(0, +\infty)$ for each $\varepsilon > 0$ and we use a similar argument to conclude that for each $\varepsilon > 0$ the inequality below holds

$$u(x) \geq \xi_\varepsilon(x) \quad \text{for all } x \in (0, +\infty).$$

The above estimate leads us to

$$u(x) \geq \lim_{\varepsilon\to 0} \xi_\varepsilon(x) = u_0(x) \quad \text{for all } x \in (0,+\infty),$$

from where it follows that u_0 is a minimal solution. Once u_0 is at the same time a maximal and minimal solution, we can conclude that $u(x) = Ax^{-\alpha}$, for $x > 0$ is the unique solution of (4.1), finishing the proof of the lemma. □

Proof of Lemma 4.3. In this proof, our first step is to show that $u_0(x) = u_0(x_1,\ldots,x_n) = Ax_1^{-\alpha}$ is a solution of (4.3), where $A > 0$ is the unique solution of (4.2). Below, we prove that (4.3) admits a minimal and a maximal solutions depending on just x_1. In fact, we will begin showing the existence of the maximal solution, which we will be denoted by $u_{\max}$.

To do this, let $\{D_k\}$ be a sequence of smooth bounded domains $D_k \subset\subset D_{k+1}$ such that $D = \bigcup_{k=1}^{\infty} D_k$. Related to $\{D_k\}$, we consider the problem

$$\begin{cases} -\Delta_p u = Rx_1^{-\eta}u^m - Qx_1^{-\gamma}u^q & \text{in } D_k,\\ u > 0 & \text{in } D_k,\\ u = +\infty & \text{on } \partial D_k. \end{cases} \tag{5.16}$$

By Theorem 1.2, there exists a solution $u_k \in C^1(D_k)$ of (5.16) satisfying

$$u_0(x) \leq u_{k+1}(x) \leq u_k(x), \quad x \in D_k,$$

because we are able to apply Lemma 3.2. Thus, there is $w \in C^1(D)$ such that $u_k \to w$ in $C^1_{\rm loc}(D)$, w is a solution of (4.3) and $w(x) \geq u_0(x)$ for all $x \in D$.

Let $v \in C^1(D)$ be another solution of (4.3). By Lemma 3.2, $v \leq u_k$ in D_k for all k. Then, $v \leq w$ in D, showing that w is a maximal solution for (4.3). In the sequel, we denote by $u_{\max}$ the function w and set

$$\widetilde{w}(x) = u_{\max}(x_1, x' + t), \quad \text{for } x_1 > 0 \text{ and } x' \in \mathbb{R}^{N-1},$$

for each $t \in \mathbb{R}^{N-1}$ given. Since, $\widetilde{w}$ is a solution of (4.3) as well, it follows that $\widetilde{w} \leq u_{max}$ in D, or equivalently,

$$u_{max}(x_1, x' + t) \leq u_{max}(x_1, x') \text{ for each } x_1 > 0 \text{ and } t, x' \in \mathbb{R}^{N-1}.$$

Once $t \in \mathbb{R}^{N-1}$ is arbitrary, the above inequality implies that

$$u_{\max}(x_1, x') = u_{\max}(x_1, y') \quad \text{for all } x', y' \in \mathbb{R}^{N-1},$$

showing that $u_{\max}$ depends just on x_1. Thereby, $u_{\max}$ is a solution of problem (4.1), and by Lemma 4.2,

$$u_{\max}(x_1,\ldots,x_n) = Ax_1^{-\alpha}, \quad x_1 > 0 \text{ and } (x_2,\ldots,x_n) \in \mathbb{R}^{N-1}.$$

To finish the proof, our next step is showing the existence of a minimal solution for (4.3), denoted by u_{min}, which will also depend on just x_1. To do this, setting $D'_k = B_{2k}(0) \cap D$, we have that

$$D'_k \subset D'_{k+1}, \quad B_k(0) \cap \partial D \subset \partial D'_k \quad \text{and} \quad D = \bigcup_{k=1}^{\infty} D'_k.$$

From now on, for each $k \in \mathbb{N}$, we fix $\psi_k \in C^\infty(D'_k)$ satisfying $0 \leq \psi_k \leq 1-1/(2k)$ on $\partial D'_k$, $\psi_k = 1 - 1/(2k)$ on $\partial D \cap \overline{B}_k(0)$, $\psi_k = 0$ in $\partial D'_k \setminus (B_{2k}(0) \cap \partial D)$, $0 < \psi_k < 1-1(2k)$ on $\partial D \cap (B_{2k}(0) \setminus \overline{B}_k(0))$ and $\psi_{k+1} > \psi_k$ on $\partial D'_k \cap \partial D'_{k+1} \cap \partial D$.

