Topological Methods in Nonlinear Analysis
Volume 53, No. 2, 2019, 731–746
DOI: 10.12775/TMNA.2019.021

EXISTENCE OF POSITIVE SOLUTIONS FOR HARDY NONLOCAL FRACTIONAL ELLIPTIC EQUATIONS INVOLVING CRITICAL NONLINEARITIES

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Abstract. In this paper, we have used variational methods to study existence of solutions for the following critical nonlocal fractional Hardy elliptic equation

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(b)-2}u}{|x|^b} + \lambda f(x,u), \quad \text{in } \mathbb{R}^N,$$

where $N > 2s$, $0 < s < 1$, γ, λ are real parameters, $(-\Delta)^s$ is the fractional Laplace operator, $2_s^*(b) = 2(N-b)/(N-2s)$ is a critical Hardy–Sobolev exponent with $b \in [0, 2s)$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

1. Introduction

This paper is devoted to the study of the following critical nonlocal fractional Hardy elliptic equation

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(b)-2}u}{|x|^b} + \lambda f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N > 2s$, $0 < s < 1$, γ, λ are real parameters, $2_s^*(b) = 2(N-b)/(N-2s)$ is a critical Hardy–Sobolev exponent with $b \in [0, 2s)$, $2_s^* = 2_s^*(0) = 2N/(N-2s)$

2010 *Mathematics Subject Classification.* Primary: 35R11, 35A15; Secondary: 35J60, 47G20.

Key words and phrases. Nonlocal Laplacian operators; Hardy coefficients; Critical exponents; Mountain pass theorem; Variational methods.

and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $(-\Delta)^s$ is the fractional Laplace operator, which, up to normalization factors, may be defined as

$$(-\Delta)^s u(x) = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(y)}{|x-y|^{N+2s}} \, dy,$$

for $x \in \mathbb{R}^N$. Moreover, let $\mathfrak{L}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$, then, for any $u \in \mathfrak{L}$ and $s \in (0,1)$, $(-\Delta)^s$ is defined as

$$\begin{aligned}(-\Delta)^s u(x) &= C(N,s)\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy \\ &= C(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathfrak{L}B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy\end{aligned}$$

where $\mathfrak{L}B_\varepsilon(x) = \mathbb{R}^N \setminus B_\varepsilon(x)$ and the symbol P.V. stands for the Cauchy principal value and $C(N,s)$ is a dimensional constant that depends on N, s, precisely given by

$$C(N,s) := \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\varsigma_1)}{|\varsigma|^{N+2s}} d\varsigma \right)^{-1}.$$

Fractional and nonlocal operators in addition to their applications are very interesting, we refer the readers to [3], [8], [9], [11], [12], [16], [18]–[21], [23], [25]–[29], [31] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the readers to [6], [20]. In [23], Molica Bisci and Vilasi studied a class of Kirchhoff nonlocal fractional equation in a bounded domain Ω and obtained three solutions by using three critical point theorem. Pucci and Saldi [25] established the existence and multiplicity of nontrivial solutions for a Kirchhoff type eigenvalue problem in $\mathbb{R}^N$ involving a critical nonlinearity and the nonlocal fractional Laplacian. We refer also to [10], [14], [20], [22] for related problems. Note that in [3], [12], in order to consider the existence of solutions to Kirchhoff fractional p-Laplacian equations involving critical Hardy–Sobolev nonlinearities. Fiscella and Pucci in [11] studied the existence, multiplicity and the asymptotic behavior of nontrivial solutions for nonlinear problems driven by the fractional Laplace operator $(-\triangle)^s$ and involving a critical Hardy potential. In [3], Filippucci, Pucci and Robert proved that there exists a positive solution for a p-Laplacian problem with critical Sobolev and Hardy–Sobolev terms. In [13], Fiscella, Pucci and Saldi dealt with the existence of nontrivial nonnegative solutions of Schrödinger–Hardy systems driven by two possibly different fractional p-Laplacian operators, also including critical nonlinear terms, where the nonlinearities do not necessarily satisfy the Ambrosetti–Rabinowitz condition. In [30], the authors study a fractional Laplacian system in $\mathbb{R}^N$, which involves critical Sobolev-type nonlinearities and critical Hardy–Sobolev-type nonlinearities by using variational methods. They have investigated the extremals of the corresponding best fractional Hardy–Sobolev constant and establish the existence of solutions. In [7] the authors prove the existence and qualitative properties of a solution for a fractional problem with a critical nonlinearity and still a Hardy potential by combining a variational approach and the moving plane method. Also, we refer to [3], [12], [24] for existence results concerning different Kirchhoff–Hardy problems and Hardy–Schrödinger–Kirchhoff equations driven by the fractional Laplacian.

Inspired by the above works, in the present paper we shall study the existence of nontrivial solutions for problem (1.1).

Now, we state our main results.

THEOREM 1.1. *Assume that $\gamma < \gamma_1$ (γ_1 is defined in (2.3)) and the following conditions hold*:

(F1) *there exists a positive function η of class $L^\infty(\mathbb{R}^N)$ such that $\eta(x) = o(1)$ as $|x| \to \infty$ and $|f(x,t)| \le \eta(x)|t|^{2_s^*(b)-1}/|x|^b$ for almost every $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$.*

(F2)
$$\lim_{t\to 0^+} \frac{f(x,t)}{t} = +\infty \quad \text{uniformly in } x \in \mathbb{R}^N.$$

Then there exists $\overline{\lambda} > 0$ such that the problem (1.1) has at least one positive weak solution u_λ for any $\lambda \in (0, \overline{\lambda})$.

Now, we assume that $f(x,t) := K(x)f(t)$, where

(F3) $K \in L^\infty(\mathbb{R}^N) \cap L^{2_s^*/(2_s^*-2)}(\mathbb{R}^N)$ such that $K(x) = o(1)$ as $|x| \to \infty$, $K(x) > 0$ for every $x \in \mathbb{R}^N$.

(F4) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \le R$ for all $n \in \mathbb{N}$ and some $R > 0$, then
$$\lim_{r\to+\infty} \int_{A_n \cap B_r^c(0)} K(x)\,dx = 0, \quad \text{uniformly in } n \in \mathbb{N}.$$

(F5) $\lim_{t\to 0} f(t)/t = 0$ and $\lim_{|t|\to+\infty} f(t)/|t|^{2_s^*(b)-1)} = 0$.

(F6) $tf(t) - 2F(t) \ge stf(st) - 2F(st)$ for all $t \in \mathbb{R}$ and $s \in [0,1]$, where $F(t) = \int_0^t f(s)\,ds$.

(F7) $tf(t) > 0$ for all $t \neq 0$.

THEOREM 1.2. *Assume that $\gamma < \gamma_1$ and (F3)–(F7) hold. Then the problem (1.1) has a positive weak solution for large λ.*

2. Preliminary and proof of main results

In this section, we need to present some preliminaries of variational frame work, definitions and lemmas which will play an important role to solve the problem (1.1).

We first give some useful notations and basic results of fractional Sobolev space that will be used in proof of the main results. Let $0 < s < 1 < p < \infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N) = \left\{u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy\right\},$$

equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy\right)^{1/p}.$$

We know that, if $p = 2$, then $W^{s,2}(\mathbb{R}^N) := H^s(\mathbb{R}^N)$. Also, $H^s(\mathbb{R}^N)$ denotes the fractional Sobolev space of the functions $g \in L^2(\mathbb{R}^N)$ such that the map $(x, y) \mapsto (g(x) - g(y))/|x - y|^{(N+2s)/2}$ is in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$. So, let consider $H^s(\mathbb{R}^N)$ with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\|u\|^2_{L^2(\mathbb{R}^N)} + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy\right)^{1/2}.$$

We introduce the space $H^s_0(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$[u]_s = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy\right)^{1/2}.$$

We know that $(H^s_0(\mathbb{R}^N), [u]_s)$ is a uniformly convex Banach space. By Theorem 1 of [17]

$$\|u\|^2_{2^*_s} \leq C_N \frac{s(1-s)}{N-2s} [u]^2_s, \quad \text{for all } u \in H^s_0(\mathbb{R}^N), \tag{2.1}$$

and, by Theorem 1.1 of [15],

$$\gamma_1 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\, dx \leq [u]^2_s, \quad \text{for all } u \in H^s_0(\mathbb{R}^N), \tag{2.2}$$

where

$$\gamma_1 := 2\pi^{N/2} \frac{\Gamma\left(\dfrac{N+2s}{4}\right)^2 |\Gamma(-s)|}{\Gamma\left(\dfrac{N-2s}{4}\right)^2 \Gamma\left(\dfrac{N+2s}{2}\right)}. \tag{2.3}$$

Moreover, the constant γ_1 is optimal. If $\gamma < \gamma_1$, it follows from the Hardy inequality (2.2) that

$$\|u\| := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy - \gamma \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\, dx\right)^{1/2},$$

is well defined on $H^s_0(\mathbb{R}^N)$. Since

$$\left(1 - \frac{\lambda_+}{\gamma_1}\right)[u]^2_s \leq \|u\|^2 \leq \left(1 + \frac{\lambda_-}{\gamma_1}\right)[u]^2_s, \quad \text{for all } u \in H^s_0(\mathbb{R}^N) \tag{2.4}$$

where $\lambda_+ = \max\{\gamma, 0\}$ and $\lambda_- = \max\{-\gamma, 0\}$, then $\|u\|$ is comparable to $[u]_s$. Thus, (2.1) and (2.2) imply that the fractional Sobolev embedding $H^s_0(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ and the fractional Hardy embedding $H^s_0(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, |x|^{-2s})$

are continuous, but not compact. However, we are able to introduce the best fractional critical Sobolev constant $S(N,s,\gamma,0)$ given by

$$(2.5)\quad S(N,s,\gamma,0) = \inf_{u\in H_0^s(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}\,dx}{\displaystyle\Big(\int_{\mathbb{R}^N}|u(x)|^{2_s^*}\,dx\Big)^{2/2_s^*}}.$$

Combining the Hardy inequality and the Sobolev inequality, we obtain the Hardy–Sobolev inequality. Indeed, let $b\in[0,2s)$ be a real number: then $H_0^s(\mathbb{R}^N)$ is continuously embedded in the weighted space $L^{2_s^*(b)}(\mathbb{R}^N,|x|^{-b})$. Here again, taking the smallest constant associated to this embedding, let

$$(2.6)\quad S(N,s,\gamma,b) = \inf_{u\in H_0^s(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}\,dx}{\displaystyle\Big(\int_{\mathbb{R}^N}|u(x)|^{2_s^*}|x|^{-b}\,dx\Big)^{2/(2_s^*(b))}}.$$

The weak solution of problem (1.1) is as follow:

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,(\varphi(x)-\varphi(y))\,dx\,dy - \gamma\int_{\mathbb{R}^N}\frac{u(x)}{|x|^{2s}}\,\varphi(x)\,dx \\ = \int_{\mathbb{R}^N}\frac{|u(x)|^{2_s^*(b)-2}}{|x|^b}\,u(x)\varphi(x)\,dx + \lambda\int \_\mathbb{R}^N f(x,u(x))\varphi(x)\,dx,\end{aligned}$$

for all $u,\varphi\in H_0^s(\mathbb{R}^N)$, and the energy functional $I\colon H_0^s(\mathbb{R}^N)\to\mathbb{R}$, which is defined by the formula

$$(2.7)\quad \begin{aligned}I(u) = \frac12\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}\,dx\Big) \\ - \frac{1}{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(b)}}{|x|^b}\,dx - \lambda\int_{\mathbb{R}^N}F(x,u(x))\,dx,\end{aligned}$$

where $F(x,u)=\int_0^u f(x,s)\,ds$.