By a result found in [8], there exists a unique solution $\underline{u}_{k,n} \in C^1(\overline{D'_k})$ of the problem

$$\begin{cases} -\Delta_p u = -Q x_1^{-\gamma} u^q & \text{in } D'_k, \\ u > 0 & \text{in } D'_k, \\ u = n\psi_k & \text{on } \partial D'_k, \end{cases}$$

and $\underline{u}_{k,n}$ is increasing with k and n. That is, $\underline{u}_{k,n}$ is a sub solution of the problem

$$\begin{cases} -\Delta_p u = R x_1^{-\eta} u^m - Q x_1^{-\gamma} u^q & \text{in } D^r_k, \\ u > 0 & \text{in } D^r_k, \\ u = \underline{u}_{k,n} & \text{on } \partial D^r_k, \end{cases} \tag{5.17}$$

where $D^r_k = B_{4k}(0) \cap \{x \in D,\ x_1 > r\} \subset D'_k$ for each $r \in \big(0, (A/n)^{1/\alpha}\big)$.

Since u_0 is a super solution of (5.17) with $\underline{u}_{k,n} < u_0$ on ∂D^r_k, it follows from Lemma 3.2 that $\underline{u}_{k,n} \leq u_0$ in D^r_k. So, by a result in [12], there exists a $v^r_{k,n} \in C^1(\overline{D^r_k})$ solution of the problem (5.17) satisfying $\underline{u}_{k,n} \leq v^r_{k,n} \leq u_0$ in D^r_k. Then, after a diagonal process, there is $v_{k,n} \in C^1(\overline{D'_k})$ such that $v^r_{k,n} \to v_{k,n}$ in $C^1(\overline{D'_k})$ as $r \to 0$. Moreover, $\underline{u}_{k,n} \leq v_{k,n} \leq u_0$ in D'_k and $v_{k,n}$ is a solution of the problem

$$\begin{cases} -\Delta_p u = R x_1^{-\eta} u^m - Q x_1^{-\gamma} u^q & \text{in } D'_k, \\ u > 0 & \text{in } D'_k, \\ u = n\psi_k & \text{on } \partial D'_k. \end{cases}$$

Applying the Lemma 3.2, we deduce that $v_{k,n}$ satisfies $v_{k,n} \leq v_{k+1,n}$, and $v_{k,n} \leq u_0$ in D'_k. Thus, $v_{k,n} \to v_n$ and $\underline{u}_{k,n} \to \underline{u}_n$ in $C^1_{\text{loc}}(D)$ with $\underline{u}_n \leq v_n \leq u_0$ in D, where v_n satisfies

$$\begin{cases} -\Delta_p u = R x_1^{-\eta} u^m - Q x_1^{-\gamma} u^q & \text{in } D, \\ u > 0 & \text{in } D, \\ u = n & \text{on } \partial D, \end{cases}$$

and $\underline{u}_n \in C^1_{\mathrm{loc}}(D)$ satisfies

$$\begin{cases} -\Delta_p u = -Q x_1^{-\gamma} u^q & \text{in } D, \\ u > 0 & \text{in } D, \\ u = n & \text{on } \partial D. \end{cases}$$

That is, after a diagonal process, we have $v_n \to v := u_{\min}$ in $C^1_{\mathrm{loc}}(D)$. Besides this, following the arguments concerning to $u_{\max}$, we show that $u_{\min}$ is a minimal solution for (4.3), which depends on just x_1. So, $u_{\min}$ is a solution of problem (4.1) and from Lemma 4.2, we have that

$$u_{\min}(x_1, \ldots, x_n) = A x_1^{-\alpha}, \quad x_1 > 0 \text{ and } (x_2, \ldots, x_n) \in \mathbb{R}^{N-1},$$

with $A > 0$ being the unique solution of (4.2). Hence, given a $u \in C^1(D)$ solution of (4.3), we must to have

$$u(x_1, \ldots, x_n) = A x_1^{-\alpha}, \quad x_1 > 0 \text{ and } (x_2, \ldots, x_n) \in \mathbb{R}^{N-1}. \qquad \square$$

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