Under our assumptions the functional I is of class $C^1(H_0^s(\mathbb{R}^N),\mathbb{R})$ and

$$(2.8)\quad \begin{aligned}\langle I'(u),\varphi\rangle = \iint\limits_{\mathbb{R}^{2N}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}(\varphi(x)-\varphi(y))\,dx\,dy - \gamma\int_{\mathbb{R}^N}\frac{u(x)}{|x|^{2s}}\,\varphi(x)\,dx \\ - \int_{\mathbb{R}^N}\frac{|u(x)|^{2_s^*(b)-2}u(x)}{|x|^b}\varphi(x)\,dx - \lambda\int_{\mathbb{R}^N}f(x,u(x))\varphi(x)\,dx,\end{aligned}$$

for all $u,\varphi\in H_0^s(\mathbb{R}^N)$.

Definition 2.1. Let X be a real Banach space with its dual space X^* and $I \in C^1(X,\mathbb{R})$. For $c \in \mathbb{R}$ we say that I satisfies the $(\mathrm{C})_c$ condition if for any sequence $\{x_n\} \subset X$ with

$$I(x_n) \to c \quad \text{and} \quad (1+\|x_n\|_X)\|I'(x_n)\|_{X^*} \to 0,$$

then there exists a subsequence $\{x_{n_k}\}$ that converges strongly in X.

Proof of Theorem 1.1. Since we consider the existence of positive solutions of the problem (1.1), set

$$\overline{f}(x,t) = \begin{cases} f(x,t) & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases} \quad \text{and} \quad \overline{F}(x,t) = \int_0^t \overline{f}(x,s)\,ds.$$

In view of (F1), (2.5) and (2.6), we can get

$$\begin{aligned} \overline{I}(u) &= \frac{1}{2}\bigg(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy - \gamma\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}\,dx\bigg) \\ &\quad - \frac{1}{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|u^+|^{2_s^*(b)}}{|x|^b}\,dx - \lambda\int_{\mathbb{R}^N}\overline{F}(x,u^+(x))\,dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2_s^*(b)}\,(S(N,s,\gamma,b))^{2_s^*(b)/2}\|u\|^{2_s^*(b)} \\ &\quad - \frac{\lambda}{2}\,\|\eta\|_\infty(S(N,s,\gamma,b))^{2_s^*(b)/2}\|u\|^{2_s^*(b)}, \end{aligned}$$

so, for any $\lambda \in [0,1)$, there exist $\rho > 0$, $\theta \in [0,1)$ such that

$$\overline{I}(u) > c_1, \quad \|u\| = \rho \quad \text{and} \quad \overline{I}(u) > -c_1, \quad \|u\| \le \rho,$$

where $c_1 = (\theta+1)(S(N,s,\gamma,b))^{-2_s^*(b)/2}\rho^{2_s^*(b)}$. Choosing $u_0 \in H_0^s(\mathbb{R}^N)$ such that $u_0^+ \ne 0$ and $\varrho := \|u_0\|^2/\lambda\|u_0^+\|^2_{L^2(\mathbb{R}^N)}$. In view of (F2), there exists $\delta > 0$ such that $|\overline{F}(x,t)| \ge \varrho|t|^2$, $|t| < \delta$. So, we can get

$$\begin{aligned} \overline{I}(tu_0) &= \frac{t^2}{2}\bigg(\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_{\mathbb{R}^N}\frac{|u_0(x)|^2}{|x|^{2s}}\,dx\bigg) \\ &\quad - \frac{t^{2_s^*(b)}}{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|u_0^+|^{2_s^*(b)}}{|x|^b}\,dx - \lambda\int_{\mathbb{R}^N}\overline{F}(x,tu_0^+(x))\,dx \\ &\le \frac{t^2}{2}\,\|u_0\|^2 - \lambda\varrho t^2\|u_0^+\|^2_{L^2(\mathbb{R}^N)} = -\frac{t^2}{2}\,\|u_0\|^2 < 0 \end{aligned}$$

for any $0 < \lambda < \theta$ and $0 < t < \min\{\rho, \delta/\|u_0^+\|_\infty\}$. Then there exists $u \in H_0^s(\mathbb{R}^N)$ such that $\overline{I}(u) < 0$ and we have

$$\inf_{u\in\overline{B_\rho(0)}}\overline{I}(u) < 0 < \inf_{u\in\overline{\partial B_\rho(0)}}\overline{I}(u).$$

Set $1/n \in (0, \inf_{u\in\partial\overline{B_\rho(0)}} \overline{I}(u) - \inf_{u\in\overline{B_\rho(0)}} \overline{I}(u))$, $n \in \mathbb{N}$, by Ekeland's variational principle, there is a $\{u_n\} \subset \overline{B_\rho(0)}$, such that

$$\overline{I}(u_n) \le \inf_{u\in\overline{B_\rho(0)}} \overline{I}(u) + \frac{1}{n}, \qquad \overline{I}(u_n) \le \overline{I}(w) + \frac{1}{n}\,\|w-u_n\|, \tag{2.9}$$

for all $w \in \overline{B_\rho(0)}$. Also, one has

$$\overline{I}(u_n) \le \inf_{u\in\overline{B_\rho(0)}} \overline{I}(u) + \frac{1}{n} < \inf_{u\in\partial B_\rho(0)} \overline{I}(u).$$

So $\{u_n\} \subset B_\rho(0)$. Now, we define $\varphi_n\colon H_0^s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\varphi_n(u) := \overline{I}(u) + \frac{1}{n}\,\|w-u_n\|.$$

In view of (2.9), one can get $\{u_n\} \subset B_\rho(0)$ minimizes of $\varphi_n(u)$ on $\overline{B_\rho(0)}$. Hence, for all $v \in H_0^s(\mathbb{R}^N)$ with $\|v\| = 1$, take $h > 0$ such that $u_n + hv \in \overline{B_\rho(0)}$, so one has

$$\frac{\varphi_n(u_n+hv) - \varphi_n(u_n)}{h} \ge 0,$$

which this concludes

$$\frac{\overline{I}(u_n+hv) - \overline{I}(u_n)}{h} + \frac{1}{n} \ge 0.$$

So $\langle \overline{I}'(u_n), v\rangle \ge -1/n$, which implies that

$$\|\overline{I}'(u_n)\|_{(H_0^s(\mathbb{R}^N))^*} \le \frac{1}{n}. \tag{2.10}$$

Then, from (2.9) and (2.10), we obtain that

$$\|\overline{I}'(u_n)\|_{(H_0^s(\mathbb{R}^N))^*} \to 0, \quad \overline{I}(u_n) \to \inf_{u\in\overline{B_\rho(0)}} \overline{I}(u) \quad \text{as } n\to\infty. \tag{2.11}$$

Since $\{u_n\}$ is bounded, $\overline{B_\rho(0)}$ is a closed convex set and by Lemma 2.1 in [3], there exists $u_\lambda \in \overline{B_\rho(0)} \subset H_0^s(\mathbb{R}^N)$ such that, going if necessary to a subsequence,

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{weakly in } H_0^s(\mathbb{R}^N),\\ u_n \rightharpoonup u_\lambda & \text{weakly in } L^{2_s^*(b)}(\mathbb{R}^N, |x|^{-b}),\\ u_n \to u_\lambda & \text{a.e. in } \mathbb{R}^N. \end{cases}$$

So, by (F1), (F2), Vitali's theorem and

$$\frac{|u_n|^{2_s^*(b)-1}}{|x|^b} \rightharpoonup \frac{|u_\lambda|^{2_s^*(b)-1}}{|x|^b}$$

weakly in $L^{2_s^*(b)/(2_s^*(b)-1)}(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u_n(x))\,v(x)\,dx = \int_{\mathbb{R}^N} f(x,u_\lambda(x))v(x)\,dx.$$

Passing to limit in $\langle \overline{I}'(u_n), v\rangle$ and by Lemma 2.4 in [12], one can get

$$\iint_{\mathbb{R}^{2N}} \frac{u_\lambda(x) - u_\lambda(y)}{|x-y|^{N+2s}} (v(x) - v(y))\, dx\, dy - \gamma \int_{\mathbb{R}^N} \frac{u_\lambda(x)}{|x|^{2s}} v(x)\, dx \\ - \int_{\mathbb{R}^N} \frac{(u_\lambda^+(x))^{2_s^*(b)-1}}{|x|^b} v(x)\, dx - \lambda \int_{\mathbb{R}^N} \overline{f}(x, u_\lambda^+(x)) v(x)\, dx = 0$$

for all $v \in H_0^s(\mathbb{R}^N)$, and this implies that $\langle \overline{I}'(u_\lambda), v\rangle = 0$, So u_λ is a critical point of $\overline{I}$. Set $u_\lambda^- = \min\{u_\lambda, 0\}$. Then, u_λ is a nontrivial weak solution for the following problem

$$(2.12)\qquad (-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = \frac{|u^+|^{2_s^*(b)-2} u^+}{|x|^b} + \lambda \overline{f}(x, u^+), \quad \text{in } \mathbb{R}^N.$$

By using u_λ^- as a test function in (2.12) and integrating by parts, one has

$$\int_{\mathbb{R}^N} (-\Delta)^s u_\lambda(x) \cdot u_\lambda^-(x)\, dx = -\gamma \int_{\mathbb{R}^N} \frac{(u_\lambda^-(x))^2}{|x|^{2s}} \leq 0.$$

Also, one can get

$$\begin{aligned} &\int_{\mathbb{R}^N} (-\Delta)^s u_\lambda(x) \cdot u_\lambda^-(x)\, dx \\ &= \frac{C(N,s)}{2} \iint_{\mathbb{R}^{2N}} \frac{(u_\lambda(x) - u_\lambda(y))(u_\lambda^-(x) - u_\lambda^-(y))}{|x-y|^{N+2s}}\, dx\, dy \\ &= \frac{C(N,s)}{2} \iint_{\mathbb{R}^{2N}} \frac{-u_\lambda^+(x) u_\lambda^-(y) - u_\lambda^+(y) u_\lambda^-(x) + |u_\lambda^-(x) - u_\lambda^-(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \geq 0. \end{aligned}$$

So, $\gamma \int_{\mathbb{R}^N} (u_\lambda^-(x))^2/|x|^{2s} = 0$, and then $u_\lambda \geq 0$. By (F1) and (F2), there exist $\delta_1, \delta_2 > 0$ such that

$$(2.13)\qquad \begin{aligned} &|\overline{f}(x,t)| < \frac{1}{\lambda} \frac{|t|^{2_s^*(b)-1}}{|x|^b} && \text{for } |x| > \delta_1, \\ &\overline{f}(x,t) > 0 && \text{for } x \in \mathbb{R}^N,\ 0 < t < \delta_2. \end{aligned}$$

Also, by continuity of f, there exist $c_2 > 0$ and $\delta_3 > \delta_2 > 0$ such that

$$(2.14)\qquad |\overline{f}(x,t)| < c_2 \quad \text{for } |x| \leq \delta_1,\ t \in [\delta_2, \delta_3].$$

So, in view of (2.13) and (2.14), we have

$$(2.15)\qquad \overline{f}(x,t) \geq -\frac{1}{\lambda} \frac{t^{2_s^*(b)-1}}{|x|^b} - c_2 t \delta_2^{-1} \quad \text{for } x \in \mathbb{R}^N,\ t \in \mathbb{R}^+.$$

Hence, from (1.1) and (2.15), we only can get

$$(-\Delta)^s u_\lambda - \gamma \frac{u_\lambda}{|x|^{2s}} + \lambda c_2 \delta_2^{-1} u_\lambda \geq 0,$$

here γ is a real parameter. Therefore, from the strong maximum principle (see [2, Remark 4.14.]), we can easily get $u_\lambda > 0$. $\square$

We will use the following version of the Mountain Pass theorem to prove Theorem 1.2.

THEOREM 2.2 ([4], [5]). *Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$ satisfies the* $(\mathrm{C})_c$ *condition for any $c \in \mathbb{R}$, $I(0) = 0$ and*

(a) *There exist $\beta, \rho > 0$ such that $I(u) \geq \beta$ for all $\|u\| = \rho$.*
(b) *There is an $u_1 \in X$ with $\|u_1\| > \rho$ such that $I(u_1) \leq 0$.*

Then $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)) \geq \beta$ is a critical value of I, where

$$\Gamma = \big\{\gamma \in C^0([0,1], X) : \gamma(0) = 0, \gamma(1) = u_1\big\}.$$

LEMMA 2.3. *Assume that* (F3) *and* (F5) *hold. Then the energy functional I satisfies the following conditions:*

(a) *There exist $\beta, \rho > 0$ such that $I(u) \geq \beta$ for all $\|u\| = \rho$.*
(b) *There exist an $e_0 \in H^s_0(\mathbb{R}^N)$ with $\|e_0\| > \rho$ such that $I(e_0) < 0$.*

PROOF. (a) In view of (F5), there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^{2^*_s(b)-1}, \quad \text{for all } t \in \mathbb{R}.$$

So, for $u \in H^s_0(\mathbb{R}^N)$ with $\|u\| = \rho$, Lemma 2.1 in [3], Hölder inequality, (2.4) and the continuity of the embeddeding $H^s_0(\mathbb{R}^N)$ in $L^{2^*_s(b)}(\mathbb{R}^N, |x|^{-b})$ implies that

$$\begin{aligned}
I(u) \geq{}& \frac{1}{2}\bigg(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\,dx\bigg) \\
&- \frac{1}{2^*_s(b)}\int_{\mathbb{R}^N} \frac{|u^+|^{2^*_s(b)}}{|x|^b}\,dx - \lambda\int_{\mathbb{R}^N} F(x,u(x))\,dx \\
\geq{}& \frac{1}{2}\bigg(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\,dx\bigg) \\
&- \frac{1}{2^*_s(b)}\int_{\mathbb{R}^N} \frac{|u^+|^{2^*_s(b)}}{|x|^b}\,dx - \lambda\int_{\mathbb{R}^N} K(x)(\varepsilon|u|^2 + C_\varepsilon|u|^{2^*_s(b)})\,dx \\
\geq{}& \frac{1}{2}\bigg(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\,dx\bigg) \\
&- \frac{1}{2^*_s(b)}\int_{\mathbb{R}^N} \frac{|u^+|^{2^*_s(b)}}{|x|^b}\,dx \\
&- \lambda\varepsilon\bigg(\int_{\mathbb{R}^N} (K(x))^{2^*_s/(2^*_s-2)}\bigg)^{2^*_s/(2^*_s-2)}\bigg(\int_{\mathbb{R}^N} |u|^{2^*_s}\,dx\bigg)^{2/2^*_s} \\
&- \lambda C_\varepsilon\int_{\mathbb{R}^N} K(x)|u|^{2^*_s(b)}\,dx \\
\geq{}& \frac{1}{2}\rho^2 - \frac{1}{2^*_s(b)}(S(N,s,\gamma,b))^{-2^*_s(b)/2}\rho^{2^*_s(b)}
\end{aligned}$$

$$- \lambda\varepsilon\|K\|_{2_s^*/(2_s^*-2)}(S(N,s,\gamma,0))^{-1}\bigg(\frac{\gamma_1}{\gamma_1-\lambda_+}\bigg)\rho^2$$
$$- \lambda C_\varepsilon\|K\|_\infty(S(N,s,\gamma,b))^{-2_s^*(b)/2}\rho^{2_s^*(b)} := \beta > 0$$

for small $\varepsilon > 0$ and $\rho > 0$.

(b) From (F5), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon|t|^{2_s^*(b)-1} + C_\varepsilon|t|, \quad \text{for all } t \in \mathbb{R}. \tag{2.16}$$

Then

$$I(t\overline{u}) \le \frac{t^2}{2}\,\|\overline{u}\|^2 - \frac{t^{2_s^*(b)}}{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|\overline{u}^+|^{2_s^*(b)}}{|x|^b}\,dx$$
$$+ \lambda C\varepsilon\|K\|_\infty t^{2_s^*(b)}\int_{\mathbb{R}^N}|\overline{u}(x)|^{2_s^*(b)}\,dx + \lambda C_\varepsilon\|K\|_\infty t^2\|\overline{u}\|^2 < 0$$

for large t and small ε. So, we can take $e_0 := \overline{t}\overline{u}$ for some large $\overline{t}$. Therefore (b) holds. $\square$

Also, to prove the main result (Theorem 1.2), we need the following definitions, lemmas and theorems.

Lemma 2.4. *Assume that* (F3)–(F7) *hold. Then the I satisfies the* $(\mathrm{C})_{c_\lambda}$ *condition for all $c_\lambda \in \mathbb{R}$.*

Proof. Let $\{u_n\}$ be a sequence in $H_0^s(\mathbb{R}^N)$ such that

$$I(u_n) \to c_\lambda \quad\text{and}\quad (1+\|u_n\|)\|I'(u_n)\|_{(H_0^s(\mathbb{R}^N))^*} \to 0,$$

this is equivalent to

$$(1+\|u_n\|)\sup_{\|\varphi\|}\langle I'(u_n),\varphi\rangle \to 0, \tag{2.17}$$

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|u_n|^{2_s^*(b)}}{|x|^b}\,dx - \lambda\int_{\mathbb{R}^N}F(x,u_n(x))\,dx \to c_\lambda \tag{2.18}$$

as $n \to \infty$.

Let $t_n \in [0,1]$ be such that

$$I(t_nu_n) = \max_{t\in[0,1]} I(tu_n). \tag{2.19}$$

Since $I(0) = 0$, we can assume that $t_n \in (0,1)$ and by (2.19), one can get

$$\frac{d}{dt}\,I(tu_n)|_{t=t_n} = 0.$$

Consequently,

$$\langle I'(t_nu_n), t_nu_n\rangle = 0. \tag{2.20}$$

So, in view of (2.20) and (F6), we have

$$\begin{aligned}
I(t_nu_n) &= I(t_nu_n) - \frac{1}{2}\langle I'(t_nu_n), t_nu_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{2_s^*(b)}\right) t_n^{2_s^*(b)} \int_{\mathbb{R}^N} \frac{|u_n|^{2_s^*(b)}}{|x|^b}\,dx \\
&\quad + \lambda \int_{\mathbb{R}^N} K(x)\left[\frac{1}{2}\,t_nu_nf(t_nu_n) - F(t_nu_n)\right]dx \\
&\leq \left(\frac{1}{2} - \frac{1}{2_s^*(b)}\right) \int_{\mathbb{R}^N} \frac{|u_n|^{2_s^*(b)}}{|x|^b}\,dx + \lambda \int_{\mathbb{R}^N} K(x)\left[\frac{1}{2}\,u_nf(u_n) - F(u_n)\right]dx \\
&= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle = c_\lambda + o_n(1).
\end{aligned}$$

Therefore, $\{I(t_nu_n)\}$ is bounded from above.

Now, we prove that $\{u_n\}$ is bounded in $H_0^s(\mathbb{R}^N)$. By a contradiction, we assume that $\|u_n\| \to \infty$. Set $v_n = u_n/\|u_n\|$. By Lemma 2.1 in [3], their exists $v \in H_0^s(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $H_0^s(\mathbb{R}^N)$. From $I(u_n) \to c$, one has

$$o_n(1) + \frac{1}{2} = \frac{1}{2_s^*(b)} \frac{\displaystyle\int_{\mathbb{R}^N} \frac{|u_n|^{2_s^*(b)}}{|x|^b}}{\|u_n\|^2}\,dx + \lambda \int_{\mathbb{R}^N} \frac{K(x)F(u_n)}{\|u_n\|^2}\,dx.$$

Let $\Omega := \{x \in \mathbb{R}^N : v(x) \neq 0\}$. If $|\Omega| > 0$, and by the definition of v_n, we get

$$|u_n(x)| \to +\infty, \quad \text{for a.e. } x \in \Omega,$$

then from (F7) and Fatous's Lemma, we have

$$\begin{aligned}
o_n(1) + \frac{1}{2} &= \frac{1}{2_s^*(b)} \frac{\displaystyle\int_{\mathbb{R}^N} \frac{|u_n|^{2_s^*(b)}}{|x|^b}}{\|u_n\|^2}\,dx + \lambda \int_{\mathbb{R}^N} \frac{K(x)F(u_n)}{\|u_n\|^2}\,dx \\
&\geq \frac{1}{2_s^*(b)} \int_\Omega v_n^2 \frac{|u_n|^{2_s^*(b)-2}}{|x|^b}\,dx \to +\infty,
\end{aligned}$$

as $n \to \infty$, which is a contradiction. Then, $|\Omega| = 0$ and so $v = 0$ almost everywhere on $\mathbb{R}^N$.

For any $R > 0$, since $R/\|u_n\| \in [0,1]$ for n large enough, then we have

$$\begin{aligned}
I(t_nu_n) &\geq I\left(\frac{R}{\|u_n\|}\,u_n\right) = I(Rv_n) \\
&= \frac{R^2}{2} - \frac{R^{2_s^*(b)}}{2_s^*(b)} \int_{\mathbb{R}^N} \frac{|v_n|^{2_s^*(b)}}{|x|^b}\,dx - \lambda \int_{\mathbb{R}^N} K(x)F(Rv_n)\,dx.
\end{aligned} \tag{2.21}$$

Also, by Lemma 2.1 in [3] and the conditions (F4) and (F5), as Lemma 2.2 in [1], we can prove

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)F(Rv_n)\,dx = \int_{\mathbb{R}^N} K(x)F(Rv)\,dx = 0. \tag{2.22}$$

Moreover, in view of (2.4), (2.6), (2.16) and Hölder inequality, we have

$$\begin{aligned}
\bigg|\frac{1}{\|u_n\|^{2_s^*(b)}}\int_{\mathbb{R}^N} K(x)f(u_n)u_n\,dx\bigg|
&\leq \frac{1}{\|u_n\|^{2_s^*(b)}}\int_{\mathbb{R}^N} K(x)\big(\varepsilon|u_n|^{2_s^*(b)}+C_\varepsilon|u_n|^2\big)\,dx\\
&\leq \frac{1}{\|u_n\|^{2_s^*(b)}}\bigg[\varepsilon\int_{\mathbb{R}^N} K(x)|u_n|^{2_s^*(b)}\,dx\\
&\quad + C_\varepsilon\bigg(\int_{\mathbb{R}^N}(K(x))^{2_s^*/(2_s^*-2)}\bigg)^{2_s^*/(2_s^*-2)}\bigg(\int_{\mathbb{R}^N}|u_n|^{2_s^*}\,dx\bigg)^{2/2_s^*}\bigg]\\
&\leq \frac{1}{\|u_n\|^{2_s^*(b)}}\bigg[\varepsilon\|K\|_\infty(S(N,s,\gamma,b))^{-2_s^*(b)/2}\|u_n\|^{2_s^*(b)}\\
&\quad + C_\varepsilon\|K\|_{2_s^*/(2_s^*-2)}(S(N,s,\gamma,0))^{-1}\bigg(\frac{\gamma_1}{\gamma_1-\lambda_+}\bigg)\|u_n\|^2\bigg]\\
&= \varepsilon\|K\|_\infty(S(N,s,\gamma,b))^{-2_s^*(b)/2}\\
&\quad + C_\varepsilon\|K\|_{2_s^*/(2_s^*-2)}(S(N,s,\gamma,0))^{-1}\bigg(\frac{\gamma_1}{\gamma_1-\lambda_+}\bigg)\frac{1}{\|u_n\|^{2_s^*(b)-2}},
\end{aligned}$$

consequently

$$\lim_{n\to\infty}\frac{1}{\|u_n\|^{2_s^*(b)}}\int_{\mathbb{R}^N} K(x)f(u_n)u_n\,dx = 0. \tag{2.23}$$

Since $\langle I'(u_n),u_n\rangle = o_n(1)$, then (2.23) implies that

$$\begin{aligned}
\int_{\mathbb{R}^N}\frac{|v_n|^{2_s^*(b)}}{|x|^b}\,dx = {}& \frac{1}{\|u_n\|^{2_s^*(b)-2}}\\
&- \frac{\lambda}{\|u_n\|^{2_s^*(b)}}\int_{\mathbb{R}^N} K(x)f(u_n)u_n\,dx + o_n(1)\to 0
\end{aligned}\tag{2.24}$$

as $n\to\infty$. Hence, (2.21), (2.22) and (2.24) imply that

$$\lim_{n\to\infty} I(t_nu_n)\geq \frac{R^2}{2},\quad \text{for all } R>0,$$

which contradicts the fact $\{I(t_nu_n)\}$ is bounded from above. So, $\{u_n\}$ is bounded in $H_0^s(\mathbb{R}^N)$. Therefore, by very similar method in Lemma 2.4 of [3], we can complete the proof. □

In the following, we will estimate the mountain pass level c_λ.

Lemma 2.5. *Assume that* (F5) *and* (F7) *hold. Then, there exists* $\lambda_*>0$ *such that*

$$0<c_\lambda<\frac{2s-b}{2(N-b)}(S(N,s,\gamma,b))^{(N-b)/(2s-b)}\quad \textit{for all } \lambda>\lambda_*.$$

PROOF. By a contradiction argument, assume that the conclusion is false, then there exists a sequence $\{\lambda_n\}$ with $\lambda_n \to +\infty$ such that

$$c_{\lambda_n} \geq \frac{2s-b}{2(N-b)}(S(N,s,\gamma,b))^{(N-b)/(2s-b)}.$$

Choose $v_0 \in H_0^s(\mathbb{R}^N) \setminus \{0\}$. Set $I = I_{\lambda_n}$, so from (F5) and (F7), there exists a unique $t_{\lambda_n} > 0$ such that $\max_{t\geq 0} I_{\lambda_n}(tv_0) = I_{\lambda_n}(t_{\lambda_n}v_0)$. So

$$t_{\lambda_n}^2\|v_0\|^2 = t_{\lambda_n}^{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|v_0|^{2_s^*(b)}}{|x|^b}\,dx + \lambda_n\int_{\mathbb{R}^N}K(x)f(t_{\lambda_n}v_0)t_{\lambda_n}v_0\,dx.$$

In view of (F7), we have

$$\|v_0\|^2 \geq t_{\lambda_n}^{2_s^*(b)-2}\int_{\mathbb{R}^N}\frac{|v_0|^{2_s^*(b)}}{|x|^b}\,dx,$$

that implies that t_{λ_n} is bounded. Then, there exists a subsequence t_{λ_n} and $\widehat{t} \geq 0$ such that $t_{\lambda_n} \to \widehat{t}$ as $n \to \infty$. We claim that $\widehat{t} = 0$. To this end, we assume that $\overline{t} > 0$, so by (F7) and Fatou's Lemma, one has

$$\lim_{n\to\infty}\left[t_{\lambda_n}^{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|v_0|^{2_s^*(b)}}{|x|^b}\,dx + \lambda_n\int_{\mathbb{R}^N}K(x)f(t_{\lambda_n}v_0)t_{\lambda_n}v_0\,dx\right] = +\infty.$$

But

$$t_{\lambda_n}^{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|v_0|^{2_s^*(b)}}{|x|^b}\,dx + \lambda_n\int_{\mathbb{R}^N}K(x)f(t_{\lambda_n}v_0)\,t_{\lambda_n}v_0\,dx = t_{\lambda_n}\|v_0\|^2 \to \widehat{t}^2\|v_0\|^2$$

as $n \to \infty$, which is a contradiction and so $\widehat{t} = 0$. Choose $v_1 = t_{\lambda_n}v_0$. Hence,

$$I_{\lambda_n}(v_1) = I_{\lambda_n}(t_{\lambda_n}v_0) = \max_{t\geq 0} I(tv_0).$$

Consequently, by (F7) we get

$$\max_{t\geq 0} I(tv_0) = I_{\lambda_n}(t_{\lambda_n}v_0) \leq \frac{1}{2}\,t_{\lambda_n}^2\|v_0\|^2 - \frac{1}{2_s^*(b)}\,t_{\lambda_n}^{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|v_0|^{2_s^*(b)}}{|x|^b}\,dx \to 0$$

as $n \to \infty$. Therefore,

$$\begin{aligned}0 &< \frac{2s-b}{2(N-b)}(S(N,s,\gamma,b))^{(N-b)/(2s-b)}\\ &\leq c_{\lambda_n} \leq \inf_{u\in H_0^s(\mathbb{R}^N)\setminus\{0\}}\sup_{t\geq 0} I(tu) \leq \sup_{t\geq 0} I_{\lambda_n}(tv_0) \to 0\end{aligned}$$

as $n \to \infty$, which it is a contradiction. □

PROOF OF THEOREM 1.2. By applying the Theorem 2.2 and combining Lemmas 2.3 and 2.4, we deduce that problem (1.1) has a nontrivial weak solution. We now set $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. Replacing I with the functional

$$I^+(u) = \frac{1}{2}\bigg(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}\,dx\bigg) - \frac{1}{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|u^+|^{2_s^*(b)}}{|x|^b}\,dx - \lambda\int_{\mathbb{R}^N} K(x)F(u^+(x))\,dx.$$

By the above argument, we can get a nontrivial weak solution u for the following problem

$$(2.25)\qquad (-\Delta)^s u + \gamma\,\frac{u}{|x|^{2s}} = \frac{|u^+|^{2_s^*(b)-2}u^+}{|x|^b} + \lambda K(x)f(u^+), \quad \text{in } \mathbb{R}^N.$$

By using u^- as a test function in (2.25) and integrating by parts, one has

$$\int_{\mathbb{R}^N} (-\Delta)^s u(x)\cdot u^-(x)\,dx = -\gamma\int_{\mathbb{R}^N}\frac{(u^-(x))^2}{|x|^{2s}} \le 0.$$

Also, one can get

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^s u(x)\cdot u^-(x)\,dx \\
&\quad = \frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad = \frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}} \frac{-u^+(x)u^-(y)-u^+(y)u^-(x)+|u^-(x)-u^-(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \ge 0.
\end{aligned}$$

So, $\gamma\int_{\mathbb{R}^N}(u^-(x))^2/|x|^{2s} = 0$, and then $u\ge 0$. By Lemmas 2.4 and 2.5, we note that

$$\begin{aligned}
c_\lambda + o_n(1) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2_s^*(b)}\int_{\mathbb{R}^N}\frac{|u_n|^{2_s^*(b)}}{|x|^b}\,dx - \lambda\int_{\mathbb{R}^N} K(x)F(u_n(x))\,dx,\\
o_n(1) &= \|u_n\|^2 - \int_{\mathbb{R}^N}\frac{|u_n|^{2_s^*(b)}}{|x|^b}\,dx - \lambda\int_{\mathbb{R}^N} K(x)f(u_n(x))u_n(x)\,dx.
\end{aligned}$$

If $u=0$, then by similar argument of (2.22), one can get

$$(2.26)\qquad c_\lambda + o_n(1) = \bigg(\frac{1}{2}-\frac{1}{2_s^*(b)}\bigg)\|u_n\|^2.$$

Also, in view of definition of $S(N,s,\gamma,b)$ and (2.26), we have

$$\begin{aligned}
\|u_n\|^2 &\ge S(N,s,\gamma,b)\bigg(\int_{\mathbb{R}^N}|u(x)|^{2_s^*}|x|^{-b}\,dx\bigg)^{2/2_s^*(b)}\\
&= S(N,s,\gamma,b)\big(\|u_n\|^2+o_n(1)\big)^{2/2_s^*(b)} = S(N,s,\gamma,b)\|u_n\|^{4/2_s^*(b)} + o_n(1).
\end{aligned}$$

Since $c_\lambda > 0$, then $\|u_n\|^{2-(4/2_s^*(b))} \ge S(N,s,\gamma,b)+o_n(1)$, so,

$$\|u_n\|^2 \ge (S(N,s,\gamma,b)+o_n(1))^{(N-b)/(2s-b)} = (S(N,s,\gamma,b))^{(N-b)/(2s-b)} + o_n(1).$$

Therefore,

$$c_\lambda = \lim_{n\to\infty}\bigg(\frac{1}{2}-\frac{1}{2_s^*(b)}\bigg)\|u_n\|^2 \ge \frac{2s-b}{2(N-b)}(S(N,s,\gamma,b))^{(N-b)/(2s-b)},$$

which is a contradiction and so $u \neq 0$. Therefore, by the maximum principle we know that $u > 0$. □

Acknowledgments. The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

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Manuscript received January 18, 2018

accepted May 25, 2018

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